Optimization
A Journal of Mathematical Programming and Operations Research

ISSN: 0233-1934 (Print) 1029-4945 (Online) Journal homepage: www.tandfonline.com/journals/gopt20

Lipschitz upper semicontinuity of linear inequality systems under full perturbations

J. Camacho, M.J. Cánovas, H. Gfrerer & J. Parra

To cite this article: J. Camacho, M.J. Cánovas, H. Gfrerer & J. Parra (03 Feb 2026): Lipschitz upper semicontinuity of linear inequality systems under full perturbations, Optimization, DOI: 10.1080/02331934.2026.2615775

To link to this article: https://doi.org/10.1080/02331934.2026.2615775

© 2026 The Author(s). Published by Informa UK Limited, trading as Taylor & Francis Group.

Published online: 03 Feb 2026.

Submit your article to this journal

Article views: 182

View related articles

View Crossmark data

Full Terms & Conditions of access and use can be found at
https://www.tandfonline.com/action/journalInformation?journalCode=gopt20

OPTIMIZATION
https://doi.org/10.1080/02331934.2026.2615775

Taylor & Francis
Taylor & Francis Group

OPEN ACCESS Check for updates

Lipschitz upper semicontinuity of linear inequality systems under full perturbations

J. Camacho [a], M.J. Cánovas [a], H. Gfrerer [b,c] and J. Parra [a]

[a]Center of Operations Research, Miguel Hernández University of Elche, Elche (Alicante), Spain; [b]Johann Radon Institute for Computational and Applied Mathematics (RICAM), Linz, Austria; [c]Institute of Information Theory and Automation, Czech Academy of Sciences, Prague, Czech Republic

ABSTRACT

The present paper is focused on the computation of the Lipschitz upper semicontinuity modulus of the feasible set mapping in the context of fully perturbed linear inequality systems, i.e. where all coefficients are allowed to be perturbed. The direct antecedent comes from the framework of right-hand side (RHS, for short) perturbations. The difference between both parametric contexts, full versus RHS perturbations, is emphasized. In particular, the polyhedral structure of the graph of the feasible set mapping in the latter framework enables us to apply classical results as those of Hoffman [On approximate solutions of systems of linear inequalities. J Res Natl Bur Stand. 1952;49:263–265] and Robinson [Some continuity properties of polyhedral multifunctions. Math Progr Study. 1981;14:206–214]. In contrast, the graph of the feasible set mapping under full perturbations is no longer polyhedral (not even convex). This fact requires *ad hoc* techniques to analyse the Lipschitz upper semicontinuity property and its corresponding modulus.

ARTICLE HISTORY
Received 5 May 2025
Accepted 3 January 2026

KEYWORDS
Calmness constants; Lipschitz upper semicontinuity; linear inequality systems; feasible set mapping

2020 MATHEMATICS SUBJECT CLASSIFICATIONS
90C31; 49J53; 49K40; 90C05

1. Introduction

This paper is focused on the upper Lipschitzian behaviour of the feasible set associated with a parameterized linear inequality system written in the form

$$\sigma\,(A,b) := \left\{a_t' x \leq b_t,\ t \in T := \{1,\ldots,m\}\right\}, \tag{1}$$

where $x \in \mathbb{R}^n$ is the vector of variables, $(A,b) \in \mathbb{R}^{m\times n} \times \mathbb{R}^m$ is the parameter to be perturbed, a_t' represents the tth row of matrix A and b_t is the tth coordinate of vector b. Vectors in $\mathbb{R}^n$ are understood as column vectors and the prime represents transposition. Associated with the parameterized family of systems (1), we consider the feasible set mapping $\mathcal{F} : \mathbb{R}^{m\times n} \times \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ given by

$$\mathcal{F}\,(A,b) := \left\{x \in \mathbb{R}^n \mid Ax \leq b\right\}. \tag{2}$$

CONTACT J. Parra parra@umh.es
Dedicated to Michel Théra on the occasion of his 80th birthday

© 2026 The Author(s). Published by Informa UK Limited, trading as Taylor & Francis Group.
This is an Open Access article distributed under the terms of the Creative Commons Attribution License (http://creativecommons.org/licenses/by/4.0/), which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited. The terms on which this article has been published allow the posting of the Accepted Manuscript in a repository by the author(s) or with their consent.

Given a fixed $\bar{A} \in \mathbb{R}^{m\times n}$, we also consider the multifunction $\mathcal{F}_{\bar{A}} : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ defined as

$$\mathcal{F}_{\bar{A}}(b) := \mathcal{F}\left(\bar{A}, b\right), \quad \text{for all } b \in \mathbb{R}^m.$$

In this way, $\mathcal{F}_{\bar{A}}$ is set in the framework of right-hand side (RHS, in brief) perturbations, while $\mathcal{F}$ formalizes the variation of the feasible set under full perturbations.

The main goal of this paper is to analyse the Lipschitz upper semicontinuity property of multifunction $\mathcal{F}$ at a nominal (given) element $(\bar{A}, \bar{b}) \in \text{dom}\mathcal{F}$ (where 'dom' stands for domain, i.e. $(\bar{A}, \bar{b}) \in \text{dom}\mathcal{F} \Leftrightarrow \mathcal{F}(\bar{A}, \bar{b}) \neq \emptyset$) as well as to compute the corresponding modulus of $\mathcal{F}$ at $(\bar{A}, \bar{b})$, denoted by $\text{Lipusc}\,\mathcal{F}(\bar{A}, \bar{b})$. We are appealing to the terminology of Klatte and Kummer [3] or Uderzo [4], although this property was introduced by Robinson [2] under the name of upper Lipschitz continuity and can be found in the monograph of Dontchev and Rockafellar [5] as outer Lipschitz continuity. The immediate antecedent to the present work can be traced out from [6], where a formula for computing the Lipschitz upper semicontinuity modulus of $\mathcal{F}_{\bar{A}}$ at a nominal $\bar{b} \in \text{dom}\mathcal{F}_{\bar{A}}$, $\text{Lipusc}\,\mathcal{F}_{\bar{A}}(\bar{b})$, is provided. Roughly speaking, these moduli constitute the 'tightest' measures of the expansion of feasible sets with respect to data perturbations, around a nominal $(\bar{A}, \bar{b}) \in \text{dom}\mathcal{F}$ or $\bar{b} \in \text{dom}\mathcal{F}_{\bar{A}}$, depending on the context; see the following paragraphs for some preliminary details and Section 2 for the definitions and first results about generic multifunctions between normed linear spaces.

At this moment, we point out the fact that there are notable differences between both contexts of RHS and full perturbations. To start with, $\text{gph}\mathcal{F}_{\bar{A}}$ (where 'gph' means graph) is a convex polyhedral cone as the solution set of system '$-b + Ax \leq 0_m$' in the variable $(b, x) \in \mathbb{R}^m \times \mathbb{R}^n$. Hence appealing to a classical result by Robinson [2] (which applies for mappings whose graphs are finite union of convex polyhedra), $\mathcal{F}_{\bar{A}}$ is Lipschitz upper semicontinuous at any element $\bar{b} \in \text{dom}\mathcal{F}_{\bar{A}}$, which reads as the existence of a constant $\kappa \geq 0$ along with a neighbourhood $V \subset \mathbb{R}^m$ of $\bar{b}$ such that

$$\text{d}\left(x, \mathcal{F}_{\bar{A}}\left(\bar{b}\right)\right) \leq \kappa \left\|b - \bar{b}\right\| \quad \text{for all } b \in V \text{ and all } x \in \mathcal{F}_{\bar{A}}(b), \tag{3}$$

where 'd' denotes the distance (associated to the norm) in the space of variables and $\|\cdot\|$ the norm in the space of parameters. By definition, $\text{Lipusc}\,\mathcal{F}_{\bar{A}}(\bar{b})$ is the infimum of all constants $\kappa \geq 0$ verifying (3). Hence $\text{Lipusc}\,\mathcal{F}_{\bar{A}}(\bar{b})$ is finite for all $\bar{b} \in \text{dom}\mathcal{F}_{\bar{A}}$. However, $\text{gph}\mathcal{F}$ is no longer convex (and not a cone) which leads to a different scenario. Indeed, we advance that $\text{Lipusc}\,\mathcal{F}(\bar{A}, \bar{b})$ is finite if and only if one of the following conditions holds: (*i*) $\mathcal{F}(\bar{A}, \bar{b})$ is bounded, (*ii*) $\mathcal{F}(\bar{A}, \bar{b}) = \mathbb{R}^n$ or (*iii*) $n = 1$ (see Proposition 3.2 and Theorem 3.2 below).

