Transportation Research Part E 210 (2026) 104768

Contents lists available at ScienceDirect

Transportation Research Part E

journal homepage: www.elsevier.com/locate/tre

Under-capacitated p-median location problem

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ARTICLE INFO

Keywords:
Facility location
p-median location problem
Lack of capacity
Fair allocation

ABSTRACT

In many Location Science applications, a key challenge is the fair allocation of scarce resources among clients, especially in contexts such as disaster response, transportation planning, and healthcare systems. This challenge becomes particularly critical under severe capacity shortages, where not all demand can be satisfied, an aspect that has received limited attention in the existing literature.

This paper studies the joint problem of locating p facilities from a set of candidates with heterogeneous, limited capacities and allocating indivisible resources (e.g., hospital beds) among clients. We propose a generalized capacitated p-median model that integrates location decisions with a fair allocation of insufficient capacity. Fairness is defined using concepts from bankruptcy theory in cooperative game theory, which are adapted to explicitly represent capacity scarcity.

Several optimization models based on bankruptcy-inspired allocation rules are introduced and their theoretical properties are analyzed. We examine how different weights in the objective function affect the trade-offs between efficiency, equity, and network costs, and we compare bankruptcy-based allocations with alternative fairness rules. Extensions to divisible resources (e.g., water) are also discussed, and computational experiments are used to assess the impact of the proposed models on location decisions.

The proposed framework is well suited for strategic and tactical planning under static and deterministic settings in which total capacity is insufficient to meet demand. Relative to classical p-median formulations and existing fairness-based benchmarks, the model explicitly represents demand shortfalls through bankruptcy-based allocation rules, leading to substantially fairer allocations while preserving high capacity utilization and competitive network costs. From a managerial perspective, the approach allows decision-makers to embed equity considerations directly into the planning stage, rather than relying on ad hoc rationing after facility locations are fixed. The practical relevance of the framework is illustrated through a real-world case study on food insecurity in Todd County, South Dakota, demonstrating its effectiveness in designing fair and efficient service networks under severe capacity shortages.

1. Introduction

Insufficient facility capacity is a critical issue in many location problems, yet it has received limited attention in traditional models. In real-world settings, total demand often exceeds available capacity, so not all clients can be fully served. This creates a dual decision

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https://doi.org/10.1016/j.tre.2026.104768
Received 29 June 2025; Received in revised form 14 January 2026; Accepted 11 February 2026
Available online 2 March 2026

challenge: determining where to locate facilities and how to allocate scarce capacity among clients in a fair and effective manner. Classical location models typically assume sufficient capacity or penalize unmet demand, but they do not explicitly model how limited resources should be distributed when shortages are unavoidable.

Such under-capacity situations arise naturally in a wide range of real-world applications, particularly in public and humanitarian contexts. Examples include food distribution networks in regions affected by food insecurity, emergency shelters after natural disasters, vaccination hubs during supply shortages, or primary healthcare facilities in underserved areas. In these settings, demand is usually estimated from population data and vulnerability indicators, while the available resources (e.g., food, medical supplies, or service time) are clearly insufficient to meet all needs. As a result, decision-makers must not only choose facility locations, but also adopt explicit and transparent fairness principles to guide how scarce resources are allocated.

In this paper, we propose a novel approach to tackle the under-capacity allocation problem by using a game-theoretic framework inspired by bankruptcy problems, which naturally arise whenever limited resources must be allocated among multiple claimants. A bankruptcy problem (also known as a claims problem) is a well-known concept in cooperative game theory that deals with dividing a limited resource among several claimants when the resource is not enough to satisfy all claims. In the facility location context we have that the facility's capacity is the "asset" to be allocated, and the demands of customers are the "claims" on that asset. By applying principles and solution concepts from bankruptcy theory (such as proportionality and egalitarianism that are widely used to operationalize fairness in real-world allocation problems Gómez-Limón et al., 2021; Itani and Shakya, 2017), we develop models for location problems with capacity shortfalls. Thus, the proposed approach ensures that the limited service capacity is allocated according to transparent fairness criteria derived from bankruptcy theory.

Although numerous location models have considered capacity constraints, traditional approaches emphasize either maximizing served demand or minimizing costs related to unmet demand, without explicitly incorporating distributive fairness criteria. On the other hand, cooperative game theory has been successfully applied in operations research problems involving equitable cost and resource allocation, ensuring outcomes that are fair and stable. Nevertheless, the specific integration of fairness principles derived from bankruptcy problems into facility location models remains unexplored. Our work addresses precisely this intersection, explicitly introducing bankruptcy-inspired fairness rules to handle capacity shortages within location models.

In particular, we revisit the capacitated p-median location problem by explicitly modeling scenarios in which customer allocations are unsplittable, and the total available facility capacity is insufficient to meet aggregate demand. Client allocations are treated as discrete units, such as individuals, vehicles, or products, as is common in the resource allocation literature (Katoh et al., 2013). Our models are designed not only to determine optimal facility locations and capacity allocations, but also to distribute scarce resources equitably, thereby addressing a notable gap in existing research. As we will demonstrate, incorporating strategies inspired by bankruptcy problems leads to location decisions that differ significantly from those obtained when scarcity is handled only after selecting facility locations based purely on distance minimization.

The proposed models are intended to support strategic and tactical planning decisions under capacity scarcity, rather than real-time operational deployment. Accordingly, demand, facility locations, and transportation costs are assumed to be deterministic and static.

An illustrative example of the type of problem addressed here is presented in Example 1, which highlights the complexity arising from simultaneously optimizing costs, capacity distribution, and fairness in resource allocation decisions. The following example is intended to illustrate the structural features of the problem in a simplified setting; more realistic, data-driven numerical experiments motivated by humanitarian and public-service applications are presented in Section 10.

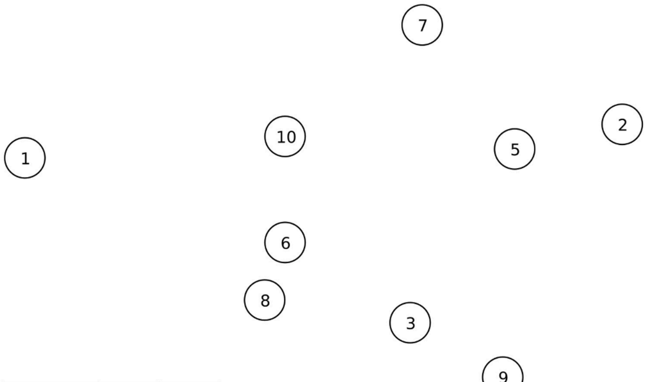
Example 1. Given the location problem of Fig. 1, find a solution that minimises the network cost while attaining a certain level of fairness in resource distribution.

In this example, ten clients (circles) are also potential facility locations (though this is not mandatory). Given that two facilities can be installed, one gray with a capacity of 70 and one blue with a capacity of 50, which are clearly insufficient to meet the total demand of 249, the problem emphasizes the critical decisions regarding facility location, capacity assignment, and allocation rules to balance efficiency and fairness.

1.1. Literature review

Recent contributions in facility location have incorporated fairness to ensure that service provision is not only efficient but also equitable. For example, Shehadeh and Snyder (2023) study stochastic healthcare facility location where fairness aims to avoid disparities in coverage under uncertain demand, while Blanco and Gázquez (2023) embed fairness in maximal covering models to balance service provision across regions. Blanco et al. (2024) introduce the notion of intra-facility equity, focusing on reducing disparities among users assigned to the same facility, and Liu and Salari (2024) define fairness in terms of equitable spatial accessibility to public services. Blanco et al. (2025) extend fairness by incorporating regional preferences into location models, while Digehsara et al. (2025) propose a robust two-stage model that embeds explicit equity metrics into facility location under uncertain demand. In general, these studies conceptualize fairness mainly in relation to distances, accessibility, or coverage. In contrast, our work applies fairness in a different context: the allocation of a scarce supply among demanders, ensuring both balanced distribution and spatial accessibility. To this end, we incorporate concepts from bankruptcy theory into facility location problems.

A growing trend in location analysis involves the integration of game theory to address scenarios with multiple decision-makers whose actions are interdependent. This perspective is particularly valuable in contexts where limited resources must be allocated among competing agents or users. For instance, Ergün et al. (2023) propose a cooperative flow game to model logistics networks



Capacity	Value	Color
A	70	
B	50	

		Cost matrix									
Client	Demand	1	2	3	4	5	6	7	8	9	10
1	3	0	62	61	98	59	22	48	11	76	29
2	4	62	0	36	55	3	51	25	65	39	34
3	10	61	36	0	36	36	17	53	18	15	33
4	15	98	55	36	0	57	49	79	38	22	67
5	20	59	3	36	57	0	22	22	21	40	27
6	23	22	51	17	49	22	0	36	14	27	18
7	31	48	25	53	79	22	36	0	42	61	27
8	42	11	65	18	38	21	14	42	0	19	31
9	45	76	39	15	22	40	27	61	19	0	45
10	56	29	34	33	67	27	18	27	31	45	0

Fig. 1. Instance used in Example 1. Nodes 1 to 10 represent the spatial distribution of clients and potential facility locations. The table on the right reports client demands and transportation costs between clients. The table on the lower left shows the two available facility capacities (70 and 50)..

after an earthquake, while Ma et al. (2023) and Bae et al. (2020) apply game-theoretic methods to the placement of electric vehicle charging stations, aiming to balance costs between users and operators or to achieve Nash-stable user partitions. Similarly, Bosek et al. (2018) use the game of cops and robbers for wireless network localization, and other studies explore cost-allocation and cooperative frameworks for location problems (Puerto et al., 2012; Mallozzi, 2007, 2011).

Game-theoretic approaches are especially relevant when facility capacity is insufficient to meet demand. In disaster management, rapid response requires prioritizing allocation to affected clients (Jain and Bharti, 2023), while in environmental planning, equitable water distribution under scarcity becomes a critical issue (Xevi and Khan, 2005). In healthcare, resource allocation problems such as vaccine distribution (Medlock and Galvani, 2009; Lim et al., 2022), hospital location (Gul and Guneri, 2021), or hospital bed availability (Delgado et al., 2022) naturally involve the balance between efficiency and fairness under scarcity. This is where bankruptcy theory can support fair and efficient outcomes. Fairness in resource allocation problems has been modeled through several well-established strategies. A common approach is max–min fairness, which seeks to maximize the minimum service level or minimize the maximum disadvantage across clients (Rawls, 1971). Another widely used strategy is to minimize disparity measures, such as pairwise differences, variance, or inequality indices like the Gini coefficient (Gini, 1912; Atkinson et al., 1970). Proportional allocation rules and the Constrained Equal Awards (CEA) and Constrained Equal Losses (CEL) rules, inspired by the bankruptcy literature, provide alternative ways to distribute scarce resources among agents with competing claims. While proportional rules allocate according to demand shares, CEA and CEL aim at equalizing awards or losses under feasibility constraints (Thomson, 2019).

Capacity constraints in location problems have been addressed in several ways. Some models expand available capacity, such as the Dynamic Capacitated Plant Location Problem (DCPLP) proposed by Shulman (1991), its modular extensions (Jena et al., 2015), or heuristic approaches (Silva et al., 2021). Other approaches allow transferring capacity between open facilities (Corberán et al., 2020) or tolerating small capacity violations when unexpected failures occur (Albareda-Sambola et al., 2017). From the supplier's point of view, equity has been addressed by balancing the number of customers per facility (Marín, 2011), minimizing the maximum demand assigned to each facility (Berman et al., 2009), or workload balancing in long-term care (Kim and Kim, 2010). From the client's perspective, alternative allocation rules have been explored, such as distributing demand shares among successive closest facilities (Weaver and Church, 1985), allowing clients to rank preferred suppliers (Hanjoul and Peeters, 1987), or explicitly modeling that some customers remain unserved, as in maximal covering problems (Church et al., 2018; Xu et al., 2023).

In most of these works, scarcity is addressed by either expanding capacities or restricting which clients can be served, often privileging efficiency over fairness. Although such approaches may reduce transportation costs, they can generate inequities in service allocation. Building on this growing body of research, our study extends the literature by explicitly integrating fairness principles into capacity-constrained settings, where limited resources must be distributed among competing demands. We address this gap by revisiting the capacitated p-median problem and proposing an adaptation for resource scarcity, interpreted through the lens of bankruptcy theory, where fairness governs the allocation of an insufficient supply among competing demanders.

The remainder of this paper is organized as follows. We first establish in Section 2 the notation and preliminary definitions required for the development of the paper. In Section 3, we introduce a baseline model for the under-capacitated p-median location problem. Sections 4–6 extend this formulation by incorporating constraints that reflect different fairness principles. We address the allocation of resources from three perspectives, which we refer to as *proportional*, *equal gain*, and *equal loss*. Additional constraints are introduced in terms of the model's variables to ensure symmetry, order preservation within facilities, and adherence to welfarism principles. In Section 7, we present a heuristic algorithm to compute the solutions of the previous models. A comprehensive numerical study is carried out in Section 8, illustrating the performance of the models developed in Sections 3 through 6, comparing them with benchmark problems in the literature and performing a sensitivity analysis of the main parameters of the models. In Section 9 the managerial implications of the different elements that have been used to model the under-capacitated p-median location problem are analyzed, including the implications of using fairness principles and the parameters that balance between cost and fairness. In Section 10, we apply our models to a case study related to food insecurity in Todd County (South Dakota). Finally, Section 11 concludes with a summary of our findings. Additional information on the numerical experiments, the details of the continuous case and the mathematical proofs of all the results together with the linearizations of some constraints can be found in the supplementary material for this paper.

2. Notation and standard model for the capacitated *p*-median location problem with unsplittable demands

Let $I = \{1, \dots, |I|\}$ be a set of clients, each with a demand $d_i \in \mathbb{N}$ for all $i \in I$, and let $J = \{1, \dots, |J|\}$ be a set of potential facility locations, each with a capacity $Q_j \in \mathbb{N}$ for all $j \in J$. Let c_{ij} denote the transportation cost between facility j and client i, for all $i \in I$, $j \in J$.

The capacitated p-median location problem (CpMP) with unsplittable demands consists of selecting p facilities to open (with $p \le |J|$) and assigning each client to one of the open facilities, such that the entire demand is satisfied and the total transportation cost is minimized.

The standard mixed integer linear model considers two families of variables, x_{ij} and y_j, where:

$$x_{ij} = \begin{cases} 1 & \text{if } i \text{ is assigned to } j, \\ 0 & \text{otherwise,} \end{cases} \quad \forall i \in I,\, j \in J, \qquad \text{and} \qquad y_j = \begin{cases} 1 & \text{if a facility is open at } j, \\ 0 & \text{otherwise,} \end{cases} \quad \forall j \in J.$$

Then, the CpMP is usually modeled as mixed integer linear model as follows:

$$\begin{aligned} \min \quad & \sum_{i\in I}\sum_{j\in j} c_{ij} d_i x_{ij} && && (1)\\ s.t. \quad & \sum_{j\in J} y_j = p && && (2)\\ & \sum_{j\in J} x_{ij} = 1 && \forall i \in I && (3)\\ & x_{ij} \le y_j && \forall i \in I, j \in J && (4)\\ & \sum_{i\in I} d_i x_{ij} \le Q_j && \forall j \in J && (5)\\ & x_{ij}, y_j \in \{0,1\} && \forall i, j && (6) \end{aligned}$$

The objective function (1) expresses the total cost of supplying the clients' demand. In this paper, we do not consider the fixed costs of opening a facility in the models; therefore, opening a facility costs the same independently of the potential location where it is opened. The first family of constraints (2) states that exactly p facilities are opened in the network. The second family of constraints (3) indicates that each client i can only be allocated to one facility j, for all $i \in I$ and $j \in J$. Constraints (4) prevent clients from being allocated to a potential facility locations that do not open in the solution of the problem. Constraint (5) ensures that the capacity demanded to a facility by the clients allocated to it does not exceed the facility limit. Finally, the binary nature of x_{ij} and y_j is explicitly constrained in (6). It is worth noting that, in the optimal solution to the CpMP, it is no true that each client goes to their nearest open facility, the client assignment to an open facility is influenced by the capacities of the facilities.

3. The under-capacitated *p*-median location problem

The under-capacitated p-median location problem entails four interconnected decision components: (i) selecting locations for a fixed number of service centers from a predefined set of candidates; (ii) allocating the total available capacity among the selected centers; (iii) assigning clients to the operational centers; and (iv) distributing each center's limited capacity among its assigned clients.

In the under-capacitated p-median location model, a distinction is made between the demand of each client, denoted by d_i, and the actual amount of demand satisfied, represented by a decision variable z_i. This variable reflects the portion of the demand that is effectively covered, and it is typically assumed that $z_i \le d_i$ for all $i \in I$.

The total available capacity is distributed among the p facilities to be opened, with capacities denoted by Q_t for $t \in \{1, \dots, p\}$. Without loss of generality, we assume that the capacities are ordered such that $Q_1 \ge Q_2 \ge \cdots \ge Q_p$.

To capture the allocation of capacity from facility t to client j, we introduce a new set of decision variables, s_{tj}, defined as follows:

$$s_{tj} = \begin{cases} 1 & \text{if facility installed at location } j \text{ has capacity } Q_t \\ 0 & \text{otherwise} \end{cases} \qquad \forall t \in \{1, \dots, p\}, j \in J.$$

Therefore, the under-capacitated p-median location problem (UCpMP) can be modeled as follows:

$$\min \quad \sum_{i\in I}\sum_{j\in j} c_{ij} z_i x_{ij} \tag{7}$$

$$\max \quad \sum_{i\in I} z_i \tag{8}$$

$$s.t. \quad (2)-(4)$$

$$s_{tj} \le y_j \qquad \forall t \in \{1,\dots,p\}, j \in J \tag{9}$$

$$\sum_{j\in J} s_{tj} = 1 \qquad \forall t \in \{1,\dots,p\} \tag{10}$$

$$\sum_{t=1}^{p} s_{tj} \le 1 \qquad \forall j \in J \tag{11}$$

$$z_i \le d_i \qquad \forall i \in I \tag{12}$$

$$\sum_{i\in I} z_i x_{ij} \le \sum_{t=1}^{p} Q_t s_{tj} \qquad \forall j \in J \tag{13}$$

$$\sum_{i\in I} z_i x_{ij} \ge \min\left\{\sum_{i\in I} d_i x_{ij}, \sum_{t=1}^{p} Q_t s_{tj}\right\} \qquad \forall j \in J \tag{14}$$

$$s_{tj}, x_{ij}, y_j \in \{0,1\}, z_i \in \{0\} \cup \mathbb{N} \qquad \forall i \in I, \forall j \in J \tag{15}$$

In this model, constraints (9) ensure that only locations chosen to host a facility can be granted a capacity. Constraints (10) indicate that one certain capacity can only be granted once. On the other hand, constraints (11) specify that each potential facility location can only hold one capacity (if finally chosen as facility location) or none (if not chosen). On the allocations side, constraint (12) stipulates that the maximum supply allocated to each client is their individual demands. Constraints (13) ensure that the allocation given to clients assigned to a specific facility never surpasses the facility's capacity. In the same line, constraint (14) forces that, for each specific facility, either the clients' demands are fully met or the facility allocates its whole capacity. This constraint prevents situations where clients with unmet demands are assigned to a factory that does not allocate all its capacity, which we consider inefficient. The allocation is defined as integer values including zero in (15).

