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Apportionment when seats are allocated in lots. The D'Hondt method case and political implications

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ABSTRACT

The apportionment problem involves determining how to distribute a given (non-negative) integer number among a group of individuals based on their respective sizes. In electoral systems with proportional representation, this problem arises in two situations: assigning seats to constituencies, if applicable, and distributing seats to political parties within each constituency. This paper addresses the scenario where seats are grouped into lots, extending the standard apportionment problem. We propose and analyze various apportionment methods based on the D'Hondt method for this new problem. Additionally, we examine the political implications of allocating seats not individually but in groups of varying sizes.

1. Introduction

Electoral systems are of significant social importance as they determine the formation of governments and institutions that govern countries, directly impacting the lives of their citizens. Consequently, there exists an extensive body of literature dedicated to electoral systems and their structuring (see, for instance, Taagepera and Shugart (1989) or Horowitz (2003)). In the case of representative electoral systems, one of the foundations of fair representation is ensuring proportionate representation based on specific numerical criteria. In the political context, this is typically measured by the votes obtained by each political party, representing different sensitivities in an electoral process (Balinski and Young, 1982). This process of converting votes into seats is known as the *apportionment problem*, a topic that has sparked considerable interest, both from a mathematical perspective and from the lens of political analysis. However, it is not solely the allocation of seats within an electoral system that demands attention; other factors must also be considered. This includes the capacity to form majorities, which is crucial for the establishment of stable governments.

In recent decades, it has been observed that proportional electoral systems often produce results leading to significant fragmentation of the parliament or chamber of representatives, making the formation of stable governments difficult, as many political parties must reach an agreement. For this reason, countries such as Greece and Italy have introduced mechanisms to reward the parties with the most votes and thus facilitate the formation of more homogeneous governments. Therefore, achieving a balance between governability and proportionality should be an important objective in the design of an electoral system, although both will be difficult to achieve simultaneously under all possible circumstances.

This work attempts to address precisely the problem of the balance between governability and proportionality, but directly using the design of an apportionment mechanism in a single tier, although the possibility of multiple constituencies is accepted. Specifically, in this work it is considered that in the design of the electoral system the seats are assigned in lots of different sizes and not one by one as in the standard apportionment problem. Therefore, the way in which the size of the lots is configured will determine the more or less proportional distribution of the seats between the parties, and the parties with the most votes may be more favored, thus facilitating the formation of more stable and lasting governments, and less conditioned by minority parties in their government action. In this sense, the new approach tries to provide an alternative response to the solutions proposed in some countries to reward the parties with the most votes. At the same time, the proposal made in this work is an extension of the apportionment problem to the case in which the allocation of resources (not perfectly divisible) is not carried out resource by resource but in lots of different sizes in general, therefore, in this sense we introduce a new mathematical extension of the apportionment problem in the literature. The analysis of apportionment methods for lots problems will be conducted from an optimization and algorithmic perspective, proposing models that provide allocations as proportional as possible according to some measure of disproportionality (see Gallagher (1991))

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and that, in addition, have good properties related to fairness. Among all the apportionment methods that can be used as optimization problems, we will focus on the D'Hondt method, for which algorithms for its calculation will also be provided. One of these algorithms provides an allocation of seats grouped into lots that satisfies several non-arbitrariness properties, including order preservation, and a weak lower quota property which guarantees a good level of proportionality in the context of apportionment problems with lots.

In the literature you can find numerous methods to transform votes into seats, all seeking fair representation (see, for instance, (Pukelsheim, 2017). However, given the variety of existing apportionment methods, the question of which method to use arises. For this purpose, based on the properties that each method satisfies, this should be chosen; however, unfortunately, there is no one method that satisfies all desirable properties (see, for example, the "impossibility theorem" from Balinski and Young (1982). Balinski and Young (1975, 1977a,b, 1979, 1980) analyzed several of the most well-known apportionment methods.

Moreover, other apportionment methods have been proposed that try to maintain proportionality but that, at the same time, incorporate other elements that can be considered relevant in the design of an electoral system. For example, Balinski and Demange (1989a,b) first introduced *biproportional apportionment methods*. These methods seek to achieve double proportionality, on the one hand, in the constituencies relative to their populations and, on the other hand, in the political parties relative to their total votes. In this case, the fact that in an electoral system the representatives to be elected are divided among different representative constituencies is incorporated. In Maier et al. (2010) biproportional apportionment methods based on divisors are evaluated with respect to the possibility of the appearance of different anomalous phenomena, such as quota deviation, reversal ordering, or the occurrence of ties. Sánchez-Soriano et al. (2016) studied the apportionment problem as a bankruptcy problem. Furthermore, they introduce some apportionment methods based on comparison standards that, respecting the lower allotment, favor the parties with the most votes and with it the possibility of longer-lasting governments. Balinski and Ramírez (1999) studied parametric apportionment methods that range from those that most favor the least voted parties to those that most favor the most voted parties. Nonetheless, some of the methods proposed in Sánchez-Soriano et al. (2016) favor the parties with the most votes more than the most favorable ones in Balinski and Ramírez (1999) and, therefore, are more oriented towards the possibility of forming lasting governments than towards obtaining a more proportional distribution of seats.

On the other hand, the apportionment problem is closely related to the fair allocations of goods, in particular to "weighted" fair allocation, where agents have different entitlements (see, Suksompong (2024)). Indeed, the presented model of apportionment is a special case of weighted fair allocation where all agents have uniform valuations for the goods. Chakraborty et al. (2021) studied picking sequences for weighted fair allocation problems when there are indivisible goods. Roughly speaking, picking sequences are allocation methods based on the orders of the agents, who may appear several times in the ordering, to choose the indivisible goods. In this paper, the authors analyzed picking sequences obtained by means of apportionment methods, and what fairness and monotonicity properties the corresponding allocations obtained satisfied. This approach differs from the one used in this paper in two respects. First, this paper analyzes the problem from an optimization perspective, and second, when applying the D'Hondt method, the allocation order also involves the set of indivisible goods, in our case, the seat lots. Chingoma et al. (2024) studied the apportionment problem when the seats are weighted. They analyzed the fairness (in terms of approximately proportional outcomes) of the allocations obtained by using the picking sequences generated by the Adams and D'Hondt method but including the weights in the determination of the picking order. The latter coincides with our approach for the D'Hondt algorithms introduced in this paper and differs from the approach in Chakraborty et al. (2021). However, there are several differences from our approach. First, there is a conceptual difference: while Chingoma et al. (2024) focuses on weighted seats, this work focuses on lots of seats. This difference influences the meaning and definition of the properties for apportionment methods. Second, they use picking sequences, whereas this work uses optimization problems, besides algorithms for applying the D'Hondt method. One of these algorithms provides an allocation satisfying additional properties related to non-arbitrariness. Finally, we use both the properties of weak lower quota introduced in Aziz et al. (2020) and Chingoma et al. (2024), and disproportionality measures to obtain the most proportional allocation possible.

The remainder of the paper is organized as follows. In Section 2, we will present the basic elements of classic apportionment problem. The mathematical model that describes apportionment problems will be discussed in Section 2.1. In Section 2.2, we will present how to solve apportionment problems by using optimization models which seek the most proportional allocation of seats possible. The relationship between these optimization models and well-known apportionment methods in the literature will also be presented. Some basic properties of apportionment methods will be given in Section 2.3. Fo the case of the D'Hondt method, an algorithm will be given in Section 2.4. We will discuss the new apportionment problem in Section 3 following the same structure as in Section 2. In Section 3.1, the mathematical model for apportionment problems when seats are grouped into lots will be defined. Apportionment methods based on optimization programs for apportionment problems with lots will be discussed in Section 3.2. Section 3.3 discusses how to extend the basic properties for apportionment problems presented in Section 2.3 to the context of apportionment problems when seats are grouped into lots. Next, in Section 3.4, we will study how to extend D'Hondt's algorithm to the case with lots so that it continues to satisfy good properties related to non-arbitrariness and proportionality of the allocation. In Section 4, two families of seat divisions into lots will be introduced. For these families, the behavior of the D'Hondt method will be studied from both the optimization and algorithmic perspectives, including the properties it satisfies. In Section 5, an experimental study will be presented on the impact of the apportionment problem with lots on the results of different electoral processes, carrying out a comparative study of both the proportionality of the results and the sizes of the possible governments that could be formed. Finally, Section 6 concludes.

2. The apportionment problem

In this section, we will present the basic elements of apportionment problems, starting with the description of these problems. Apportionment problems involve the allocation of a fixed amount of resources to a group of individuals in integer portions, with each individual making non-negative claims. The objective is to find an allocation that is proportionally fair relative to the individuals' claims. However, multiple solutions exist for this problem, each with a set of specific desirable properties, and thus, no single solution can be considered superior to others. In this paper, we examine apportionment problems within the realm of political scenarios, specifically addressing the distribution of seats among a set of political parties based on their respective vote counts.

2.1. Mathematical model

In this subsection, we will present the mathematical model that describes apportionment problems. Let $M = \{1, 2, \ldots, m\}$ denote a set of political parties. Each party $i \in M$ is associated with a positive vote fraction v_i. Without loss of generality, we assume that $v_1 \geq v_2 \geq v_3 \geq \cdots \geq v_m$. An integer number of seats, denoted by S and satisfying $S > 1$, must be distributed among the parties, where s_i represents the total number of seats allocated to party i for each $i \in M$. An apportionment problem is defined by the tuple $(v;S)$. Consequently, we introduce the concept of an apportionment method, denoted by A, from $\Delta^M \times \mathbb{N}$ to $\mathbb{N}^M$, such that $A(v;S) \subset \mathbb{N}^M$. For every $(s_1,\dots,s_m) \in A(v;S)$, it holds that $\sum_{i=1}^{m} s_i = S$.

Ensuring a proportional allocation of seats to parties based on their vote share is a crucial aspect of a representative electoral system. The ideal scenario implies a direct proportionality between the number of seats and votes. Specifically, for each party $i \in M$, the ideal number of seats should be determined by $q_i = v_i \cdot S$, where S denotes the total number of seats. However, since these values are typically non-integer, rounding effects must be carefully considered. In the following, we illustrate the impact of rounding effects through an illustrative example.

Example 2.1. Consider the set of parties $M = \{1,2,3,4\}$ with corresponding vote fractions $v_1 = 0.51$, $v_2 = 0.20$, $v_3 = 0.15$ and $v_4 = 0.14$. Let the total number of seats be $S = 10$. In this case, the actual proportion of seats allocated to each party is given by

$$0.51 \times 10 = 5.10; \quad 0.20 \times 10 = 2.00; \quad 0.15 \times 10 = 1.50; \quad 0.14 \times 10 = 1.40.$$

Considering the presence of rounding effects, various methods can be employed. We adopt the floor function $\lfloor x \rfloor = \max\{k \in \mathbb{Z} : k \leq x\}$ and the ceiling function $\lceil x \rceil = \min\{k \in \mathbb{Z} : x \leq k\}$ to address rounding. Now, turning our attention back to the example at hand.

The floor function is applied to the first and fourth parties, while the ceiling function is applied to the third party. Consequently, the resulting apportionment is $(5,2,2,1)$. Alternatively, employing the ceiling function for the first party and the floor function for the third and fourth parties yields a different allocation of seats, resulting in $(6,2,1,1)$.

As illustrated in the previous example, it is imperative to consider the effects of rounding in calculating quotas. Furthermore, there are multiple criteria for allocating the total number of seats. Consequently, there is no single mapping function A to solve apportionment problems, highlighting the need to address the non-integer nature of proportions through appropriate methodologies.

2.2. Apportionment methods as optimization problems

In this subsection, we will present some optimization-based allocation methods and their relationship to known apportionment methods. Numerous of these methods have been proposed in the existing literature, and many of them can be obtained as solutions to the following optimization problem.

$$\begin{aligned} \min \quad & \varphi(v;S) \\ \text{s.t.} \quad & \sum_{i=1}^{m} s_i = S \\ & s_i \in \mathbb{Z}_+, \qquad \forall i \in \{1,\dots,m\}, \end{aligned} \tag{1}$$

where the cost function $\varphi(v;S)$ is a measure of disproportionality and where it generally satisfies $\varphi(v;S) = 0$ if and only if $s_i = q_i$ for all $i \in M$.[1] Some disproportionality measures that give rise to different apportionment methods are formulated as follows (see, for details, Table 6.1 in Grilli di Cortona et al. (1999)):

$$\varphi(v,s) = \begin{cases} \sum_{i=1}^{m} |s_i - q_i|^p,\ p \geq 1 & \text{Greatest remainder method} \\ \max_{i=1,2,\dots,m} \dfrac{s_i}{v_i} & \text{D'Hondt method} \\ \sum_{i=1}^{m} \dfrac{(s_i - q_i)^2}{q_i} & \text{Sainte-Laguë's method} \\ \sum_{i=1}^{m} \dfrac{(s_i - q_i)^2}{s_i} & \text{Equal proportions method} \\ \max_{i=1,2,\dots,m} \dfrac{v_i}{s_i} & \text{Smallest divisor method} \end{cases} \tag{2}$$

However, there is no definitive criterion for determining the superiority of one apportionment method over another. The selection of an appropriate method depends on the specific contextual factors and the desired properties to be fulfilled.

2.3. Some properties for apportionment methods

In this subsection, a set of basic properties or principles for apportionment methods will be presented, which can be taken into account in the evaluation of suitability of one allocation method or another.

The first principle, *scale invariance*, relates to the scale used to measure the votes of each party and asserts that when the vote fraction of each party is multiplied by the same constant, the resulting allocation of seats remains unaffected.

Scale invariance. For every apportionment problem $(v;S)$ and all $\lambda > 0$, a map A satisfies scale invariance if and only if $A(\lambda v;S) = A(v;S)$.

Anonymity asserts that the allocation concerns the size of the votes obtained by the parties, not their names.

Anonymity. For every apportionment problem $(v;S)$ and all permutation π of the set of parties M, a map A satisfies anonymity if and only if $A((v_{\pi(1)}, v_{\pi(2)},\dots,v_{\pi(m)});S) = \pi(A((v_1,v_2,\dots,v_m);S))$.[2]

Order preservation states that when one party has a higher vote fraction than another, the former will achieve a greater representation in the allocation of seats.