Looking at (3), the Lipschitz upper semicontinuity property can be seen as a semilocal Lipschitzian property in the sense that it involves the whole image set (all points $x \in \mathcal{F}_{\bar{A}}(b)$) but only parameters around the nominal $\bar{b}$ (recall $b \in V$). For completeness, and with the aim of emphasizing the difference between the contexts of RHS and full perturbations, let us comment what happens with the so-called Hoffman constant. The celebrated Hoffman lemma [1] entails the existence of $\kappa \geq 0$ such that

$$\text{d}\left(x_1, \mathcal{F}_{\bar{A}}(b_2)\right) \leq \kappa \|b_1 - b_2\| \quad \text{for all } b_1, b_2 \in \text{dom}\mathcal{F}_{\bar{A}} \text{ and all } x_1 \in \mathcal{F}_{\bar{A}}(b_1). \tag{4}$$

This last property, which is known as the Lipschitz continuity of $\mathcal{F}_{\bar{A}}$ on its domain, is obviously stronger than the Lipschitz upper semicontinuity. Observe that condition (4)

provides a global (instead of semilocal) Lipschitzian property as it involves all parameters in the whole $\mathrm{dom}\mathcal{F}_{\bar{A}}$ (instead of parameters around a fixed $\bar{b}$). The (sharp) Hoffman constant of $\mathcal{F}_{\bar{A}}$, denoted by $\mathrm{Hof}\mathcal{F}_{\bar{A}}$, is the infimum of all $\kappa \geq 0$ verifying (4). Constant $\mathrm{Hof}\mathcal{F}_{\bar{A}}$ is a relevant measure of the stability of $\mathcal{F}_{\bar{A}}$ and different authors have provided computable expressions for it; let us cite Azé and Corvellec [7, Theorem 3.3], Burke and Tseng [8, Theorem 8], Klatte and Thiere [9, Theorem 2.7], Li [10], Peña et al. [11, Formula (3)] and Zălinescu [12], among others. However, the counterpart of (4) in the context of full perturbations is no longer true and the Hoffman constant of $\mathcal{F}$ is infinite. In fact, in the case $n \geq 2$, just taking any $(\bar{A}, \bar{b}) \in \mathrm{dom}\mathcal{F}$ such that $\mathcal{F}(\bar{A}, \bar{b})$ is unbounded and different from $\mathbb{R}^n$, it is clear from the definitions that this Hoffman constant is greater than or equal to $\mathrm{Lipusc}\,\mathcal{F}(\bar{A}, \bar{b}) = +\infty$. In the case $n = 1$, for the system consisting of the single inequality $x \leq \alpha$, letting $\alpha \to +\infty$, Proposition 3.2 below also ensures that the Hoffman constant of $\mathcal{F}$ is infinite in this case.

Apart from the announced differences between the stability of $\mathcal{F}$ and $\mathcal{F}_{\bar{A}}$, let us advance that, via ad-hoc techniques, this paper brings to light a remarkable similarity. Specifically, Theorem 3.2, together with Corollary 1, provides a point-based formula (based on the nominal data) for $\mathrm{Lipusc}\,\mathcal{F}(\bar{A}, \bar{b})$ when $\mathcal{F}(\bar{A}, \bar{b})$ is bounded, which reminds of the one given in [6, Theorem 5] for $\mathrm{Lipusc}\,\mathcal{F}_{\bar{A}}(\bar{b})$ (recalled here in Theorem 2.2). In both cases, the Lipschitz upper semicontinuity moduli are expressed in terms of the supremum of certain local stability measures (called calmness moduli) evaluated in the set of extreme points of the nominal feasible set.

Regarding the structure of this paper, Section 2 recalls the necessary concepts and previous results which are appealed to in the current paper. Section 3 contains the main results of this work, devoted to the computation of $\mathrm{Lipusc}\,\mathcal{F}(\bar{A}, \bar{b})$, with $(\bar{A}, \bar{b}) \in \mathrm{dom}\mathcal{F}$, by distinguishing two cases focused on the boundedness or not of $\mathcal{F}(\bar{A}, \bar{b})$. Finally, Section 4 presents some conclusions, illustrations and comments on further research.

2. Preliminaries

Let us start by fixing the notation and introducing the main definitions used throughout the paper. Given $X \subset \mathbb{R}^p$, $p \in \mathbb{N}$, we denote by $\mathrm{int}X$, $\mathrm{ri}X$, $\mathrm{conv}X$ and $\mathrm{span}X$ the interior, the relative interior, the convex hull and the linear hull of X respectively. Hereafter, the convention $\mathrm{conv}\emptyset = \emptyset$ is adopted. Moreover, if X is convex, its set of extreme points is denoted by $\mathrm{extr}X$.

We appeal to several Lipschitz-type properties that are recalled now for a generic multifunction between normed linear spaces (in fact, these properties can be stated in the more general context of metric spaces); the reader is addressed to the monographs [3, 5, 13, 14] for extra information about Lipschitz-type properties among other topics of variational analysis. Let $\mathcal{M} : Y \rightrightarrows X$ be a set-valued mapping between normed linear spaces with both norms denoted by $\|\cdot\|$ indistinctly. This paper focuses on the Lipschitz upper semicontinuity property of $\mathcal{M}$ at $\bar{y} \in \mathrm{dom}\,\mathcal{M}$, which is defined as the existence of a neighbourhood V of $\bar{y}$ together with a constant $\kappa \geq 0$ such that

$$\mathrm{d}\big(x, \mathcal{M}\left(\bar{y}\right)\big) \leq \kappa \left\|y - \bar{y}\right\| \quad \text{for all } y \in V \text{ and all } x \in \mathcal{M}\left(y\right), \tag{5}$$

where $\mathrm{d}(x,\Omega) := \inf\{\|x-\omega\| \mid \omega \in \Omega\}$, for $x \in X$ and $\Omega \subset X$. For convenience we assume $\inf \emptyset := +\infty$; in particular, $\mathrm{d}(x,\emptyset) = +\infty$. As commented in the previous section, the Lipschitz upper semicontinuity property is considered to be a semilocal measure in the sense that only y's are required to be around the nominal value, while x may vary in the whole image set $\mathcal{M}(y)$.

Concerning local measures, the calmness property holds for $\mathcal{M}$ at $(\bar{y},\bar{x}) \in \mathrm{gph}\mathcal{M}$ if there exist a constant $\kappa \geq 0$ and a neighbourhood of $(\bar{y},\bar{x})$, $V \times U \subset Y \times X$, such that

$$\mathrm{d}\big(x, \mathcal{M}(\bar{y})\big) \leq \kappa \left\|y-\bar{y}\right\| \quad \text{for all } x \in \mathcal{M}(y) \cap U \text{ and all } y \in V. \tag{6}$$

It can be equivalently read in terms of the inverse function, $\mathcal{M}^{-1}$, i.e. $y \in \mathcal{M}^{-1}(x) \iff x \in \mathcal{M}(y)$, appealing to the well-known metric subregularity property (cf. [5, Theorem 3H.3 and Exercise 3H.4]) of $\mathcal{M}^{-1}$ at $(\bar{x},\bar{y})$. Namely, the existence of $\kappa \geq 0$ and a (possibly smaller) neighbourhood U of $\bar{x}$ such that

$$\mathrm{d}\big(x, \mathcal{M}(\bar{y})\big) \leq \kappa\, \mathrm{d}\big(\bar{y}, \mathcal{M}^{-1}(x)\big) \quad \text{for all } x \in U. \tag{7}$$

The infimum of constants κ appearing in (5) and (6), or equivalently (7), for some associated neighbourhoods, are the corresponding moduli of those properties. Specifically, $\mathrm{Lipusc}\,\mathcal{M}(\bar{y})$ is the Lipschitz upper semicontinuity modulus of $\mathcal{M}$ at a given $\bar{y}$ and $\mathrm{clm}\,\mathcal{M}(\bar{y},\bar{x})$ denotes the calmness modulus of $\mathcal{M}$ at $(\bar{y},\bar{x})$. These constants may be rewritten using upper limits instead of infima, as the following expressions show:

$$\begin{aligned}
\mathrm{Lipusc}\,\mathcal{M}(\bar{y}) &= \limsup_{y\to\bar{y}} \left(\sup_{x\in\mathcal{M}(y)} \frac{\mathrm{d}\big(x,\mathcal{M}(\bar{y})\big)}{\|y-\bar{y}\|} \right), \quad \bar{y} \in \mathrm{dom}\,\mathcal{M}, \\
\mathrm{clm}\,\mathcal{M}(\bar{y},\bar{x}) &= \limsup_{(y,x)\overset{\mathrm{gph}\,\mathcal{M}}{\longrightarrow}(\bar{y},\bar{x})} \frac{\mathrm{d}\big(x,\mathcal{M}(\bar{y})\big)}{\|y-\bar{y}\|}, \quad (\bar{y},\bar{x}) \in \mathrm{gph}\,\mathcal{M},
\end{aligned} \tag{8}$$

where $(y,x) \overset{\mathrm{gph}\,\mathcal{M}}{\longrightarrow} (\bar{y},\bar{x})$ means $(y,x) \to (\bar{y},\bar{x})$ with $(y,x) \in \mathrm{gph}\,\mathcal{M}$. The first equality, (8), was established in [6, Proposition 2], while the second one comes directly from the definition. Here, $\limsup_{z\to\bar{z}}$ represents the supremum (maximum, indeed) over all possible sequences $\{z_r\}_{r\in\mathbb{N}} \to \bar{z}$ under the convention $\frac{0}{0} := 0$. Given $\bar{y} \in \mathrm{dom}\,\mathcal{M}$, it is straightforward to check that

$$\mathrm{Lipusc}\,\mathcal{M}(\bar{y}) \geq \sup_{x\in\mathcal{M}(\bar{y})} \mathrm{clm}\,\mathcal{M}(\bar{y},x) \tag{9}$$

General multifunctions may present strict inequality in (9), as illustrated by the following example.

Example 2.1 ([6, Example 2]): Let $\mathcal{M}: \mathbb{R} \to \mathbb{R}$ be defined by $\mathcal{M}(y) := 0$ for $y \leq 0$ and $\mathcal{M}(y) := 1$ for $y > 0$. Then $\mathrm{clm}\,\mathcal{M}(y,x) = 0$ for all $(y,x) \in \mathrm{gph}\,\mathcal{M}$, while $\mathrm{Lipusc}\,\mathcal{M}(0) = +\infty$ by just considering the sequence $y_r := \frac{1}{r}$.