The objective function (7)-(8) evaluates the impact between two different terms as part of the optimization, thus making the problem biobjective. The first term represents the transportation cost of the whole network, where the real allocation z_i is employed to calculate this cost instead of the demand d_i. However, while employing z_i is more realistic towards calculating the real cost of the network, it also causes the forcing of minimal allocations to achieve the minimum cost of the network, favoring situations of unmet client demands and underused capacities. To counteract this, the second term is introduced: it favors as much allocation as possible of the available capacity. From now on, we will use the weighted sum method to address this multi-objective problem. For this purpose, we consider hereinafter the following objective function:

$$\min \sum_{i\in I}\sum_{j\in j} c_{ij} z_i x_{ij} - \alpha \sum_{i\in I} z_i$$

where α is a positive parameter. So, α is converted into a decision parameter that balances between cost and efficiency. Decision-makers prioritizing the cost of the network will choose low values of α, while decision-makers interested in allocating as much resources as possible will choose high values of α. Once all available capacity is employed, the operand $\sum_{i\in I} z_i$ will be equal to $\sum_{t=1}^{p} Q_t$, hence the network solution will stay the same for higher values of α.

Therefore, two main metrics can be employed to evaluate the optimal solutions of this type of biobjective location problems. On the one side, the network cost as defined by the first term of the objective function, $\sum_{i\in I}\sum_{j\in j} c_{ij} z_i x_{ij}$, represents the economic perspective. On the other side, efficiency in allocation of resources is represented by the unused capacity, $\sum\limits_{t=1}^{p} Q_t - \sum\limits_{i\in I} z_i$, instead of employing the second term of the objective function. In this work, our philosophy for choosing α is preferring to allocate as much capacity as possible.

Now, we apply this new model to the examples shown.

Example 2.

We now solve the instance from Example 1 using the UCpMP model under four different α values. The corresponding solutions are illustrated in Fig. 2. The two clients where a facility is potentially installed are represented with colored squares and circles, following the legend in Fig. 1. The remaining eight clients are depicted with colored circles only, indicating the facility to which they are assigned. Individual allocation values are shown in purple, to the left of each client.

When $\alpha = 0$, the model achieves a network cost of zero by installing $Q_A = 70$ at client 5 and assigning the rest of the clients to location 10, where $Q_B = 50$ is installed. Since $d_{10} = 56$ and $Q_B < d_{10}$, the most cost-effective solution is for client 10 to consume all of Q_B, with no allocation to the other clients assigned to that facility. As for Q_A, client 5 can fully satisfy its demand ($d_5 = 20$), leaving 50 units of unused capacity. As α increases, this unused capacity decreases until it reaches zero, at the expense of incrementing the cost of the network.

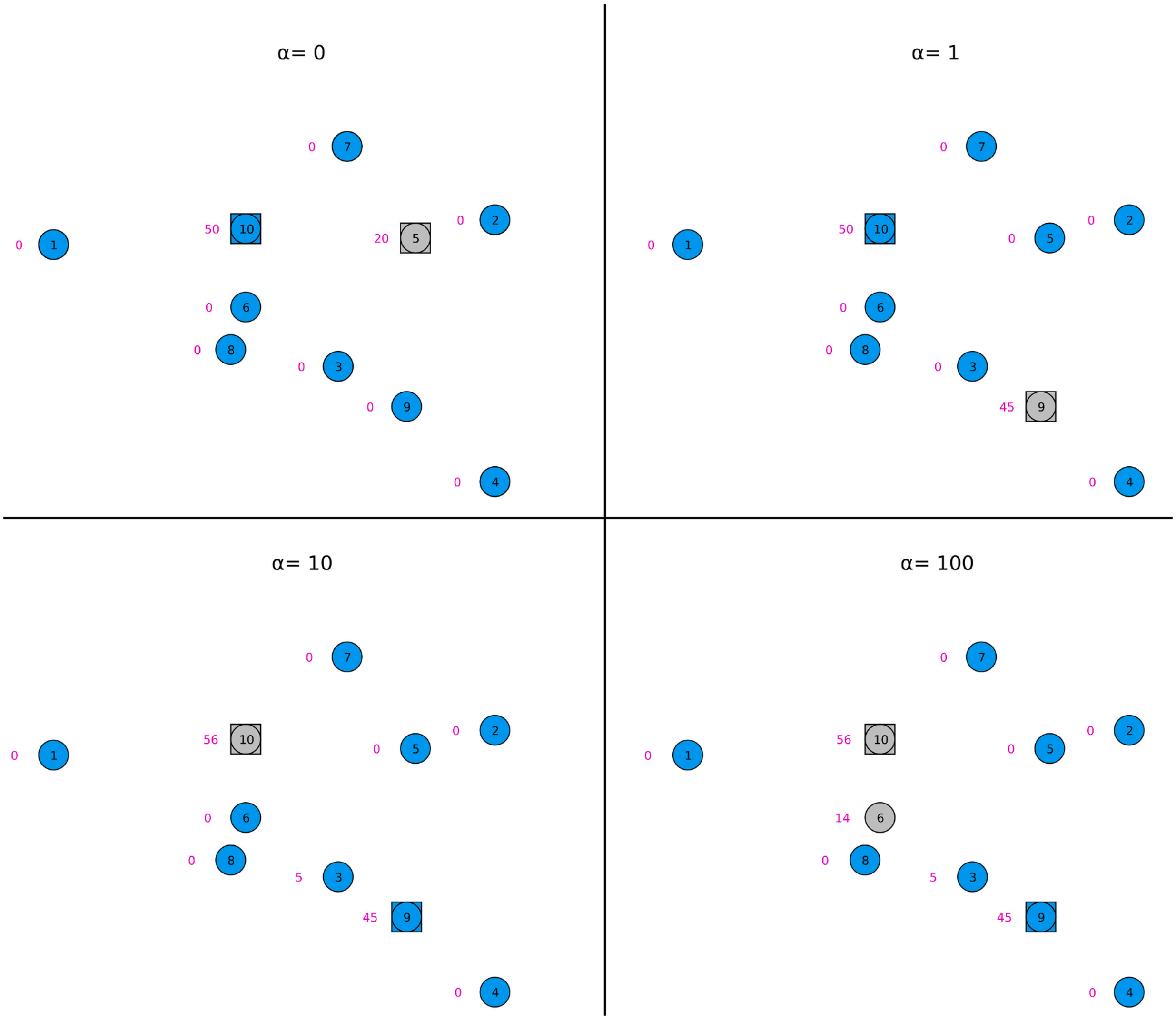


Fig. 2. Solutions of Example 1 obtained with the UCpMP model for different values of α. Squares indicate facility locations, colors identify the facility serving each client, and purple numbers denote the allocated demand z_i.. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

For $\alpha = 1$, the solution changes slightly. Since the allocated amount now contributes to the objective function, Q_A is installed at client 9, the second most demanding client ($d_9 = 45$). Although the network cost remains zero, as Q_A supplies only client 9, the unused capacity is reduced to 25 units.

At $\alpha = 10$, the solution changes again, and the network cost increases to $z_3 c_{39} = 5 \times 15 = 75$ units. Here, Q_A is installed at client 10, which fully satisfies d_{10}. The remaining clients are assigned to client 9. In this case, client 9 consumes most of Q_B to meet d_9, leaving 5 units unused. These 5 units are allocated to client 3 ($d_3 = 10$), as it is the closest to client 9. Although this solution incurs a higher network cost than the previous one, the total objective value is lower due to the increased allocation:

$$0 - \alpha \times (z_{10} + z_9) > z_3 c_{39} - \alpha \times (z_{10} + z_9 + z_3) \Leftrightarrow 0 - 10 \times (50 + 45) > 75 - 10 \times (56 + 45 + 5) \Leftrightarrow -950 > -985.$$

At $\alpha = 100$, the same principle applies. The allocation on the Q_B side remains unchanged, but now client 6 is assigned to client 10, where Q_A is located. Client 6 ($d_6 = 23$) receives the unused capacity left by client 10, as it is the nearest client. The total network cost rises to 327 units, and all available capacity ($Q_A + Q_B$) is utilized. Beyond this value of α, the solution remains unchanged.

As we have seen in Example 2, increasing the value of α increases the total amount of capacity allocated among the different clients. In the following proposition, we establish a necessary condition for the existence of a solution that fully distributes the available capacity, as well as a sufficient condition on the value of α to guarantee such a solution.

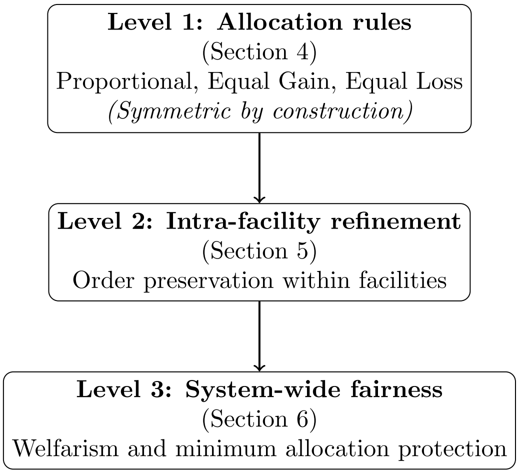


Fig. 3. Conceptual structure of the fairness extensions in the UCpMP model. Fairness is introduced progressively, starting from symmetric allocation rules, then refined through order preservation within facilities, and finally extended to system-wide equity considerations..

Proposition 1. *Let us consider the UCpMP problem. If there exist a partition P of I with $|P| = p$ such that $\sum_{i\in P_t} d_i \geq Q_t$ for all $t \in \{1,\dots,p\}$ and $P_t \in P$, then*

1. *There exists a feasible solution with $\sum_{i\in I} z_i = \sum_{t=1}^{p} Q_t$.*
2. *If $\alpha > (\max_{i\in I, j\in J} c_{ij}) \sum_{t=1}^{p} Q_t$, then every optimal solution satisfies $\sum_{i\in I} z_i = \sum_{t=1}^{p} Q_t$.*

From now on, we consider $\alpha^+ = (\max_{i\in I, j\in J} c_{ij}) \sum_{t=1}^{p} Q_t + 1$ to ensure full utilization of the available capacity. Note that we must assume the existence of a partition $P = \{P_1, \dots, P_p\}$ of I such that $\sum_{i\in P_t} d_i \geq Q_t$ for all $t \in \{1,\dots,p\}$. Even under a global shortage ($\sum_i d_i \geq \sum_t Q_t$), full utilization of the total capacity may be impossible because each customer must be assigned to exactly one facility. The following example illustrates this phenomenon and shows that, without the partition condition, it is possible to have $\sum_i z_i < \sum_t Q_t$ despite the shortage.

Example 3. Consider $I = \{1,2,3\}$ with $d_1 = d_2 = d_3 = 8$ and two facilities with capacities $Q_1 = Q_2 = 10$. There is no feasible way to fully utilize Q_1 and Q_2 when each customer must visit exactly one facility.

The UCpMP is the first step towards making the difference between allocations z_i and demands d_i, enabling which $j \in J$ facilities to open and at what Q_t capacities. However, the solutions it provides are highly unbalanced. As seen, the UCpMP is biased towards satisfying as much as possible the demand of the client nearest to the facility, or the client where the facility has been directly placed, to the detriment of other clients. For $\alpha = 100$ of Example 1, this means fully satisfying the high demands of clients 9 and 10, where facilities are opened, while leaving 6 clients with zero allocation. Thus, the model does not follow any allocation criteria beyond distributing resources as given by the minimization of the objective function. To overcome this limitation, allocation rules for allocating resources must be added to the UCpMP.

Before introducing the fairness-based extensions of the UCpMP model, we briefly outline the conceptual structure that links Sections 4–6. These sections address fairness at three complementary and progressively refined levels (see Fig. 3).

Section 4 introduces fairness at the allocation-rule level, by specifying how scarce capacity is distributed among clients assigned to the same facility through proportional, equal gain, or equal loss principles. These allocation rules are inherently symmetric, ensuring that clients with identical characteristics are treated equally within the same facility. Section 5 builds on these allocation rules by incorporating an additional intra-facility fairness property, in particular order preservation, which ensures consistency in the treatment of clients with different demand levels assigned to the same facility. Finally, Section 6 introduces a system-wide notion of fairness grounded in welfarism, where equity is evaluated across all clients by ensuring that the worst-served client attains an acceptable level of resources.

Taken together, these three sections define a layered framework in which fairness is progressively strengthened: from the choice of an allocation principle, to consistency within facilities, and ultimately to equity at the network level. This structure allows decision-makers to select the degree of fairness that is appropriate for their application while maintaining control over network cost and capacity utilization.

4. First level of fairness: allocation principles and symmetry

When distributing scarce resources among clients in the most balanced way possible, allocation principles of different natures may be adopted. There is a large body of literature devoted to the formal analysis of such problems; see, for example, Thomson

(2019). Here, we present three methods for allocating these resources. The well-known method is the proportional rule, which can be based on allocating resources proportionally to the total demand for a particular facility. Another allocation method would allocate resources as equally as possible until facility capacities are reached. A third allocation method, based on the opposite approach, would allocate resources so that what clients stop receiving from their demands is as equal as possible until the facility capacities are met. The first approach, which we refer to as Proportional Allocation, has been studied in works such as Christodoulou et al. (2016). The ideas behind the second and third approaches, which we refer to as Equal Gain and Equal Loss, respectively, have their origin in bankruptcy problems (Herrero, 2003). We propose three models based on these allocation rules, aimed at improving the UCpMP. At the same time, we consider a property that captures an aspect of equity.

Symmetry within facilities. For each problem and each pair of clients $i_1, i_2 \in I$ with demands $d_{i_1} = d_{i_2}$ assigned to the same facility $j \in J$ then $|z_{i_1} - z_{i_2}| \leq 1$.

The symmetry within facilities property establishes a minimum standard of impartiality. It requires that symmetric clients, those with identical demands, be treated equally when assigned to the same facility, ensuring that the amounts allocated to them differ by no more than one unit. This requirement does not assume global symmetry among all clients; it only applies conditionally to those pairs of clients who (i) have equal demands and (ii) are assigned to the same facility. This one-unit difference arises from the discrete nature of the problem. It is important to note that this principle is local, meaning it applies only when both clients are assigned to the same facility.

Note that since the allocations are discrete, a correcting margin $\epsilon < 1$[1] is included to handle the decimal remainders when calculating the allocation rules.

4.1. Proportional allocation

The UCpMP with a Proportional Allocation distribution (UCpMP-PA-1) allocates resources proportionally to the demands of the clients assigned to the same facility. Given a set of clients I with demands d_i for all $i \in I$, and a total capacity Q_t to be allocated among them, a Proportional Allocation is equal to

$$z_i = \frac{d_i Q_t}{\sum_{j \in I} d_j}. \tag{16}$$

The UCpMP-PA-1 can be modeled as follows, where, without loss of generality, we assume $d_i \geq d_j$ if $i \geq j$:

$$\begin{aligned}
\min \quad & \sum_{i \in I} \sum_{j \in J} c_{ij} z_i x_{ij} - \alpha \sum_{i \in I} z_i \\
s.t. \quad & (2)-(4) \\
& (9)-(15) \\
& z_i \sum_{n \in I} d_n x_{nj} \leq d_i x_{ij} \sum_{t=1}^{p} Q_t s_{tj} + \epsilon \sum_{n \in I} d_n x_{nj} + M(1 - x_{ij}) \qquad \forall i \in I, j \in J \qquad (17) \\
& z_i \sum_{n \in I} d_n x_{nj} \geq \min \left\{ d_i x_{ij} \sum_{t=1}^{p} Q_t s_{tj} - \epsilon \sum_{n \in I} d_n x_{nj}, d_i \sum_{n \in I} d_n x_{nj} \right\} \qquad \forall i \in I, j \in J \qquad (18)
\end{aligned}$$

Constraints (17)–(18) enforce that when customer i is assigned to facility j ($x_{ij} = 1$) the allocation variable z_i satisfies

$$z_i \leq d_i \frac{\sum_{t=1}^{p} Q_t s_{tj}}{\sum_{n \in I} d_n x_{nj}} + \epsilon \qquad \forall i \in I, j \in J$$

$$z_i \geq \min \left\{ d_i \frac{\sum_{t=1}^{p} Q_t s_{tj}}{\sum_{n \in I} d_n x_{nj}} - \epsilon, d_i \right\} \qquad \forall i \in I, j \in J.$$

Since $\sum_{t=1}^{p} Q_t s_{tj} / \sum_{n \in I} d_n x_{nj}$ can be fractional, we use a tolerance parameter $\epsilon = 0.99$ to approximate integer rounding in the model. In particular, constraint (17) sets an upper bound corresponding to the *rounded-up* value $\lceil d_i \sum_{t=1}^{p} Q_t s_{tj} / \sum_{n \in I} d_n x_{nj} \rceil$, while constraint (18) sets a lower bound corresponding to the *rounded-down* value $\lfloor d_i \sum_{t=1}^{p} Q_t s_{tj} / \sum_{n \in I} d_n x_{nj} \rfloor$, additionally limited by the client demand d_i. In this way, z_i is restricted to lie within the interval $[\lfloor d_i \sum_{t=1}^{p} Q_t s_{tj} / \sum_{n \in I} d_n x_{nj} \rfloor, \lceil d_i \sum_{t=1}^{p} Q_t s_{tj} / \sum_{n \in I} d_n x_{nj} \rceil]$, which captures the integer rounding of the proportion. For instance, if $d_i \sum_{t=1}^{p} Q_t s_{tj} / \sum_{n \in I} d_n x_{nj} = 3.5$ and $\epsilon = 0.99$, then constraint (17) yields $z_i \leq 4$ while constraint (18) yields $z_i \geq 3$, so that $z_i \in \{3, 4\}$.