Order preservation. For every apportionment problem $(v;S)$ and all $i,j \in M$ with $v_i > v_j$, a map A satisfies order preservation if and only if $A_i(v;S) \geq A_j(v;S)$.

The *lower quota condition* establishes that each party is assigned at least as many seats as the largest integer less than or equal to the ideal number of seats, $q_i = v_i \cdot S$ where $i \in M$. This condition is usually considered a minimum requirement of proportionality.

Lower quota condition. For every apportionment problem $(v;S)$ a map A satisfies lower quota condition if and only if $A_i(v;S) \geq \lfloor q_i \rfloor$ for all $i \in M$.

The *majority condition* stipulates that if a party secures more than half of the total votes in a scenario where the number of seats is odd, it will be allocated more than half of the seats.

Majority condition. For every apportionment problem $(v;S)$, where S is odd, and such that $i \in M$ with $v_i > \frac{\sum_{j\in M} v_j}{2}$, a map A satisfies majority condition if and only if $A_i(v;S) > \frac{S}{2}$.

The *weak majority condition* states that if a party obtains more than half of the total votes, it will be allocated at least half of the seats. It is noteworthy that the weak majority condition implies the fulfillment of the majority condition in situations where the number of seats is odd.

Weak majority condition. For every apportionment problem $(v;S)$ such that $i \in M$ with $v_i > \frac{\sum_{j\in M} v_j}{2}$, a map A satisfies weak majority condition if and only if $A_i(v;S) \geq \frac{S}{2}$.

[1] Note that in the case of the D'Hondt and the Smallest Divisor methods, the conventional form of the cost function is not normalized to vanish at perfect proportionality. Instead, under perfect proportionality it takes the values S and $1/S$, respectively. Indeed, in the D'Hondt method we have $\max_{i=1,\dots,m} \frac{s_i}{v_i} = \max_{i=1,\dots,m} \frac{v_i S}{v_i} = S$, while in the Smallest Divisor method $\max_{i=1,\dots,m} \frac{v_i}{s_i} = \max_{i=1,\dots,m} \frac{v_i}{v_i S} = \frac{1}{S}$.

[2] When there are ties, this property cannot be formulated in terms of permutations as it stands because the ties could be resolved differently in each of the two situations. However, the important thing about this property is that seat allocations depend only on the vote share and not on any other factor.

Table 1
The D'Hondt quotients in Example 2.2.

	1	2	3	4	5	6	7	8	9	10
0.51	1.96	3.92	5.88	7.84	9.80	11.77	13.73	15.69	17.65	19.61
0.20	5	10	15	20	25	30	35	40	45	50
0.15	6.67	13.33	20	26.67	33.33	40	46.67	53.33	60	66.67
0.14	7.14	14.29	21.42	28.57	35.71	42.86	50	57.14	64.29	71.42

The *house monotonicity* states that if the total number of seats increases by one unit, then no party gets a lower allocation.

House monotonicity. For every apportionment problem $(v;S)$ a map A satisfies house monotonicity if and only if $A_i(v;S+1) \geq A_i(v;S)$ for all $i \in M$.

The *population monotonicity* states that if one party increases its vote fraction while another decreases, then the former should receive at least the same allocation, while the latter should receive at most the same allocation.

Population monotonicity. For every pair of apportionment problems $(v;S)$ and $(v';S)$ such that for some parties $i,j \in M$ with $i \neq j$, $v'_i \geq v_i$ and $v'_j \leq v_j$, a map A satisfies population monotonicity if and only if $A_i(v';S) \geq A_i(v;S)$ and $A_j(v';S) \leq A_j(v;S)$.

The apportionment methods mentioned in the previous section satisfy many of the properties listed above. In particular, the D'Hondt method satisfies all of them (see Balinski and Young (1982) and Niemeyer and Niemeyer (2008), among others).

2.4. An algorithm for the D'Hondt method

In this subsection, we will present an algorithm to calculate the allocation of seats using the D'Hondt method. This apportionment method is widely employed as one of the most commonly used methods in practical applications (see, for instance, Pukelsheim (2017)). Although the optimization problems involved are integer linear, with solutions obtainable within reasonable timeframes for a moderate number of parties and seats, an alternative algorithm has been proposed to achieve a polynomial-time solution. Additionally, this algorithm appears to offer greater transparency to the general public compared to an assignment derived solely from being the optimal solution of an optimization problem. The pseudocode is described in Algorithm 1 below.

Algorithm 1 The lowest quotient.

```
fix s_i = 0 for all i ∈ {1,2,...,m}.                      ▷ Initialization.
while ∑_{i=1}^m s_i < S do
    calculate w_i = (s_i+1)/v_i for all i ∈ {1,2,...,m}    ▷ The quotients.
    choose z such that w_z = min_{i=1,...,m} w_i
    update s_z = s_z + 1
end while
```

Balinski and Young (1982) established that the set of optimal solutions derived from the optimization problem is equivalent to the set of allocations obtained using Algorithm 1. We refer to set because when there are ties in vote fractions, multiple solutions and also allocations are possible. Nevertheless, the case of ties in vote fractions is very unlikely in practice. In any case, since the algorithm is simpler and more visual than the optimization problem, it is often used much more in practical situations. The following example illustrate how it works.

Example 2.2. Let $M = \{1,2,3,4\}$ be the set of parties with $v_1 = 0.51$, $v_2 = 0.20$, $v_3 = 0.15$ and $v_3 = 0.14$ and the total number of seats is $S = 10$. All the possible quotients are given in Table 1:

In the second step, we allocate seats to the parties by selecting the minimum quotients until the 10 seats are reached. Thus, the seats are allocated in the following order: $(1,0,0,0) \to (2,0,0,0) \to (2,1,0,0) \to (3,1,0,0) \to (3,1,1,0) \to (3,1,1,1) \to (4,1,1,1) \to (5,1,1,1) \to (5,2,1,1) \to (6,2,1,1)$.

3. The apportionment problem when seats are grouped into lots

In this section, we introduce the extension of the apportionment problem to the case in which the seats are grouped into and allocated by lots. We address the theoretical analysis of this problem focusing on the generalization of the D'Hondt method. Likewise, this section follows a similar structure to Section 2 and includes the main results obtained. From a theoretical point of view, a possible generalization of the apportionment problem involves considering the grouping of seats into lots. This concept allows for a more nuanced distribution of seats, considering not only the total number of seats but also their grouping within specific units.

3.1. Mathematical model for the apportionment problem with lots

In this subsection, we will define a new model to address the situation in which seats are allocated in lots. Let $M = \{1,2,\ldots,m\}$ be a set of parties. Each party $i \in M$ has associated a positive vote fraction v_i, w.l.o.g., we assume that $v_1 \geq v_2 \geq v_3 \geq \cdots \geq v_m$. An integer number of S (> 1) seats must be distributed among the parties, where S is distributed into r lots of sizes $S_1, S_2, \ldots, S_r$, so that $\sum_{j=1}^{r} S_j = S$ and $S_1 \geq S_2 \geq \cdots \geq S_r$.

A new set of allocation variables are defined as follows

$$x_{ij} = \begin{cases} 1 & \text{if lot } j \text{ is allocated to party } i, \\ 0 & \text{otherwise,} \end{cases}$$

for all $i \in M$ and $j \in \{1,2,\ldots,r\}$. The total number of seats allocated to party i is equal to $s_i = \sum_{j=1}^{r} S_j x_{ij}$ for each $i \in M$.

An apportionment problem with lots is defined by $\left(v;(S_1,S_2,\ldots,S_r)\right)$. We also define an apportionment method for lots as a map A from $\Delta^M \times \mathbb{N}^r$ to $\{0,1\}^{M\times r}$ such that $A(v;(S_1,\ldots,S_r)) \subset \{0,1\}^{M\times r}$ and for each $\left(x_{ij}\right)_{i=1,2,\ldots,m}^{j=1,2,\ldots,r} \in A(v;(S_1,\ldots,S_r))$, $\sum_{i=1}^{m} x_{ij} = 1$ for all $j \in \{1,2,\ldots,r\}$. Note that this condition implies that each lot can only be allocated to a party but a party can be allocated more than one lot. Let us see an example of this new situation.

Example 3.1. Let $M = \{1,2,3,4\}$ be the set of parties with $v_1 = 0.51$, $v_2 = 0.20$, $v_3 = 0.15$ and $v_4 = 0.14$ and the total number of seats is equal to $S = 10$ divided into four lots of 3, 3, 3 and 1 seats, respectively. An allocation of seats can be the following

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

where the final distribution of seats is equal to $(4,3,3,0)$.

Faced with this problem, there may be very different ways to allocate the lots of seats to parties, depending on the fairness criteria or approach used.

3.2. Apportionment methods with lots as optimization problems

In this subsection, we will approach the problem of allocating lots of seats to parties from the perspective of optimizing a program that meets certain fairness criteria. In this sense, the optimization problem

Table 2
Allocations by measure of disproportionality.

Disproportionality measure	Allocation	Optimal value
D'Hondt method: $\max_{i=1,2,\ldots,m} \frac{s_i}{v_i}$	(7, 3, 0, 0)	15
Greatest remainder method: $\sum_{i=1}^{m} \lvert s_i - q_i \rvert^2$	(6, 3, 1, 0)	4.02
Sainte-Laguë method: $\sum_{i=1}^{m} \frac{(s_i-q_i)^2}{q_i}$	(6, 3, 1, 0)	12.23
Equal proportions method: $\sum_{i=1}^{m} \frac{(s_i-q_i)^2}{s_i}$	(3, 3, 3, 1)	20.85
Smallest divisor method: $\max_{i=1,2,\ldots,m} \frac{v_i}{s_i}$	(3, 3, 3, 1)	0.17

(1) has to be rewritten as follows to take into account the distribution into lots of all seats:

$$
\begin{aligned}
\min \quad & \varphi(v,s) \\
s.t. \quad & \sum_{i=1}^{m} x_{ij} = 1, j = 1,2,\ldots,r \\
& s_i = \sum_{j=1}^{r} S_j x_{ij}, i = 1,2,\ldots,m \\
& s_i \geq \min\left\{ \frac{(v_i - v_j)}{\min_{h<k}\{v_h - v_k : v_h - v_k \neq 0\}}; 1 \right\} s_j, \text{for all } i < j \\
& x_{ij} \in \{0,1\}, i = 1,2,\ldots,m, j = 1,2,\ldots,r;
\end{aligned}
\tag{3}
$$

and the objective functions of the optimization problem can be the different ones presented in (2) which are different ways of measuring the degree of disproportionality of the allocation of the lots of seats obtained. Therefore, the allocation of seats obtained is as proportional as possible in terms of these measures. The set of constraints $\{\sum_{i=1}^{m} x_{ij} = 1, j = 1,2,\ldots,r\}$ guarantees that all lots are allocated. The set of constraints $\{s_i = \sum_{j=1}^{r} S_j x_{ij}, i = 1,2,\ldots,m\}$ is simply the number of seats obtained by each party. And finally, the set of constraints $\left\{ s_i \geq \min\left\{ \frac{(v_i-v_j)}{\min_{h<k}\{v_h-v_k : v_h-v_k\neq 0\}}; 1 \right\} s_j, \text{for all } i<j \right\}$ ensures the preservation of order in the allocation of seats. Furthermore, when $v_1 > v_2 > \cdots > v_m$, this set of constraints can be changed for $\{s_i \geq s_{i+1}, i = 1,2,\ldots,m-1\}$. Next, let us consider Example 3.1 again to illustrate the results of lot allocations for each disproportionality measure.

Example 3.2. Consider again the situation described in Example 3.1. Table 2 shows the resulting allocations for each disproportionality measure presented in Section (2), as well as the corresponding disproportionality value.

Note that, as in the case of apportionment without lots, when there are ties in the vote fractions there will be multiple optimal solutions to the optimization problem (3). However, the division into lots can cause that, even without ties in the vote fractions, there may be multiple optimal solutions to the optimization problem (3). This is particularly true in the D'Hondt method and the smallest divisor method, because only one of the terms appears in the objective function, and the remaining terms could be changed without altering the optimal value. However, this is also difficult to occur in practice because the vote fractions must be chosen appropriately based on the specific division into lots to obtain multiple optimal solutions.

Furthermore, the optimization problem (3) always has some feasible solution and, therefore, there is at least an optimal allocation, as shown in the following proposition. For sake of exposition, the proofs are deferred to Appendix.

Proposition 1. *The optimization problem (3) has a nonempty feasible set.*

3.3. Some properties for apportionment methods with lots

In this subsection, the adaptation to the case of seats grouped into lots of the properties for apportionment problems will be presented and analyzed. Of the properties listed in Section 2.3, several have an identical formulation for the apportionment problem with lots. In particular, the properties of scale invariance, anonymity, order preservation, majority condition, the weak majority condition, and population monotonicity have the same definition. However, the properties of the lower quota condition and house monotonicity must be adapted to the context of the existence of lots.

Proposition 2. *All optimization-based methods of apportionment given by (3) with the disproportionality measures given in Table (2) satisfy scale invariance, anonymity and order preservation*

Nevertheless, the properties of the majority condition, the weak majority condition and population monotonicity cannot be guaranteed since their fulfillment will depend on the specific division of seats into lots, as we can see in the following examples.

Example 3.3 (*Violation of the Weak Majority Condition*)**.** Let $M = \{1,2,3\}$ be the set of parties with $v_1 = 0.51, v_2 = 0.31$ and $v_3 = 0.18$. The total number of seats is equal to $S = 9$ divided into one lot of 4 seats, one lot of 3 seats and one lot of 2 seats. Now, it is easy to check that the allocation obtained, using any measure of disproportionality, is equal to $s = (4,3,2)$, but $v_1 > \frac{\sum_{i\in M} v_i}{2}$ and $s_1 < \frac{S}{2}$.

Example 3.4 (*Violation of the Population Monotonicity*)**.** Let $M = \{1,2,3\}$ be the set of parties with $v_1 = 0.35, v_2 = 0.34$ and $v_3 = 0.31$. The total number of seats is equal to $S = 10$ divided into one lot of 4 seats and three lots of 2 seat. It is easy to check that the allocation obtained, using any disproportionality measure, is equal to $s = (4,4,2)$.

Now, suppose that $v'_1 = 0.29, v'_2 = 0.41$ and $v'_3 = 0.30$. We observe that $v'_2 > v_2$ and $v'_3 < v_3$. The resulting allocation, using any disproportionality measure, is $s' = (2,4,4)$. However, even though $v'_3 < v_3$, it holds that $s'_3 > s_3$.