At this point, let us recall an important class of mappings that do satisfy the equality in (9), as stated in [6, Theorem 4]. We recover this result in the following theorem for completeness.

Theorem 2.1: *Let $\mathcal{M} : Y \rightrightarrows X$, with Y being a normed space and X being a reflexive Banach space. Assume that* gph $\mathcal{M}$ *is a nonempty convex set. For any $\bar{y} \in \operatorname{dom} \mathcal{M}$ such that $\mathcal{M}(\bar{y})$ is closed, we have*

$$\operatorname{Lipusc} \mathcal{M}(\bar{y}) = \sup_{x \in \mathcal{M}(\bar{y})} \operatorname{clm} \mathcal{M}(\bar{y}, x). \tag{10}$$

Indeed, the feasible set mapping of linear – possibly infinite – inequality systems with respect to right-hand side perturbations fits into this class. Moreover, certain subclasses of mappings attain the equality (10) as a maximum in some specific points. In fact, whenever the system is confined to finitely many constraints, one may select an extension of the set of extreme points of the original feasible set (see [15, p. 142] and [16, Section 2.2] for an in-depth discussion on this extension)

$$\mathcal{E}(A, b) := \operatorname{extr}\left(\mathcal{F}(A, b) \cap \operatorname{span}\{a_t,\ t \in T\}\right), \quad (A, b) \in \operatorname{dom} \mathcal{F}, \tag{11}$$

and the following theorem shows how we may confine to this finite subset of points.

Theorem 2.2 (cf. [6, Theorem 5]): *Let $(\bar{b}, \bar{x}) \in$ gph $\mathcal{F}_{\bar{A}}$, then*

$$\operatorname{Lipusc} \mathcal{F}_{\bar{A}}\left(\bar{b}\right) = \max_{x \in \mathcal{E}\left(\bar{A}, \bar{b}\right)} \operatorname{clm} \mathcal{F}_{\bar{A}}\left(\bar{b}, x\right).$$

The aim of this paper is to look for the counterpart results of the previous theorem in the context of full perturbations, where the feasible set mapping presents a nonconvex graph. Theorem 3.1 fulfils this goal, providing a new context in which equality in (9) holds. In a second step, the supremum is replaced by a maximum over appropriate sets of extreme points, captured in Theorem 3.2. Indeed, the ability to compute the Lipschitz upper semi-continuity modulus is translated into the possibility of determining the calmness modulus at a fixed point of the graph of the multifunction (see Section 4 for some illustrations).

The second half of this section provides some background on the calmness moduli for multifunction $\mathcal{F}$. First, let us fix the topology of the corresponding spaces. The space of variables, $\mathbb{R}^n$, is endowed with an arbitrary norm $\|\cdot\|$, whose dual norm, $\|\cdot\|_*$, is given by $\|u\|_* = \max_{\|x\| \leq 1} |u'x|$. Elements b in the right-hand side parameter space, $\mathbb{R}^m$, work with the norm $\|b\|_\infty := \max_{t \in T} |b_t|$. The full parameter space, $\mathbb{R}^{m \times n} \times \mathbb{R}^m$, is endowed with the norm $\|(A, b)\| := \max_{t \in T} \|(a_t, b_t)\|$, where $\|(a_t, b_t)\| = \max\{\|a_t\|_*, |b_t|\}$. Here a_t is identified with the functional $x \mapsto a_t' x$, hence the use of the dual norm.

In relation to the feasible set mapping $\mathcal{F}$, the set of active indices at $x \in \mathcal{F}(A, b)$ is denoted by

$$T_{A,b}(x) := \left\{t \in T \mid a_t' x = b_t\right\}. \tag{12}$$

Furthermore, we appeal to the family of sets $\mathcal{D}_{A,b}(x)$ (introduced in [17]) conformed by all $D \subset T_{A,b}(x)$ such that the following system in the variable $d \in \mathbb{R}^n$ is consistent:

$$\left\{ \begin{array}{ll} a_t' d = 1, & t \in D, \\ a_t' d < 1, & t \in T_{A,b}(x) \setminus D \end{array} \right\}. \tag{13}$$

Note that, for any solution $d \in \mathbb{R}^n$ of (13) with associated index set D, the hyperplane $\{z \in \mathbb{R}^n \mid d'z = 1\}$ contains the set $\{a_t,\ t \in D\}$ and leaves $\{0_n\} \cup \{a_t,\ t \in T(x) \setminus D\}$ on the same open half-space.

To finish this section, we introduce the main preliminary results that will be used in the sequel. The first one, which deals with the set of parameters that make a given point feasible, simply adapts [17, Lemma 1] to the notation and framework of this paper.

Lemma 2.1: *Let* $(\bar{A}, \bar{b}) \in \operatorname{dom} \mathcal{F}$. *For any* $x \in \mathbb{R}^n$ *we have*

$$\mathrm{d}\left(\left(\bar{A}, \bar{b}\right), \mathcal{F}^{-1}(x)\right) = \frac{\left\| \left(\bar{A}x - \bar{b}\right)_+ \right\|_\infty}{\|x\| + 1},$$

where $(\bar{A}x - \bar{b})_+$ *denotes the vector with components* $(\bar{a}_t' x - \bar{b}_t)_+$, *the nonnegative part of* $\bar{a}_t' x - \bar{b}_t$, *for* $t \in T$.

Remark 2.1: The reader can easily check that

$$\left\| \left(\bar{A}x - \bar{b}\right)_+ \right\|_\infty = \mathrm{d}\left(\bar{b}, \bar{A}'x + \mathbb{R}^m_+\right) = \mathrm{d}\left(\bar{b}, \mathcal{F}^{-1}_{\bar{A}}(x)\right)$$

for all $\bar{b} \in \operatorname{dom} \mathcal{F}_{\bar{A}}$ and all $x \in \mathbb{R}^n$.

These expressions for the respective distances are considered to be point-based in the sense that they only depend on the nominal parameter $(\bar{A}, \bar{b})$ and a fixed $x \in \mathbb{R}^n$. In the same spirit, the next result provides a computable formula for the calmness modulus of the feasible set mapping at a given point. In it, $\operatorname{end} C$ denotes the end set of the convex set $C \subset \mathbb{R}^n$ (see [18]) defined as

$$\operatorname{end} C := \{u \in \operatorname{cl} C \mid \nexists \mu > 1 \text{ such that } \mu u \in \operatorname{cl} C\}.$$

In particular, $\operatorname{end} \emptyset = \emptyset$ and $0_n \notin \operatorname{end} C$ for any convex subset $C \subset \mathbb{R}^n$.

Theorem 2.3 ([17, Theorems 4 and 5]): *Let* $((\bar{A}, \bar{b}), \bar{x}) \in \operatorname{gph} \mathcal{F}$. *Then*

$$\operatorname{clm} \mathcal{F}\left(\left(\bar{A}, \bar{b}\right), \bar{x}\right) = (\|\bar{x}\| + 1) \operatorname{clm} \mathcal{F}_{\bar{A}}\left(\bar{b}, \bar{x}\right)$$

and

$$\begin{aligned} \operatorname{clm} \mathcal{F}_{\bar{A}}\left(\bar{b}, \bar{x}\right) &= \max_{D \in \mathcal{D}_{\bar{A}, \bar{b}}(\bar{x})} \left[\mathrm{d}_*\left(0_n, \operatorname{conv}\{\bar{a}_t,\ t \in D\}\right)\right]^{-1} \\ &= \mathrm{d}_*\left(0_n, \operatorname{end} \operatorname{conv}\left\{\bar{a}_t,\ t \in T_{\bar{A}, \bar{b}}(\bar{x})\right\}\right)^{-1}. \end{aligned}$$

The reader is addressed to [19, Corollary 2.1, Remark 2.3 and Corollary 3.2] for an extension of the previous theorem to semi-infinite systems (with infinitely many constraints) under a certain regularity condition. Regarding semi-infinite systems, the monograph [20] contains a comprehensive development of theory and applications of these models. See also [21] for an approach of the calmness modulus for inequality systems in terms of the subdifferential of an associated supremum function.

3. Lipschitz upper semicontinuity of the feasible set mapping under full perturbations

This section tackles the main goal of this paper, which is to establish the relationship between the Lipschitz upper semicontinuity modulus and the calmness moduli of the feasible set mapping under full perturbations, closing the gap shown in (9). In a second step, a computable point-based expression for the Lipschitz upper semicontinuity modulus is provided through the calmness moduli at certain points of the domain. The section is divided into two since the case of unbounded feasible set has a pathological behaviour in the context of full perturbations.

The following proposition reformulates the expression of the Lipschitz upper semicontinuity modulus given in (8) in the context of full perturbations of the system's parameters.