The following proposition shows that the UCpMP-PA-1 model satisfies the property of symmetry within the facilities.

Proposition 2. *Any feasible solution of the UCpMP-PA-1 model satisfies symmetry within facilities.*

Finally, note that the results obtained in Proposition 1 are also valid for the UCpMP-PA-1 model, as they do not depend on the constraints used in the demand allocations.

[1] This margin is modeled as $\epsilon = 0.99$ wherever it appears.

4.2. Equal gain

The UCpMP with an Equal Gain distribution (UCpMP-EG-1) aims to allocate resources equally among the clients assigned to the same facility. The idea is to increase all allocations simultaneously and equally, until either the total capacity is exhausted or a client's demand is fully satisfied. In this way, no client receives more than requested, and the available capacity is shared as evenly as possible.

Mathematically, Equal Gain is defined as follows. Given a set of clients I with demands d_i for all $i \in I$, and a total capacity Q_t to be allocated among them, the allocation is

$$z_i = \min\{d_i, \lambda_t\}, \tag{19}$$

where λ_t is chosen such that

$$\sum_{i \in I} \min\{d_i, \lambda_t\} = Q_t.$$

In Alcalde and Peris (2022), the authors show that the Equal Gain allocation coincides with the outcome of the following iterative redistribution procedure. Start by splitting the facility capacity Q_t equally among the $|I|$ clients. Any client whose demand is below the current equal split is fully satisfied and removed from the pool; the remaining capacity is then divided equally among the remaining clients. This continues until capacity is exhausted. For the closed-form computation, let $d_1 \le \cdots \le d_n$ be the non-decreasing demands. For each $i = 1, \ldots, |I|$, set

$$\lambda_{ti} = \frac{Q_t - \sum_{j=1}^{i-1} d_j}{|I| - i + 1}.$$

For $i = 1$, we assume $\sum_{j=1}^{i-1} d_j = 0$. Therefore, the λ_t of Eq. (19) is given by $\max_{i \in I} \lambda_{ti}$.

In our setting, however, the facility assigned to each client, and the group of clients served by the same facility, are decision variables within the model. Therefore, this maximum must be reformulated using model-specific variables. Following the approach of Alcalde and Peris (2022), we introduce a parameter λ_t for each facility, representing the common allocation granted to the clients it serves. This common value is assigned provided it does not exceed a client's individual demand; otherwise, the client's demand is fully satisfied. So, the UCpMP-EG-1 can be modeled as follows, where, without loss of generality, we assume $d_i \ge d_j$ if $i \ge j$:

$$\min \quad \sum_{i \in I} \sum_{j \in j} c_{ij} z_i x_{ij} - \alpha \sum_{i \in I} z_i$$

$$s.t. \quad (2) - (4)$$

$$(9) - (15)$$

$$\lambda_t \ge 0 \qquad \forall t \in \{1, \ldots, p\} \tag{20}$$

$$\lambda_t = \sum_{j \in J} s_{tj} \max_{k \in I} \frac{Q_t - \sum_{i=1}^{k-1} d_i x_{ij}}{\sum_{i \in I} x_{ij} - \sum_{i=1}^{k} x_{ij} + 1} \qquad \forall t \in \{1, \ldots, p\} \tag{21}$$

$$z_i x_{ij} s_{tj} \le \lambda_t + \epsilon \qquad \forall i \in I, \forall j \in J, t \in \{1, \ldots, p\} \tag{22}$$

$$z_i \ge \min\left\{\lambda_t x_{ij} s_{tj} - \epsilon, d_i x_{ij} s_{tj}\right\} \qquad \forall i \in I, \forall j \in J, t \in \{1, \ldots, p\} \tag{23}$$

The parameter λ_t is fixed as positive in constraint (20), and it is calculated as the maximum value of the increasing sequence portrayed in constraint (21). The sum in the constraint (21) has only a positive term because only one facility has been assigned the capacity Q_t. The allocation for the clients going to a specific facility is superiorly bounded by the rounded-up value of λ_t in constraint (22). Finally, the allocation for the clients going to a specific facility is lower bounded in constraint (23) by the minimum between the rounded-down value of λ_t and the demand of the client.

The following proposition shows that the UCpMP-EG-1 model satisfies the property of symmetry within the facilities.

Proposition 3. *Any feasible solution of the UCpMP-EG-1 model satisfies symmetry within facilities.*

Moreover, the results presented in Proposition 1 are also applicable to the UCpMP-EG-1 model, since they are independent of the constraints involved in the demand allocations.

4.3. Equal loss

The UCpMP with an Equal Loss distribution (UCpMP-EL-1) aims to reduce the clients' demands in an equal manner when the available capacity is insufficient. The idea is to decrease all demands simultaneously and equally, until either the total capacity matches the reduced sum of demands or some client's allocation reaches zero. In this way, no client receives less than zero, and the loss is shared as evenly as possible.

Mathematically, Equal Loss is defined as follows. Given a set of clients I with demands d_i for all $i \in I$, and a total capacity Q_t to be allocated among them, the allocation is

$$z_i = \max\{0, d_i - \mu_t\}, \tag{24}$$

Fig. 4. Solutions of Example 1 obtained with the UCpMP model under different allocation rules with $\alpha = \alpha^+$. The three panels correspond to Equal Gain (UCpMP-EG-1), Equal Loss (UCpMP-EL-1), and Proportional Allocation (UCpMP-PA-1). Squares indicate facility locations, colors identify the facility serving each client, and purple numbers denote the allocated demand $z_{i\cdot}$. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

where μ_t is chosen such that

$$\sum_{i\in I} \max\{0,\ d_i - \mu_t\} = Q_t.$$

In Alcalde and Peris (2022), the authors show that the Equal Loss allocation coincides with the outcome of the following iterative reduction procedure. Start by subtracting the same amount from each client's demand. Any client whose demand reaches zero is removed from the pool, they are allocated an amount of 0, and the remaining loss is then distributed equally among the other clients. This continues until all capacity is allocated. For the closed-form computation, let $d_1 \geq \cdots \geq d_n$ be the non-increasing demands. For each $i = 1, \ldots, |I|$, set

$$\mu_{ti} = \frac{\sum_{j=i}^{|I|} d_j - Q_t}{|I| - i + 1}.$$

Therefore, the μ_t of Eq. (24) is given by $\max_{i\in I} \mu_{ti}$.

In our model, the facility serving each set of clients is determined endogenously, and the corresponding deduction must be expressed in terms of the model variables. As adapted from Alcalde and Peris (2022), the parameter μ_t represents, for each facility, the common loss applied to the clients assigned to that facility. So, the UCpMP-EL-1 can be modeled as follows, where, without loss of generality, we assume $d_i \geq d_j$ if $i \geq j$:

$$\min \quad \sum_{i\in I}\sum_{j\in j} c_{ij} z_i x_{ij} - \alpha \sum_{i\in I} z_i$$

$$s.t. \quad (2)-(4)$$

$$(9)-(15)$$

$$\mu_t \geq 0 \qquad \forall t \in \{1,\dots,p\} \tag{25}$$

$$\mu_t = \sum_{j\in J} s_{tj} \max_{k\in I} \frac{\sum_{i\in I: i\geq k} d_i x_{ij} - Q_t}{\sum_{i\in I} x_{ij} - \sum_{i=1}^{k} x_{ij} + 1} \qquad \forall t \in \{1,\dots,p\} \tag{26}$$

$$z_i x_{ij} s_{tj} \leq \max\{d_i x_{ij} s_{tj} - \mu_t x_{ij} s_{tj} + \epsilon, 0\} \qquad \forall i \in I, \forall j \in J, t \in \{1,\dots,p\} \tag{27}$$

$$z_i \geq d_i x_{ij} s_{tj} - \mu_t x_{ij} s_{tj} - \epsilon \qquad \forall i \in I, \forall j \in J, t \in \{1,\dots,p\} \tag{28}$$

The parameter μ_t is fixed as positive in constraint (25), and it is calculated as the maximum value of the increasing sequence portrayed in constraint (26). The allocation for the clients going to a specific facility is superiorly bounded in constraint (27) by the maximum between the rounded-up value of demand minus μ_t, and zero. Finally, the allocation is inferiorly bounded by the rounded-down value of demand minus μ_t.

The following proposition shows that the UCpMP-EL-1 model satisfies the property of symmetry within the facilities.

Proposition 4. *Any feasible solution of the UCpMP-EL-1 model satisfies symmetry within facilities.*

Likewise, the results established in Proposition 1 also hold for the UCpMP-EL-1 model.

Example 4. We now solve the instance from Example 1 using the UCpMP-EG-1, UCpMP-EL-1 and UCpMP-PA-1 models. Fig. 4 shows the solution obtained for α given by Proposition 1, α^+. The same display conventions used in the previous figures apply here. As observed, the solution varies across distribution rules, affecting the facilities that are opened, the allocation of capacities, the assignment of clients to facilities, and the individual allocations received by each client.

The network cost according to UCpMP-EG-1 is 1448 units, and two clusters are formed around client 5 (bearing Q_B) and client 9 (bearing Q_A). Regarding the capacity Q_A, we obtain $\lambda_A = 45.0$. Given this value, it is sufficient to fully supply d_3 and d_4 using an equal awards approach, and it satisfies 45 units of demand for client 9. On the other hand, as more clients are assigned to Q_B, $\lambda_B = 8.6$ is lower. Therefore, demands d_1 and d_2 are fully met, whilst the rest of the clients are allocated 8 units and the closest ones to the open facility are allocated the remaining 3 units.

In contrast, UCpMP-EL-1 opts for installing Q_B in client 10 which, together with the different client assignments and allocations, results in a decreased network cost of 978 units. The clients supplied by Q_B experience a common loss of $\mu_B = 20.0$ that leaves up to 5 of them with zero allocation. As less clients are assigned to Q_A, $\mu_A = 8.5$, and the additional unit taken off to complete the capacity is subtracted from client 8, which is more optimal in terms of transport cost. These set of models are already complex enough for some seemingly counterintuitive assignments to happen: for instance, while client 3 is closer to Q_A facility than client 8, it is assigned to Q_B. If both client 8 and client 3 were assigned to Q_A, then $\mu'_A = 9.0$, which would increase the network cost by assigning one unit to client 3 that is currently assigned to client 9 at transport cost zero. Conversely, if only client 3 was assigned to Q_A, then client 3 and client 9 would not consume all Q_A by themselves.

The UCpMP-PA-1 yields a different solution: Q_B is installed at client 10, while the remaining clients are assigned to client 8, where Q_A is located. Since allocations are based on proportions, the distribution of resources is more varied compared to the previous one-step allocation rules. The cost of the network is equal to 1370 units.

With the Proportional Allocation, Equal Gain and Equal Loss rules, the UCpMP already achieves a certain level of fairness, as seen by the diversity of allocation solutions. At this point, nonetheless, there is an additional allocation detail that can be improved. In many problems concerning the allocating rules of the UCpMP model, due to the discrete conditions of the allocation, remainders that must be allocated among the clients assigned to one facility exist. In the current models, the remaining resources are not distributed according to any specific criterion; rather, they are assigned to the client closest to the facility. This can potentially cause that a client demanding less resources to a facility receives more than another client who is demanding more resources to that same facility. For instance, in Example 1 as solved by UCpMP-EG-1, while client 5 has a lower demand than client 10, it receives the remainder resources because Q_B is located there. Therefore, a coherent approach to distributing these remainders would be to allocate them preferentially to clients with higher resource demands for the same facility.

5. Second level of fairness: order preservation rules

When only the proposed allocation rules are applied in the discrete UCpMP, the presence of leftover resources at facilities can lead to situations in which clients with higher demand receive less than others with lower demand, despite being assigned to the same facility. Ideally, these remainder resources should be assigned to clients that have a higher demand within the same facility. This stated property is known as order preservation, and has been analyzed in other works like Kułakowski (2015), Martínez and Sánchez-Soriano (2023). This principle is adapted to our context as follows.

Order preservation within facilities. For each problem and each pair of clients $i_1, i_2 \in I$ assigned to the same facility $j \in J$, it holds that if $d_{i_1} > d_{i_2}$ then $z_{i_1} \geq z_{i_2}$.

The previous principle requires that clients assigned to the same facility with higher demands receive higher allocations. New models, namely UCpMP-EG-2, UCpMP-EL-2, and UCpMP-PA-2, that respect this principle have been developed.

5.1. Proportional allocation and equal gain

To ensure order preservation within facilities in the Proportional Allocation and Equal Gain methods, additional constraints must be considered. Hence, UCpMP-EG-2 and UCpMP-PA-2 are formulated as follows:

$$\begin{aligned} \min \quad & \sum_{i\in I}\sum_{j\in j} c_{ij} z_i x_{ij} - \alpha \sum_{i\in I} z_i \\ s.t. \quad & (2)-(4) \\ & (9)-(15) \\ & (17)-(18)\ if\ Proportional\ Allocation \\ & (20)-(23)\ if\ Equal\ Gain \\ & z_n \geq z_i\left(x_{ij}+x_{nj}-1\right) \qquad \forall i,n\in I, \forall j\in J, d_n > d_i \qquad (29) \end{aligned}$$

Constraint (29) ensures that, when assigned to a same facility, a client demanding more resources will receive at least the same allocation as the client demanding less resources.

5.2. Equal loss

Since allocations are restricted to integers, we introduce a small parameter $\epsilon \in (0,1)$ to make rounding explicit. Under the Equal Loss (EL) rule, all clients assigned to the same facility j incur the same loss μ_j. Thus, for any two clients i_1, i_2 with $d_{i_1} > d_{i_2}$ (hence $d_{i_1} - d_{i_2} \geq 1$), their targets differ by at least one unit:

$$(d_{i_1} - \mu_j) - (d_{i_2} - \mu_j) = d_{i_1} - d_{i_2} \geq 1.$$

Each integer allocation must then satisfy

$$z_i \in \left[d_i - \mu_j - \epsilon,\ d_i - \mu_j + \epsilon\right] \cap \mathbb{Z}_+ \qquad \forall i \in \{i_1, i_2\},$$

so the rounding window has width strictly smaller than 1 and cannot reverse the order: a client with higher initial demand keeps an allocation no smaller than that of a client with lower initial demand (and if truncation at 0 occurs, order is trivially preserved). Therefore, order preservation holds under integer allocations within each facility. Consequently, UCpMP-EL-1 needs no additional constraint, and we may set UCpMP-EL-2 := UCpMP-EL-1. The following proposition formalizes this result.

Proposition 5. *Any feasible solution of the UCpMP-EL-2 model satisfies order preservation within facilities.*

Example 5. For the resolution of Example 1 employing UCpMP-EG-2 and UCpMP-PA-2, there are only two changes of the allocations per model with respect to Fig. 4. With respect to UCpMP-EG-1, UCpMP-EG-2 allocates eight units to client 5 and nine units to client 10. This allocation change makes the network cost rise to 1475, which is 27 monetary units higher than with UCpMP-EG-1. With respect to UCpMP-PA-1, UCpMP-PA-2 allocates one unit to client 1 and four units to client 3. This allocation change accounts for a network cost of 1377, which is 7 monetary units higher than the solution with UCpMP-PA-1. Once again, a fairer distribution comes at a cost. As for UCpMP-EL-2, it bears the same results as UCpMP-EL-1.

The UCpMP-EG-2, UCpMP-EL-2 and UCpMP-PA-2 models satisfy both symmetry within facilities and order preservation within facilities. The statement of Proposition 1 also applies to these models. However, there is a last step to further account for fairness that can be included in the model. In conditions where clients demand very similar number of resources and transportation costs are alike, certain similar situations might happen with different level of balance in the resource allocation. Some correcting factors to the objective function and some additional constraints can be added to break these similar situations towards more balanced optimal solutions.

6. Third level of fairness: principle of welfarism

Even though the UCpMP models presented satisfy both symmetry and order preservation within each facility, they lack mechanisms to promote fairness in situations where clients are in comparable positions, for example, when their demands and associated costs are similar, and a more balanced allocation would be desirable. Works such as Jacobs (2012), Rawls (1971), Delorme et al. (2019) and Csató (2023) highlight the importance of implementing prioritization rules for such cases in resource allocation. Therefore, the UCpMP-EG-3, UCpMP-EL-3, and UCpMP-PA-3 models are proposed, incorporating welfare principles, specifically, the Rawlsian maximin criterion, which aims to maximize the allocation received by the worst-off client (Rawls, 1971).

To implement the welfare approach, the objective function is modified and additional constraints are introduced. Accordingly, the UCpMP-EG-3, UCpMP-EL-3, and UCpMP-PA-3 models are formulated as follows:

$$\begin{aligned}
\min \quad & \sum_{i\in I}\sum_{j\in j} c_{ij} z_i x_{ij} - \alpha \sum_{i\in I} z_i - \beta w && (30)\\
s.t. \quad & (2)-(4) \\
& (9)-(15) \\
& (17)-(18),(29)\ if\ Proportional\ Allocation \\
& (20)-(23),(29)\ if\ Equal\ Gain \\
& (25)-(28)\ if\ Equal\ Loss \\
& w = \min_{i\in I}\{z_i\} && (31)
\end{aligned}$$

In these models, the term $-\beta w$ is added to the objective function (30) as a mechanism to promote welfarism, where w represents the minimum allocation z_i across all clients, as defined in constraint (31). This term represents a Rawlsian maximin objective, aiming to improve the outcome for the worst-off client. The influence of this fairness-driven component can be adjusted via the parameter β, allowing control over the trade-off between overall cost and equity. By introducing this term, the models aim to reduce disparities in allocation, particularly in instances where the problem structure is highly symmetric.

As with α, there exists a threshold value of β beyond which increasing it no longer affects the solution or the final network cost. In some instances, this occurs once the demand of the least-demanding client is fully satisfied. In other cases, such full satisfaction is infeasible due to problem constraints, and therefore variations in β have no impact. In yet other situations, the solution stabilizes once the fairest allocation achievable under the constraints has been reached. Accordingly, decision-makers who wish to emphasize fairness across a wider range of scenarios may choose higher values of β, whereas those who prefer to prioritize efficiency except in the most symmetric or equitable cases may opt for lower values. In fact, setting $\beta = 0$ makes the models equivalent to UCpMP-EG-2, UCpMP-EL-2, and UCpMP-PA-2, respectively. The following proposition makes this intuition precise by showing that, for sufficiently large values of β, the model enforces the fairest possible allocation, namely when the common satisfaction level w attains its maximum feasible value $w^\star$.