In the case of the properties of the lower quota condition and house monotonicity, it will depend on the adaptation made of these properties to the context of the grouping of seats into lots. The property of house monotonicity cannot be directly transposed from the apportionment problem without lots to the apportionment problem with lots because adding a new seat could lead to changes in the sizes of the lots. Next, we propose two possible extensions.

The first extension of house monotonicity establishes that if there are two division into the same number of lots, such that in one of them each lot has more seats than in the other, so all parties will receive at least as many seats in the first as in the second.

House monotonicity with the same number of lots. For every pair of apportionment problems $(v;(S_1,S_2,\ldots,S_r))$ so that $\sum_{j=1}^{r} S_j = S$ and $S_1 \geq S_2 \geq \cdots \geq S_r$, and $(v;(S'_1,S'_2,\ldots,S'_r))$ so that $\sum_{j=1}^{r} S'_j = S'$ and $S'_1 \geq S'_2 \geq \cdots \geq S'_r$, such that $S'_j \geq S_j$ for all $j = 1,\ldots,r$, with at least one strict inequality, a map A satisfies house monotonicity with the same number of lots if and only if $A_i(v;(S'_1,S'_2,\ldots,S'_r)) \geq A_i(v;(S_1,S_2,\ldots,S_r))$ for all $i \in M$.

The second extension of house monotonicity says that if we add a new lot, then all parties will receive at least as many seats in the new situation than in the initial one.

House monotonicity with extra lots. For every pair of apportionment problems $(v; (S_1, S_2, \ldots, S_r))$ so that $\sum_{j=1}^{r} S_j = S$ and $S_1 \geq S_2 \geq \cdots \geq S_r$, and $(v; (S'_1, S'_2, \ldots, S'_{r+1}))$ so that $\sum_{j=1}^{r+1} S'_j = S'$ and $S'_1 \geq S'_2 \geq \cdots \geq S'_{r+1}$, such that there is j_0 such that $S'_j = S_j$ for all $j = 1, \ldots, j_0 - 1$ and $S'_{j+1} = S_j$ for all $j = j_0, \ldots, r$, a map A satisfies house monotonicity with the same number of lots if and only if $A_i(v; (S'_1, S'_2, \ldots, S'_{r+1})) \geq A_i(v; (S_1, S_2, \ldots, S_r))$ for all $i \in M$.

Note that the property of house monotonicity with extra lots coincides with the property of house monotonicity when the sizes of all lots are 1. However, this is not the case for the property of house monotonicity with the same number of lots. In this sense, the second property is an extension of the house monotonicity property but the first one is not, so it is a new property only meaningful in the context of the apportionment problem with lots. As in the classic case, the apportionment methods with lots will fail to satisfy these properties in general, since the lot structure must be added to the usual difficulty. The case of the lower quota condition will be analyzed in the next subsection.

Example 3.5 (*Violation of the House Monotonicity with the Same Number of Lots*). Let $M = \{1, 2\}$ be the set of parties with votes $v_1 = 0.59$ and $v_2 = 0.41$. The total number of seats is $S = 13$, divided into one lot of 7 seats, one lot of 4 seats and one lot of 2 seats, i.e., $(7, 4, 2)$. It is easy to check that the resulting allocation, using any measure of disproportionality, is $s = (7, 6)$.

Now, consider $S' = 14$, divided into one lot of 7 seats, one lot of 5 seats and one lot of 2 seats, i.e., $(7, 5, 2)$. Thus, $S'_j \geq S_j$ for all $j \in \{1, 2, 3\}$, with at least one strict inequality. However, the resulting allocation, using any measure of disproportionality, is $s' = (9, 5)$, and we observe that $s'_2 < s_2$.

Example 3.6 (*Violation of the House Monotonicity with Extra Lots*). Let $M = \{1, 2, 3\}$ be the set of parties with votes $v_1 = 0.55, v_2 = 0.32$ and $v_3 = 0.13$. The total number of seats is $S = 10$, divided into one lot of 6 seats and two lots of 2 seats, i.e., $(6, 2, 2)$. It is easy to check that the resulting allocation, using D'Hondt method, is $s = (6, 4, 0)$.

Now, consider $S' = 13$, divided into one lot of 6 seats, one lot of 3 seats and two lots of 2 seats, i.e., $(6, 3, 2, 2)$. Thus, $S'_1 = S_1$ and $S'_{j+1} = S_j$ for all $j \in \{2, 3\}$. However, the resulting allocation, using D'Hondt method, is $s' = (8, 3, 2)$, and we observe that $s'_2 < s_2$.

Example 3.7 (*Violation of the House Monotonicity with Extra Lots*). Let $M = \{1, 2, 3\}$ be the set of parties with votes $v_1 = 0.50, v_2 = 0.35$ and $v_3 = 0.15$. The total number of seats is $S = 7$, divided into one lot of 3 seats and two lots of 2 seats, i.e., $(3, 2, 2)$. It is easy to check that the resulting allocation, using any disproportionality measure other than D'Hondt method, is $s = (3, 2, 2)$.

Now, consider $S' = 8$, divided into one lot of 3 seats, two lots of 2 seats and one lot of 1 seat, i.e., $(3, 2, 2, 1)$. Thus, $S'_j = S_j$ for all $j \in \{1, 2, 3\}$. However, the resulting allocation, using any disproportionality measure other than D'Hondt method, is $s' = (4, 3, 1)$, and we observe that $s'_3 < s_3$.

3.3.1. Lower quota condition with lots

In electoral systems, one of the key proportionality criteria is the lower quota condition. This principle requires that each party receives at least the integer part of its ideal number of seats. This section delves into the concept of the lower quota and examines its implications in scenarios where seats are not allocated individually, but rather in lots. This setting introduces additional complexities, and some relaxations of the lower condition can be found in the literature.

The *lower quota condition* stipulates that each party must receive no fewer seats than the greatest integer less than or equal to its ideal allocation, given by $q_i = v_i S$. This property is satisfied by Algorithm 1 (see Balinski and Young (1982)), which shows that the D'Hondt method assigns to each party at least $\lfloor q_i \rfloor$ seats for all $i \in M$. However, when seats are allocated in lots, this property may fail to hold under any such a division as it is shown in the following example.

Example 3.8 (*Violation of the Lower Quota Condition*). Let $M = \{1, 2, 3, 4, 5\}$ be the set of parties with $v_1 = 0.22, v_2 = 0.21, v_3 = 0.20, v_4 = 0.19, v_5 = 0.18$. The total number of seats is equal to $S = 6$ divided into one lot of 3 seats and three lots of 1 seat. Now, it is easy to check that any possible allocation can satisfy the lower quota of all parties. In fact, the most proportional allocation is $s = (3, 1, 1, 1, 0)$, but $\lfloor q_5 \rfloor = 1 > 0$.

Although the classical lower-quota condition ensures a minimum level of representation in standard apportionment settings, its direct application may fail when seats are distributed by lots. In the fair division literature, one can find extensions of the lower quota condition to the case of indivisible goods. The first of these is the weak proportionality condition introduced in Aziz et al. (2020), whose definition in the context of apportionment problems with lots would be as follows:

Weak proportionality condition. For every apportionment problem with lots $(v; (S_1, \ldots, S_r))$, a map A satisfies weak proportionality condition if and only if for every party $i \in M$, there exists S_k not allocated to i, such that $A_i(v; (S_1, \ldots, S_r)) + S_k \geq q_i$.

Other three extensions of the lower quota, in this case adapted to the context of weighted seats, were introduced in Chingoma et al. (2024), whose definitions in the context of apportionment problems with lots would be as follows:

Weak lower quota up to one lot condition. For every apportionment problem $(v; (S_1, \ldots, S_r))$, a map A satisfies weak lower quota up to one lot condition if and only if for every party $i \in M$, either $A_i(v; (S_1, \ldots, S_r)) \geq q_i$ or there exists S_k not allocated to i, such that $A_i(v; (S_1, \ldots, S_r)) + S_k > q_i$.

Note that the weak lower quota up to one lot condition is completely similar to the weak proportionality condition.

Weak lower quota up to any lot condition. For every apportionment problem $(v; (S_1, \ldots, S_r))$, a map A satisfies weak lower quota up to any lot condition if and only if for every party $i \in M$, either $A_i(v; (S_1, \ldots, S_r)) \geq q_i$ or for each S_k not allocated to i, it holds that $A_i(v; (S_1, \ldots, S_r)) + S_k > q_i$.

Weak lower quota up to any lot from sufficiently represented party condition. For every apportionment problem $(v; (S_1, \ldots, S_r))$, a map A satisfies weak lower quota up to any lot from sufficiently represented party condition if and only if for every party $i \in M$, either $A_i(v; (S_1, \ldots, S_r)) \geq q_i$ or for each S_k allocated to each party j such that $A_j(v; (S_1, \ldots, S_r)) > q_j$, it holds that $A_i(v; (S_1, \ldots, S_r)) + S_k > q_i$.

These weak lower quota conditions can be incorporated in the optimization problem (3). In this sense, the weak lower condition up to one lot condition when incorporated, we obtain the optimization problem (4); when the weak lower condition up to any lot condition is considered, then the optimization problem (5) is obtained; and when weak lower quota up to any lot from sufficiently represented party condition is included, then it is obtained the optimization problem (6).

$$
\begin{aligned}
\min \quad & \varphi(v, s) \\
s.t. \quad & \sum_{i=1}^{m} x_{ij} = 1, \quad j = 1, 2, \ldots, r \\
& s_i = \sum_{j=1}^{r} S_j x_{ij}, \quad i = 1, 2, \ldots, m \\
& s_i \geq \min \left\{ \frac{(v_i - v_j)}{\min_{h<k}\{v_h - v_k \,:\, v_h - v_k \neq 0\}}; 1 \right\} s_j, \quad \text{for all } i < j \\
& s_i + \max_{j=1,2,\ldots,r} \{(1 - x_{ij}) S_j\} \geq v_i S, \quad i = 1, 2, \ldots, m \\
& x_{ij} \in \{0, 1\}, \quad i = 1, 2, \ldots, m, j = 1, 2, \ldots, r;
\end{aligned}
\tag{4}
$$

$$
\begin{aligned}
\min \quad & \varphi(v,s) \\
s.t. \quad & \sum_{i=1}^{m} x_{ij} = 1, \quad j = 1,2,\dots,r \\
& s_i = \sum_{j=1}^{r} S_j x_{ij}, \quad i = 1,2,\dots,m \\
& s_i \geq \min\left\{ \frac{(v_i - v_j)}{\min_{h<k}\{v_h - v_k \,:\, v_h - v_k \neq 0\}} ; 1 \right\} s_j, \quad \text{for all } i < j \\
& s_i + (1 - x_{ij}) S_j \geq (1 - x_{ij}) v_i S, \quad i = 1,2,\dots,m; j = 1,2,\dots,r \\
& x_{ij} \in \{0,1\}, \quad i = 1,2,\dots,m, j = 1,2,\dots,r;
\end{aligned} \tag{5}
$$

$$
\begin{aligned}
\min \quad & \varphi(v,s) \\
s.t. \quad & \sum_{i=1}^{m} x_{ij} = 1, \quad j = 1,2,\dots,r \\
& s_i = \sum_{j=1}^{r} S_j x_{ij}, \quad i = 1,2,\dots,m \\
& s_i \geq \min\left\{ \frac{(v_i - v_j)}{\min_{h<k}\{v_h - v_k \,:\, v_h - v_k \neq 0\}} ; 1 \right\} s_j, \quad \text{for all } i < j \\
& s_i - v_i S \geq -M(1 - y_i), \quad i = 1,2,\dots,m; j = 1,2,\dots,r \\
& s_i - v_i S \leq M y_i - \epsilon, \quad i = 1,2,\dots,m; j = 1,2,\dots,r \\
& s_i + S_j x_{lj} y_l \geq (1 - y_i) x_{lj} y_l v_i S, \quad i,l = 1,2,\dots,m; l \neq i; j = 1,2,\dots,r \\
& x_{ij} \in \{0,1\}, \quad i = 1,2,\dots,m, j = 1,2,\dots,r \\
& y_i, z_i \in \{0,1\}, \quad i = 1,2,\dots,m
\end{aligned} \tag{6}
$$

where $M \geq S + 1$ and $\epsilon > 0$.

On the other hand, the weak lower quota up to any lot condition is also very demanding and there is no apportionment method that satisfies it in general as the next example shows.[3]

Example 3.9 (*Violation of the Weak Lower Quota Up to Any Lot Condition*)**.** Let $M = \{1,2,3\}$ be the set of parties with $v_1 = 0.34$, $v_2 = 0.34$ and $v_3 = 0.32$. The total number of seats is equal to $S = 50$ divided into one lot of 35, one lot of 10, and one lot of 5. The lower quotas for each party are: $q_1 = 17, q_2 = 17, q_3 = 16$. Now, it is easy to check that there is no possible allocation that satisfies the weak lower quota up to any lot condition.

After studying the properties of apportionment methods in the context of seat allocation by lot, it has been established that apportionment methods can be derived that preserve the properties of scale invariance, anonymity, and order preservation that guarantee non-arbitrariness in determining the allocation; and a weak lower quota condition that guarantees a minimum requirement of proportionality in the allocation.

3.4. Algorithm for the D'Hondt method with lots

In this subsection, we will focus in the extension of the algorithm of the D'Hondt method to the case of seats grouped into lots. First, note that most apportionment methods belong to the class of divisor methods. By their very construction, there are an infinite number of them, as many as there are non-decreasing sequences of positive divisors. Three relevant examples of these methods are the D'Hondt method, the Saint-Laguë's method and the equal proportions method. The first one uses the divisors $1,2,3,4,\dots,n,\dots$; the second the divisors $1,3,5,7,\dots,(2n-1),\dots$; and the third uses the divisors $0, 1.41, 2.45, \dots, \sqrt{(n-1)n}, \dots$. For all these methods, Algorithm 1 can be used by simply changing the way of determining the numerators in Step 3.

We now present a modified version of Algorithm 1 adapted to accommodate the concept of dividing seats into lots. This adapted algorithm follows a similar idea to the adaptation of the algorithm to the case of weighted seats in Chingoma et al. (2024) and the latter, in turn, was intended to be an extension of the idea of picking sequences introduced in Chakraborty et al. (2021). The pseudocode can be viewed in Algorithm 2 below.