Proposition 3.1: *Let* $(\bar{A}, \bar{b}) \in \operatorname{dom} \mathcal{F}$. *Then*

$$\operatorname{Lipusc} \mathcal{F}\left(\bar{A}, \bar{b}\right) = \lim_{\delta \downarrow 0} \left(\sup \left\{ \frac{(\|x\|+1)\, \mathrm{d}\left(x, \mathcal{F}\left(\bar{A}, \bar{b}\right)\right)}{\left\| \left(\bar{A}x - \bar{b}\right)_+ \right\|_\infty} \,\middle|\, \left\| \left(\bar{A}x - \bar{b}\right)_+ \right\|_\infty \leq \delta\,(\|x\|+1) \right\} \right). \quad (14)$$

Proof: First, observe that the supremum in the right-hand side of (14) is a (real-extended) decreasing function of $\delta > 0$, so that the limit of the supremum exists in $[0, +\infty]$. One can check, according to the definitions and appealing to Lemma 2.1 that, when the right-hand side of (14) is infinite, then $\operatorname{Lipusc} \mathcal{F}(\bar{A}, \bar{b}) = +\infty$. Assume now that this right-hand side is finite and consider a paired sequence $(\delta_k, x_k) \in \mathbb{R} \times \mathbb{R}^n$ for which the limit of the supremum is attained. More precisely, consider a sequence $\{\delta_k\}_{k\in\mathbb{N}} \downarrow 0$ and a corresponding sequence $\{x_k\}_{k\in\mathbb{N}}$ satisfying conditions

$$\left\| \left(\bar{A}x_k - \bar{b}\right)_+ \right\|_\infty \leq \delta_k\,(\|x_k\|+1) \quad (15)$$

and

$$\frac{(\|x_k\|+1)\, \mathrm{d}\left(x_k, \mathcal{F}\left(\bar{A}, \bar{b}\right)\right)}{\left\| \left(\bar{A}x_k - \bar{b}\right)_+ \right\|_\infty} \geq \sup \left\{ \frac{(\|x\|+1)\, \mathrm{d}\left(x, \mathcal{F}\left(\bar{A}, \bar{b}\right)\right)}{\left\| \left(\bar{A}x - \bar{b}\right)_+ \right\|_\infty} \,\middle|\, \left\| \left(\bar{A}x - \bar{b}\right)_+ \right\|_\infty \leq \delta_k\,(\|x\|+1) \right\} - \frac{1}{k}. \quad (16)$$

For each point x_k, $\mathcal{F}^{-1}(x_k)$ is a closed subset (a polyhedral convex cone, indeed) of $\mathbb{R}^{m\times n} \times \mathbb{R}^m$, so there exists a pair of parameters $(A_k, b_k) \in \mathcal{F}^{-1}(x_k)$ such that

$$\mathrm{d}\left(\left(\bar{A}, \bar{b}\right), \mathcal{F}^{-1}(x_k)\right) = \left\| \left(\bar{A} - A_k, \bar{b} - b_k\right) \right\|,$$

and, as a consequence of Lemma 2.1, we have

$$\left\|\left(\bar{A}-A_k,\bar{b}-b_k\right)\right\|=\frac{\left\|\left(\bar{A}x_k-\bar{b}\right)_+\right\|_\infty}{\|x_k\|+1}\leq\delta_k. \tag{17}$$

Thus we have

$$\begin{aligned}\operatorname{Lipusc}\mathcal{F}\left(\bar{A},\bar{b}\right)&\geq\limsup_{k\to\infty}\frac{\mathrm{d}\left(x_k,\mathcal{F}\left(\bar{A},\bar{b}\right)\right)}{\left\|\left(\bar{A}-A_k,\bar{b}-b_k\right)\right\|}\\&=\limsup_{k\to\infty}\frac{(\|x_k\|+1)\,\mathrm{d}\left(x_k,\mathcal{F}\left(\bar{A},\bar{b}\right)\right)}{\left\|\left(\bar{A}x_k-\bar{b}\right)_+\right\|_\infty},\end{aligned}$$

where the inequality comes from (8) and the equality from (17). Hence due to the construction of the sequences $\{\delta_k\}_{k\in\mathbb{N}}$ and $\{x_k\}_{k\in\mathbb{N}}$ we have

$$\begin{aligned}&\operatorname{Lipusc}\mathcal{F}\left(\bar{A},\bar{b}\right)\\&\geq\lim_{\delta\downarrow0}\left(\sup\left\{\frac{(\|x\|+1)\,\mathrm{d}\left(x,\mathcal{F}\left(\bar{A},\bar{b}\right)\right)}{\left\|\left(\bar{A}x-\bar{b}\right)_+\right\|_\infty}\;\middle|\;\left\|\left(\bar{A}x-\bar{b}\right)_+\right\|_\infty\leq\delta\,(\|x\|+1)\right\}\right).\end{aligned}$$

On the other hand, to prove the converse inequality, consider sequences $\{(A_k,b_k)\}_{k\in\mathbb{N}}\to(\bar{A},\bar{b})$ and $\{x_k\}_{k\in\mathbb{N}}$ with $x_k\in\mathcal{F}(A_k,b_k)$ for each k such that

$$\operatorname{Lipusc}\mathcal{F}\left(\bar{A},\bar{b}\right)=\lim_{k\to\infty}\frac{\mathrm{d}\left(x_k,\mathcal{F}\left(\bar{A},\bar{b}\right)\right)}{\left\|\left(\bar{A}-A_k,\bar{b}-b_k\right)\right\|}.$$

In such a case, we may define the corresponding sequence $\{\delta_k\}_{k\in\mathbb{N}}\to0$ by

$$\delta_k:=\left\|\left(\bar{A}-A_k,\bar{b}-b_k\right)\right\|\geq\mathrm{d}\left(\left(\bar{A},\bar{b}\right),\mathcal{F}^{-1}(x_k)\right)=\frac{\left\|\left(\bar{A}x_k-\bar{b}\right)_+\right\|_\infty}{\|x_k\|+1},$$

appealing again to Lemma 2.1. Hence, we can conclude that

$$\begin{aligned}
&\operatorname{Lipusc}\mathcal{F}\left(\bar{A},\bar{b}\right)\\
&\leq \limsup_{k\to\infty}\frac{(\|x_k\|+1)\,\mathrm{d}\left(x_k,\mathcal{F}\left(\bar{A},\bar{b}\right)\right)}{\left\|\left(\bar{A}x_k-\bar{b}\right)_+\right\|_\infty}\\
&\leq \limsup_{k\to\infty}\left(\sup\left\{\frac{(\|x\|+1)\,\mathrm{d}\left(x,\mathcal{F}\left(\bar{A},\bar{b}\right)\right)}{\left\|\left(\bar{A}x-\bar{b}\right)_+\right\|_\infty}\,\middle|\,\left\|\left(\bar{A}x-\bar{b}\right)_+\right\|_\infty\leq\delta_k\,(\|x\|+1)\right\}\right)\\
&\leq \lim_{\delta\downarrow 0}\left(\sup\left\{\frac{(\|x\|+1)\,\mathrm{d}\left(x,\mathcal{F}\left(\bar{A},\bar{b}\right)\right)}{\left\|\left(\bar{A}x-\bar{b}\right)_+\right\|_\infty}\,\middle|\,\left\|\left(\bar{A}x-\bar{b}\right)_+\right\|_\infty\leq\delta\,(\|x\|+1)\right\}\right).
\end{aligned}$$

■

3.1. Unbounded feasible set

This section is devoted to the case where the nominal feasible set, $\mathcal{F}(\bar{A},\bar{b})$, is unbounded. Indeed, the Lipschitz upper semicontinuity modulus is infinite in this framework of full perturbations, excluding a trivial case $\mathcal{F}(\bar{A},\bar{b})=\mathbb{R}^n$ and the special case $n=1$, as opposed to the RHS perturbations one. The following result formalizes this statement.

Proposition 3.2: *Let $(\bar{A},\bar{b})\in\operatorname{dom}\mathcal{F}$ with $\mathcal{F}(\bar{A},\bar{b})$ unbounded. Then*

$$\operatorname{Lipusc}\mathcal{F}\left(\bar{A},\bar{b}\right)=\begin{cases}0 & \text{if } \mathcal{F}(\bar{A},\bar{b})=\mathbb{R}^n,\\ \dfrac{\|\overline{x}\|+1}{\max\{\|\overline{a}_t\|_*,\ t\in T_{\bar{A},\bar{b}}(\overline{x})\}} & \text{if } n=1 \text{ and } \mathcal{F}(\bar{A},\bar{b}) \text{ is}\\ &]-\infty,\overline{x}] \text{ or } [\overline{x},+\infty[,\\ +\infty & \text{otherwise.}\end{cases}$$

Proof: The case $\mathcal{F}(\bar{A},\bar{b})=\mathbb{R}^n$ follows trivially from the definition. If $n=1$, we will consider the case $\mathcal{F}(\bar{A},\bar{b})=]-\infty,\overline{x}]$, since the case $[\overline{x},+\infty[$ is completely analogous. It is a well-known fact that in the case $n=1$ the feasible set mapping $\mathcal{F}$ is upper semicontinuous in the sense of Berge (see, e.g. [20, Exercise 6.6]). This means that for all $\varepsilon>0$ there exists $\delta>0$ such that $\|(A,b)-(\bar{A},\bar{b})\|<\delta$ implies $\mathcal{F}(A,b)\subset]-\infty,\overline{x}+\varepsilon[$. Accordingly, looking at Definitions (5) and (6), the Lipschitz upper semicontinuity of $\mathcal{F}$ at $(\bar{A},\bar{b})$ turns out to be equivalent to the calmness of $\mathcal{F}$ at $((\bar{A},\bar{b}),\overline{x})$ and both moduli do coincide. Since the case $\mathcal{F}(\bar{A},\bar{b})=]-\infty,\overline{x}]$ obviously implies $\overline{a}_t\geq 0$ for all $t\in T$, one has

$$\mathrm{d}_*\left(0_n,\operatorname{end}\operatorname{conv}\left\{\overline{a}_t,\ t\in T_{\bar{A},\bar{b}}\left(\overline{x}\right)\right\}\right)=\max\left\{\|\overline{a}_t\|_*,\ t\in T_{\bar{A},\bar{b}}\left(\overline{x}\right)\right\},$$

so that, recalling Theorem 2.3, we have

$$\operatorname{Lipusc}\mathcal{F}\left(\bar{A},\bar{b}\right)=\operatorname{clm}\mathcal{F}\left(\left(\bar{A},\bar{b}\right),\bar{x}\right)=\frac{\|\bar{x}\|+1}{\max\left\{\|\bar{a}_t\|_*,\ t\in T_{\bar{A},\bar{b}}(\bar{x})\right\}}.$$