Proposition 6. *Consider UCpMP-EG-3, UCpMP-EL-3 or UCpMP-PA-3 and assume $\alpha \geq 0$. Let $w^\star = \max\{w : (x,y,s,z,w)$ is UCpMP-EG-3 (UCpMP-EL-3 or UCpMP-PA-3) feasible$\}$. If*

$$\beta > |I| \max_{i\in I, j\in J} c_{ij} + \alpha\Big(\sum_{t=1}^{p} Q_t\Big),$$

then every optimal solution satisfies $w = w^\star$.

From now on, we consider $\beta^+ = |I| \max_{i\in I,j\in J} c_{ij} + \alpha\Big(\sum_{t=1}^{p} Q_t\Big) + 1$ to ensure that the minimum value of w is as large as possible. To visualize the impact of UCpMP-EG-3, UCpMP-EL-3 and UCpMP-PA-3, we discontinue Example 1 and propose Example 6, as it is more illustrating.

Example 6. Consider a location problem where there are ten clients, who are candidates for the placement of $p = 2$ facilities. The facility capacities to be located are 31 units and 20 units, and all clients have the same demand of 9 units. Additionally, the unitary supply cost is the same for all clients, at 7 units.

The resolution of Example 6 employing UCpMP-EG-3, UCpMP-EL-3 and UCpMP-PA-3 is represented in Fig. 5 for an α that completely allocates all resources and three representative β values. When $\beta = 1$, solutions provided by UCpMP-EG-3 and UCpMP-EL-3 improve with respect to $\beta = 0$ from having a minimum allocation of 3 to having a minimum allocation of 4, without modifying the cost of 273 units. These solutions are now equivalent to the solution provided already by UCpMP-PA-3 with $\beta = 0$. For $\beta = \beta^+$, the models find the fairest solution possible: nine clients supplied with 5 units and one client with an allocation of 6, incurring in a total network cost of 280 units. So, β is illustrated as the welfarism-inducing parameter.

The next result extends the bound on α given in Proposition 1 to the case in which the welfarism term involving β is present in the objective function.

Corollary 1. *Let us consider UCpMP-EG-3, UCpMP-EL-3 or UCpMP-PA-3. If there exists a partition P of I with $|P| = p$ such that $\sum_{i\in P_t} d_i \geq Q_t$ for all $t \in \{1,\dots,p\}$ and $P_t \in P$ and $\alpha > \sum_{t=1}^{p} Q_t \max_{i\in I,j\in J} c_{ij} + \beta \min_{i\in I} d_i$, then every optimal solution satisfies $\sum_{i\in I} z_i = \sum_{t=1}^{p} Q_t$.*

From now on, we consider $\alpha^{++} = \sum_{t=1}^{p} Q_t \max_{i\in I,j\in J} c_{ij} + \beta \min_{i\in I} d_i + 1$ to ensure full utilization of the available capacity under the welfarism term.

7. A heuristic approach for the UCpMP

The heuristic approach proposed in this section is not purely rule-based, but is explicitly designed to exploit structural properties of the UCpMP formulation. In particular, our numerical experiments reveal two key problem-specific features that motivate the

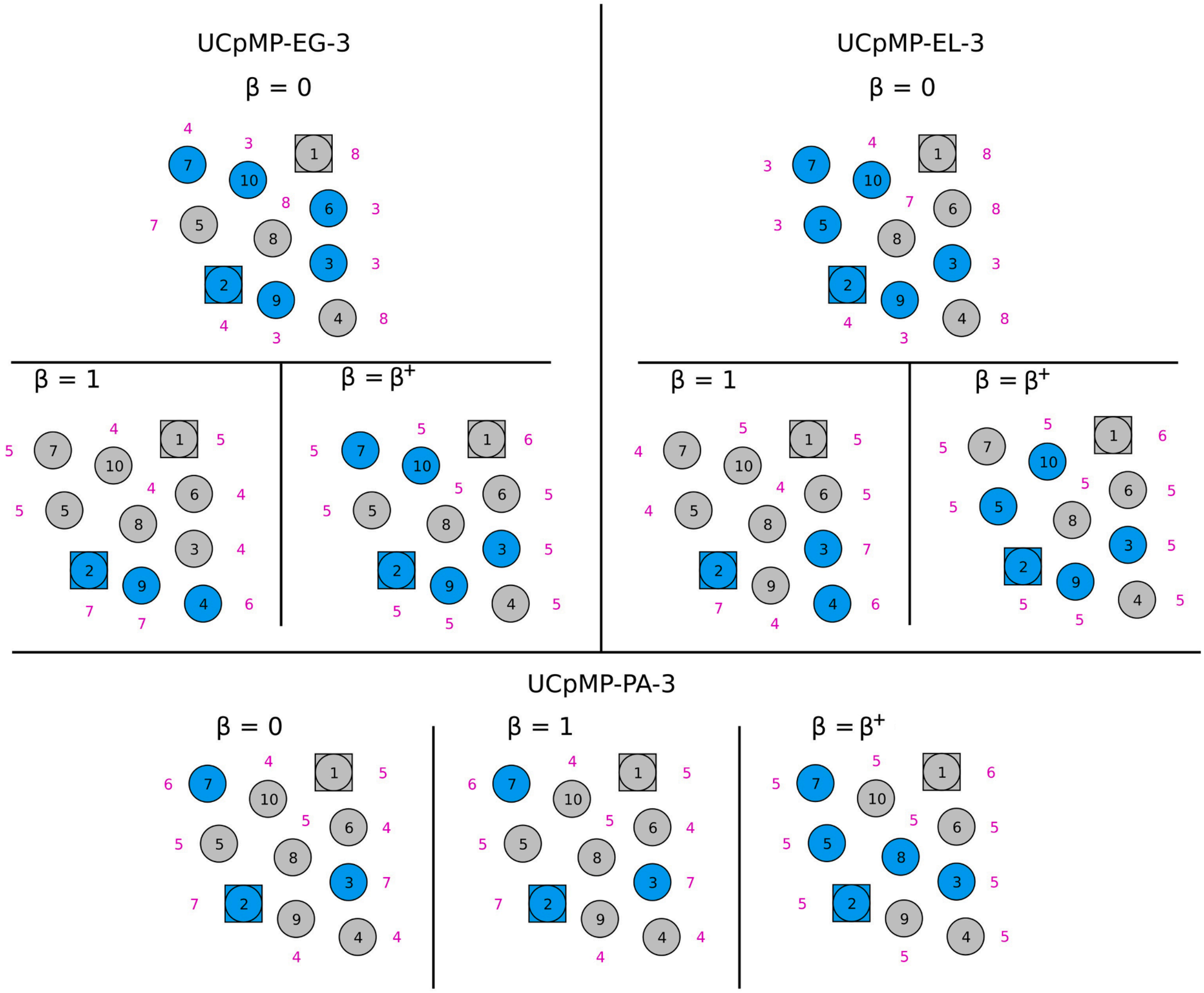


Fig. 5. Solutions of Example 6 obtained with the UCpMP model under welfarism-based formulations for $\alpha = \alpha^+$. Panels correspond to Equal Gain (UCpMP-EG-3), Equal Loss (UCpMP-EL-3), and Proportional Allocation (UCpMP-PA-3), shown for increasing values of β ($\beta = 0$, $\beta = 1$, and $\beta = \beta^+$). Squares indicate facility locations, colors identify the facility serving each client, and purple numbers denote the allocated demand z_i.. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

heuristic design. First, optimal or near-optimal facility locations under capacity scarcity are often very close to those obtained from the uncapacitated p-median problem (UpMP). Second, a substantial portion of the computational complexity of the UCpMP arises from the binary location variables and their associated linearization constraints, while fixing facility locations drastically simplifies the remaining allocation problem.

Building on these observations, the proposed heuristic leverages the structural link between the UCpMP and the UpMP by using the latter as a starting point, and then explores a carefully restricted neighborhood of alternative facility locations. This neighborhood is defined using spatial proximity, which is consistent with the metric structure of transportation costs and allows meaningful candidate solutions to be evaluated efficiently. For each candidate set of facility locations explored in the neighborhood, the heuristic solves the UCpMP with locations fixed, optimizing client-to-facility assignments, client allocations, and facility capacity usage, while fully enforcing all fairness rules, order preservation, and welfarism.

The resulting procedure, denoted UCpMP-NB, is a deterministic neighborhood-based heuristic designed to obtain solutions to the under-capacitated p-median problem in reasonable computational times. The method is detailed in Algorithm 1.

The method begins by solving the classical UpMP for a given p, obtaining a solution (x^{UpMP}, y^{UpMP}) and identifying the set $J' = \{j \in J : y_j^{UpMP} \neq 0\}$ that contains the location of the p facilities. Next, the UCpMP-EG-3 (UCpMP-EL-3 or UCpMP-PA-3) model is solved with the facility-opening variables fixed to y^{UpMP}, which yields an initial feasible solution $\mathcal{S} = (x^{UCpMP}, y^{UpMP}, s^{UCpMP}, z^{UCpMP}, w^{UCpMP})$ and its associated objective value $\mathcal{F} = \sum_{i\in I}\sum_{j\in J} c_{ij} z_i^{UCpMP} x_{ij}^{UCpMP} - \alpha \sum_{i\in I} z_i^{UCpMP} - \beta w^{UCpMP}$. To explore alternative facility locations, the n closest nodes to each $j \in J'$ are computed according to the Euclidean distance

Algorithm 1 Procedure UCpMP-NB.

Require: $I, J, \alpha, \beta, p, n$ and c_{ij} for all $i \in I$ and $j \in J$

Solve UpMP $\rightarrow \left(x^{UpMP}, y^{UpMP}\right)$
Calculate UCpMP-EG-3 (UCpMP-EL-3 or UCpMP-PA-3) fixing $y^{UpMP} \rightarrow \left(x^{UCpMP}, y^{UpMP}, s^{UCpMP}, z^{UCpMP}, w^{UCpMP}\right)$
Set $\mathcal{S} = \left(x^{UCpMP}, y^{UpMP}, s^{UCpMP}, z^{UCpMP}, w^{UCpMP}\right)$
Calculate $\mathcal{F} = \sum\limits_{i\in I}\sum\limits_{j\in J} c_{ij} z_i^{UCpMP} x_{ij}^{UCpMP} - \alpha \sum_{i\in I} z_i^{UCpMP} - \beta w^{UCpMP}$
Let $J' = \{j \in J : y_j^{UpMP} \neq 0\}$
for all $j \in J'$ **do**
  Obtain the n closest nodes, denoted by N_j
end for
$C \leftarrow \{(j_1, \dots, j_{|J'|}) \in \prod\limits_{j\in J'} N_j \;:\; j_r \neq j_s \quad \forall r \neq s\}$
for all $C \in \mathcal{C}$ **do**
  Calculate UCpMP-EG-3 (UCpMP-EL-3 or UCpMP-PA-3) fixing $y = C \rightarrow \left(x^{UCpMP}, y, s^{UCpMP}, z^{UCpMP}, w^{UCpMP}\right)$
  if $\sum\limits_{i\in I}\sum\limits_{j\in J} c_{ij} z_i^{UCpMP} x_{ij}^{UCpMP} - \alpha \sum_{i\in I} z_i^{UCpMP} - \beta w^{UCpMP}$ is lower than $\mathcal{F}$:
    $\mathcal{S} = \left(x^{UCpMP}, y, s^{UCpMP}, z^{UCpMP}, w^{UCpMP}\right)$
    $\mathcal{F} = \sum\limits_{i\in I}\sum\limits_{j\in J} c_{ij} z_i^{UCpMP} x_{ij}^{UCpMP} - \alpha \sum_{i\in I} z_i^{UCpMP} - \beta w^{UCpMP}$
end for
return $\mathcal{S}$

Table 1
Resolution with UCpMP-NB heuristic of Example 1 for $\alpha = 100$, $\beta = 0$.

n	Nodes	PA				EG				EL			
		Obj	$\sum\limits_{i\in I}\sum\limits_{j\in J} c_{ij}x_{ij}z_i$	%dif	time	Obj	$\sum\limits_{i\in I}\sum\limits_{j\in J} c_{ij}x_{ij}z_i$	%dif	time	Obj	$\sum\limits_{i\in I}\sum\limits_{j\in J} c_{ij}x_{ij}z_i$	%dif	time
n = 0	[9,10]	-10614	1386	0.65	1.58	-10415	1585	7.46	0.67	-11022	978	0.00	0.38
n = 1	[9,6]	-10593	1407	2.18	1.19	-10462	1538	4.27	0.81	-10612	1388	41.92	0.70
	[3,10]	-10376	1624	17.94	1.28	-10153	1847	25.22	0.77	-10584	1416	44.79	1.86
	[3,6]	-9948	2052	49.02	1.86	-9805	2195	48.81	0.76	-10122	1878	92.02	2.58
	[9,5]	-10527	1473	6.97	0.99	-10525	1475	0.00	0.63	-10291	1709	74.74	2.01
	[8,10]	-10623	1377	0.00	1.39	-10467	1533	3.93	0.83	-10036	1964	100.82	2.25
n = 2	[3,5]	-9908	2092	51.92	1.64	-9895	2105	42.71	0.61	-10446	1554	58.90	2.70
	[8,6]	-10235	1765	28.18	1.82	-10068	1932	30.98	0.90	-9801	2199	124.85	3.06
	[8,5]	-10173	1827	32.68	1.28	-10179	1821	23.46	0.64	-10965	1035	5.83	1.12

and collected in the sets N_j. A neighborhood of candidate solutions is then defined as the Cartesian product $\prod_{j\in J'} N_j$, restricted to tuples with no repeated elements, which results in the set $\mathcal{C}$ of feasible combinations where one alternative location is chosen for each facility in J'. For every $C \in \mathcal{C}$, the Calculate UCpMP-EG-3 (UCpMP-EL-3 or UCpMP-PA-3) problem is solved again where $y = C$ is fixed, generating a candidate new feasible solution $(x^{UCpMP}, y, s^{UCpMP}, z^{UCpMP}, w^{UCpMP})$ with its corresponding objective value. Whenever an improvement is found, i.e., the objective value of the solution $(x^{UCpMP}, y, s^{UCpMP}, z^{UCpMP}, w^{UCpMP})$ is less than $\mathcal{F}$, the current solution is updated as $\mathcal{S} = (x^{UCpMP}, y, s^{UCpMP}, z^{UCpMP}, w^{UCpMP})$ and $\mathcal{F} = \sum\limits_{i\in I}\sum\limits_{j\in J} c_{ij} z_i^{UCpMP} x_{ij}^{UCpMP} - \alpha \sum_{i\in I} z_i^{UCpMP} - \beta w^{UCpMP}$. This exhaustive evaluation of the neighborhood ensures that all meaningful reallocations of the initial p facilities within their n nearest neighbors are tested, while avoiding redundancies by excluding repeated locations, and the procedure finally returns the best solution $\mathcal{S}$ identified throughout the search.

The application of UCpMP-NB of Example 1 is as follows.

Example 7. The resolution of Example 1 with UCpMP-NB is represented in Table 1. The Table represents how much time (in seconds) it takes for the evaluation of each node combination, together with the $\sum\limits_{i\in I}\sum\limits_{j\in J} c_{ij}x_{ij}z_i$ cost and the objective function value, Obj. The gap with respect to the optimal solution (%dif) is calculated with $\sum\limits_{i\in I}\sum\limits_{j\in J} c_{ij}x_{ij}z_i$ cost, and not Obj, as it is more realistic and unfavorable.

For the first step, the UpMP is solved for Example 1 and provides the selection of nodes [9,10]. These nodes are correctly chosen as facilities for EL. For the rest of cases, an exploration of neighbors is needed. The closest node to 9 is node 3, followed by node 8. The closest node to 10 is node 6, followed by node 5. When exploring the $n = 1$ closest neighbor, this yields the combinations [9,6], [3,10] and [3,6]. When exploring $n = 2$ closest neighbors, in addition to these combinations, there is [9,5], [8,10], [3,5], [8,6] and [8,5]. For the remaining cases, optimality is reached when exploring $n = 2$ neighbors: PA finds [10,8] and EG finds [9,5].

When compared to the exact UCpMP models, UCpMP-NB is very effective. UCpMP-PA-3, UCpMP-EG-3 and UCpMP-EL-3 took 35.2, 6.1 and 7.2 seconds respectively. All the UCpMP-NB simulations outperform the UCpMP models time-wise. When the facility nodes match those obtained by the first-step UpMP, it typically lasts less than 1.6 seconds overall. When neighborhood exploration is required, PA and EG take 13.0 s and 6.0 s, respectively, outperforming the exact UCpMP models.

Before concluding this section, we briefly comment on the applicability of the proposed models and heuristic to settings in which resources and demands are divisible rather than discrete. This extension is relevant in applications such as water or energy and can be accommodated with minimal changes to the formulation.

Remark 1. The mathematical model for the UCpMP does not change significantly when $z_i \in \mathbb{R}_+$ and $d_i \in \mathbb{R}_+$ for all $i \in I$, and $Q_t \in \mathbb{R}_+$ for all $t \in \{1, \dots, p\}$, since the only modification concerns the constraint defining z_i.

The α-threshold in the continuous allocation case does not coincide with that of the discrete case. However, a correct threshold can still be obtained by following a procedure similar to the discrete setting. Analogously, the β-threshold differs as well but can be derived using the same type of reasoning.

Moreover, the different allocation rules can be incorporated with only minor modifications, which mainly involve removing certain constraints and adjusting small ϵ-values in some of them. It is noteworthy that the optimal locations of the facilities may change depending on whether the goods are discrete or continuous. Indeed, using the same Example 1 from the paper, this change in location can be easily observed.

Finally, the developed heuristic operates without difficulty in the continuous case. Detailed theoretical analysis and numerical examples for the case of divisible goods can be found in the supplementary material.

8. Numerical experiments

The numerical experiments aim to illustrate the behavior and performance of the proposed models on different types of instances. Since these models are new to the literature and require location problems with capacity shortages, benchmark problems from standard OR libraries were adapted to fit the UCpMP formulation. Specifically, Beasley's CpMP instances (Beasley, 1990) were modified so that the total available capacity across the p facilities is lower than the total client demand. The adapted instances are publicly available at Martín Melero (2025). All experiments were run on an HP Z4 G4 workstation (Intel(R) Xeon(R) W-2235 CPU @ 3.80 GHz, 6 cores/12 threads, 64 GB RAM). The models were implemented in Python using Pyomo and solved with Gurobi 11.0.3.