Algorithm 2 The lowest quotient in lots.

```
fix s_i = 0 for all i ∈ {1,2,...,m}, j = 0.                ▷ Initialization.
reorder the lots of seats: S_1 > S_2 > ... > S_r.
while there are non-allocated lots do
    j = j + 1
    calculate w_i = (s_i + S_j)/v_i for all i ∈ {1,2,...,m}    ▷ The quotients.
    choose z such that w_z = min_{i=1,...,m} w_i
    update s_z = s_z + S_j
end while
```

First, note that Algorithm 2 can be adapted to other divisor methods in a manner analogous to that of Algorithm 1. On the other hand, even though Algorithm 2 is simpler than the corresponding optimization problem, several aspects must be taken into account to avoid unwanted situations in its application as will be shown below. First, let us see how this algorithm works in the following example.

Example 3.10. Consider once again the situation described in Example 3.1. Algorithm 2 for this situation runs as follows

- $s = (0,0,0,0)$ and $S_1 = 3$ then $w = (\frac{3}{0.51}, \frac{3}{0.20}, \frac{3}{0.15}, \frac{3}{0.14}) = (5.88, 15.00, 20.00, 21.43)$ and $s = (3,0,0,0)$.
- $s = (3,0,0,0)$ and $S_2 = 3$ then $w = (\frac{6}{0.51}, \frac{3}{0.20}, \frac{3}{0.15}, \frac{3}{0.14}) = (11.76, 15.00, 20.00, 21.43)$ and $s = (6,0,0,0)$.
- $s = (6,0,0,0)$ and $S_3 = 3$ then $w = (\frac{9}{0.51}, \frac{3}{0.20}, \frac{3}{0.15}, \frac{3}{0.14}) = (17.65, 15.00, 20.00, 21.43)$ and $s = (6,3,0,0)$.
- $s = (6,3,0,0)$ and $S_4 = 1$ then $w = (\frac{7}{0.51}, \frac{4}{0.20}, \frac{1}{0.15}, \frac{1}{0.14}) = (13.72, 20, 6.67, 7.14)$ and $s = (6,3,1,0)$.

Note that the solution obtained in Example 3.10 is also an optimal solution to the optimization problem when using the D'Hondt method as the measure of disproportionality different from the one obtained in Example 3.2. Therefore, in this new context of apportionment problems, alternative global optima can be obtained even in the case where there are no ties in the votes received by the parties as mentioned above. On the other hand, Algorithm 2 could result in an allocation that is not a global optimum of the optimization problem as shown in the following example.

Example 3.11. Let $M = \{1,2\}$ be the set of parties with $v_1 = 0.5294$, $v_2 = 0.4706$ and the total number of seats is equal to $S = 19$ divided into four lots of 9, 5, 3 and 2 seats, respectively. Algorithm 2 for this situation runs as follows

- $s = (0,0)$ and $S_1 = 9$ then $w = (\frac{9}{0.5294}, \frac{9}{0.4706}) = (17.000, 19.125)$ and $s = (9,0)$.
- $s = (9,0)$ and $S_2 = 5$ then $w = (\frac{14}{0.5294}, \frac{5}{0.4706}) = (26.445, 10.625)$ and $s = (9,5)$.
- $s = (9,5)$ and $S_3 = 3$ then $w = (\frac{12}{0.5294}, \frac{8}{0.4706}) = (22.667, 17.000)$ and $s = (9,8)$.
- $s = (9,8)$ and $S_4 = 2$ then $w = (\frac{11}{0.5294}, \frac{10}{0.4706}) = (20.778, 21.249)$ and $s = (11,8)$.

However, it is easy to check that allocation obtained with the optimization problem (3) considering the objective function that leads to the D'Hondt method is equal to $s = (10,9)$.

[3] This is also proved for the case of weighted seats in Proposition 7 of Chingoma et al. (2024).

We have seen that the solution obtained by Algorithm 2 is not always a global optimal solution. But it is even possible that Algorithm 2 gives an infeasible solution to the optimization problem (3) as the following example shows.

Example 3.12 (*Violation of the Order Preservation*)**.** Let $M = \{1,2\}$ be the set of parties with $v_1 = 0.501$, $v_2 = 0.499$ and the total number of seats is equal to $S = 11$ divided into three lots of 5, 4, and 2 seats, respectively. Algorithm 2 for this situation runs as follows

- $s = (0,0)$ and $S_1 = 5$ then $w = (\frac{5}{0.501}, \frac{5}{0.499}) = (9.98, 10.02)$ and $s = (5,0)$.
- $s = (5,0)$ and $S_2 = 4$ then $w = (\frac{9}{0.501}, \frac{4}{0.499}) = (17.96, 8.02)$ and $s = (5,4)$.
- $s = (5,4)$ and $S_3 = 2$ then $w = (\frac{7}{0.501}, \frac{6}{0.499}) = (13.97, 12.02)$ and $s = (5,6)$.

Therefore, Algorithm 2 leads to an allocation that does not satisfy order preservation. This also happens for the case of weighted seats in Chingoma et al. (2024). To solve this problem, a modification of Algorithm 2 is proposed by introducing a readjustment phase to guarantee the order preservation property. The pseudocode can be viewed in Algorithm 3:

Algorithm 3 The lowest quotient in lots readjusted.

fix $s_i = 0$ for all $i \in \{1,2,...,m\}$, $j = 0$. ⊳ Initialization.
reorder the lots of seats: $S_1 > S_2 > ... > S_r$.
while there are non-allocated lots **do**
  $j = j + 1$
  calculate $w_i = \frac{s_i + S_j}{v_i}$ for all $i \in \{1,2,...,m\}$ ⊳ The quotients.
  choose z such that $w_z = \min_{i=1,...,m} w_i$
  update $s_z = s_z + S_j$
end while
if $s_z \geq s_{z+1}$ for all $z \in \{1,2,\dots,m\}$ **then**
  This is the D'Hont allocation
else
  Readjust the allocation ⊳ Readjustment.
  for $z = 1$ to m **do**
    if $s_z < s_{z+1}$ **then**
      reallocate $s_z := s_{z+1}$ and $s_{z+1} := s_z$
    else
      no reallocate $s_z := s_z$ and $s_{z+1} := s_{z+1}$
    end if
  end for
end if

Theorem 1. *The allocation obtained by the application of Algorithm 3 satisfies scale invariance, anonymity, order preservation, and the weak lower quota up to one lot condition.*

However, the weak lower quota up to any from sufficiently represented party condition is not satisfied by the allocation obtained with Algorithm 3 as the following example shows.

Example 3.13 (*Violation of the Weak Lower Quota Up to Any from Sufficiently Represented Party Condition*)**.** Let us consider the following situation: $M = \{1,2,3\}$ with $v = (0.50, 0.41, 0.09)$. $S = 25$ divided into 5 lots $(10,7,6,1,1)$, and $q = (12.5, 10.25, 2.25)$. The allocation obtained by Algorithm 2 is $s = (11,13,1) = (10+1, 7+6, 1)$. Now, if we exchange s_1 and s_2, we obtain $s' = (13,11,1) = (7+6, 10+1, 1)$. Therefore, $s'_2 = 10 + 1 > q_2 = 10.25$, but $s'_3 + S_5 = 1 + 1 < 2.25$. So the allocation s' does not satisfy the weak lower quota up to any from sufficiently represented party condition.

The first implication of Theorem 1 is that it is possible to find seat allocations in the context of seats grouped into lots that satisfy three properties related to non-arbitrariness, while being as proportional as possible. Second, since the readjustment procedure can be computed in polynomial time, Algorithm 3 can be also computed in polynomial time. Finally, this theorem implies that the optimization problem (4) has a nonempty feasible set, as stated in the following corollary.

Corollary 1. *The optimization problem (4) has a nonempty feasible set.*

Although Algorithm 3 provides a feasible solution for the optimization problem (4), it does not have to be a global optimum, as shown in Example 3.11. Therefore, in theory, it is even possible to obtain a more proportional allocation of seats than those produced by Algorithm 3 using the disproportionality measure in the objective function for the D'Hondt method. However, in practice, for medium-sized problems, obtaining a solution to the optimization problem (4) may be computationally difficult given the combinatorial nature of the problem. In any case, Algorithm 3 provides a seat allocation that has very good properties related to non-arbitrariness and proportionality, and is also computable in polynomial time. Therefore, we believe that the allocation given by Algorithm 3 is a good solution by itself for the context of apportionment problems when seats are grouped into lots.

4. Two families of division into lots

In this section, we will discuss concrete ways of dividing seats into lots. In particular, we consider two families of division into lots. The first family is the one that divides the lots into one lot of any size but larger than one, and the rest of the lots of size one. The largest size lot represents the minimum number of seats a party is guaranteed upon winning the elections, while the remaining lots are distributed proportionally according to some criterion. This family, in a sense, reflects the idea of electoral systems that reward the winner of elections so that they can more easily form a government. Formally, the definition is the following.

Definition 1. Let M be a set of political parties. Given an apportionment problem $(v; S)$, $S > 1$, the *α-reward division (α-RD) method*, $0 < \alpha \leq 1$, consists of the following division into lots of the seats

$$S_1 = \lfloor \alpha S \rfloor; \quad S_2 = 1; \quad \dots \quad S_i = 1; \quad \dots \quad S_r = 1,$$

where $S = \sum_{k=1}^{r} S_k$.

The second family is one in which the seat lots are formed systematically by multiplying by some factor. In this type of lot division, a system of "reward" is established for parties based on their position in the election results. Formally, the definition is the following.

Definition 2. Let M be a set of political parties. Given an apportionment problem $(v; S)$, $S > 1$, the *α-fractional division (α-FD) method*, $0 < \alpha \leq 1$, consists of the following division into lots of the seats

$$S_1 = \lfloor \alpha S \rfloor,$$

$$S_i = \begin{cases} \lfloor \alpha S_{i-1} \rfloor, & \text{if } \sum_{l=1}^{i-1} S_l + \lfloor \alpha S_{i-1} \rfloor \leq S \text{ and } \lfloor \alpha S_{i-1} \rfloor > 1, \\ 1, & \text{otherwise} \end{cases} \quad \text{for } i = 2, \dots, r,$$

where the process continues until $\sum_{l=1}^{r} S_l = S$.

Two special members of these families are the *Half-and-ones-division (HOD)* and the *Half division (HD)* methods which use $\alpha = \frac{1}{2}$ as the reward factor or fraction, respectively. In these cases, the party with the highest number of votes will be able to easily govern by obtaining an absolute majority of seats or only requiring a coalition with another party in order to govern. To avoid the government of the most voted party in a situation where it only gets the S_1 lot, all the other parties would have to agree to form a government together. This can be interpreted in terms of winning coalitions, those formed by the minimum number of parties that can establish a government. These methods always results in the following two situations

- $W = \{1\}$, if the most voted party obtains $s_1 = S_1 + k$ seats with $k > 0$,
- $W = \{(1, i), (2, 3, \dots, m)\}$ with $i \in \{2, \dots, m\}$, if the most voted party obtains $s_1 = S_1$ seats,

where the power of all minority parties is the same (see, Shapley and Shubik (1954) and Banzhaf (1965)).

Consider the following real example to illustrate how the HD and HOD methods work together the D'Hondt method.

Example 4.1 (*Alicante, 2015*)**.** The set of parties with representation was M={PP, PSOE, Guayem, C's, Compromis} with $v_1 = 0.2558$, $v_2 = 0.2029$, $v_3 = 0.1871$, $v_4 = 0.1870$ and $v_5 = 0.0903$. The rest of the political parties that participated in the elections obtained insufficient percentages of votes to obtain councilors. The total number of seats is equal to $S = 29$; then the solution obtained by the Algorithm 1 is equal to $s = (8, 6, 6, 6, 3)$. If we consider the HD method, we obtain the following lots $S_1 = 14$, $S_2 = 7$, $S_3 = 3$ and $S_4 = S_5 = S_6 = S_7 = S_8 = 1$. In this case, Algorithm 2 gives the allocation $s = (14, 7, 4, 3, 1)$. If we consider the Half-and-ones-division method, we obtain the following lots $S_1 = 14$, $S_2 = S_3 = \cdots S_{15} = S_{16} = 1$. In this case, Algorithm 2 gives the allocation $s = (14, 5, 4, 4, 2)$. In the real case, a government supported by three left-wing political parties (PSOE, Guanyem, Compromis) was formed, but halfway through the legislature it failed and another government formed by two right-wing political parties (PP, C's) was formed and concluded the legislature.

Next, we theoretically analyze the α-RD and the α-FD methods when the D'Hondt method is applied.

Theorem 2. *The allocation obtained by Algorithm 2 when the seats are grouped according to an α-RD method is an optimal solution of the optimization problem (3) with $\varphi(v, s) = \max_{i=1,2,\dots,m} \frac{s_i}{v_i}$.*

The following proposition demonstrates that Algorithm 2 with a division into lots given by an α-RD method satisfies some properties given in Section 3.3.

Proposition 3. *The allocation obtained by Algorithm 2 with the α-RD method satisfies order preservation, the weak majority condition, house monotonicity with the same number of lots and house monotonicity with extra lots.*

From Proposition 3 it can be concluded that the allocation obtained by Algorithm 3 is an optimal solution to the optimization problem (6) when the seats are grouped according to the α-RD method. This is established in the following corollary.

Corollary 2. *The allocation obtained by Algorithm 3 when the seats are grouped according to an α-RD method is an optimal solution of the optimization problem (6) with $\varphi(v, s) = \max_{i=1,2,\dots,m} \frac{s_i}{v_i}$.*

Theorem 3. *The allocation obtained by Algorithm 3 when the seats are grouped according to an α-RD method satisfies scale invariance, anonymity, order preservation, weak lower quota up to one lot condition, weak lower quota up to any lot from sufficiently represented party condition, weak majority condition, house monotonicity with the same number of lots and house monotonicity with extra lots.*

In the case of α-FD methods, a general result has not been achieved. However, for some cases, it is easy to verify that it is fulfilled, for example, when α is large enough to transform the FD-type method into an RD-type method. Furthermore, for the HD method, we obtain the following result.