Assume now the situation $\mathcal{F}(\bar{A},\bar{b})\subsetneq\mathbb{R}^n$ with $n\geq 2$. In such a case, every unbounded closed convex set different from $\mathbb{R}^n$ is known to have an unbounded boundary. So, consider an unbounded sequence $\{z_k\}_{k\in\mathbb{N}}\subset \operatorname{bd}\mathcal{F}(\bar{A},\bar{b})$. It is not restrictive to assume $\|z_k\|\to+\infty$. Then, for any $k\in\mathbb{N}$, $\operatorname{clm}\mathcal{F}_{\bar{A}}(\bar{b},z_k)>0$ by Theorem 2.3. More precisely, the only possibility enabling $\operatorname{clm}\mathcal{F}_{\bar{A}}(\bar{b},\bar{x})=0$ is $\operatorname{end}\operatorname{conv}\{\bar{a}_t,\ t\in T_{\bar{A},\bar{b}}(\bar{x})\}=\emptyset$, and this never happens if $\bar{x}\in\operatorname{bd}\mathcal{F}(\bar{A},\bar{b})$ because the separation theorem ensures the existence of $t\in T_{\bar{A},\bar{b}}(\bar{x})$ with $\bar{a}_t\neq 0_n$. Since there are only finitely many subsets of T, there exists some constant $\gamma>0$ such that $\operatorname{clm}\mathcal{F}_{\bar{A}}(\bar{b},z_k)\geq\gamma$ for all $k\in\mathbb{N}$. Hence,

$$\begin{aligned}\operatorname{lipusc}\mathcal{F}\left(\bar{A},\bar{b}\right)&\geq\sup_{k\in\mathbb{N}}\operatorname{clm}\mathcal{F}\left(\left(\bar{A},\bar{b}\right),z_k\right)\\&=\sup_{k\in\mathbb{N}}\left[\left(\|z_k\|+1\right)\operatorname{clm}\mathcal{F}_{\bar{A}}\left(\bar{b},z_k\right)\right]=+\infty.\end{aligned}$$

■

3.2. Bounded feasible set

This section deals with systems whose feasible set is bounded at the nominal parameters. The main results are Theorem 3.1, which provides the equality between the Lipschitz upper semicontinuity modulus and the supremum of calmness moduli at nominal feasible points, as well as Theorem 3.2, which replaces the supremum with a maximum over the set of extreme points of the feasible set. To achieve the second goal, the technical Lemma 3.1 exploits the polyhedral structure of the feasible set as an essential tool.

Theorem 3.1: *Let $(\bar{A},\bar{b})\in\operatorname{dom}\mathcal{F}$ such that $\mathcal{F}(\bar{A},\bar{b})$ is bounded. Then*

$$\operatorname{Lipusc}\mathcal{F}\left(\bar{A},\bar{b}\right)=\sup_{x\in\mathcal{F}\left(\bar{A},\bar{b}\right)}\operatorname{clm}\mathcal{F}\left(\left(\bar{A},\bar{b}\right),x\right).\tag{18}$$

Proof: According to (9), we only have to prove inequality $\leq$. Appealing to Proposition 3.1, there exists a paired sequence $(\delta_k,x_k)_{k\in\mathbb{N}}$ satisfying $\delta_k\downarrow 0$ as well as (15) and (16) such that

$$\operatorname{Lipusc}\mathcal{F}\left(\bar{A},\bar{b}\right)=\lim_{k\to\infty}\frac{\left(\|x_k\|+1\right)\operatorname{d}\left(x_k,\mathcal{F}\left(\bar{A},\bar{b}\right)\right)}{\left\|\left(\bar{A}x_k-\bar{b}\right)_+\right\|_\infty}.$$

From (4) and Remark 2.1 (see alternatively [1, p. 263]), there exists some constant $\kappa\geq 0$ fulfilling

$$\|x_k\|-\beta\leq\operatorname{d}\left(x_k,\mathcal{F}\left(\bar{A},\bar{b}\right)\right)\leq\kappa\left\|\left(\bar{A}x_k-\bar{b}\right)_+\right\|_\infty\leq\kappa\delta_k\left(\|x_k\|+1\right),\tag{19}$$

where $\beta := \max\{\|x\| \mid x \in \mathcal{F}(\bar{A}, \bar{b})\}$. Thus $\|x_k\|(1 - \kappa\delta_k) \leq \beta + \kappa\delta_k$ and we can conclude that the sequence $\{x_k\}_{k\in\mathbb{N}}$ is bounded. Then it is clear that $\kappa\delta_k(\|x_k\| + 1) \to 0$ and, as a consequence of inequality (19), we have $\mathrm{d}(x_k, \mathcal{F}(\bar{A}, \bar{b})) \to 0$ too. Now consider for every $k \in \mathbb{N}$ a projection of x_k onto $\mathcal{F}(\bar{A}, \bar{b})$, denoted by $z_k \in \mathcal{F}(\bar{A}, \bar{b})$. In other words, $\|x_k - z_k\| = \mathrm{d}(x_k, \mathcal{F}(\bar{A}, \bar{b})) \to 0$. Then we also have

$$\operatorname{Lipusc} \mathcal{F}\left(\bar{A}, \bar{b}\right) = \lim_{k\to\infty} \frac{(\|z_k\| + 1)\,\mathrm{d}\left(x_k, \mathcal{F}\left(\bar{A}, \bar{b}\right)\right)}{\left\|\left(\bar{A}x_k - \bar{b}\right)_+\right\|_\infty}. \tag{20}$$

We may consider any convex combination between x_k and z_k, say $x_k^\lambda := \lambda x_k + (1 - \lambda)z_k$ for $\lambda \in [0, 1]$. Indeed, z_k is still a projection of x_k^λ onto $\mathcal{F}(\bar{A}, \bar{b})$ and

$$\mathrm{d}\left(x_k^\lambda, \mathcal{F}\left(\bar{A}, \bar{b}\right)\right) = \lambda\,\mathrm{d}\left(x_k, \mathcal{F}\left(\bar{A}, \bar{b}\right)\right).$$

Moreover, we have $\|(\bar{A}x_k^\lambda - \bar{b})_+\|_\infty \leq \lambda\|(\bar{A}x_k - \bar{b})_+\|_\infty$, implying

$$\frac{\mathrm{d}\left(x_k, \mathcal{F}\left(\bar{A}, \bar{b}\right)\right)}{\left\|\left(\bar{A}x_k - \bar{b}\right)_+\right\|_\infty} \leq \frac{\mathrm{d}\left(x_k^\lambda, \mathcal{F}\left(\bar{A}, \bar{b}\right)\right)}{\left\|\left(\bar{A}x_k^\lambda - \bar{b}\right)_+\right\|_\infty} \quad \forall\, \lambda \in [0, 1].$$

Passing to the limit on $\lambda \to 0$, we obtain

$$\frac{\mathrm{d}\left(x_k, \mathcal{F}\left(\bar{A}, \bar{b}\right)\right)}{\left\|\left(\bar{A}x_k - \bar{b}\right)_+\right\|_\infty} \leq \limsup_{\lambda\to 0} \frac{\mathrm{d}\left(x_k^\lambda, \mathcal{F}\left(\bar{A}, \bar{b}\right)\right)}{\left\|\left(\bar{A}x_k^\lambda - \bar{b}\right)_+\right\|_\infty} \leq \operatorname{clm} \mathcal{F}\left(\left(\bar{A}, \bar{b}\right), z_k\right).$$

Finally, combining the last inequality with expression (20), we conclude that

$$\operatorname{Lipusc} \mathcal{F}\left(\bar{A}, \bar{b}\right) \leq \sup_{x\in\mathcal{F}\left(\bar{A},\bar{b}\right)} \operatorname{clm} \mathcal{F}\left(\left(\bar{A}, \bar{b}\right), x\right),$$

as desired. ■

This result completes our first goal, which was to find the counterpart of Theorem 2.1 in our framework. The following remark shows that (18) is also true when $\mathcal{F}(\bar{A}, \bar{b})$ is unbounded, although in this case the supremum may not be attained. In particular, the extreme points of $\mathcal{F}(\bar{A}, \bar{b})$, if any, do not play the prominent role that we will observe in the bounded case.