Our UCpMP variants allocate scarce resources using different fairness rules and determine facility capacities. We assess their contribution to the fairness-in-facility-location literature by comparing them with seven benchmark models ($BM1$–$BM7$) from previous studies, which we describe below. Each expression indicates the term added to the objective function, weighted by a parameter ω to control the trade-off between fairness and efficiency.

- $BM1$: Two-step approach–solve the uncapacitated p-median problem, assign capacities proportionally to facility demand, then allocate resources to clients via EG, EL, or PA rules with order preservation.
- $BM2$: Maximize the allocation to each facility's least-served client: $-\sum_{j\in J} \min_{i\in I}(z_i x_{ij})$.
- $BM3$: Minimize squared differences between client demand and allocation: $+\sum_{i\in I}(d_i - z_i)^2$.
- $BM4$: Minimize sum of pairwise absolute allocation differences (*Equity I*, Digehsara et al., 2025): $+\sum_{i\in I}\sum_{k\in I}|z_i - z_k|$.
- $BM5$: Minimize sum of maximum pairwise differences (*Equity II*, Digehsara et al., 2025): $+\sum_{i\in I}\max_{k\in I}|z_i - z_k|$.
- $BM6$: Minimize maximum pairwise difference (*Equity III*, Digehsara et al., 2025): $+\max_{i,k}|z_i - z_k|$.
- $BM7$: Maximize sum of per-facility Gini indices (Drezner et al., 2009): $+\sum_{j\in J} \frac{\sum_{i,k}|z_i - z_k| x_{ij} x_{kj}}{2\sum_i x_{ij}\sum_i z_i x_{ij}}$.

The main elements contrasted in the simulations are the individual allocations (from the logistic perspective), the total network cost (from the economic perspective), the unused capacity of the facilities (from the efficiency perspective), and the difference between the maximum and minimum allocations (from the fairness perspective). In addition, symmetry and order preservation are examined to assess the internal consistency and fairness properties of the allocation rules. Part of this information is analyzed in this section, while the remainder is discussed in the following section on managerial insights.

Within the wide variety of location problems with lack of capacity, among others, we have identified two types of problems whose solutions are of particular interest. The first type, referred to as heterogeneous problems (see Example 1), involves highly diverse demands and transportation costs. The second type, homogeneous problems (see Example 6), arises when clients have similar or identical demands and the transportation costs to potential facilities are similar or equal. The computational study includes experiments on both heterogeneous and homogeneous cases. The computational experiments are structured in three parts.

In the first part, small instances of the problem are solved. Specifically, simulations are performed with 10 clients. The UCpMP-EG-2, UCpMP-EL-2, and UCpMP-PA-2 models are tested on both heterogeneous and homogeneous datasets for several values of α. After fixing α to the bound obtained in Proposition 1, the UCpMP-EG-3, UCpMP-EL-3, and UCpMP-PA-3 models are evaluated on the same heterogeneous and homogeneous cases to analyze how solutions vary for different values of β. Finally, the effect of varying the weighting parameter ω is studied using benchmarks $BM1$ through $BM7$, and the results are compared with those obtained from the UCpMP-EG-3, UCpMP-EL-3, and UCpMP-PA-3 models.

In the second part, medium-sized instances including 15, 20, and 25 clients are studied for both homogeneous and heterogeneous cases. From the fairness perspective, benchmarks $BM1$ through $BM7$ are compared with the UCpMP-EG-3, UCpMP-EL-3, and UCpMP-PA-3 models. From the heuristic perspective, the UCpMP models are compared with the heuristic UCpMP-NB, with $n = 2$ for the

Table 2
Summary of changes in y_j, s_{tj}, x_{ij}, z_i for small instances, as α and β change.

α/β	Cases	p	EG, change in…				EL, change in…				PA, change in…			
			y_j	s_{tj}	x_{ij}	z_i	y_j	s_{tj}	x_{ij}	z_i	y_j	s_{tj}	x_{ij}	z_i
$\alpha \in \{1, \ldots, \alpha^+\}$	Heterogeneous	2		✓		✓		✓	✓	✓		✓		✓
		3		✓		✓	✓	✓	✓	✓	✓	✓	✓	✓
		4	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
	Homogeneous	2			✓	✓			✓	✓			✓	✓
		3			✓	✓			✓	✓			✓	✓
		4			✓	✓			✓	✓			✓	✓
$\beta \in \{1, \ldots, \beta^+\}$ $\alpha = \alpha^+$	Heterogeneous	2					✓	✓	✓	✓	✓	✓	✓	✓
		3							✓	✓				✓
		4					✓	✓	✓	✓				✓
	Homogeneous	2			✓	✓			✓	✓			✓	✓
		3			✓	✓							✓	✓
		4			✓	✓			✓	✓				

heterogeneous cases and $n = 1$ for the homogeneous cases, and further compared against Hexaly, a solver that combines traditional global optimization with heuristic techniques such as column generation and local search (Hexaly, 2025).

In the third part, large-scale instances involving 40, 60, 80, and 100 clients are simulated using the heuristic UCpMP-NB. Throughout all simulations, different values of p are also considered. Detailed results are presented in a supplementary material due to space limitations.

8.1. Small instances

We consider both the heterogeneous and homogeneous instances with 10 clients for $p = 2, 3, 4$ and $\alpha \in \{0, 1, 10, 10^2, 10^3, 10^4, \alpha^+\}$. Then, we solve again the same heterogeneous and homogeneous instances with 10 clients for $p = 2, 3, 4$, fixing $\alpha = \alpha^+$ and varying $\beta \in \{0, 1, 10, 10^2, 10^3, 10^4, \beta^+\}$. These analyses allow us to study the sensitivity of the model to parameters α and β. Table 2 summarizes the main changes observed in the solution configuration as these parameters vary. Specifically, the table indicates whether:

- there are changes in the locations where facilities are opened (column y_j);
- whether the same or a different capacity level is chosen (column s_{tj});
- the assignment of clients to facilities changes (column x_{ij}); and
- the amount of allocation received by each client changes (column z_i).

Changes of all four types are observed in the heterogeneous case. In the homogeneous case, the facilities always remain in the same locations, and the capacities associated with them are always the same, although the other decisions vary.

Two examples of these changes in the heterogeneous case, when α varies, are as follows. For $p = 2$ and the Equal Loss rule, facilities are opened at clients 7 and 10. For low values of α, no clients other than client 10 are assigned to the facility located at this node. In contrast, for high values of α, the network becomes more balanced, with a total of four clients assigned to the facility at client 10. This behavior results from a greater distribution of network capacity among the different clients. For $p = 4$, when α is low, facilities are opened at clients 7, 8, 9, and 10 under the Equal Gain and Equal Loss rules, and at clients 5, 6, 8, and 10 under the Proportional Allocation rule. Conversely, for high values of α, facilities are opened at clients 3, 4, 7, and 10 across all three models, with slight variations under the Equal Loss rule compared to the other two.

Other examples, when β varies and we solve the heterogeneous instances, are as follows. For $p = 2$ and the Proportional Allocation rule, facilities are opened at clients 4 and 8 instead of at clients 7 and 10. Similarly, under the Equal Loss rule with $p = 4$, one of the opened facilities is relocated from client 3 to client 5. In the homogeneous setting, from $\beta = 1$ onwards, the facility locations are already determined and remain unchanged.

Lastly, we analyze and compare the results for the heterogeneous and homogeneous cases with 10 clients, obtained using the UCpMP-EG-3, UCpMP-EL-3, and UCpMP-PA-3 models, along with benchmarks $BM1--BM7$. For $BM2$–$BM7$, we consider $\omega \in \{0, 1, 10, 10^2, 10^3, 10^4, 10^5, 10^6\}$; in the case of $BM7$ only, $\omega = 10^7$ is also included. The results of these experiments are summarized in Table 3. This table has the same headers as Table 2. In this case, a tick mark appears whenever there are changes with respect to solving UCpMP-3 for any of ω.

For $p = 2$ in the heterogeneous case, the approaches most similar to UCpMP-3 are BM2 and BM3, which differ only in the clients' received demands. For $p = 3$, only BM2 coincides with UCpMP-3. Although this is not shown in the table, the solutions also differ among the various configurations BM1 and BM3–BM7. In the homogeneous instances, when $p = 4$, BM2 coincides completely with UCpMP-3. In the remaining homogeneous instances, the differences are mainly observed in the allocation and in the amounts received by the clients.

Some opened facilities resemble those of the UCpMP-EG-3, UCpMP-EL-3, and UCpMP-PA-3 models, for example, facilities are established at clients 7 and 10 for $p = 2$ in the heterogeneous cases, while in other configurations, such as $BM3$ with $p = 3$, completely different facilities are opened.

Table 3
Summary of changes in y_j, s_{tj}, x_{ij}, z_i of BM1-BM7 with respect to UCpMP-3 models.

Cases	p	BM1				BM2				BM3				BM4				BM5				BM6				BM7			
		y_j	s_{tj}	x_{ij}	z_i	y_j	s_{tj}	x_{ij}	z_i	y_j	s_{tj}	x_{ij}	z_i	y_j	s_{tj}	x_{ij}	z_i	y_j	s_{tj}	x_{ij}	z_i	y_j	s_{tj}	x_{ij}	z_i	y_j	s_{tj}	x_{ij}	z_i
Heterogeneous	2	✓	✓	✓	✓				✓				✓			✓	✓			✓	✓			✓	✓	✓	✓	✓	✓
	3	✓	✓	✓	✓				✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
	4	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
Homogeneous	2			✓	✓			✓	✓			✓	✓			✓	✓			✓	✓			✓	✓				
	3			✓	✓			✓	✓			✓	✓			✓	✓			✓	✓			✓	✓				
	4			✓	✓							✓	✓			✓	✓			✓	✓			✓	✓				

Table 4
Medium sized, heterogeneous cases for α^{++}, $\beta = 1$, $\omega = 10^6$.

$\lvert I \rvert$	p	Metric	UCpMP			UCpMP-NB, n = 2			Hexaly		
			EG-3	EL-3	PA-3	EG-3	EL-3	PA-3	EG-3	EL-3	PA-3
15	2	%dif	0.00	0.00	0.00	0.00	0.00	0.00	67.34	83.16	32.07
		time	42	55	354	41	46	64	1800	1800	1800
15	3	%dif	0.00	0.00	0.00	0.00	0.00	0.00	57.95	170.76	50.92
		time	58	69	238	315	184	961	1800	1800	1800
20	2	%dif	0.00	0.00	0.00	0.00	0.60	0.00	80.40	72.68	76.08
		time	306	492	3442	153	166	229	1800	1800	1800
20	3	%dif	0.00	0.00	0.00	1.40	8.18	5.32	87.82	203.01	99.96
		time	652	3027	2779	635	924	1005	1800	1800	1800
25	2	%dif	0.00	0.00	0.00	0.29	0.26	0.00	96.69	71.58	87.33
		time	1281	12500	18071	273	404	572	1800	1800	1800
25	3	%dif	0.00	0.00	0.00	1.27	0.00	0.00	127.09	137.17	164.01
		time	7244	11808	10892	1115	1425	2219	1800	1800	1800

Moreover, it is observed that for high ω values $BM7$ based on Gini index applies Equal Gain distribution rule, while $BM3$ based on MSE applies Equal Loss distribution rule. For example, in heterogeneous cases and $BM7$, $\omega = 10^5$ in $p = 2$ matches with UCpMP-EG-3. For homogeneous cases and $\omega = 10^4$, for every p the solution of UCpMP-EG-3 and $BM7$ match.

The UCpMP-EG-3, UCpMP-EL-3, and UCpMP-PA-3 models consistently preserve symmetry and order preservation within facilities. In contrast, benchmarks $BM2$–$BM7$ frequently break symmetry or/and order preservation, allocating unequal amounts to customers with identical conditions, or larger amounts to those whose demand is lower than that of others. Although some benchmarks recover these properties for very large ω values (typically $\omega \geq 10^6$), this occurs at the expense of higher network costs or greater allocation disparities than the UCpMP-3 models.

It is important to note that, theoretically, the Gini index corresponds to the Constrained Equal Awards (CEA) principle (the Equal Gain rule adapted in this context), while the MSE corresponds to the Constrained Equal Losses (CEL) principle (the Equal Loss rule adapted in this context) in bankruptcy problems. However, in this context, they do not always yield identical solutions. This divergence arises from the modeling approach: in UCpMP-3 models, the CEA and CEL principles are incorporated as constraints that govern the resource allocation process, while the objective function minimizes the total transportation cost and maximizes the total distributed capacity. In contrast, in the benchmark models, the Gini index and MSE are included directly in the objective function, weighted by a parameter ω, rather than as constraints. This integration causes an inherent trade-off between fairness and efficiency, since increasing ω amplifies the fairness term but simultaneously conflicts with the distance minimization and capacity utilization terms.

8.2. Medium instances

The next step is further comparing these benchmarks with medium instances of the UCpMP, where the heuristics employed for large instances of the UCpMP will also be tested. In these simulations, an $\omega = 10^6$ is selected for the benchmarks, as it is where $BM2 - BM7$ are already stable.

The instances with $p = 2$ and $p = 3$ for 15, 20, and 25 clients have been solved. The analysis is now twofold: comparing $BM1$–$BM7$ on larger instances with the proposed models, and assessing how well the heuristics perform in order to use them for large instances where the UCpMP-EG-3, UCpMP-EL-3, and UCpMP-PA-3 models require excessive computation time. For both the proposed models and the heuristics, the runtime (in seconds) and the gap with respect to the optimal network cost, %dif,[2] are also compared. Finally,

[2] We define $\%\text{dif} := 100 \cdot \frac{\text{cost}_{\text{UCpMP-NB}} - \text{cost}_{\text{UCpMP-EG-3}}}{\text{cost}_{\text{UCpMP-EG-3}}}$, and analogously for the UCpMP-EL-3 and UCpMP-PA-3 models.

Table 5
Main metrics for medium sized, heterogeneous cases for α^{++}, $\beta = 1$, $\omega = 10^6$.

Metric	$\|I\|$	p	Benchmarks									UCpMP		
			BM2	BM3	BM4	BM5	BM6	BM7	BM1			EG-3	EL-3	PA-3
									EG	EL	PA			
$\max_{i\in I} z_i - \min_{i\in I} z_i$	15	2	19	21	14	14	14	18	19	24	20	18	22	19
	15	3	21	21	17	17	17	19	19	22	19	19	20	20
	20	2	34	38	15	15	15	36	33	42	35	36	41	34
	20	3	34	39	22	22	22	30	39	48	43	36	47	37
	25	2	32	35	19	19	19	32	35	39	36	34	39	36
	25	3	34	36	28	28	28	33	36	40	36	33	36	36
$\sum_{i\in I}\sum_{j\in J} c_{ij} x_{ij} z_i$	15	2	5581	5084	5857	5338	5298	5102	3727	3985	3775	4869	4661	4733
	15	3	6941	5007	6324	6324	6232	7695	3398	3103	3175	4205	4193	4193
	20	2	13687	11989	11581	11556	11556	12816	10855	10644	10706	11187	11075	11179
	20	3	15508	11498	12534	12012	11994	12143	8320	8359	8302	10768	10242	10437
	25	2	10294	10457	10127	10103	10103	11727	8897	9496	9152	9811	10017	10002
	25	3	12424	9762	10029	9744	9744	1294	7582	7222	7318	9649	9586	9624

we compare Hexaly (30-minute limit) with our heuristic. A summary of the results for the heterogeneous instances is presented in Tables 4 and 5.

Table 4 shows that our models solves instances with 15 clients and $p = 2$ in under six minutes, but requires over two hours for 25 clients and $p = 3$. When performing the time-wise comparison with heuristics, UCpMP-NB with $n = 2$ is faster than UCpMP-EG-3, UCpMP-EL-3, and UCpMP-PA-3 in all heterogeneous cases except for 15 clients and $p = 3$. This is because 15 clients is still relatively fast to simulate with the exact models (less than 6 minutes), while for UCpMP-NB with $n = 2$ it requires 27 node combinations to be tested. From the 18 possible comparisons (3 numbers of clients $\times$ 2 p values $\times$ 3 distribution rules), UCpMP-NB obtains exactly the same solution in 11 of them. The largest deviations are found in 20 clients and $p = 3$, with a maximum of %dif $=$ 8.18%. One advantage of UCpMP-NB is the possibility to tune n for reaching the desired level of network cost deviation (reducing %dif), at the expense of increasing the combinations and computational time, especially as p increases. In any case, for the current simulations, $n = 2$ achieves an acceptable level of optimum deviation with reasonable time: for the rest of clients and p, all %dif values are lower than 1.50%. A good example illustrating the efficiency of UCpMP-NB is found in the cases with 25 clients. From those 6 comparisons, the largest deviation in network cost is only %dif $=$ 1.27%, while the time saving reaches even one order of magnitude for some $p = 2$ cases. As for the heuristic benchmark, Hexaly, it significantly underperforms in the heterogeneous cases. It should also be noted that Hexaly leaves unused capacity and does not conserve symmetry and order preservation properties within the facilities.

Table 5 shows the allocation gaps and network costs. In general, the benchmarks that most consistently minimize this gap are $BM4$, $BM5$, and $BM6$, but they do so at a higher cost than UCpMP-3. On the other hand, this range merely indicates that the approaches differ in nature. A closer look at the solutions reveals that the most evident effect of $BM1$–$BM7$ is a lower degree of symmetry in what clients receive and higher network costs.

Considering both homogeneous and heterogeneous cases, we observe that the EL rule is usually the most time-consuming in homogeneous instances, whereas PA tends to be the slowest in heterogeneous ones. In both settings, EG is the fastest and scales better as problem complexity increases. Overall, the gap between the UCpMP-EG-3, UCpMP-EL-3, and UCpMP-PA-3 models and the benchmarks is smaller in the homogeneous setting, owing to the enforced symmetry in demands and inter-client distances.

8.3. Large instances

The computation times for applying UCpMP-NB in large heterogeneous and homogeneous cases are reported in Fig. 6. We observe that the evolution of simulation times when the instance size increases is similar for the three rules, being slightly worse for the proportional rule. Furthermore, in the cases of all three rules, the times corresponding to the heterogeneous cases perform better than the homogeneous cases; that is, in the heterogeneous cases, the increase in simulation times with respect to instance size is much less pronounced than in the homogeneous cases.