Lemma 1. *Given a total of S seats and a division into lots $S_1, \dots, S_r$ belonging to the α-FD family the following holds:*

$$S_j \geq \sum_{l=j+1}^{t} S_l,$$

for all $j \in \{1, \dots, r\}$ where $S_j > 1$ and $S_l > 1$ for all $l \in \{j+1, \dots, t\}$ and $t \leq r$.

Lemma 2. *Given a total of S seats and a division into lots $S_1, \dots, S_r$ with the HD method the following holds:*

$$S_j = \sum_{l=j+1}^{t} S_l,$$

for all $j \in \{1, \dots, r\}$ where $S_j > 1$ and $t \leq r$.

Theorem 4. *The allocation obtained by Algorithm 2 with the HD method is an optimal solution of the optimization problem (3) with $\varphi(v, s) = \max_{i=1,2,\dots,m} \frac{s_i}{v_i}$.*

Proposition 4. *The allocation obtained by Algorithm 2 with the α-FD method satisfies order preservation.*

In view of Proposition 4, we can also conclude that Algorithm 2 and Algorithm 3 coincide when the set of lots is obtained by an α-FD method. Thus, we have the following result.

Theorem 5. *The allocation obtained by Algorithm 3 when the seats are grouped according to an α-FD method satisfies scale invariance, anonymity, order preservation, weak lower quota up to one lot condition, and weak lower quota up to any lot from sufficiently represented party condition.*

5. Case studies

In this section, will examine how electoral results highlight the complexities of the political landscape and underscore the importance of coalition-building to achieve governability in a multiparty system. To do so, we will present various results obtained in the Spanish municipal elections held in 2023, the Dutch general elections held in 2023, and an extensive study of the results in general elections in several countries. We will then compare them with the allocation of seats by lots to show how this new approach can achieve greater political stability.

5.1. Alcorcón municipal elections held in 2023

This case study examines the election results in the municipality of Alcorcón (a city in the province of Madrid). The percentage of votes obtained by each party was as follows. The PP secured 38.79% of the votes, PSOE obtained 25.89%, Ganar Alcorcón (a left-wing party) received 15.70%, VOX garnered 9.16%, and Más Madrid (another left-wing party) received 7.31%. The distribution of the 27 seats was 11, 8, 4, 2, and 2, respectively. The coalition of left-wing parties, which includes PSOE, Ganar Alcorcón and Más Madrid, obtained a combined total of 14 seats and successfully formed a governing alliance. In contrast, the right-wing parties, with the PP emerging as the most voted party (securing 12.9% more votes than the second-largest party, PSOE), and a total of 13 seats, was in the opposition, without participating in the government.

If we split the seats with the HD method we obtain the lots $S_1 = 13$, $S_2 = 6$, $S_3 = 3$, $S_4 = S_5 = S_6 = S_7 = S_8 = 1$. With this procedure, the PP would obtain 13 seats, PSOE 7, Ganar Alcorcón 4, VOX 2 and Más Madrid 1. Under this new allocation, the PP with VOX would reach the majority needed to govern. So the government is reduced from three to two parties and likely greater stability because there is one less party in the coalition structure. The Half-and-ones-division method yields the same result as the HD method.

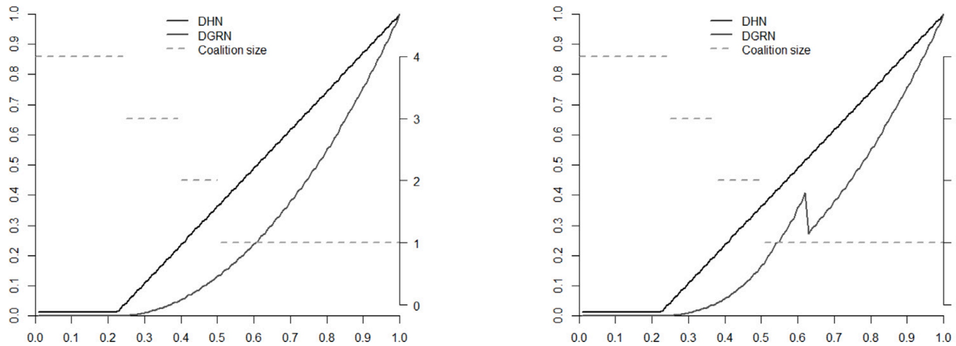

Fig. 1. Evolution of disproportionality under the α-RD method (left) and α-FD method (right) with the minimum coalition size required to achieve a majority.

5.2. General elections in the Netherlands held in 2017

The Dutch electoral system, which is based on proportional representation with open lists and uses the D'Hondt method for seat allocation, offers an ideal framework for analyzing our extension of the apportionment problem. Specifically, we examine the 2017 general election, in which a total of 28 parties ran, 13 of which won representation in the 150-seat House of Representatives. This case can help illustrate the challenges of achieving both proportional equity and political governance.

Following the elections, cabinet formation began to determine which parties would form the next government. Seven months later, a coalition agreement was reached between the VVD, CDA, D66, and CU, and the new government was due to take office in October 2017. The four parties that formed the government belong to the center-right and right-wing ideological spectrum.

In Table 3, the 13 parties that secured representation, along with their corresponding vote shares are shown. Likewise, for this scenario, the 150 seats have been distributed according to different distribution of the seats into lots, in particular, we have considered the following nine sets of lots:

- Allocation scheme L1: 150 lots of size one (without lots).
- Allocation scheme α-RD: The 150 items are divided into lots according to the α-RD method, with $\alpha \in \{0.2, 0.3, 0.4, 0.5\}$.
- Allocation scheme α-FD: The 150 items are divided into lots according to the α-FD method, with $\alpha \in \{0.2, 0.3, 0.4, 0.5\}$.

Note that if $\alpha \geq 0.51$, under both methods, the party with the largest number of votes obtains an absolute majority. Therefore, we restrict our attention to values below this threshold in Table 3.

In addition, the Banzhaf Power Index (BPI) for each party has been calculated, based on each distribution of lots and the corresponding seat allocation. Finally, the normalized[4] disproportionality measures corresponding to the D'Hondt method (DHN) and the greatest remainder method with $p = 2$ (DGRN) have been added for each resulting allocation and the minimum number of parties (C.S.), without ideology, needed to obtain a majority of seats.

In Table 3, the first thing to note is that, when neither ideology nor party affinity are considered, the number of parties needed to form a majority coalition varies from 2 to 4 across different seat distributions, highlighting that the division of seats into lots significantly affects the chances of forming a government. Furthermore, it can be seen that a certain level of proportionality is guaranteed with respect to the disproportionality measure used. In line with this, the Banzhaf Power Index values show a strong concentration of power in a few parties in certain lot distributions, especially in 0.5−RD and 0.5−HD, where the VVD alone achieves a near-absolute majority.

Fig. 1 shows how disproportionality increases as α rises, using both the normalized disproportionality measures corresponding to the D'Hondt method (DHN) and the greatest remainder method with $p = 2$ (DGRN), under the α-FD method (left) and the α-RD method (right). One can also observe that as the value of α increases, the number of party coalitions required to govern decreases. In both methods, disproportionality as measured by DHN behaves identically, although the seat distributions differ slightly, as reported in Table 3. This disproportionality is lowest under L1 (single-size lots), which is very close to perfect proportionality, and remains unchanged until $\alpha = 0.23$. On the other hand, disproportionality as measured by DGRN shows some differences under both methods. In particular, according to the DGRN measure, the α-RD method tends to be slightly more proportional than the α-FD method. Note that, under the α-FD method, for $\alpha = 0.62$ there are only two lots of 93 and 57 seats, while for $\alpha = 0.63$ the division into lots is reduced to a single lot of 94 seats, with the remainder filled by individual seats. As a result, disproportionality according to DGRN decreases at this point, converging to the same disproportionality obtained under the α-RD method from then on. The number of parties required to form a majority differs slightly between the two methods. Under the α−RD method, a majority requires four parties when $\alpha \leq 0.24$, three parties for $0.25 \leq \alpha \leq 0.39$, and two parties for $0.40 \leq \alpha \leq 0.50$. Under the α−FD method, four parties are also required for $\alpha \leq 0.24$, but only three parties for $0.25 \leq \alpha \leq 0.37$, and two parties for $0.38 \leq \alpha \leq 0.50$. Finally, when $\alpha = 0.51$, the largest party obtains an absolute majority under both methods, and further increases in α only raise disproportionality.

In the real case, a government was formed with the support of four political parties. Under some of the alternative allocation schemes considered here, the coalition size could have been reduced to two parties: the largest one and one additional partner. This could be regarded as a way of reducing tensions within the government, since reaching agreements between two parties is generally easier than among four. Furthermore, considering the lowest alpha value for both methods to obtain a two-party coalition government, the DHN disproportionality are 0.2376 and 0.2122 for the α-RD and α-FD methods, respectively. In both cases, the result remains fairly close to proportional representation, with the α-FD method yielding a slightly more proportional result.

5.3. Extensive analysis of elections

In order to assess the impact of grouping seats into lots, the last four elections in the Netherlands, Greece, Slovakia, and Greenland were considered. Greece was chosen because it has an electoral system that rewards the parties with the most votes, making it easier to form a stable government. The other three countries were chosen because they have single-constituency proportional representation electoral systems. Therefore, a total of 16 election results were taken, and for each

[4] The measure is given normalized to $0-1$, where 0 corresponds to perfect proportionality and 1 corresponds to the case all-for-the winner.

Table 3
The Netherlands 2017, 150 seats.

Party	Votes (%)	L1		0.2-RD		0.3-RD		0.4-RD	
		Seats	BPI	Seats	BPI	Seats	BPI	Seats	BPI
VVD	21.3	33	0.251	33	0.251	45	0.402	60	0.731
PVV	13.1	20	0.131	20	0.131	18	0.100	16	0.042
CDA	12.4	19	0.123	19	0.123	17	0.095	15	0.042
D66	12.2	19	0.123	19	0.123	17	0.095	14	0.042
GL	9.1	14	0.090	14	0.090	13	0.076	11	0.036
SP	9.1	14	0.090	14	0.090	13	0.076	11	0.036
PVDA	5.7	9	0.051	9	0.051	8	0.051	7	0.018
CU	3.4	5	0.032	5	0.032	4	0.022	4	0.013
PVDD	3.2	5	0.032	5	0.032	4	0.022	3	0.010
50+	3.1	4	0.026	4	0.026	4	0.022	3	0.010
SGP	2.1	3	0.018	3	0.018	3	0.017	2	0.007
DENK	2.0	3	0.018	3	0.018	2	0.011	2	0.007
FVD	1.8	2	0.014	2	0.014	2	0.011	2	0.007
DHN	–	0.0142		0.0142		0.1106		0.2376	
DGRN	–	0.0002		0.0002		0.0117		0.0552	
C.S.	–	4		4		3		2	

Party	0.5-RD		0.2-FD		0.3-FD		0.4-FD		0.5-FD	
	Seats	BPI	Seats	BPI	Seats	BPI	Seats	BPI	Seats	BPI
VVD	75	0.997	33	0.251	45	0.402	60	0.678	75	0.983
PVV	13	0.000	20	0.131	18	0.100	24	0.054	37	0.002
CDA	13	0.000	19	0.123	17	0.095	13	0.052	18	0.002
D66	12	0.000	19	0.123	17	0.095	13	0.052	9	0.002
GL	9	0.000	14	0.090	13	0.076	10	0.042	4	0.002
SP	9	0.000	14	0.090	13	0.076	10	0.042	3	0.002
PVDA	5	0.000	9	0.051	8	0.051	6	0.020	1	0.002
CU	3	0.000	5	0.032	4	0.022	3	0.013	1	0.002
PVDD	3	0.000	5	0.032	4	0.022	3	0.013	1	0.002
50+	3	0.000	4	0.026	4	0.022	3	0.013	1	0.002
SGP	2	0.000	3	0.018	3	0.017	2	0.008	0	0.000
DENK	2	0.000	3	0.018	2	0.011	2	0.008	0	0.000
FVD	1	0.000	2	0.014	2	0.011	1	0.004	0	0.000
DHN	0.3647		0.0142		0.1106		0.2376		0.3647	
DGRN	0.1312		0.0002		0.0117		0.0588		0.1657	
C.S.	2		4		3		2		2	

Table 4
Basic statistics of the election results studied.

	The Netherlands		Greece		Slovakia		Greenland		Total	
	C	D	C	D	C	D	C	D	C	D
Real	3.5	.017	1.66[5]	.192	3	.121	2.75	.035	2.72[5]	.083
L1	3	.017	2	.076	2.25	.121	2	.035	2.31	.062
0.2–RD	3	.017	2	.076	2.25	.121	2	.035	2.31	.062
0.3–RD	2.75	.086	2	.076	2	.127	2	.035	2.19	.081
0.4–RD	2	.217	2	.082	1.75	.197	2	.072	2	.142
0.5–RD	2	.347	2	.191	1.75	.298	2	.188	2	.256
0.2–FD	3	.017	2	.076	2.25	.121	2	.035	2.31	.062
0.3–FD	2.75	.086	2	.076	2	.127	2	.035	2.31	.081
0.4–FD	2	.217	2	.082	1.75	.197	2	.072	2	.142
0.5–FD	2	.347	2	.194	1.75	.409	2	.188	2	.285

country and for those four elections the average number of parties that actually formed a government,[5] the minimum average number of parties required to form a government when ideology is not taken into account, and the average DHN disproportionality for each of the sets of lots defined in the previous section were calculated. The results obtained are presented in Table 4. In particular, Column C of Table 4 presents the average number of parties required to form a government, and column D presents the average disproportionality.

In Table 4, the first thing we notice is that, in general, there is a negative correlation between the number of parties that can form a government and the disproportionality in the distribution of seats. Therefore, if we want to reduce the number of parties needed to form a government, we must partially abandon proportionality. Furthermore, as expected, the number of parties that actually form a government is higher than the minimum required when other aspects such as ideology and political affinity are not taken into account. On the other hand, in view of the results, one could conclude that, in general, when no party obtains an absolute majority, two parties are sufficient to form a government, regardless of the seat allocation system chosen. This is true, but there is a nuance: these two parties tend to be focal representatives of the left–right ideological spectrum and, therefore, on many occasions it will be difficult for them to reach a government agreement. Thus, roughly speaking, the ultimate objective of grouping seats into lots would be to sufficiently reward the winner of the elections to facilitate reaching a governing agreement with another party close on the ideological spectrum, thus avoiding the need to bring many parties together in the governing agreement.