Remark 3.1: Indeed, formula (18) holds true for every $(\bar{A}, \bar{b}) \in \operatorname{dom} \mathcal{F}$. More specifically, in the case $\mathcal{F}(\bar{A}, \bar{b}) = \mathbb{R}^n$ both sides in (18) are trivially zero from the definitions; in the case $n = 1$ equality follows from the previous theorem and Proposition 3.2; and the case $n \geq 2$ with $\mathcal{F}(\bar{A}, \bar{b})$ unbounded and different from $\mathbb{R}^n$ we may proceed as in the proof of

Proposition 3.2, finding a sequence $\{z_k\}_{k\in\mathbb{N}} \subset \operatorname{bd} \mathcal{F}(\bar{A},\bar{b})$ with $\|z_k\| \to +\infty$ and considering $\gamma > 0$ with $\operatorname{clm} \mathcal{F}_{\bar{A}}(\bar{b}, z_k) \geq \gamma$ for all k. Then $\operatorname{clm} \mathcal{F}((\bar{A},\bar{b}), z_k) \geq (\|z_k\|+1)\gamma \to +\infty$. Accordingly, both sides of (18) are infinite.

The following lemma relies on the polyhedral structure of the feasible set mapping $\mathcal{F}_{\bar{A}}$ to provide some hierarchy among the calmness moduli at different points of the image set. It follows the same spirit of the proof of [6, Proposition 5 and Theorem 6], but coupled here in a single result for completeness.

Lemma 3.1: *Let F be a face of $\mathcal{F}(\bar{A},\bar{b})$ for $(\bar{A},\bar{b}) \in \operatorname{dom}\mathcal{F}$. Then $\operatorname{clm} \mathcal{F}_{\bar{A}}(\bar{b},\cdot)$ is constant on $\operatorname{ri} F$, i.e. there exists $\gamma_F \geq 0$ such that $\operatorname{clm} \mathcal{F}_{\bar{A}}(\bar{b},x) = \gamma_F$ for all $x \in \operatorname{ri} F$. Further,*

$$\operatorname{clm} \mathcal{F}_{\bar{A}}\left(\bar{b},x\right) \geq \gamma_F \quad \textit{for all } x \in F \setminus \operatorname{ri} F.$$

Proof: Take any $x \in \operatorname{ri} F$ and set $I := T_{\bar{A},\bar{b}}(x)$. Consider the associated set

$$F_I := \left\{ \tilde{x} \in F \;\middle|\; \begin{array}{ll} \bar{a}_t'\tilde{x} = \bar{b}_t, & t \in I, \\ \bar{a}_t'\tilde{x} \leq \bar{b}_t, & t \in T \setminus I. \end{array} \right\}.$$

By construction, F_I is a face of F and $x \in \operatorname{ri} F_I$ (see, e.g. [20, Theorem A.7]). Hence, appealing to [22, Corollary 18.1.2], this means that $F = F_I$ and so

$$\operatorname{ri} F = \left\{ x \in F \;\middle|\; \begin{array}{ll} \bar{a}_t'x = \bar{b}_t, & t \in I, \\ \bar{a}_t'x < \bar{b}_t, & t \in T \setminus I \end{array} \right\}.$$

Therefore, $T_{\bar{A},\bar{b}}(x) = I$ for all $x \in \operatorname{ri} F$ and, as a consequence of Theorem 2.3, we conclude that $\operatorname{clm} \mathcal{F}_{\bar{A}}(\bar{b},\cdot)$ is constant on $\operatorname{ri} F$. For the second part of the statement, consider any $\hat{x} \in F \setminus \operatorname{ri} F$ and any index set $D \in \mathcal{D}_{\bar{A},\bar{b}}(x)$ with an associated solution, $d \in \mathbb{R}^n$, of the system (13). Since $D \subset I \subset T_{\bar{A},\bar{b}}(\hat{x})$, we obtain that

$$\left\{ \begin{array}{ll} \bar{a}_t'(x-\hat{x}) = 0, & t \in D, \\ \bar{a}_t'(x-\hat{x}) < 0, & t \in T_{\bar{A},\bar{b}}(\hat{x}) \setminus D \end{array} \right\},$$

and consequently

$$\left\{ \begin{array}{ll} \bar{a}_t'(d+\lambda(x-\hat{x})) = \bar{a}_t'd = 1, & t \in D, \\ \bar{a}_t'(d+\lambda(x-\hat{x})) = \bar{a}_t'd < 1, & t \in I \setminus D, \\ \bar{a}_t'(d+\lambda(x-\hat{x})) < \bar{a}_t'd < 1, & t \in T_{\bar{A},\bar{b}}(\hat{x}) \setminus I \end{array} \right\},$$

for any $\lambda > 0$. Thus $D \in \mathcal{D}_{\bar{A},\bar{b}}(\hat{x})$ and $\operatorname{clm} \mathcal{F}_{\bar{A}}(\bar{b},\hat{x}) \geq \operatorname{clm} \mathcal{F}_{\bar{A}}(\bar{b},x)$ follows from Theorem 2.3. ■

Theorem 3.2: *Let $(\bar{A},\bar{b}) \in \operatorname{dom}\mathcal{F}$ with $\mathcal{F}(\bar{A},\bar{b})$ bounded. Then*

$$\operatorname{Lipusc} \mathcal{F}\left(\bar{A},\bar{b}\right) = \max_{x \in \mathcal{E}\left(\bar{A},\bar{b}\right)} \operatorname{clm} \mathcal{F}\left(\left(\bar{A},\bar{b}\right),x\right).$$

Proof: By combining Theorems 2.3 and 3.1, we have that

$$\operatorname{Lipusc}\mathcal{F}\left(\bar{A},\bar{b}\right)=\sup_{x\in\mathcal{F}\left(\bar{A},\bar{b}\right)}\operatorname{clm}\mathcal{F}\left(\left(\bar{A},\bar{b}\right),x\right)=\sup_{x\in\mathcal{F}\left(\bar{A},\bar{b}\right)}\left(\|x\|+1\right)\operatorname{clm}\mathcal{F}_{\bar{A}}\left(\bar{b},x\right). \tag{21}$$

So it is enough to show that the supremum is attained at some extreme point of the feasible set. Indeed the supremum is known to be a maximum because of Theorem 2.3 and the finiteness of T (or, alternatively, from Lemma 3.1 and the finiteness of the number of faces of $\mathcal{F}(\bar{A},\bar{b})$; see [22, Theorem 19.1]). Assume that the maximum is reached at $\bar{x}\in\mathcal{F}(\bar{A},\bar{b})$ and apply the Minkowsky–Weyl theorem (cf. [23, Theorems 3.13 and 3.37]) to write $\mathcal{F}(\bar{A},\bar{b})$ as the finitely generated convex hull of its extreme points. Hence, (21) reads as

$$\left(\|\bar{x}\|+1\right)\operatorname{clm}\mathcal{F}_{\bar{A}}\left(\bar{b},\bar{x}\right)=\left(\|\sum_{j=1}^{k}\lambda_j e_j\|+1\right)\operatorname{clm}\mathcal{F}_{\bar{A}}\left(\bar{b},\sum_{j=1}^{k}\lambda_j e_j\right),$$

with $\bar{x}=\sum_{j=1}^{k}\lambda_j e_j$, $\sum_{j=1}^{k}\lambda_j=1$, $\lambda_j\geq 0$ and $e_j\in\operatorname{extr}\mathcal{F}(\bar{A},\bar{b})$ for all $j\in\{1,\ldots,k\}$ (we may assume $k\leq n+1$ by Carathéodory's theorem). Based on Lemma 3.1 together with the well-known fact that a nonempty convex subset of $\mathbb{R}^n$ is the (disjoint) union of the relative interiors of its nonempty faces (cf. [22, Theorem 18.2]), the function $\operatorname{clm}\mathcal{F}_{\bar{A}}(\bar{b},\cdot)$ is convex on $\mathcal{F}(\bar{A},\bar{b})$. Moreover, one can verify the following tighter bound for all $\lambda\in[0,1]$ and any $x_0,x_1\in\mathcal{F}(\bar{A},\bar{b})$:

$$\operatorname{clm}\mathcal{F}_{\bar{A}}\left(\bar{b},(1-\lambda)\,x_0+\lambda x_1\right)\leq\min\left\{\operatorname{clm}\mathcal{F}_{\bar{A}}\left(\bar{b},x_0\right),\operatorname{clm}\mathcal{F}_{\bar{A}}\left(\bar{b},x_1\right)\right\}. \tag{22}$$

Just consider a face F of $\mathcal{F}(\bar{A},\bar{b})$ whose relative interior contains $(1-\lambda)x_0+\lambda x_1$, hence $x_0,x_1\in F$ (not necessarily belonging to the relative interior of F), and apply Lemma 3.1. Clearly (22) works for any convex combination of finitely many elements of $\mathcal{F}(\bar{A},\bar{b})$. Therefore, the desired result holds:

$$\begin{aligned}\operatorname{Lipusc}\mathcal{F}\left(\bar{A},\bar{b}\right)&\leq\max_j\left\{\|e_j\|+1\right\}\min_j\left\{\operatorname{clm}\mathcal{F}_{\bar{A}}\left(\bar{b},e_j\right)\right\}\\&\leq\max_j\left\{\left(\|e_j\|+1\right)\operatorname{clm}\mathcal{F}_{\bar{A}}\left(\bar{b},e_j\right)\right\}.\end{aligned}$$

■

We finish this section by providing a computable expression for the Lipschitz upper semicontinuity modulus in our context of full perturbations. The following corollary is a direct consequence of Theorems 2.3 and 3.2.