9. Managerial insights

9.1. Which rule to choose and when?

In practice, the choice between Equal Gain, Equal Loss, or Proportional allocation depends on the nature of the problem and the fairness criterion that is most suitable for the problem at hands; see Thomson (2003, 2019). Each rule corresponds to a distinct notion of fairness that has been extensively studied in resource allocation, or bankruptcy problems. Moreover, each of these rules satisfies different properties related to different principles such as impartiality, equity, priority, consistency or solidarity, which are related to the idea of fairness. Thus, taking into account the principle of fairness and the properties considered relevant, a choice can be made as to which rule to implement; see Thomson (2019). Below are some ideas about the contexts in which each of the rules may be most appropriate.

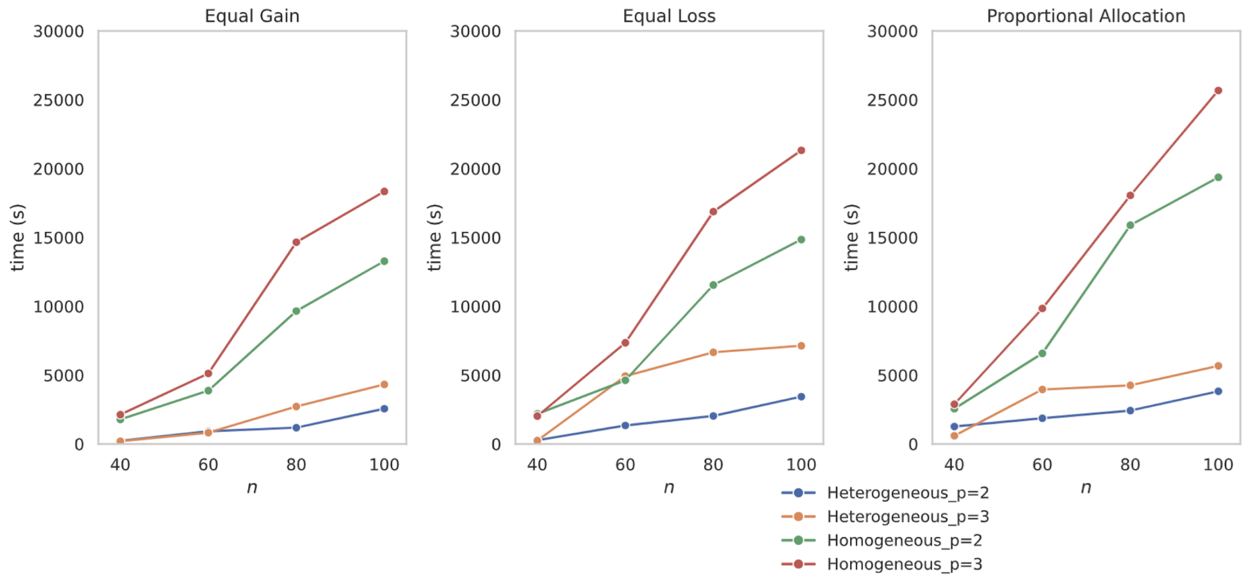


Fig. 6. Computational time of the UCpMP-NB heuristic under the different allocation rules. Panels correspond to Equal Gain, Equal Loss, and Proportional Allocation. The horizontal axis represents the number of clients n, while the vertical axis reports the solution time (in seconds). Results are shown for heterogeneous and homogeneous capacity settings and for $p = 2$ and $p = 3$ facilities..

Equal Gain Rule (Constrained Equal Awards, CEA): This rule is particularly suitable when the goal is to guarantee that every client receives the same incremental benefit from the available capacity. It allocates resources by increasing all claims equally until some claim is fully satisfied. In other words, smaller demands are fully satisfied first, and no agent receives more than their claim. This makes Equal Gain attractive in contexts where equity in access is prioritized and a fair baseline of access for all is critical. Practical examples include the distribution of medical supplies or vaccines during emergencies (Itani and Shakya, 2017), fair bandwidth allocation in communication networks (Lucas-Estañ et al., 2012), and allocation of energy in microgrids (Alyami, 2024). In such scenarios, algorithms that implement equal incremental allocations (progressive filling) are used to ensure no small claimant is left unmet. The emphasis is on leveling up everyone's allocation equally until limited resources are exhausted, reflecting a strong equity motive. The Equal Gain rule reduces inequality in allocations, as it minimizes disparities in received amounts across claimants. If allocations are represented through a Lorenz curve, the CEA solution Lorenz-dominates other feasible allocations satisfying claim constraints, that is, its Lorenz curve lies nowhere below those of alternative rules such as proportional or equal-loss.

In contrast, the Equal Loss rule (Constrained equal losses, CEL) is preferable in scarcity scenarios where the total demand exceeds available supply and the emphasis is on sharing sacrifices. Equal Loss distributes the shortfall evenly among all clients, ensuring that no one bears a disproportionate reduction. Every claimant gives up an identical amount of their claim (to the extent possible) so that the pain of shortage is spread fairly. This approach is especially appealing in situations such as water rationing during droughts (Wang et al., 2025) or budget cuts in public institutions (Giménez-Gómez et al., 2025), contexts where the legitimacy of an allocation depends on everyone sharing the sacrifice equally. For example, in a water allocation crisis, the CEL rule would reduce the water entitlement of each region by the same amount, a method that is often viewed as fair when resources are critically low. By imposing equal losses on all, this rule ensures that larger claimants do not end up shouldering an outsized cut relative to others, aligning with common fairness expectations in many public rationing policies. The Equal Loss rule can also be viewed as a least-squares (MSE) solution that minimizes the aggregate variance in losses across claimants. By imposing a common loss level, the rule minimizes the sum of squared differences between individual and collective sacrifices, leading to the most balanced distribution of reductions compatible with the capacity constraint.

The proportional rule represents a compromise approach that scales each client's allocated amount by the same factor relative to their claim, thereby preserving the original proportions of demand. Each claimant receives a share of the resource in proportion to their claim size, ensuring that relative differences among needs are maintained while total allocations remain feasible. This rule is particularly suitable in contexts where preserving proportional fairness is important and where stakeholders perceive equality of ratios as equitable. In practice, proportional allocation is one of the most frequently implemented rules in real-world scarcity problems. For example, in water management, irrigation districts often distribute available water proportionally to users' rights during droughts (Gómez-Limón et al., 2021). In energy and environmental policy, emission reduction quotas are frequently assigned by equal proportional cuts across firms or regions, ensuring that all bear the same percentage reduction in emissions or energy use (Ringius et al., 2002). Similarly, in public budgeting, government funds are often allocated using formula-based schemes that distribute resources proportionally to population size or assessed fiscal needs (Dougherty et al., 2025). By giving all claimants the same fraction of their requested amount, the proportional approach strikes a balance between absolute equality and need-based weighting.

Table 6
Allocations and shortfall percentages to reach d_i.

		Allocation z_i by rule		
i	d_i	PA	EG	EL
1	4	3 (25.00%)	4 (0.00%)	2 (50.00%)
2	8	6 (25.00%)	7 (12.50%)	6 (25.00%)
3	12	9 (25.00%)	7 (41.67%)	10 (16.67%)

Table 7
Cost increases vs. baseline (327) and two-step penalty.

Rule	One-step cost	% vs. base	× base	Two-step cost	% vs. base	Δ two-step
EL	978	199.1%	2.99×	978	199.1%	+0.00%
PA	1370	319.0%	4.19×	1377	321.1%	+0.51%
EG	1448	342.8%	4.43×	1475	351.1%	+1.86%

In summary, the Equal Gain, Equal Loss, and Proportional rules capture complementary notions of fairness in allocation under scarcity. The Equal Gain (CEA) rule promotes equality in benefits, minimizing disparities and favoring smaller claimants; the Equal Loss (CEL) rule enforces equality in sacrifices, ensuring that all claimants share shortages evenly; and the Proportional rule preserves relative differences, maintaining the structure of claims while scaling them to feasibility. Together, they define a spectrum of fairness–from egalitarian benefit to egalitarian sacrifice to proportional justice–that allows decision-makers to select the principle most aligned with policy goals and social perceptions of fairness (Thomson, 2003, 2019). The following example illustrates the effect of each of these rules in the allocation obtained by each client for the same problem.

Example 8. If we only consider demands $d = (4, 8, 12)$ and a total capacity $Q = 18$, the allocations for each rule and the corresponding shortfall to meet each d_i are shown in Table 6.

9.2. The cost of fairness: adding allocation rules

Introducing fairness allocation rules on top of the baseline UCpMP entails a quantifiable trade-off between fairness and network cost. In the illustrative instance of Example 1, the baseline UCpMP yields a network cost of 327 with zero unused capacity, whereas the one-step fairness-based variants result in substantially higher costs: 978 under Equal Loss (UCpMP-EL-1), 1370 under Proportional Allocation (UCpMP-PA-1), and 1448 under Equal Gain (UCpMP-EG-1). In particular, enforcing fairness rules may increase network cost by more than 300% relative to the baseline UCpMP in this illustrative instance. When order preservation within facilities is additionally enforced (two-step formulations), the corresponding costs become 978 (UCpMP-EL-2), 1377 (UCpMP-PA-2), and 1475 (UCpMP-EG-2). These results quantify the cost associated with enforcing increasingly stronger fairness requirements at the allocation level. Table 7 summarizes these differences, suggesting that the more fairness requirements are imposed, the higher the network costs, although this growth rate slows down with the number of requirements. In particular, the relative increase from one-step to two-step models remains below 2% in all cases, indicating that strengthening fairness requirements through order preservation does not dramatically deteriorate network efficiency.

9.3. Understanding the managerial implications of α parameter (capacity–efficiency)

Parameter α rewards capacity utilization in the UCpMP objective. As α increases, the model deploys more of the available capacity, which can change facility openings and client assignments; solutions stabilize at the bound provided by Proposition 1 or, when the welfarism term is included in the objective, by Corollary 1. Once α reaches that bound, the unused capacity becomes zero. Thus, α can be read as a per-unit penalty on unused capacity: small values of α tolerate slack (cheaper networks, more unused capacity), whereas large α discourage slack and may trigger different openings/assignments to mobilize capacity. In practice, α acts as a managerial dial trading off transportation cost against capacity utilization.

The results obtained in the numerical experiments are plotted in Fig. 7, which shows, for each allocation rule, the network cost and unused capacity as functions of α. The figure makes the trade-off explicit: as α increases, unused capacity decreases until it vanishes, at which point the network cost plateaus. In the heterogeneous setting, unused capacity drops from 72 to 0 units for $p = 2$ and $p = 3$ as α increases, from 60% (for $\sum_{t=1}^{2} Q_t = 120$) and 54.55% (for $\sum_{t=1}^{3} Q_t = 132$) to full utilization, while network cost rises by about 5× for $p = 2$ and 10–15× for $p \in \{3, 4\}$. These jumps arise from changes in openings, assignments and allocations.

Fig. 7 shows that, for the heterogeneous set, unused capacity falls in discrete steps while network cost rises until it stabilizes. The three rules has the same behavior for the unused capacity for $p = 2, 3$ since this value decreases from 72 to 0. Under Equal Gain with $p = 4$, the unused capacity drops from $72 \to 63 \to 2 \to 0$ as α moves from $0/1 \to 10 \to 10^2 \to\geq 10^3$ (plateau thereafter). Under Equal Loss or Proportional allocation with $p = 4$ the unused capacity decreases from $72 \to 2 \to 0$ when α moves from $0/1 \to 10^2 \to\geq 10^3$, and network cost exhibits the same plateauing pattern. In the homogeneous cases, all three rules show the same decreasing behavior in unused capacity for all value of p. Relative to costs with the three rules as α increases in the heterogeneous 10-client cases this implies about

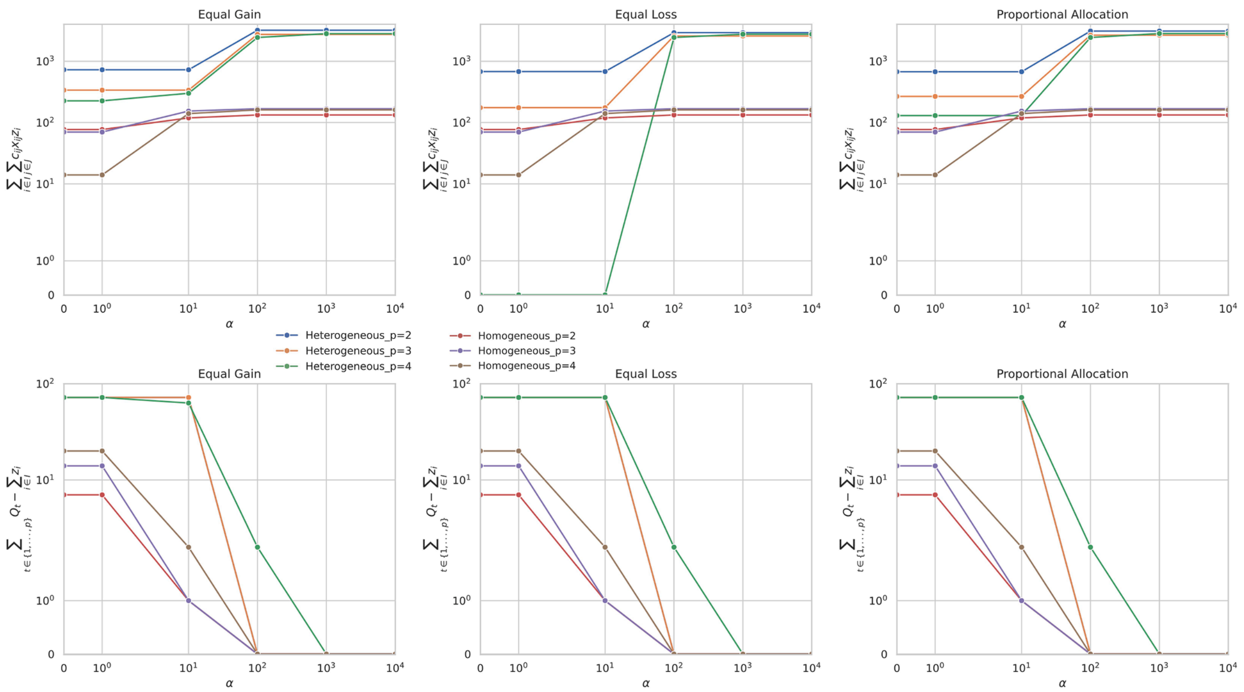


Fig. 7. Performance of the UCpMP models under different allocation rules. Panels correspond to Equal Gain, Equal Loss, and Proportional Allocation. Upper panels report the network cost $\sum_{i\in I}\sum_{j\in J} c_{ij} z_i x_{ij}$, while lower panels report the unused capacity $\sum_{t\in\{1,\dots,p\}} Q_t - \sum_{i\in I} z_i$ as a function of the parameter α. Results are shown for heterogeneous and homogeneous capacity settings and for $p = 2$, $p = 3$, and $p = 4$ facilities..

a 5× increase for p=2 and roughly 10–15× for $p \in \{3, 4\}$ relative to α=0. It is worth noting the *Equal Loss* case with $p = 4$: for $\alpha \leq 10$ the network cost is 0 (unused capacity = 72), whereas at $\alpha = 100$ it is 2441 with the unused capacity at 2.

In summary, for the UCpMP, the unused capacity is non-increasing in α, while the network cost is non-decreasing. There exists a threshold value α^+ such that the unused capacity becomes zero and the network cost remains constant thereafter.

9.4. Understanding the managerial implications of β parameter (welfarism)

The results obtained in the numerical experiments for the β analysis are presented in Fig. 8, where the network cost (top) and the difference between the maximum and minimum allocations (bottom) for each case are shown as functions of β under each allocation rule. Overall, increasing β improves the allocation received by the worst-off clients, as reflected by the reduction in the allocation gap $\max_{i\in I} z_i - \min_{i\in I} z_i$. However, these improvements tend to level off: beyond moderate values of β, the reduction in the allocation gap becomes small, indicating diminishing marginal equity gains relative to the additional cost.

In the heterogeneous setting, $\beta = 0$ and $\beta = 1$ leave solutions unchanged; from $\beta = 10$ onward most rules adjust either allocations or even opened facilities, with the difference between the maximum and minimum decreasing and the network cost increasing. For $p = 3$, the gap decreases from 22 (EL) and 17 (PA) to 16; for $p = 4$, from 21 (EL) and 17 (PA) to 16. A mild non-monotonicity appears for PA with $p = 2$, where the gap drops to 17 at $\beta = 10^2$ but bounces to 18 at $\beta = 10^3$ because the objective pushes the minimum allocation $w = \min_{i\in I} z_i$; stabilization (β^+) is reached once w attains the feasible maximum, which here coincides with fully serving the least-demanding client (6 units). Facility patterns may also change (e.g., for $p = 2$ under PA, openings move to clients 4 and 8 instead of 7 and 10; for $p = 4$ under EL, one facility shifts from client 3 to 5). In general, increasing β brings fairness, but it comes at a cost.

In the homogeneous setting, symmetry makes β especially effective: a uniform allocation already emerges at $\beta = 1$ in several cases. For $p = 2$, the minimum allocation rises from 1 to 2 (and the maximum falls 5 → 4); for $p = 3$, EL keeps the minimum allocation while EG/PA lift the minimum 2 → 3; for $p = 4$, EG and EL similarly increase the minimum allocation, decreasing the difference between the maximum and the minimum from 5 to 3. All these fairness-enhancing changes were achieved without increasing network cost.

In summary, for UCpMP-EG-3, UCpMP-EL-3, and UCpMP-PA-3, the minimum allocation $-\beta \min_{i\in I} z_i$ is non-decreasing in β. Furthermore, there exists a threshold β^+ such that, for all $\beta \geq \beta^+$, the minimum allocation attains its feasible maximum, and the optimal solution remains unchanged. The quantity $\max_{i\in I} z_i - \min_{i\in I} z_i$, shown in Fig. 8, represents the allocation disparity.