6. Conclusions

The contribution of this work is twofold. On the one hand, an extension of the apportionment problem has been introduced to the case in which resources are grouped in lots, in general, of different sizes. This extension is not only useful for its application to seat allocation problems in electoral systems, but it is also useful for its application to resource allocation problems when these are not perfectly divisible and, moreover, have different sizes (see, for example, Estañ et al. (2021)

[5] In the Greek elections in May 2023, no government was formed.

and Suksompong (2024)). In this context, a polynomial-time algorithm (Algorithm 3) has been introduced that provides allocations satisfying non-arbitrariness properties such as anonymity, scale invariance and order preservation and a weak proportionality property. This result adds the order preservation property with respect to the results obtained by Aziz et al. (2020), Chakraborty et al. (2021) and Chingoma et al. (2024).

On the other hand, apportionment methods have been introduced based on the fact that seats are not allocated one by one, but rather in lots of different sizes. This allows introducing one more element in the design of electoral systems that takes into account other factors, in addition to proportionality. Furthermore, this is present in some cases in reality, for example, electorate votes in the elections of the President of US where there is a single lot of "seat" for each constituency (state) which is assigned to the candidate with the most votes in it.

Two factors to take into account in the design of a representative electoral system are proportionality in the distribution of seats and the possibility of forming stable and long-lasting governments. Majority, plurality and proportional electoral systems are the most commons. While the first two favor the presence of few parties more, the second favors the presence of many parties more. For this reason, mixed electoral systems have also been designed. In the case of opting for a proportional electoral system, there are several elements to take into account, for example, the number of constituencies and their tiers, the imposition of thresholds or the possibility of bonuses for the parties with the most votes. In this work, a new element to be taken into account is introduced, the size of the lots of seats to be assigned. All these elements can help balance between governability and proportionality. Lijphart (1990, 1994) has already analyzed how to use the first two to achieve the objectives of governability or proportionality or a balance between the two. The electoral systems of Greece and Italy are examples of how to use bonuses to the most voted parties to favor governability to the detriment of proportionality. An alternative method to favor governability by favoring the parties with the most votes but without adding bonuses or other corrections is presented in Sánchez-Soriano et al. (2016). This work also presents a new apportionment method to promote governance without adding bonuses or other corrections, simply adding a new element to the design, the distribution of seats in lots.

Section 5 has shown the impact of the new lot-based apportionment methods introduced in this work. In our opinion, this method is appropriate when the number of constituencies is small and governability is sought without losing too much proportionality. In fact, the negative effect on proportionality with the methods introduced could be smaller than the effect that the imposition of thresholds or many constituencies can have. Therefore, apportionment methods based on lots are interesting for elections with few constituencies and seek to favor governability without losing much proportionality, measured in terms of representativeness, that is, without excluding presence of multiple parties, as may occur in the case of the imposition of thresholds o multiple constituencies.

Some final comments on the implications of applying the HD and HOD methods, which represent the line between giving the absolute majority directly to the winner or not. First, there is only one government alternative in which the most voted party is not when it does not obtain by itself an absolute majority. Second, all the parties with representation are relevant except in the case that the party with the most votes obtains an absolute majority. This aspect is interesting because if a party obtains representation, it will always be able to participate in the development of government actions and not be irrelevant throughout the legislature. However, in all cases the bargaining power of the minority parties is the same and this can be a negative aspect, since parties with more representation have the same decision-making power as others with fewer representatives. Third, the maximum number of parties with representation is less than or equal to the number of lots of seats. This is an interesting element, because a parameter is introduced that allows deciding on heterogeneity in the representative body. Fourth, the above can also be analyzed in terms of power in weighted voting system (see, for example, Lucas (1983)). A weighted majority system is given by $[q; w_1, w_2, \dots, w_m]$, where m is the number of political parties, q is the minimum number of votes necessary to win a proposal, called the quota, and w_i is the number of votes of party i. Moreover, $q > \frac{1}{2}\sum w_i$. Thus, one of the two following stylized weighted voting systems arises:

$[m-1; m-2, 1, \dots, 1]$ or $[m; m, 1, \dots, 1]$,

where m is the number of parties with representation. Note that these two systems are simplified and others but equivalent, in terms of winning coalitions, weighted voting systems can arise. In this two systems we can observe that the power of all minority parties is the same (see, Shapley and Shubik (1954) and Banzhaf (1965)). Moreover, the majority party has multiple possibilities to advance a proposal, while if there is no absolute majority, all the minority parties can block a proposal from the majority party.

CRediT authorship contribution statement

Juan Carlos Gonçalves-Dosantos: Writing – review & editing, Writing – original draft, Visualization, Validation, Supervision, Software, Resources, Project administration, Methodology, Investigation, Funding acquisition, Formal analysis, Data curation, Conceptualization. **Joaquín Sánchez-Soriano:** Writing – review & editing, Writing – original draft, Visualization, Validation, Supervision, Software, Resources, Project administration, Methodology, Investigation, Funding acquisition, Formal analysis, Data curation, Conceptualization.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix. Proofs

Proof of Proposition 1

Proof. Let $S_1 \geq S_2 \geq \cdots \geq S_r$ be a division into lots, then we can consider the following two allocations:

$$x_{ij} = \begin{cases} 1, & \text{if } j = i + km, k = 0, 1, 2, \dots \\ 0, & \text{otherwise} \end{cases} \qquad x_{ij} = \begin{cases} 1, & \text{if } i = 1, j \in \{1, \dots, r\} \\ 0, & \text{otherwise} \end{cases}$$

Now, it is easy to check that both allocations satisfy all the constraints of the optimization problem (3). □

Proof of Proposition 2

Proof. The proof of this result is obtained by taking into account the structure of the optimization problems and their objective functions. □

Proof of Theorem 1

Proof. We proceed with the demonstration property by property:

- Scale invariance is satisfied because the quotients are scale invariant.
- Anonymity is satisfied because the allocation only depends on the fraction votes not on the names of the parties. Note that in Algorithm 3 the possible ties has been resolved in favor of the party with the smaller subindex, but this is not a problem if the order for tied parties is carried out in advance by random.
- Order preservation is satisfied by the own structure of Algorithm 3.
- A consequence of Theorem 10 in Chingoma et al. (2024) is that Algorithm 2 satisfies the weak lower quota up to one lot condition. Let s be the allocation obtained by Algorithm 2 and $s_z < s_{z+1}$ for some $z \in M$, then we have the two following cases:
  - If $s_z \geq q_z$, then $s_{z+1} > q_{z+1}$. Indeed, $s_{z+1} > s_z \geq q_z \geq q_{z+1}$. If we take s' such that $s'_z = s_{z+1}$, $s'_{z+1} = s_z$ and $s'_i = s_i$ for each $i \neq z, z+1$, then we have that $s'_z = s_{z+1} > s_z \geq q_z$ and $s'_{z+1} = s_z \geq q_z \geq q_{z+1}$, and for the remainder parties nothing changes. Therefore, it is now obvious that the new allocation s' satisfies the condition.
  - $s_z < q_z$. We take s' such that $s'_z = s_{z+1}$, $s'_{z+1} = s_z$ and $s'_i = s_i$ for each $i \neq z, z+1$, and for the remainder parties nothing changes. Since s satisfies the weak lower quota up to one lot condition, there is at least a lot S_h not allocated to z such that $s_z + S_h > q_z$. Moreover, note that S_1 is always allocated to party 1 by Algorithm 2. Now, we have that either $s'_z = s_{z+1} \geq q_z$ or $s'_z + S_1 = s_{z+1} + S_1 > q_z$ such that the lot S_1 is not allocated to party z in s', where the latter follows from $s_{z+1} + S_1 > s_z + S_1 \geq s_z + S_h > q_z$. Moreover, we have that either $s'_{z+1} = s_z \geq q_{z+1}$ or $s'_{z+1} + S_h = s_z + S_h > q_z \geq q_{z+1}$ such that the lot S_h is not allocated to $z+1$ in the allocation s'. As a result, the new allocation s' also satisfies the condition.

Now suppose that we have an allocation s' obtained by successive changes of two consecutive allocations to achieve order preservation, which satisfies the weak lower quota up to one lot condition. If there is a z such that $s'_z < s'_{z+1}$, then we have again the following two cases:

- $s'_z \geq q_z$. This case is completely analogous to the same case as before. Therefore, the new allocation s'' satisfies the condition.
- $s'_z < q_z$. We take s'' such that $s''_z = s'_{z+1}$, $s''_{z+1} = s'_z$ and $s''_i = s'_i$ for each $i \neq z, z+1$, and for the remainder parties nothing changes. Since s' satisfies the weak lower quota up to one lot condition, there is at least a lot S_h not allocated to z such that $s'_z + S_h > q_z$. Moreover, note that by the readjustment procedure the lot S_1 cannot be allocated to party $z+1$ in s' because the lot S_1 included in s_1 obtained by Algorithm 2 moves from the first position to the last in the readjustment procedure, such that in each change the ones left of s_1 are always greater, so it cannot go back. Now, we have that either $s''_z = s'_{z+1} \geq q_z$ or $s''_z + S_1 = s'_{z+1} + S_1 > q_z$ such that the lot S_1 is not allocated to party z in s'', where the latter follows from $s'_{z+1} + S_1 > s'_z + S_1 \geq s'_z + S_h > q_z$. Moreover, we have that either $s''_{z+1} = s'_z \geq q_{z+1}$ or $s''_{z+1} + S_h = s'_z + S_h > q_z \geq q_{z+1}$ such that the lot S_h is not allocated to $z+1$ in the allocation s''. As a result, the new allocation s'' also satisfies the condition.

Therefore, by repeating this process as many times as necessary, but in any case a finite number of times, one would arrive at an allocation of the lots that satisfies the order preservation property and the weak lower quota up to one lot condition. □

Proof of Theorem 2

Proof. Consider a set of parties $M = \{1, \dots, m\}$, where v_i represents the vote fraction for party $i \in M$. Let S be the total number of seats allocated across r lots, $S_1, S_2, \dots, S_r$, where $S_1 \leq S$ and $S_2 = S_3 = \cdots = S_r = 1$, such that $S = \sum_{j=1}^{r} S_j$. After applying Algorithm 2, we obtain the allocation $s_1, s_2, \dots, s_m$. Since $v_1 \geq v_i$ for all $i \in M \backslash \{1\}$, w.l.o.g, we can assume that $s_1 = S_1 + k$, where $k \in \mathbb{N}$. We now consider two cases:

Case 1. If $s_1, s_2, \dots, s_m$ coincides with the allocation obtained when Algorithm 1 is applied without lots, then according to Balinski and Young (1982), it is an optimal solution of the optimization problem (1) with the objective function for the D'Hondt method. Therefore, $s_1, s_2, \dots, s_m$ is also an optimal solution of the optimization problem (3), because if this were not an optimal solution of that problem then there would be another feasible solution $s'_1, s'_2, \dots, s'_m$ such that

$$\max_{i \in M} \frac{s'_i}{v_i} < \max_{i \in M} \frac{s_i}{v_i}.$$

But $s'_1, s'_2, \dots, s'_m$ is a feasible solution of the optimization problem (1), which is a contradiction with the fact that $s_1, s_2, \dots, s_m$ is an optimal solution of the optimization problem (1).

Case 2. If $s_1, s_2, \dots, s_m$ does not coincide with the allocation obtained when Algorithm 1 is applied without lots, this means that s_1 must be exactly S_1, otherwise there would be a step in Algorithm 2 in which $\hat{s}_1 = S_1$ and for all $i \in M \backslash \{1\}$,

$$\frac{\hat{s}_1 + 1}{v_1} \leq \frac{\hat{s}_i + 1}{v_i}.$$

But this step would also appear in Algorithm 1 and then the allocations given by Algorithm 1 and Algorithm 2 would coincide, and this is a contradiction. Moreover, in the allocation obtained with Algorithm 1, party 1 must receive strictly less than S_1 by a similar argument, and therefore m must also be greater than 2.

Now, given the problem $\left((v_2, \dots, v_m); \{1, \overset{S-S_1}{\dots}, 1\}\right)$, $s_2, \dots, s_m$ is an optimal solution of the corresponding problem (3), since it coincides with the allocation obtained by applying Algorithm 1 to the problem $((v_2, \dots, v_m); S-S_1)$. In addition, since $v_1 \geq v_i$, $i = 2, \dots, m$; and what was mentioned about party 1 in the previous paragraph, $s_i < S_1$, $\forall i \in M$. If we now consider the problem $\left((v_1, v_2, \dots, v_m); \{S_1, 1, \overset{S-S_1}{\dots}, 1\}\right)$, by constraints $s_i \geq \min\left\{\frac{(v_i - v_j)}{\min_{h<k}\{v_h - v_k : v_h - v_k \neq 0\}}; 1\right\} s_j$, for all $i < j$, a feasible solution of problem (3) must allocate at least S_1 seats to a party in $\operatorname{argmax}\{v_i \,:\, i \in M\}$, in particular to party 1. Therefore, there is an optimal solution s^* of problem (3) with $s^*_1 = S_1$. Now there are two possibilities. If $\frac{S_1}{v_1} \geq \frac{s_i}{v_i}$, $i = 2, \dots, m$; then $s_1, s_2, \dots, s_m$ is trivially an optimal solution. If there is $i > 1$, such that $\frac{s_i}{v_i} \geq \frac{S_1}{v_1}$, then $s_1, s_2, \dots, s_m$ is again an optimal solution since $s_2, \dots, s_m$ is an optimal solution of the corresponding problem (3).

Therefore, the allocation obtained by means of Algorithm 2, s_1, $s_2, \dots, s_m$ is an optimal solution of the optimization problem (3), considering the objective function for the D'Hondt method. □

Proof of Proposition 3

Proof. We proceed with the demonstration property by property:

- In order to prove order preservation, we consider the following two cases.
  First, if the solution obtained by Algorithm 2 coincides with the allocation resulting from applying Algorithm 1 without drawing lots, then, according to Balinski and Young (1982), this allocation satisfies order preservation.
  Second, if the two solutions do not coincide, then s_1 must be exactly S_1. Since
  $$\frac{S_1 + 1}{v_1} \leq \frac{S_1 + 1}{v_i},$$

it follows that $s_i \leq s_1$ for all $i \in M \setminus \{1\}$; otherwise, party 1 would have been assigned that seat.
Now, consider the subproblem $\left((v_2,\ldots,v_m);\{1,\ldots,1\}\right)$ with $S-S_1$ seats. The allocation $s_2,\ldots,s_m$, obtained by applying Algorithm 2, coincides with the allocation produced by Algorithm 1. Therefore, according to Balinski and Young (1982), $s_i \geq s_j$ for all $i,j \in M\setminus\{1\}$ such that $v_i > v_j$.