Corollary 1: *Let* $(\bar{A},\bar{b})\in\operatorname{dom}\mathcal{F}$ *with* $\mathcal{F}(\bar{A},\bar{b})$ *bounded. Then*

$$\operatorname{Lipusc}\mathcal{F}\left(\bar{A},\bar{b}\right)=\max_{x\in\mathcal{E}\left(\bar{A},\bar{b}\right)}\left(\left(\|x\|+1\right)\max_{D\in\mathcal{D}_{\bar{A},\bar{b}}(x)}\left[\mathrm{d}_*\left(0_n,\operatorname{conv}\left\{\bar{a}_t,\ t\in D\right\}\right)\right]^{-1}\right).$$

4. Conclusion, illustrations and further research

For comparative purposes, let us gather here the expressions for the Lipschitz upper semicontinuity moduli of $\mathcal{F}$ and $\mathcal{F}_{\bar{A}}$. In the case $n = 1$, we have that Lipusc $\mathcal{F}(\bar{A}, \bar{b})$ is finite at any $(\bar{A}, \bar{b}) \in \operatorname{dom} \mathcal{F}$. This case constitutes an exception, since Lipusc $\mathcal{F}(\bar{A}, \bar{b}) = +\infty$ when $n \geq 2$, $\mathcal{F}(\bar{A}, \bar{b}) \subsetneq \mathbb{R}^n$ and it is unbounded. Let us confine here to the case when $\mathcal{F}(\bar{A}, \bar{b})$ is a nonempty bounded set, in which case $\mathcal{E}(\bar{A}, \bar{b}) = \operatorname{extr} \mathcal{F}(\bar{A}, \bar{b})$. Therefore,

$$\operatorname{Lipusc} \mathcal{F}_{\bar{A}} \left(\bar{b}\right) = \max_{x \in \operatorname{extr} \mathcal{F}\left(\bar{A}, \bar{b}\right)} \operatorname{clm} \mathcal{F}_{\bar{A}} \left(\bar{b}, x\right), \tag{23}$$

while

$$\begin{aligned} \operatorname{Lipusc} \mathcal{F} \left(\bar{A}, \bar{b}\right) &= \max_{x \in \operatorname{extr} \mathcal{F}\left(\bar{A}, \bar{b}\right)} \operatorname{clm} \mathcal{F} \left(\left(\bar{A}, \bar{b}\right), x\right) \\ &= \max_{x \in \operatorname{extr} \mathcal{F}\left(\bar{A}, \bar{b}\right)} (\|x\| + 1) \operatorname{clm} \mathcal{F}_{\bar{A}} \left(\bar{b}, x\right). \end{aligned} \tag{24}$$

Moreover, it was previously known that, for any $(b, x) \in \operatorname{gph} \mathcal{F}_{\bar{A}}$,

$$\operatorname{clm} \mathcal{F}_{\bar{A}} (b, x) = \mathrm{d}_* \left(0_n, \operatorname{end} \operatorname{conv} \left\{\bar{a}_t,\ t \in T_{\bar{A}, b} (x)\right\}\right)^{-1}. \tag{25}$$

It is clear that the smaller the modulus, the greater the stability. The following example shows, on the one hand, the dependence of these moduli on the representation of the feasible set $\mathcal{F}(\bar{A}, \bar{b})$ (in particular, on the size of the coefficients of $\bar{A}$ and $\bar{b}$). On the other hand, it is intended to illustrate the fact that the Lipschitz upper semicontinuity modulus may not be a robust measure in the sense that small perturbations of $(\bar{A}, \bar{b})$ may cause great differences in Lipusc $\mathcal{F}(\bar{A}, \bar{b})$, formally that function $(A, b) \mapsto$ Lipusc $\mathcal{F}(A, b)$ is not continuous at all $(\bar{A}, \bar{b}) \in \operatorname{dom} \mathcal{F}$. Hence, the study of those $(\bar{A}, \bar{b}) \in \operatorname{dom} \mathcal{F}$ where such a function is continuous could constitute stuff for future research; see [24] for counterpart results related to the continuity of the calmness modulus.

Example 4.1: Assume $\mathbb{R}^2$ endowed with the Euclidean norm and, for each $r \in \mathbb{N}$, let us consider the system

$$\sigma\ (A_r, b_r) := \left\{x_1 \leq 1,\ \frac{1}{r} x_1 + \frac{1}{r^2} x_2 \leq \frac{1}{r} - \frac{1}{r^2},\ x_2 \leq 0,\ -x_1 - x_2 \leq 0\right\}.$$

Observe that $\{(A_r, b_r)\}_{r \in \mathbb{N}}$ converges to $(\bar{A}, \bar{b})$ whose associated system is

$$\sigma \left(\bar{A}, \bar{b}\right) := \{x_1 \leq 1,\ 0x_1 + 0x_2 \leq 0,\ x_2 \leq 0,\ -x_1 - x_2 \leq 0\}.$$

It is obvious that $\mathcal{F}(A_r, b_r) = \operatorname{conv}\{(0, 0)', (1 - \frac{1}{r}, 0)', (1, -1)'\}$, for all $r \in \mathbb{N}$, and $\mathcal{F}(\bar{A}, \bar{b}) = \operatorname{conv}\{(0, 0)', (1, 0)', (1, -1)'\}$. Applying (23), (24) and (25), and assuming $r \geq 3$, one easily checks the following:

$x \in \operatorname{extr} \mathcal{F}(A_r, b_r)$	$T_{A_r,b_r}(x)$	$\operatorname{clm} \mathcal{F}_{A_r}(b_r, x)$	$\operatorname{clm} \mathcal{F}((A_r, b_r), x)$
$(0,0)'$	$\{3,4\}$	$\sqrt{5}$	$\sqrt{5}$
$(1-\frac{1}{r},0)'$	$\{2,3\}$	$\sqrt{r^2-\frac{r^2-1}{r^2}}$	$\left(2-\frac{1}{r}\right)\sqrt{r^2-\frac{r^2-1}{r^2}}$
$(1,-1)'$	$\{1,2,4\}$	$\sqrt{r^2-\frac{r^2}{r^2+1}}$	$(\sqrt{2}+1)\sqrt{r^2-\frac{r^2}{r^2+1}}$

Consequently,

$$\operatorname{Lipusc} \mathcal{F}_{A_r}(b_r) = \operatorname{clm} \mathcal{F}_{A_r}\left(b_r, \left(1-\frac{1}{r}, 0\right)'\right) = \sqrt{r^2-\frac{r^2-1}{r^2}},$$

$$\operatorname{Lipusc} \mathcal{F}(A_r, b_r) = \operatorname{clm} \mathcal{F}\left((A_r, b_r), (1,-1)'\right) = \left(\sqrt{2}+1\right)\sqrt{r^2-\frac{r^2}{r^2+1}}.$$

Thus

$$\lim_{r\to\infty} \operatorname{Lipusc} \mathcal{F}_{A_r}(b_r) = \lim_{r\to\infty} \operatorname{Lipusc} \mathcal{F}(A_r, b_r) = +\infty.$$

However, the situation regarding the limit system $\sigma(\bar{A}, \bar{b})$ is notably different:

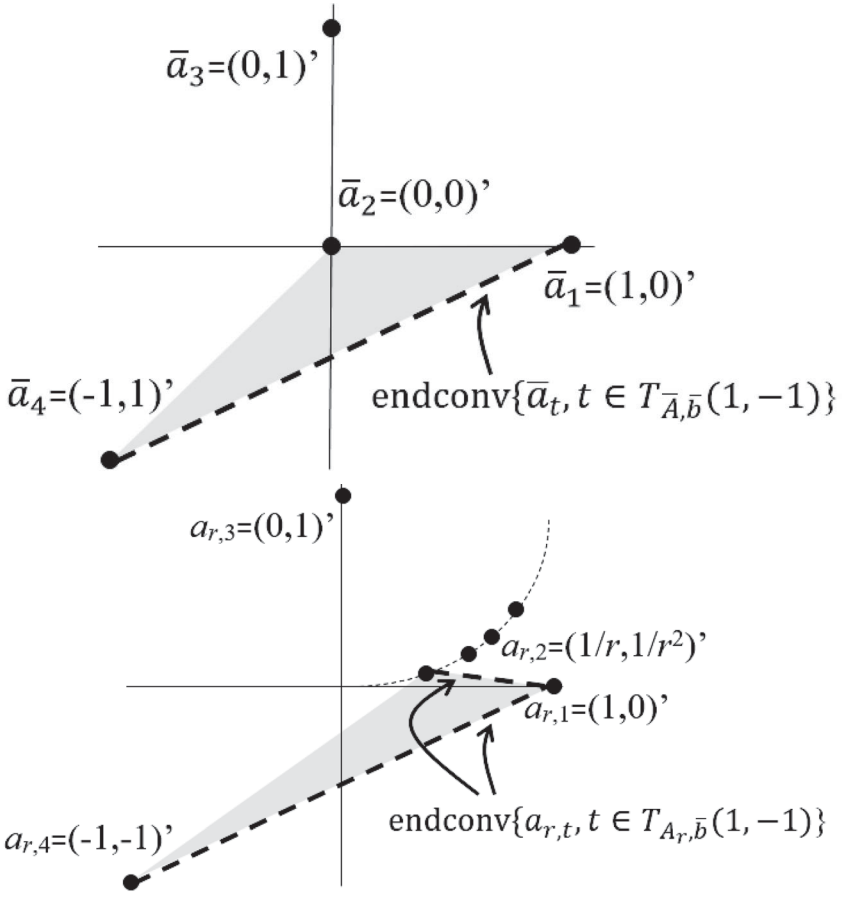
$x \in \operatorname{extr} \mathcal{F}(\bar{A}, \bar{b})$	$T_{\bar{A},\bar{b}}(x)$	$\operatorname{clm} \mathcal{F}_{\bar{A}}(\bar{b}, x)$	$\operatorname{clm} \mathcal{F}((\bar{A}, \bar{b}), x)$
$(0,0)'$	$\{2,3,4\}$	$\sqrt{5}$	$\sqrt{5}$
$(1,0)'$	$\{1,2,3\}$	$\sqrt{2}$	$2\sqrt{2}$
$(1,-1)'$	$\{1,2,4\}$	$\sqrt{5}$	$(\sqrt{2}+1)\sqrt{5}$