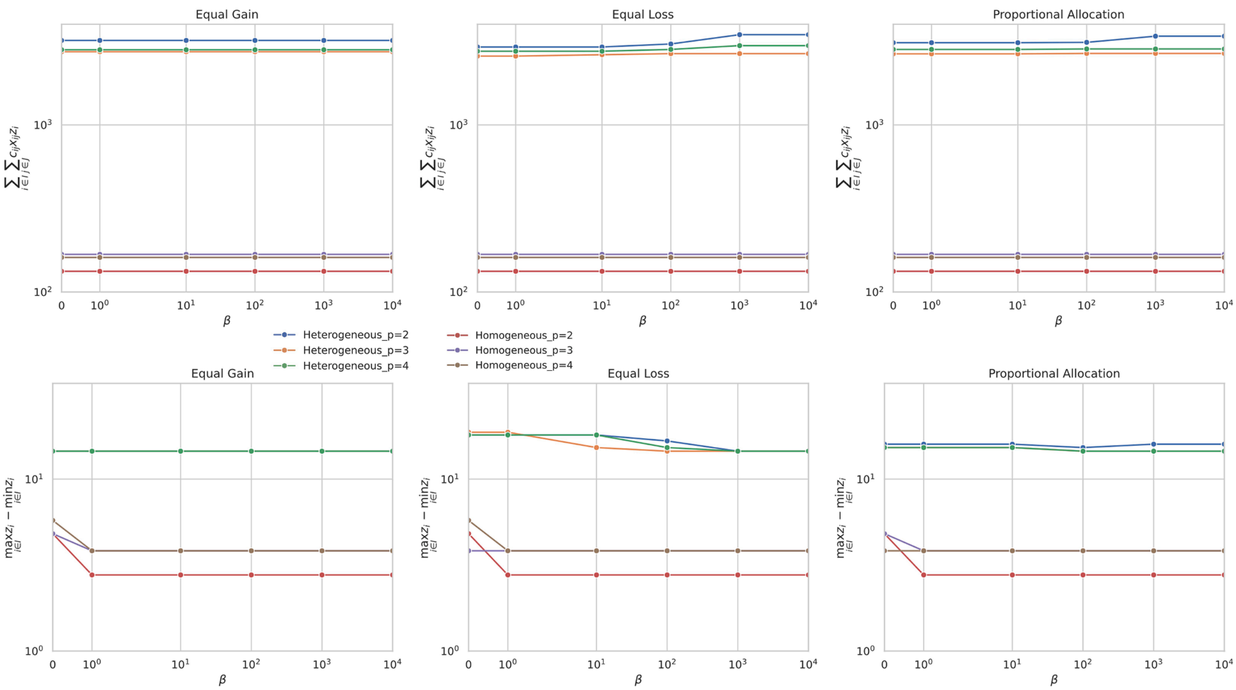


Fig. 8. Performance of the UCpMP models under different allocation rules. Panels correspond to Equal Gain, Equal Loss, and Proportional Allocation. Upper panels report the network cost $\sum_{i\in I}\sum_{j\in J} c_{ij} z_i x_{ij}$, while lower panels report the allocation gaps $\max_{i\in I} z_i - \min_{i\in I} z_i$ as a function of the parameter β. Results are shown for heterogeneous and homogeneous capacity settings and for $p = 2$, $p = 3$, and $p = 4$ facilities..

9.5. Understanding the managerial implications of ω parameter

Fig. 9 plots the network cost under benchmarks BM2–BM7 as the fairness weight ω increases. As ω grows, fairness terms dominate more strongly and the network cost rises, stabilizing around $\omega \approx 10^3$ in heterogeneous cases and $\omega \approx 10^2$ in homogeneous ones; BM7 typically requires larger ω to stabilize due to the smaller magnitude of the Gini term. No single benchmark is consistently the most expensive across p: BM7 is highest for p=2, BM3 for p=3, and (almost) all except BM3 for p=4.

The ω-benchmarks yield solutions that meet the fairness criteria (symmetry and order preservation) only at the expense of higher transportation cost than our proposed models. For some ω values, they even violate symmetry or order preservation; several recover these properties only for very large ω (typically $\omega \geq 10^6$). In heterogeneous instances, UCpMP models are cost-competitive or cheaper at high ω (e.g., it outperforms BM2–BM6 when p=4, and for p=3 only BM2 is slightly cheaper but less fair). BM1 is a special case: it can yield low costs by leaving capacity unused (e.g., 28 units, 21.21%, for p=3; 49 units, 35%, for p=4), underscoring why allocating capacity and clients must be part of the optimization. At high ω, BM7 (Gini) in some cases converges to EG-like allocations and BM3 (MSE) to EL-like allocations, though they are not identical because EG/EL are enforced as constraints in UCpMP whereas BM3/BM7 embed fairness in the objective.

In summary, assuming a fixed α, when the objectives in (BM2–BM7) are incorporated via a weight ω, the empirical behavior is as follows: (i) the network cost is non-decreasing in ω; (ii) fairness improves as ω increases (e.g., the allocation disparity is non-increasing); and (iii) there exists a stabilization threshold ω^+ beyond which both the solutions and the cost remain constant.

The UCpMP-EG-3, UCpMP-EL-3, and UCpMP-PA-3 models (with $\alpha = \alpha^{++}$ and $\beta = 1$) achieve equal or lower costs than the BM2–BM7 models while ensuring symmetry, order preservation within facilities, and full capacity utilization. In contrast, the BM2–BM7 models recover these properties only for very large ω values, which results in higher network costs.

9.6. Scaling up: practical runtimes and when to use heuristics

The growth in computational effort with problem size is evident. In heterogeneous cases, UCpMP-EG-3, UCpMP-EL-3, and UCpMP-PA-3 models with solves 15-client instances with $p = 2$ in under six minutes, but requires over two hours for 25 clients with $p = 3$; in homogeneous cases, UCpMP-EL-3 takes up to five hours for 25 clients with $p = 3$. Across models, EL is usually the slowest in homogeneous settings, PA tends to be the slowest in heterogeneous ones, and EG scales best.

The neighborhood heuristic UCpMP-NB is a robust time/speed trade-off. With $n = 2$ in heterogeneous cases, it is faster than the optimal models in all medium instances except the 15-client, $p = 3$ cases. Across the 18 possible comparisons UCpMP-NB reaches exactly the optimal solutions in 11 cases; its largest deviation is 8.18% (20 clients, $p = 3$). In the remaining cases, $n = 2$ keeps all deviations below 1.50%. For 25-client instances, the largest deviation is only 1.27% and time savings reach up to an order of magnitude

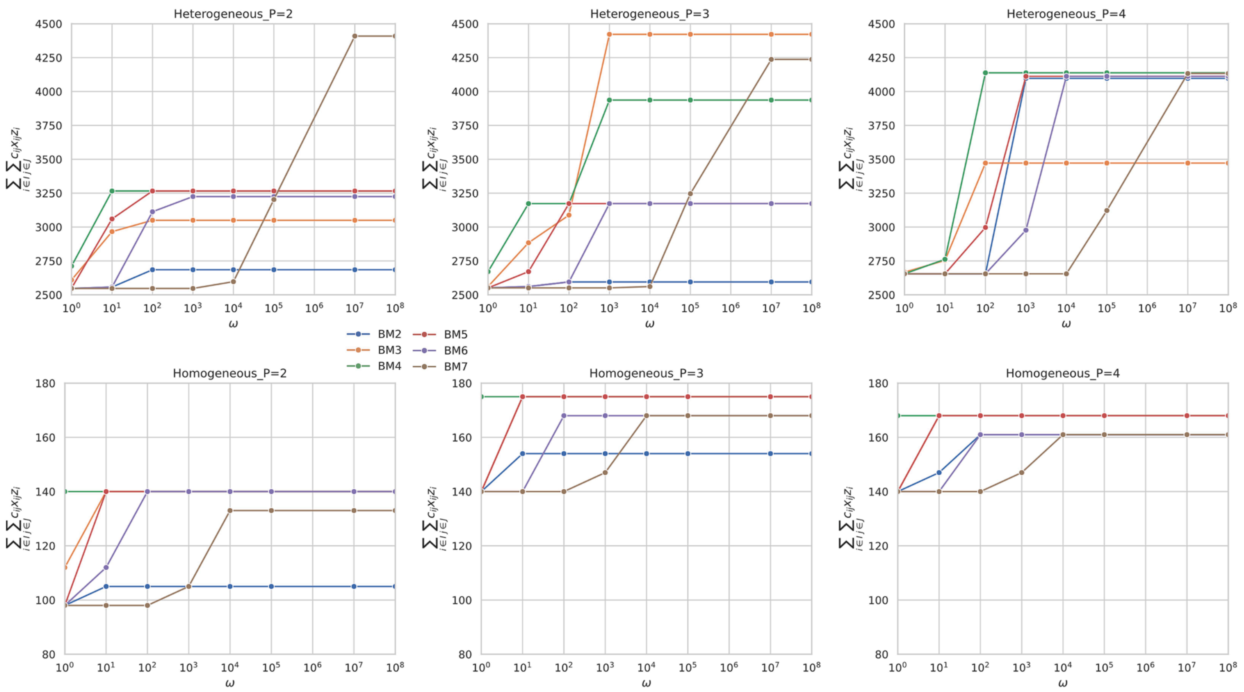


Fig. 9. Performance of the UCpMP models under different benchmark formulations. Upper panels report the network cost $\sum_{i\in I}\sum_{j\in J} c_{ij} z_i x_{ij}$ for heterogeneous capacity settings, while lower panels report the corresponding network cost for homogeneous capacity settings, as a function of the parameter ω. Results are shown for $p = 2$, $p = 3$, and $p = 4$ facilities.

for some $p = 2$ runs. In homogeneous cases, the heuristic already attains the optimal solutions with $n = 0$ (we report $n = 1$ to register the time); the only instance where UCpMP-NB takes longer is under UCpMP-PA-3 model with 15 clients and $p = 2$. For 25-client sets, UCpMP-NB typically runs in about half the time of UCpMP-EG-3, UCpMP-EL-3, and UCpMP-PA-3 models.

As a point of reference, Hexaly underperforms on heterogeneous cases: in 11 of 18 medium comparisons it leaves unused capacity, and even when it fully distributes capacity its cost deviations are large (best case 32.07% for 15 clients, PA, $p = 2$; worst 96.69% for 25 clients, EG, $p = 2$). It fares better in some homogeneous settings (e.g., with 15 clients and $p = 2$, where it obtains the optimal solution), but other $p = 3$ cases again leave capacity unused. Overall, UCpMP-NB is the preferred heuristic: it preserves full capacity use and achieves much smaller cost gaps.

At large instances scale, only UCpMP-NB can provide feasible solutions within practical computational times. In heterogeneous instances the times stay moderate–for 40 clients, p=2 and p=3 take 231–1270 s and 204–598 s, respectively; for 60 clients, 921–1873 s (p=2) and 822–4932 s (p=3); for 80 clients, 1184–2426 s (p=2) and 2722–6660 s (p=3). In homogeneous instances the heuristic is slower: for 40 clients, 1786–2564 s (p=2) and 2028–2890 s (p=3); for 60 clients, 3875–6581 s (p=2) and 5124–9843 s (p=3); and for 80 clients, 9656–15890 s (p=2) and 14652–18054 s (p=3). Overall, large *heterogeneous* instances finish within two hours, whereas *homogeneous* ones may require up to seven hours on the available hardware.

In summary, for large-scale decision support, UCpMP-NB serves as the workhorse model: it preserves the fairness constraints of the UCpMP-EG-3, UCpMP-EL-3, and UCpMP-PA-3 formulations (i.e., full capacity utilization and symmetry/order within facilities) while delivering near-optimal costs with adjustable runtime.

10. An application of food insecurity in Todd County, South Dakota

One example of resource allocation in undercapacity situations is food insecurity; tackling it is directly related to Sustainable Development Goal 2: Zero Hunger (Blesh et al., 2019). In the US, food insecurity affects up to 14.3% of the population (Feeding America, 2025). However, food insecurity is not distributed equally among racial and ethnic groups: it is higher among non-Hispanic Black (19%) and Hispanic (16%) households with respect to non-Hispanic White (8%) (Nikolaus et al., 2022). The group formed by Native Americans is particularly struck, as one in three children of this group live in poverty and this hinders their health, educational and employment opportunities (Empey et al., 2021). In addition, works like Davis et al. (2016) highlight the durable character of poverty for this group, from a historical perspective.

One of the US counties with a highest proportion of Native American population (83.7% of the total) is Todd County, in South Dakota[3] (U. S. Census Bureau, 2025). At the same time, it scores around 27% for food insecurity and is among the poorest counties

[3] Not to be confused with Todd County from Minnesota or Todd County from Kentucky.

Table 8
Number, abbreviation, full name, population and food demand of towns in Todd County.

Abbrv.	Town	Inhabitants	Demand
OK	Okreek	190	51
SO	Soldier Creek	205	55
WH	White Horse	234	63
SP	Spring Creek	236	64
SI	Sicangu Village	276	75
TW	Two Strike	282	76
ST	St Francis	469	127
PA	Parmelee	606	164
AN	Antelope	830	224
MI	Mission	1156	312
RO	Rosebud	1455	393

Table 9
Distances between the towns in Todd County, in kilometers.

	OK	SO	WH	SP	SI	TW	ST	PA	AN	MI	RO
OK	0.0	44.0	20.3	72.3	58.8	48.5	57.5	59.7	24.6	25.1	44.8
SO	44.0	0.0	24.3	37.8	53.6	15.9	29.1	15.6	20.1	19.0	12.4
WH	20.3	24.3	0.0	52.6	39.1	28.8	31.2	39.9	5.0	5.5	25.1
SP	72.3	37.8	52.6	0.0	51.2	24.5	14.8	35.1	48.3	47.2	27.4
SI	58.8	53.6	39.1	51.2	0.0	46.0	39.6	69.2	34.9	34.6	43.0
TW	48.5	15.9	28.8	24.5	46.0	0.0	9.8	20.8	24.5	23.3	3.7
ST	57.5	29.1	31.2	14.8	39.6	9.8	0.0	29.8	33.6	32.5	12.7
PA	59.7	15.6	39.9	35.1	69.2	20.8	29.8	0.0	35.7	34.6	20.0
AN	24.6	20.1	5.0	48.3	34.9	24.5	33.6	35.7	0.0	1.1	20.9
MI	25.1	19.0	5.5	47.2	34.6	23.3	32.5	34.6	1.1	0.0	19.8
RO	44.8	12.4	25.1	27.4	43.0	3.7	12.7	20.0	20.9	19.8	0.0

Table 10
Food bank network in Todd County considering $p = 2$ and α^{++}. The UCpMP-3 models are solved with $\beta = 1$, and the benchmark models BM1–BM7 with $\omega = 10^6$.

		UCpMP						BM1															
		EG-3		EL-3		PA-3			EG	EL	PA	BM2		BM3		BM4		BM5		BM6		BM7	
I	D	J	z_i	J	z_i	J	z_i	J	z_i	z_i	z_i	J	z_i	J	z_i	J	z_i	J	z_i	J	z_i	J	z_i
OK	51	AN	51	MI	0	MI	23	MI	51	0	21	RO	51	TW	14	MI	15	AN	51	AN	51	AN	51
SO	55	RO	55	RO	49	RO	53	RO	55	55	55	RO	55	MI	18	RO	32	RO	55	RO	55	RO	55
WH	63	AN	63	RO	57	RO	60	MI	62	0	26	RO	51	MI	26	MI	63	RO	63	RO	63	RO	63
SP	64	RO	64	RO	58	RO	61	RO	64	64	64	RO	51	TW	27	RO	64	RO	64	RO	64	RO	63
SI	75	RO	75	MI	0	MI	33	MI	62	0	31	RO	51	MI	38	MI	22	AN	75	AN	75	AN	73
TW	76	RO	76	RO	70	RO	73	RO	76	76	76	RO	76	TW	40	RO	76	RO	76	RO	76	RO	76
ST	127	RO	127	RO	121	RO	121	RO	127	127	127	RO	70	TW	91	RO	127	RO	127	RO	127	RO	127
PA	164	RO	164	RO	158	RO	156	RO	164	164	164	RO	51	TW	128	RO	163	RO	164	RO	163	RO	164
AN	224	AN	93	MI	106	MI	102	MI	62	106	92	RO	51	MI	187	RO	45	AN	174	AN	174	AN	176
MI	312	AN	93	MI	194	MI	142	MI	63	194	130	MI	300	MI	275	MI	200	RO	175	RO	176	RO	176
RO	393	RO	339	RO	387	RO	376	RO	393	393	393	RO	393	MI	356	RO	393	RO	176	RO	176	RO	176
$\sum_{i\in I}\sum_{j\in J} c_{ij} z_i$		12506.6		8699.8		10592.7			11444.2	7726.3	9453.6	11093.4		13949.0		9729.2		16528.1		16527.9		16450.7	
$\sum_{t\in T} Q_t - \sum_{i\in I} D_i$		0		0		0			21	21	21	0		0		0		0		0		0	
$\max_{i\in I} z_i - \min_{i\in I} z_i$		288		387		353			342	393	372	342		342		378		125		125		125	

in the whole US (DePietro, 2022). In this case study, we apply our models and fairness rules to build a food bank network for Todd County, assuming undercapacity and relevant from a humanitarian, economic and logistic point of view. We emphasize that this case study is framed as a strategic planning exercise: demands are estimated from available aggregate indicators and transportation costs are approximated by static inter-town distances, as commonly done in network design studies when real-time dynamics are out of scope.

Todd County comprehends a series of 11 towns. The largest ones, Mission and Rosebud, have more than 1000 inhabitants. The smallest ones, Soldier Creek and Okreek, have around 200 inhabitants. Table 8 represents the ID for this study of each town in Todd County, as well as the number of inhabitants and food demand (discrete) if we consider the 27% average food insecurity per town, as detailed data on food insecurity in each specific town is unavailable. Table 9 acts as a matrix that expresses the distances, in kilometers, between the different towns in Todd County; hence, the transportation costs.

Table 11
Food bank network in Todd County considering $p = 3$ and α^{++}. The UCpMP-3 models are solved with $\beta = 1$, and the benchmark models BM1–BM7 with $\omega = 10^6$.