- To prove weak majority condition, consider $v_1 > \frac{\sum_{i\in M} v_i}{m}$ and the following two cases.
First, if $S_1 \geq \frac{S}{2}$ since S_1 is allocated to party 1 we have that $s_1 \geq S_1 \geq \frac{S}{2}$.
Second, if $S_1 < \frac{S}{2}$, then, since Algorithm 1 satisfies the weak majority condition, the allocation s' obtained from this algorithm must satisfy $s'_1 \geq \frac{S}{2}$. Now, since $s'_1 > S_1$, this implies that at some step of the algorithm,
$$\frac{S_1+1}{v_1} \leq \frac{\hat{s}'_i+1}{v_i},$$
where $\hat{s}'_i$ denotes the number of seats assigned to party $i\in M\setminus\{1\}$ at that point in the algorithm. However, this would also occur in Algorithm 2. Therefore, the solution obtained by both algorithms coincides, and hence Algorithm 2 also satisfies the weak majority condition.
- To examine house monotonicity with the same number of lots, consider a pair of apportionment problems $(v;(S_1,S_2,\ldots,S_r))$ such that $\sum_{j=1}^r S_j = S$ and $S_1 \geq S_2 \geq \cdots \geq S_r$, and another problem $(v;(S'_1,S'_2,\ldots,S'_r))$ such that $\sum_{j=1}^r S'_j = S'$ and $S'_1 \geq S'_2 \geq \cdots \geq S'_r$, with $S'_j \geq S_j$ for all $j=1,\ldots,r$.
We assume that at least one strict inequality holds. Furthermore, $(S_1,S_2,\ldots,S_r)$ and $(S'_1,S'_2,\ldots,S'_r)$ represent two divisions into lots of the α-RD. Then it must be that $S'_1 > S_1 \geq 1$ and $S'_j = S_j = 1$ for all $j\in\{2,\ldots,r\}$.
Let $s_1,\ldots,s_m$ be the allocation corresponding to $(v;(S_1,S_2,\ldots,S_r))$, and let $s'_1,\ldots,s'_m$ be the allocation corresponding to $(v;(S'_1,S'_2,\ldots,S'_r))$ after applying Algorithm 2. We now consider the following three cases.
  - If $s_1,\ldots,s_m$ coincides with the solution obtained after applying Algorithm 1 to $(v;\sum_{j=1}^r S_j)$, and $s'_1,\ldots,s'_m$ coincides with the solution obtained after applying Algorithm 1 to $(v;\sum_{j=1}^r S'_j)$, then, according to Balinski and Young (1982), $s'_i \geq s_i$ for all $i\in M$.
  - If $s_1,\ldots,s_m$ does not coincide with the solution obtained after applying Algorithm 1 to $(v;\sum_{j=1}^r S_j)$, and $s'_1,\ldots,s'_m$ does not coincide with the solution obtained after applying Algorithm 1 to $(v;\sum_{j=1}^r S'_j)$, then $s_1 = S_1$ and $s'_1 = S'_1$, so $s'_1 > s_1$. On the other hand, $s_2,\ldots,s_m$ is the solution obtained by applying Algorithm 1 to $\left((v_2,\ldots,v_m);\sum_{j=2}^r S_j\right)$, and $s'_2,\ldots,s'_m$ is the solution obtained by applying Algorithm 1 to $\left((v_2,\ldots,v_m);\sum_{j=2}^r S'_j\right)$. However, since $(S_2,\ldots,S_r) = (S'_2,\ldots,S'_r)$, it follows that $s'_i = s_i$ for all $i\in M\setminus\{1\}$.
  - If $s_1,\ldots,s_m$ coincides with the solution obtained after applying Algorithm 1 to $(v;\sum_{j=1}^r S_j)$, and $s'_1,\ldots,s'_m$ does not coincide with the solution obtained after applying Algorithm 1 to $(v;\sum_{j=1}^r S'_j)$, then it has to be that $s_1 = S_1 + k < S'_1 = s'_1$ for some $k \geq 0$. On the other hand, $s'_2,\ldots,s'_m$ is the solution obtained by applying Algorithm 1 to $\left((v_2,\ldots,v_m);\sum_{j=2}^r S'_j\right)$. Since $s_1 \geq S_1$, it follows that $s_1 = S_1 + S_{j_1} + \cdots + S_{j_k}$ for some $k \geq 0$. Therefore, $s_2,\ldots,s_m$ is the solution obtained by applying Algorithm 1 to $\left((v_2,\ldots,v_m);\sum_{j=1}^r S_j - (S_1+S_{j_1}+\cdots+S_{j_r})\right)$. Since $S'_2+\cdots+S'_r > S_1+\cdots+S_r-(S_1+S_{j_1}+\cdots+S_{j_r})$, then, according to Balinski and Young (1982), $s'_i \geq s_i$ for all $i\in M\setminus\{1\}$.
  - Finally, it is not possible that $s_1,\ldots,s_m$ does not coincide with the solution obtained after applying Algorithm 1 to $(v;\sum_{j=1}^r S_j)$, while $s'_1,\ldots,s'_m$ coincides with the solution obtained after applying Algorithm 1 to $(v;\sum_{j=1}^r S'_j)$. To prove that, consider the two following cases.
    * If $s'_1 = S'_1$, since $S'_1 > S_1$ then at some step of Algorithm 1, it must happen that
    $$\frac{S_1+1}{v_1} \leq \frac{\bar{s}_i+1}{v_i}$$
    for all $i\in M\setminus\{1\}$, where $\bar{s}_i$ denotes the number of seats allocated to party i at that step of the problem $(v;\sum_{j=1}^r S'_j)$. But this would also be the case when applying Algorithm 2 to the problem $(v;(S_1,S_2,\ldots,S_r))$.
    * If $s'_1 > S'_1$, then at some step of Algorithm 2, it must happen that
    $$\frac{S'_1+S_j}{v_1} \leq \frac{\hat{s}'_i+S_j}{v_i}$$
    for all $i\in M\setminus\{1\}$ and for some $j\in\{2,\ldots,r\}$, where $\hat{s}'_i$ denotes the number of seats allocated to party i at that step of the problem $(v;(S'_1,S'_2,\ldots,S'_r))$. But, on the other hand, since
    $$\frac{S'_1+S_j}{v_1} \geq \frac{S_1+S_j}{v_1},$$
    it follows that S_j must also be allocated to party 1 in the problem $(v;(S_1,S_2,\ldots,S_r))$.
- In order to prove house monotonicity with extra lots, consider a pair of apportionment problems $(v;(S_1,S_2,\ldots,S_r))$ so that $\sum_{j=1}^r S_j = S$ and $S_1 \geq S_2 \geq \cdots \geq S_r$, and $(v;(S'_1,S'_2,\ldots,S'_{r+1}))$ so that $\sum_{j=1}^{r+1} S'_j = S'$ and $S'_1 \geq S'_2 \geq \cdots \geq S'_{r+1}$, such that there is j_0 such that $S'_j = S_j$ for all $j=1,\ldots,j_0-1$ and $S'_{j+1} = S_j$ for all $j=j_0,\ldots,r$.
Furthermore, $(S_1,S_2,\ldots,S_r)$ and $(S'_1,S'_2,\ldots,S'_r)$ represent two divisions into lots of the α-RD. Then, it must be that $S'_1 = S_1 \geq 1$, $S'_j = S_j = 1$ for all $j\in\{2,\ldots,r\}$ and $S'_{r+1} = 1$.
Let $s_1,\ldots,s_m$ be the allocation corresponding to $(v;(S_1,S_2,\ldots,S_r))$, and let $s'_1,\ldots,s'_m$ be the allocation corresponding to $(v;(S'_1,S'_2,\ldots,S'_{r+1}))$ after applying Algorithm 2. Since $S'_j = S_j$ for all $j\in\{1,\ldots,r\}$, in each step j of Algorithm 2, the quotients will be the same, and so will the corresponding allocation for $(v;(S_1,S_2,\ldots,S_r))$ and $(v;(S'_1,S'_2,\ldots,S'_{r+1}))$. For $(v;(S_1,S_2,\ldots,S_r))$, the algorithm stops at the step r, whereas for $(v;(S'_1,S'_2,\ldots,S'_{r+1}))$, the lot S'_{r+1} is allocated to some party $i\in M$. Therefore, $s'_i \geq s_i$ for all $i\in M$. □

Proof of Corollary 2

Proof. First, since the allocation obtained by Algorithm 2 directly satisfies order preservation, then it coincides with the allocation obtained by Algorithm 3. Therefore, when seats are grouped according to an α-RD method, Algorithm 2 and Algorithm 3 lead the same allocation. Now, by Theorem 10 in Chingoma et al. (2024), the allocation obtained by Algorithm 2 satisfies weak lower quota up to any lot from sufficiently represented party condition, then it is a feasible solution of the optimization problem (6). On the other hand, each feasible solution for the optimization problem (6) is also feasible for the optimization problem (3), therefore since the allocation obtained by Algorithm 2 is optimal for the optimization problem (3) and feasible for the optimization problem (6), then it is optimal for the optimization problem (6). □

Proof of Theorem 3

Proof. This result is a direct consequence of Proposition 3 and Corollary 2. □

Proof of Lemma 1

Proof. Consider S seats and a division into lots $S_1, \dots, S_r$ using the α-FD family. Fix j with $S_j > 1$. Note that for each $k \in \{1, \dots, t-j\}$ such that $S_t > 1$ and $S_{t+1} = 1$, we have that

$$S_{j+k} = \lfloor \alpha S_{j+k-1} \rfloor \leq \alpha S_{j+k-1}.$$

This implies

$$S_{j+k} \leq \alpha S_{j+k-1} \leq \alpha S_{j+k-2} \leq \cdots \leq \alpha^k S_j.$$

Therefore,

$$\sum_{l=j+1}^{t} S_l = \sum_{k=1}^{t-j} S_k \leq S_j \sum_{k=1}^{t-j} \alpha^k = S_j \frac{\alpha(1-\alpha^{t-j})}{1-\alpha}.$$

Now, let us see that

$$\frac{\alpha(1-\alpha^{t-j})}{1-\alpha} \leq 1$$

or, equivalently,

$$\alpha(1-\alpha^{t-j}) \leq 1-\alpha \Leftrightarrow 2\alpha \leq 1 + \alpha^{t-j+1}.$$

Since $0 < \alpha \leq 1$ and $t - j \geq 1$ then $\alpha^{t-j+1} \geq \alpha^2$. Therefore, it is enough to prove that

$$2\alpha \leq 1 + \alpha^2$$

or, equivalently,

$$(1-\alpha)^2 \geq 0$$

this is always true since $0 < \alpha \leq 1$. The proof is concluded. □

Proof of Lemma 2

Proof. Consider S seats and a division into lots $S_1, \dots, S_r$ using the HD method, with $S_0 = S$. Fix $j \in \{1, \dots, r\}$ such that $S_j > 1$. Consider $L = \min\{l > j : S_l = 1\}$. We have, for all $l \in \{j+1, \dots, L-1\}$, that $S_l > 1$ and $S_l = \left\lfloor \frac{S_{l-1}}{2} \right\rfloor$. We define $\epsilon_l = S_{l-1} - 2S_l = S_{l-1} - 2\left\lfloor \frac{S_{l-1}}{2} \right\rfloor$, so that $\epsilon_l \in \{0, 1\}$ for all $l \in \{j+1, \dots, L-1\}$ and $\epsilon_L = S_{L-1} - 2S_L = S_{L-1} - 2$. Therefore, $\sum_{l=j+1}^{L} \epsilon_l = p$ for some $p \in \mathbb{N}$. On the other hand,

$$\sum_{l=j+1}^{L} \epsilon_l = S_j - \sum_{l=j+1}^{L} S_l - S_L = p.$$

Since $S_L = 1$, we obtain

$$S_j - \sum_{l=j+1}^{L} S_l = p + 1.$$

Therefore, $S_1 - \sum_{l=2}^{L} S_l \geq p+1$, so there must exist at least $S_{L+1}, S_{L+2}, \dots, S_{L+p+1}$, all of size one. Finally,

$$\sum_{l=j+1}^{L+p+1} S_l = \sum_{l=j+1}^{L} S_l + \sum_{l=L+1}^{L+p+1} S_l = S_j - (p+1) + (p+1) = S_j$$

ant the proof is concluded. □

Proof of Theorem 4

Proof. Consider a set of parties $M = \{1, \dots, m\}$, where v_i represents the vote fraction for party $i \in M$. Let S be the total number of seats, and let $S_1, \dots, S_r$ be a division into lots using the HD method. Let $s_1, \dots, s_m$ be the allocation obtained by Algorithm 2, and let $s_1^j, \dots, s_m^j$ be the allocation obtained by Algorithm 2 at each step $j \in \{1, \dots, r\}$. Let $s_1^*, \dots, s_m^*$ be the optimal solution of the optimization problem (3) with $\varphi(v, s) = \max_{i=1,2,\dots,m} \frac{s_i}{v_i}$. We have the following.