So

$$\operatorname{Lipusc} \mathcal{F}\left(\bar{b}\right) = \operatorname{clm} \mathcal{F}_{\bar{A}}\left(\bar{b}, (0,0)'\right) = \operatorname{clm} \mathcal{F}_{\bar{A}}\left(\bar{b}, (1,-1)'\right) = \sqrt{5},$$

$$\operatorname{Lipusc} \mathcal{F}\left(\bar{A}, \bar{b}\right) = \operatorname{clm} \mathcal{F}\left((\bar{A}, \bar{b}), (1,-1)'\right) = \left(\sqrt{2}+1\right)\sqrt{5}.$$

The following figure is intended to illustrate the behaviour of the end sets $\operatorname{end} \operatorname{conv}\{a_{r,t},\ t \in T_{A_r,b_r}(\overline{x})\}$, where $a_{r,t}$ is the tth row of A_r, $r \in \mathbb{N}$, and $\operatorname{end} \operatorname{conv}\{\overline{a}_t,\ t \in T_{\bar{A},\bar{b}}(\overline{x})\}$ at point $\overline{x} = (1,-1)'$ which is the one where we get the maximum of calmness moduli $\operatorname{clm} \mathcal{F}((A_r, b_r), x)$ and $\operatorname{clm} \mathcal{F}((\bar{A}, \bar{b}), x)$, among the extreme points of the corresponding feasible sets. Looking at (25) the inverse of the distance from 0_2 to these end sets provides the exact value of the calmness moduli and, consequently, in the computation of the Lipschitz upper semicontinuity moduli.



The following example shows that if we remove the first inequality in the previous example, the situation is notably different

Example 4.2: Assume again $\mathbb{R}^2$ endowed with the Euclidean norm and, for each $r \in \mathbb{N}$, let us consider the system,

$$\sigma\,(A_r, b_r) := \left\{ \frac{1}{r}x_1 + \frac{1}{r^2}x_2 \leq \frac{1}{r} - \frac{1}{r^2},\ x_2 \leq 0,\ -x_1 - x_2 \leq 0 \right\},$$

$$\sigma\left(\bar{A}, \bar{b}\right) := \{0x_1 + 0x_2 \leq 0,\ x_2 \leq 0,\ -x_1 - x_2 \leq 0\}.$$

Here, for $r \geq 2$ and denoting by d_2 the Euclidean distance,

$$\begin{aligned}\operatorname{Lipusc} \mathcal{F}\,(A_r, b_r) &= \operatorname{clm} \mathcal{F}\left((A_r, b_r), (1,-1)'\right) \\ &= \frac{\sqrt{2}+1}{d_2\left(0_2, \operatorname{conv}\left\{\left(\frac{1}{r}, \frac{1}{r^2}\right)', (-1,-1)'\right\}\right)} \\ &= \frac{\left(\sqrt{2}+1\right)}{r-1}\sqrt{2r^4 + 2r^3 + 3r^2 + 1},\end{aligned}$$

and the unboundedness of $\mathcal{F}(\bar{A}, \bar{b})$ yields

$$\operatorname{Lipusc} \mathcal{F}\left(\bar{A}, \bar{b}\right) = +\infty.$$

Another direction for future research is the extension of the current results to a semi-infinite framework. In the semi-infinite RHS setting, a result similar to Theorem 3.2 was achieved under a local polyhedrality condition in [6, Theorem 6]. In general, the assumptions to be required in the semi-infinite framework to extend some results of the finite setting are quite strong. This is the case of the regularity condition and the result about the calmness property gathered in [25, Theorem 8] (see [19, Corollary 2.1, Remark 2.3, and Corollary 3.2] for the original source). To the authors' knowledge, the possible extension of these results to the semi-infinite framework with full perturbations remains as an open problem.

Acknowledgements

The authors wish to thank the anonymous referees for their valuable critical comments, which have helped to improve the original version of this paper.

Disclosure statement

No potential conflict of interest was reported by the author(s).

Funding

This work was financially supported by the CIPROM/2024/34 grant, funded by the Conselleria de Educación, Cultura, Universidades y Empleo, and also by Grant PID2022-136399NB-C22 from MICINN, Spain, and ERDF, "A way to make Europe", European Union.

ORCID

J. Camacho http://orcid.org/0000-0002-2527-7351
M.J. Cánovas http://orcid.org/0000-0002-1320-485X
H. Gfrerer http://orcid.org/0000-0001-6860-102X
J. Parra http://orcid.org/0000-0001-8864-975X

References

[1] Hoffman AJ. On approximate solutions of systems of linear inequalities. J Res Natl Bur Stand. 1952;49:263–265. doi: 10.6028/jres.049.027
[2] Robinson SM. Some continuity properties of polyhedral multifunctions. Math Progr Study. 1981;14:206–214. doi: 10.1007/BFb0120916
[3] Klatte D, Kummer B. Nonsmooth equations in optimization: regularity, calculus, methods and applications. Dordrecht, The Netherlands: Kluwer Academic; 2002. (Nonconvex optimization and its applications; vol. 60).
[4] Uderzo A. On the quantitative solution stability of parameterized set-valued inclusions. Set-Valued Var Anal. 2021;29:425–451. doi: 10.1007/s11228-020-00571-z
[5] Dontchev AL, Rockafellar RT. Implicit functions and solution mappings: a view from variational analysis. New York: Springer; 2009.
[6] Camacho J, Cánovas MJ, Parra J. From calmness to Hoffman constants for linear semi-infinite inequality systems. SIAM J Optim. 2022;32:2859–2878. doi: 10.1137/21M1418228

[7] Azé D, Corvellec J-N. On the sensitivity analysis of Hoffman constants for systems of linear inequalities. SIAM J Optim. 2002;12:913–927. doi: 10.1137/S1052623400375853
[8] Burke JV, Tseng P. A unified analysis of Hoffman's bound via Fenchel duality. SIAM J Optim. 1996;6:265–282. doi: 10.1137/0806015
[9] Klatte D, Thiere G. Error bounds for solutions of linear equations and inequalities. Z Oper Res. 1995;41:191–214.
[10] Li W. The sharp Lipschitz constants for feasible and optimal solutions of a perturbed linear program. Linear Algebral Appl. 1993;187:15–40. doi: 10.1016/0024-3795(93)90125-8
[11] Peña J, Vera JC, Zuluaga LF. New characterizations of Hoffman constants for systems of linear constraints. Math Program. 2021;187:79–109. doi: 10.1007/s10107-020-01473-6
[12] Zălinescu C. Sharp estimates for Hoffman's constant for systems of linear inequalities and equalities. SIAM J Optim. 2003;14:517–533. doi: 10.1137/S1052623402403505
[13] Mordukhovich BS. Variational analysis and generalized differentiation, I: basic theory. Berlin: Springer; 2006.
[14] Rockafellar RT, Wets RJ-B. Variational analysis. Berlin: Springer; 1998.
[15] Li W. Sharp Lipschitz constants for basic optimal solutions and basic feasible solutions of linear programs. SIAM J Control Optim. 1994;32:140–153. doi: 10.1137/S036301299222723X
[16] Gisbert MJ, Canovas MJ, Parra J, et al. Calmness of the optimal value in linear programming. SIAM J Optim. 2018;28:2201–2221. doi: 10.1137/17M112333X
[17] Cánovas MJ, López MA, Parra J, et al. Calmness of the feasible set mapping for linear inequality systems. Set-Valued Var Anal. 2014;22:375–389. doi: 10.1007/s11228-014-0272-9
[18] Hu H. Characterizations of the strong basic constraint qualifications. Math Oper Res. 2005;30:956–965. doi: 10.1287/moor.1050.0154
[19] Li MH, Meng KW, Yang XQ. On error bound moduli for locally Lipschitz and regular functions. Math Program Ser A. 2018;171:463–487. doi: 10.1007/s10107-017-1200-1
[20] Goberna MA, López MA. Linear semi-infinite optimization. Chichester (UK): John Wiley & Sons; 1998.
[21] Kruger A, Van Ngai H, Théra M. Stability of error bounds for convex constraint systems in Banach spaces. SIAM J Optim. 2010;20:3280–3296. doi: 10.1137/100782206
[22] Rockafellar RT. Convex analysis. New Jersey: Princeton; 1970.
[23] Cornuéjols G, Conforti M, Zambelli G. Integer programming. Switzerland: Springer; 2014.
[24] Camacho J, Cánovas MJ, López MA, et al. Robust and continuous metric subregularity for linear inequality systems. Comput Optim Appl. 2023;86:967–988. doi: 10.1007/s10589-022-00437-0
[25] Cánovas MJ, Parra J. Lipschitzian stability in linear semi-infinite optimization. In: Amigó JM, Cánovas MJ, L ópez-Cerdá MA, López-Pellicer M, editors. Functional analysis and continuous optimization; IMFACO 2022. Springer Proceedings in Mathematics & Statistics, Vol. 424. Cham: Springer; 2023 p. 87–111.