		UCpMP						BM1															
		EG-3		EL-3		PA-3			EG	EL	PA	BM2		BM3		BM4		BM5		BM6		BM7	
I	D	J	z_i	J	z_i	J	z_i	J	z_i	z_i	z_i	J	z_i	J	z_i	J	z_i	J	z_i	J	z_i	J	z_i
OK	51	MI	51	AN	1	AN	30	MI	51	51	51	RO	51	RO	32	WH	51	RO	51	RO	51	RO	51
SO	55	MI	55	RO	49	RO	53	RO	55	0	24	RO	55	AN	37	PA	55	RO	55	RO	55	RO	55
WH	63	MI	63	RO	57	RO	60	MI	63	63	63	RO	51	ST	45	MI	63	RO	63	RO	63	RO	63
SP	64	PA	64	RO	58	RO	61	RO	61	0	26	RO	51	ST	46	MI	64	RO	64	RO	64	RO	63
SI	75	MI	75	AN	25	AN	42	MI	75	75	75	RO	51	RO	56	WH	75	ST	75	ST	73	ST	73
TW	76	RO	76	RO	70	RO	73	RO	61	0	32	RO	76	AN	57	MI	76	AN	76	AN	76	AN	76
ST	127	MI	127	RO	121	RO	121	RO	61	17	53	RO	121	ST	109	MI	127	ST	125	ST	127	ST	127
PA	164	PA	136	RO	158	RO	156	PA	164	164	164	RO	51	RO	145	PA	145	RO	164	RO	163	RO	164
AN	224	MI	224	AN	174	AN	128	MI	224	224	224	AN	200	AN	206	WH	174	AN	224	AN	224	AN	224
MI	312	MI	305	MI	300	MI	300	MI	312	312	312	MI	300	RO	293	MI	312	RO	251	RO	252	RO	252
RO	393	RO	224	RO	387	RO	376	RO	62	283	165	RO	393	RO	374	MI	258	RO	252	RO	252	RO	252
$\sum_{i\in I}\sum_{j\in J} c_{ij} z_i$		12168.1		9480.3		10965.2			7821.8	4683.9	6269.5	10675.2		16768.0		20069.8		19383.5		19304.1		19296.7	
$\sum_{t\in T} Q_t - \sum_{i\in I} D_i$		0		0		0			211	211	211	0		0		0		0		0		0	
$\max_{i\in I} z_i - \min_{i\in I} z_i$		254		386		346			261	312	288	342		342		261		201		201		201	

The location problem would therefore be to establish a network of food banks that efficiently supplies Todd County, considering undercapacity. The main decision variables are the opened food banks y_j, the assignment of capacity to food banks s_{tj}, the town to food bank assignment x_{ij} and the food allocation z_i. We study two different cases: in the first one $p = 2$, with $Q_1 = 900$ and $Q_2 = 300$; in the second one, $p = 3$, adding $Q_3 = 200$. In these cases, Q_1 is assumed as a main food bank with larger capacity while the rest act as supporting, smaller ones.

Applying the models proposed in the main body of the paper to this case study brings similar conclusions to those obtained with Examples 1 and 6. The results for $p = 2$ are represented in Table 10. The most commonly opened food banks are Antelope, Mission and Rosebud, coinciding with the largest demanding clients. $BM1$ leaves 21 units of capacity remaining, while UCpMP models and the rest of benchmarks make full use of the available capacity. $BM2$ and $BM4$ do not satisfy order preservation, while the other benchmarks do, but at a higher network cost than the UCpMP models. Similar results are seen in Table 11 with $p = 3$, which supports the effectiveness of the proposed models in this planning setting.

11. Conclusions

Under-capacity location problems can arise in many situations like disaster management or provision of sanitary services. To tackle these situations four factors can be considered: the cost of the network; the efficiency of the allocation, that is, the usage of the available capacity; the main fairness principle to allocate the capacity to the clients (the allocation rule); and additional fairness properties that the allocation of resources should satisfy. In the related literature there is no work that simultaneously addresses these four factors, as shown in the Literature Review section.

In this paper, we fill this gap by introducing the under-capacitated p-median location problem and proposing the UCpMP model, which explicitly distinguishes between client demand d_i and allocated resources z_i. The model includes a tunable parameter α that allows decision-makers to balance network cost against capacity utilization. Building on this baseline formulation, we incorporate allocation fairness principles based on bankruptcy rules, namely egalitarianism (equal gain and equal loss) and proportionality, together with an additional fairness property, order preservation, and adherence to welfarism. This results in a holistic modeling framework for location problems under capacity scarcity. In this sense, a gap in this literature is filled.

However, the proposed framework is better suited for strategic and tactical decision support in settings with static inputs, rather than for real-time operational deployment in fully dynamic environments.

On the other hand, in order to evaluate the suitability of the approach used, numerical experiments of different sizes (small, medium and large) and characteristics (homogeneous and heterogeneous) have been carried out to compare our approach with the traditional formulation of the p-median problem and seven benchmark models that incorporate fairness principles that have been used in the literature. By explicitly accounting for unmet demand and applying bankruptcy-inspired allocation rules, our formulations achieve fairer allocations of scarce capacity while maintaining high capacity utilization and competitive network costs when compared with traditional p-median formulations and existing fairness-based benchmarks. This practical relevance is illustrated through a real-world case study on food insecurity in Todd County (South Dakota), where the models guide the design of a fair and efficient food bank network under severe capacity shortages.

While the exact UCpMP formulations provide a rigorous benchmark and allow a detailed analysis of the trade-offs between location, allocation, and fairness, they become computationally demanding as the problem size increases. To address scalability, a dedicated heuristic procedure, the UCpMP-NB heuristic, was developed to handle larger instances that are computationally intractable for exact optimization. The heuristic significantly reduces computational time at the expense of a moderate loss of optimality, achieving solutions with deviations of up to 8.18% from the optimal values in the numerical experiments.

This trade-off enables the application of the proposed framework to realistically sized instances, allowing decision-makers to balance solution quality and computational effort depending on the problem scale and available computational resources. From a managerial perspective, this means that solutions with a good balance between quality and computational effort can be obtained within practical computation times for large-scale planning settings.

Furthermore, the models introduced for the discrete case are easily adaptable to the continuous case with similar results, which adds additional practical interest to these models.

Moreover, the approach adopted allows the decision-maker to have some control over the various factors mentioned above, allowing them to be balanced using the α and β parameters and the choice of allocation rule. The implications of each of these elements are explained in the Managerial Insights section.

Finally, this study focused on a deterministic UCpMP setting, which provided a solid foundation to integrate bankruptcy-inspired fairness rules and to analyze their theoretical and computational properties. Extending these rules to stochastic or robust formulations, where demand uncertainty and random capacity shortfalls are explicitly modeled, constitutes a challenging but promising direction for future research. The proposed UCpMP-NB heuristic may also serve as a preliminary step toward solving stochastic versions of the problem. Since its first phase involves solving an un-capacitated p-median problem, demand uncertainty could be incorporated at this stage using established techniques such as scenario generation, sample average approximation, or robust optimization. Once facility locations are determined, the realized demands can then be used in a second stage to allocate clients and capacities through the bankruptcy-based rules proposed in this study.

CRediT authorship contribution statement

Iñigo Martín Melero: Writing – review & editing, Writing – original draft, Validation, Software, Project administration, Methodology, Investigation, Formal analysis, Conceptualization; **Juan Carlos Gonçalves-Dosantos:** Writing – review & editing, Writing – original draft, Supervision, Methodology, Investigation, Formal analysis, Conceptualization; **Mercedes Landete:** Writing – review & editing, Validation, Supervision, Methodology, Investigation, Conceptualization; **Joaquín Sánchez-Soriano:** Supervision, Project administration, Methodology, Investigation, Formal analysis.

Data availability

The instances are publicly available at https://github.com/inigo-martin-melero.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work.

Acknowledgements

The authors thank the grants PID2021-12403030NB-C31, PID2021-122344NB-I00 and PID2022-137211NB-I00 funded by MICIU/AEI/10.13039/501100011033 and by ERDF/EU, Spain. This research was also partially supported by the CIPROM/2024/34 grant, funded by the Conselleria de Educación, Cultura, Universidades y Empleo, Generalitat Valenciana.

Supplementary material

Supplementary material associated with this article can be found in the online version at 10.1016/j.tre.2026.104768

References

Albareda-Sambola, M., Landete, M., Monge, J.F., Sainz-Pardo, J.L., 2017. Introducing capacities in the location of unreliable facilities. Eur. J. Oper. Res. 259 (1), 175–188. https://doi.org/10.1016/j.ejor.2016.10.013
Alcalde, J., Peris, J.E., 2022. Equalizing solutions for bankruptcy problems revisited. Decis. Econ. Finance 45 (2), 481–502.
Alyami, S., 2024. Fairness and usability analysis in renewable power curtailment: a microgrid network study using bankruptcy rules. Renew. Energy 227, 120424.
Atkinson, A.B., et al., 1970. On the measurement of inequality. J. Econ. Theory 2 (3), 244–263.
Bae, S., Jang, I., Gros, S., Kulcsár, B., Hellgren, J., 2020. A game approach for charging station placement based on user preferences and crowdedness. IEEE Trans. Intell. Transp. Syst. 23 (4), 3654–3669.
Beasley, J.E., 1990. Or-library: distributing test problems by electronic mail. J. Oper. Res. Soc. 41 (11), 1069–1072.
Berman, O., Drezner, Z., Tamir, A., Wesolowsky, G.O., 2009. Optimal location with equitable loads. Ann. Oper. Res. 167 (1), 307–325.
Blanco, V., Gázquez, R., 2023. Fairness in maximal covering location problems. Comput. Oper. Res. 157, 106287.
Blanco, V., Gázquez, R., Leal, M., 2025. Incorporation of regional preferences in facility location: insights into efficiency and satisfaction trade-offs. Transp. Res. Part E Logist. Transp. Rev. 203, 104328.
Blanco, V., Marín, A., Puerto, J., 2024. Intra-facility equity in discrete and continuous p-facility location problems. Comput. Oper. Res. 162, 106487.
Blesh, J., Hoey, L., Jones, A.D., Friedmann, H., Perfecto, I., 2019. Development pathways toward “zero hunger”. World Develop. 118, 1–14.
Bosek, B., Gordinowicz, P., Grytczuk, J., Nisse, N., Sokół, J., Śleszyńska-Nowak, M., 2018. Localization game on geometric and planar graphs. Discr. Appl. Math. 251, 30–39.

Christodoulou, G., Sgouritsa, A., Tang, B., 2016. On the efficiency of the proportional allocation mechanism for divisible resources. Theory Comput. Syst. 59 (4), 600–618.
Church, R.L., Murray, A., et al., 2018. Location Covering Models. Springer.
Corberán, Á., Landete, M., Peiró, J., Saldanha-da Gama, F., 2020. The facility location problem with capacity transfers. Transp. Res. Part E Logist. Transp. Rev. 138, 101943.
Csató, L., 2023. How to avoid uncompetitive games? the importance of tie-breaking rules. Eur. J. Oper. Res. 307 (3), 1260–1269.
Davis, J.J., Roscigno, V.J., Wilson, G., 2016. American indian poverty in the contemporary united states. Sociol. Forum 31 (1), 5–28.
Delgado, E.J., Cabezas, X., Martin-Barreiro, C., Leiva, V., Rojas, F., 2022. An equity-based optimization model to solve the location problem for healthcare centers applied to hospital beds and COVID-19 vaccination. Mathematics 10 (11), 1825.
Delorme, M., García, S., Gondzio, J., Kalcsics, J., Manlove, D., Pettersson, W., 2019. Mathematical models for stable matching problems with ties and incomplete lists. Eur. J. Oper. Res. 277 (2), 426–441.
DePietro, A., 2022. Counties with the highest poverty rate in every state for 2022. https://www.forbes.com/sites/andrewdepietro/2022/02/28/counties-with-the-highest-poverty-rate-in-every-state-2022/. Accessed: 2025-12-22.
Digehsara, A.A., Ji, M., Ardestani-Jaafari, A., Bidkhori, H., 2025. Equity-driven facility location: a two-stage robust optimization approach. Comput. Oper. Res. 176, 106920.
Dougherty, S., Nebreda, A.M., Urrutia, N., 2025. Intergovernmental Fiscal Transfers and Fiscal Equalisation in a Time of Consolidation. OECD Working Papers on Fiscal Federalism .
Drezner, T., Drezner, Z., Guyse, J., 2009. Equitable service by a facility: minimizing the gini coefficient. Comput. Oper. Res. 36 (12), 3240–3246.
Empey, A., Garcia, A., Bell, S., 2021. American Indian/Alaska native child health and poverty. Acad. Pediatr. 21 (8), S134–S139.
Ergün, S., Usta, P., Alparslan Gök, S.Z., Weber, G.W., 2023. A game theoretical approach to emergency logistics planning in natural disasters. Ann. Oper. Res. 324 (1), 855–868.
Feeding America, 2025. Map the meal gap: hunger & poverty in the united states. https://map.feedingamerica.org/. Accessed: 2025-12-22.
Giménez-Gómez, J.-M., Salekpay, F., Vilella, C., 2025. The constrained equal losses family of rules for claims problems. Top 33 (1), 56–73.
Gini, C., 1912. Variabilità e Mutabilità: Contributo Allo Studio Delle Distribuzioni e Delle Relazioni Statistiche.[Fasc. I.]. Tipografia di P. Cuppini.
Gómez-Limón, J.A., Gutiérrez-Martín, C., Montilla-López, N.M., 2021. Priority water rights. are they useful for improving water-use efficiency at the irrigation district level? Agric. Water Manag. 257, 107145.
Gul, M., Guneri, A.F., 2021. Hospital location selection: a systematic literature review on methodologies and applications. Math. Probl. Eng. 2021 (1), 6682958.
Hanjoul, P., Peeters, D., 1987. A facility location problem with clients' preference orderings. Region. Sci. Urban Econ. 17 (3), 451–473.
Herrero, C., 2003. Equal awards vs. equal losses: duality in bankruptcy. In: Sertel, M.R., Koray, S. (Eds.), Advances in Economic Design. Springer Berlin Heidelberg, Berlin, Heidelberg, pp. 413–426. https://doi.org/10.1007/978-3-662-05611-0_22
Hexaly, 2025. Gurobi vs. hexaly. https://www.hexaly.com/gurobi. Accessed: 2025-04-05.
Itani, M.N., Shakya, S., 2017. A bankruptcy approach to support decisions on allocation of scarce resources in emergency operations. J. Global Bus. Advan. 10 (5), 506–526.
Jacobs, T., 2012. Analytical aspects of tie breaking. Theor. Comput. Sci. 465, 1–9.
Jain, S., Bharti, K.K., 2023. A combinatorial optimization model for post-disaster emergency resource allocation using meta-heuristics. Soft Comput. 27 (18), 13595–13611.
Jena, S.D., Cordeau, J.-F., Gendron, B., 2015. Dynamic facility location with generalized modular capacities. Transp. Sci. 49 (3), 484–499.
Katoh, N., Shioura, A., Ibaraki, T., 2013. Resource allocation problems. In: Pardalos, P.M., Du, D.-Z., Graham, R.L. (Eds.), Handbook of Combinatorial Optimization. Springer New York, New York, NY, pp. 2897–2988. https://doi.org/10.1007/978-1-4419-7997-1_44
Kim, D.-G., Kim, Y.-D., 2010. A branch and bound algorithm for determining locations of long-term care facilities. Eur. J. Oper. Res. 206 (1), 168–177.
Kułakowski, K., 2015. Notes on order preservation and consistency in AHP. Eur. J. Oper. Res. 245 (1), 333–337.
Lim, J., Norman, B.A., Rajgopal, J., 2022. Redesign of vaccine distribution networks. Int. Trans. Oper. Res. 29 (1), 200–225.
Liu, S., Salari, N., 2024. Fairness in accessibility of public service facilities. Oper. Res. Lett. 55, 107141.
Lucas-Estañ, M.C., Gozalvez, J., Sanchez-Soriano, J., 2012. Bankruptcy-based radio resource management for multimedia mobile networks. Trans. Emerg. Telecommun. Technol. 23 (2), 186–201.
Ma, H., Pei, W., Zhang, Q., Xu, D., Li, Y., 2023. Location of electric vehicle charging stations based on game theory. World Electric Veh. J. 14 (5), 128.
Mallozzi, L., 2007. Noncooperative facility location games. Oper. Res. Lett. 35 (2), 151–154.
Mallozzi, L., 2011. Cooperative games in facility location situations with regional fixed costs. Optim. Lett. 5, 173–181.
Marín, A., 2011. The discrete facility location problem with balanced allocation of customers. Eur. J. Oper. Res. 210 (1), 27–38.
Martín Melero, I., 2025. Instances for PMPLC. https://github.com/inigo-martin-melero.
Martínez, R., Sánchez-Soriano, J., 2026. Order preservation with dummies in the museum pass problem. Ann. Oper. Res. 356, 11–29.
Medlock, J., Galvani, A.P., 2009. Optimizing influenza vaccine distribution. Science 325 (5948), 1705–1708.
Nikolaus, C.J., Johnson, S., Benally, T., Maudrie, T., Henderson, A., Nelson, K., Lane, T., Segrest, V., Ferguson, G.L., Buchwald, D., et al., 2022. Food insecurity among american indian and alaska native people: a scoping review to inform future research and policy needs. Adv. Nutr. 13 (5), 1566–1583.
Puerto, J., Tamir, A., Perea, F., 2012. Cooperative location games based on the minimum diameter spanning steiner subgraph problem. Discr. Appl. Math. 160 (7–8), 970–979.
Rawls, J., 1971. A Theory of Justice. Harvard University Press.
Ringius, L., Torvanger, A., Underdal, A., 2002. Burden sharing and fairness principles in international climate policy. Int. Environ. Agreements 2 (1), 1–22.
Shehadeh, K.S., Snyder, L.V., 2023. Equity in stochastic healthcare facility location. In: Eiselt, H.A., Marianov, V. (Eds.), Uncertainty in Facility Location Problems. Springer International Publishing, Cham, pp. 303–334. https://doi.org/10.1007/978-3-031-32338-6_12
Shulman, A., 1991. An algorithm for solving dynamic capacitated plant location problems with discrete expansion sizes. Oper. Res. 39 (3), 423–436.
Silva, A., Aloise, D., Coelho, L.C., Rocha, C., 2021. Heuristics for the dynamic facility location problem with modular capacities. Eur. J. Oper. Res. 290 (2), 435–452.
Thomson, W., 2003. Axiomatic and game-theoretic analysis of bankruptcy and taxation problems: a survey. Math. Soc. Sci. 45 (3), 249–297.
Thomson, W., 2019. How to Divide When There Isn't Enough: From Aristotle, the Talmud, and Maimonides to the Axiomatics of Resource Allocation. 62, Cambridge University Press.
U. S. Census Bureau, 2025 QuickFacts: Todd County, South Dakota. https://www.census.gov/quickfacts/fact/table/toddcountysouthdakota/RHI225224. Accessed: 2025-12-22.
Wang, W., Wang, T., Liu, B., Sha, J., You, J., 2025. A two-stage game-based model for distribution of water consumption control indicators. J. Hydrol. Region. Stud. 58, 102259.
Weaver, J.R., Church, R.L., 1985. A median location model with nonclosest facility service. Transp. Sci. 19 (1), 58–74.
Xevi, E., Khan, S., 2005. A multi-objective optimisation approach to water management. J. Environ. Manag. 77 (4), 269–277.
Xu, J., Murray, A.T., Church, R.L., Wei, R., 2023. Service allocation equity in location coverage analytics. Eur. J. Oper. Res. 305 (1), 21–37.