Consider the step $j = 1$ of Algorithm 2. The quotients $w_i = \frac{S_1}{v_i}$ are calculated and we know that $1 = \arg\min_{i \in M} w_i$. Therefore, S_1 is allocated to party 1, and then $s_1^1 = S_1$, while $s_i^1 = 0$ for all $i \in M \setminus \{1\}$. Let us suppose that, in the optimal solution $s_1^*, \dots, s_m^*$, the lot S_1 is allocated to another party $k \in M \setminus \{1\}$. On the one hand, we know that

$$w_1 = \frac{S_1}{v_1} \leq \frac{S_1}{v_k} = w_k.$$

Now, by Lemma 2 there exists a subset $R \subseteq \{2, \dots, r\}$ such that $S_1 = \sum_{t \in R} S_t$. Note that, again by Lemmas 1 and 2, it is possible to allocate the S_1 seats in such a way that any remaining lots, if there are any, are of size one. Therefore, $R = \{2, \dots, q\}$ with $q \leq r$ such that $S_{q+1} = \cdots = S_r = 1$. Consider the modified allocation given by

$$\begin{aligned} s_1' &= S_1 \\ s_k' &= s_k^* - S_1 + \sum_{t \in R} S_t - \sum_{t \in R} S_t \cdot 1_{S_t \sim s_k^*} \\ s_i' &= s_i^* - \sum_{t \in R} S_t \cdot 1_{S_t \sim s_i^*} \end{aligned}$$

for all $i \in M \setminus \{1, k\}$, where we define $1_{S_t \sim s_i^*}$ to be 1 if lot S_t is allocated to party i in the solution s^*, and 0 otherwise, for all $i \in M$. We have that

$$\begin{aligned} \sum_{i=1}^{m} s_i' &= S_1 + s_k^* - S_1 + \sum_{t \in R} S_t - \sum_{t \in R} S_t \cdot 1_{S_t \sim s_k^*} + \sum_{i \in M \setminus \{1,k\}} \left(s_i^* - \sum_{t \in R} S_t \cdot 1_{S_t \sim s_i^*} \right) \\ &= s_k^* + \sum_{i \in M \setminus \{1,k\}} s_i^* + \sum_{t \in R} S_t \cdot 1_{S_t \sim s_1^*} \leq S. \end{aligned}$$

Therefore, we have two cases

- If $\sum_{i=1}^{m} s_i' = S$, let us show that $s_1', \dots, s_m'$ is an optimal solution to the optimization problem (3) with $\varphi(v, s) = \max_{i=1,\dots,m} \frac{s_i}{v_i}$. Consider $T = \max_{i \in M} \frac{s_i^*}{v_i}$. Now, since $\frac{S_1}{v_1} \leq \frac{S_1}{v_k}$, we have that $\frac{s_1'}{v_1} \leq \frac{s_k^*}{v_k}$ and therefore, it is easy to see that for all $i \in M$, we have $\frac{s_i'}{v_i} \leq T$.
- If $\sum_{i=1}^{m} s_i' < S$, let us consider the set $\Delta = \{i \in M : s_i' < s_i^*\}$. Note that $\Delta \neq \emptyset$, since otherwise we would have $s_i' \geq s_i^*$ for all i, and thus $\sum_{i=1}^{m} s_i' = S$. Let $p = S - \sum_{i=1}^{m} s_i'$. Since $R = \{2, \dots, q\}$, there must exist $S_{q+1}, \dots, S_{q+p}$, all of size one. Take any $i \in \Delta$, and update $s_i' \leftarrow s_i' + S_{q+1}$. Now, consider the new set $\Delta = \{i \in M : s_i' < s_i^*\}$, and repeat the same process until all p lots have been assigned. Finally, by construction, we know that $s_i' \leq s_i^*$ for all $i \in M \setminus \{1\}$. Moreover, if $s_1' = S_1$, then $\frac{s_1'}{v_1} \leq \frac{s_k^*}{v_k}$; and if $s_1' > S_1$, then $s_1' \leq s_1^*$ (since we only allocate lots to parties in Δ). Now, it is easy to see that $s_1', \dots, s_m'$ is an optimal solution to the optimization problem (3) with $\varphi(v, s) = \max_{i=1,\dots,m} \frac{s_i}{v_i}$,

Therefore, we update $s_i^* = s_i'$ for all $i \in M$, and we conclude that, in Step 1, the algorithm assigns the lot S_1 in the same way as in the optimization problem (3), with $\varphi(v, s) = \max_{i=1,\dots,m} \frac{s_i}{v_i}$.

Consider the step $j \in \{2, \dots, L\}$ where $L \leq r$ such that $S_L > 1$ and $S_{L+1} = 1$. By the induction hypothesis, the statement holds for all steps up to $j-1$, where we update in each step the optimal solution of the optimization problem (3), with $\varphi(v, s) = \max_{i=1,\dots,m} \frac{s_i}{v_i}$. The quotients $w_i = \frac{s_i^{j-1} + S_j}{v_i}$ are calculated and let $l = \arg\min_{i \in M} w_i$. Therefore, S_j is allocated to party l, and then $s_l^j = s_l^{j-1} + S_j$, while $s_i^j = s_i^{j-1}$ for all $i \in M \setminus \{l\}$. Let us suppose that, in the optimal solution $s_1^*, \dots, s_m^*$, the lot S_j is allocated to another party $k \in M \setminus \{l\}$. On the one hand, we have

$$w_l = \frac{s_l^{j-1} + S_j}{v_l} \leq \frac{s_k^{j-1} + S_j}{v_k} = w_k.$$

Now, by Lemma 2 there exists a subset $R \subseteq \{j+1,\dots,r\}$ such that $S_j = \sum_{t\in R} S_t$. Note that, again by Lemmas 1 and 2, it is possible to allocate the S_j seats in such a way that any remaining lots, if there are any, are of size one. Therefore, $R = \{j+1,\dots,q\}$ with $q \le r$ such that $S_{q+1} = \cdots = S_r = 1$. Consider the modified allocation given by

$$s'_l = s_l^{j-1} + S_j$$
$$s'_k = s^*_k - S_j + \sum_{t\in R} S_t - \sum_{t\in R} S_t \cdot 1_{S_t \sim s^*_k}$$
$$s'_i = s^*_i - \sum_{t\in R} S_t \cdot 1_{S_t \sim s^*_i}$$

for all $i \in M\backslash\{l,k\}$. Note that in this solution we are not modifying the allocation of the lots $S_1,\dots,S_{j-1}$. We have that

$$\sum_{i=1}^m s'_i = s_l^{j-1} + S_j + s^*_k - S_j + \sum_{t\in R} S_t - \sum_{t\in R} S_t\cdot 1_{S_t\sim s^*_k} + \sum_{i\in M\backslash\{l,k\}}\left(s^*_i - \sum_{t\in R} S_t \cdot 1_{S_t\sim s^*_i}\right)$$
$$= s_l^{j-1} + s^*_k + \sum_{i\in M\backslash\{l,k\}} s^*_i + \sum_{t\in R} S_t\cdot 1_{S_t\sim s^*_l} \le S.$$

Therefore, again, we have the following two cases

- If $\sum_{i=1}^m s'_i = S$, let us show that $s'_1,\dots,s'_m$ is an optimal solution to the optimization problem (3) with $\varphi(v,s) = \max_{i=1,\dots,m} \frac{s_i}{v_i}$. Consider $T = \max_{i\in M} \frac{s^*_i}{v_i}$. Now, since $\frac{s_l^{j-1}+S_j}{v_l} \le \frac{s_k^{j-1}+S_j}{v_k}$ and by the induction hypothesis (until step $j-1$ the Algorithm allocate the lots in the same way as the optimization problem) $\frac{s_k^{j-1}+S_j}{v_k} \le \frac{s^*_k}{v_k}$. Then, we have that $\frac{s'_l}{v_l} \le \frac{s^*_k}{v_k}$ and therefore, it is easy to see that for all $i \in M$, we have $\frac{s'_i}{v_i} \le T$.
- If $\sum_{i=1}^m s'_i < S$, let us consider the set $\Delta = \{i \in M : s'_i < s^*_i\}$. Note that $\Delta \neq \emptyset$, since otherwise we would have $s'_i \ge s^*_i$ for all i, and thus $\sum_{i=1}^m s'_i = S$. Let $p = S - \sum_{i=1}^m s'_i$. Since $R = \{j,\dots,q\}$, there must exist $S_{q+1},\dots,S_{q+p}$, all of size one. Take any $i \in \Delta$, and update $s'_i \leftarrow s'_i + S_{q+1}$. Now, consider the new set $\Delta = \{i \in M : s'_i < s^*_i\}$, and repeat the same process until all p lots have been assigned. Finally, $s'_i \le s^*_i$ for all $i \in M \setminus \{l\}$. Moreover, if $s'_l = s_l^{j-1} + S_j$, by the induction hypothesis, then $\frac{s'_l}{v_l} \le \frac{s^*_k}{v_k}$; and if $s'_l > s_l^{j-1} + S_j$, then $s'_l \le s^*_l$ (since we only allocate lots to parties in Δ). Now, it is easy to see that $s'_1,\dots,s'_m$ is an optimal solution to the optimization problem (3) with $\varphi(v,s) = \max_{i=1,\dots,m} \frac{s_i}{v_i}$.

Therefore, we update $s^*_i = s'_i$ for all $i \in M$, and we conclude that, in Step j, the algorithm assigns the lots $S_1,\dots,S_j$ in the same way as in the optimization problem (3), with $\varphi(v,s) = \max_{i=1,\dots,m} \frac{s_i}{v_i}$.

Finally, the remaining lots are of size one. Therefore, they are allocated as in the optimization problem (3), with $\varphi(v,s) = \max_{i=1,\dots,m} \frac{s_i}{v_i}$, since Algorithm 2 coincides with Algorithm 1. □

Proof of Proposition 4

Proof. We take $i,j \in M$ such that $v_i > v_j$ and let us see that $s_i \ge s_j$. Assume $s_i = s_j = 0$ and let us move to the stage in which a lot is assigned to one of the two parties. Consider the lot S_l with $l \in \{1,\dots,r\}$ then $\frac{S_l}{v_i} < \frac{S_l}{v_j}$ so S_l is assigned to party i and we have $s_i = S_l$ and $s_j = 0$. Now, by construction of lots, we have two cases.

(1) If $S_l = \sum_{k=l+1}^r S_k$ then $s_i \ge s_j$.

(2) If $S_l < \sum_{k=l+1}^r S_k$. Let us consider the subset $R = \{l+1,\dots,t\}$ such that $S_k > 1$ for all $k \in R$ (if $R = \emptyset$ then it is trivial to prove because when $s_i = s_j$ the ones break the tie in favor of the party with the largest fraction of votes). We know that $S_l \ge \sum_{k\in R} S_k$ by Lemma 1. Therefore, in the worst-case scenario, $s_j \le s_i$ if s_j was formed by all lots of R. Finally, the remaining lots have size one and in case of a tie, the party with the highest fraction of votes will receive the lot. □

Appendix B. Supplementary data

Supplementary material related to this article can be found online at https://doi.org/10.1016/j.mathsocsci.2025.102474.

Data availability

No data was used for the research described in the article.

References

Aziz, H., Moulin, H., Sandomirskiy, F., 2020. A polynomial-time algorithm for computing a Pareto optimal and almost proportional allocation. Oper. Res. Lett. 48 (5), 573–578.

Balinski, M., Demange, G., 1989a. An axiomatic approach to proportionality between matrices. Math. Oper. Res. 14, 700–719.

Balinski, M., Demange, G., 1989b. Algorithms for proportional matrices in reals and integers. Math. Program. 45, 193–210.

Balinski, M., Ramírez, V., 1999. Parametric methods of apportionment, rounding and production. Math. Social Sci. 37 (2), 107–122.

Balinski, M.L., Young, H.P., 1975. The quota method of apportionment. Am. Math. Mon. 82 (7), 701–730.

Balinski, M.L., Young, H.P., 1977a. Apportionment schemes and the quota method. Am. Math. Mon. 84 (6), 450–455.

Balinski, M.L., Young, H.P., 1977b. On Huntington methods of apportionment. SIAM J. Appl. Math. 33 (4), 607–618.

Balinski, M.L., Young, H.P., 1979. Quotatone apportionment methods. Math. Oper. Res. 4 (1), 31–38.

Balinski, M.L., Young, H.P., 1980. The webster method of apportionment. Proc. Natl. Acad. Sci. USA 77 (1), 1–4.

Balinski, M.L., Young, H.P., 1982. Fair Representation: Meeting the Ideal One Man One Vote. Yale Univ. Pr. N. Heaven CT.

Banzhaf, J.F., 1965. Weighted voting doesn't work: A mathematical analysis. Rutgers Law Rev. 19 (2), 317–343.

Chakraborty, M., Schmidt-Kraepelin, U., Suksompong, W., 2021. Picking sequences and monotonicity in weighted fair division. Artificial Intelligence 301, 103578.

Chingoma, J., Endriss, U., Haan, R.de., Haret, A., Maly, J., 2024. Apportionment with weighted seats. arXiv:2405.18102 cs.GT.

Grilli di Cortona, P., Manzi, C., Pennisi, A., Ricca, F., Simeone, B., 1999. Evaluation and optimization of electoral systems. Soc. Ind. Appl. Math..

Estañ, T., Llorca, N., Martínez, R., Sánchez-Soriano, J., 2021. On the difficulty of budget allocation in claims problems with indivisible items and prices. Group Decis. Negot. 30 (5), 1133–1159.

Gallagher, M., 1991. Proportionality, disproportionality and electoral systems. Elect. Stud. 10 (1), 33–51.

Horowitz, D.L., 2003. Electoral systems: A primer for decision makers. J. Democr. 14 (4), 115–127.

Lijphart, A., 1990. The political consequences of electoral laws, 1945-85. Am. Political Sci. Rev. 84 (2), 481–496.

Lijphart, A., 1994. Electoral Systems and Party Systems a Study of Twenty-Seven Democracies, 1945–1990. Oxford University Press.

Lucas, W.F., 1983. Measuring power in weighted voting systems. In: Brams, S.J., Lucas, W.F., Straffin, P.D. (Eds.), Political and Related Models. In: Modules in Applied Mathematics, Springer, New York, NY.

Maier, S., Zachariassen, P., Zachariasen, M., 2010. Divisor-based biproportional apportionment in electoral systems: A real-life benchmark study. Manag. Sci. 56 (2), 373–387.

Niemeyer, H.F., Niemeyer, A.C., 2008. Apportionment methods. Math. Social Sci. 56, 240–253.

Pukelsheim, F., 2017. Proportional Representation: Apportionment Methods and their Applications, second ed. Springer.

Sánchez-Soriano, J., Llorca, N., Algaba, E., 2016. An approach from bankruptcy rules applied to the apportionment problem in proportional electoral systems. Oper. Res. Decis. 26 (2), 127–145.

Shapley, L.S., Shubik, M., 1954. A method for evaluating the distribution of power in a committee system. Am. Political Sci. Rev. 48 (3), 787–792.

Suksompong, W., 2024. Weighted fair division of indivisible items: A review. Inform. Process. Lett. 187, 106519.

Taagepera, R., Shugart, M.S., 1989. Designing electoral systems. Elect. Stud. 8 (1), 49–58.