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Families of sequential priority rules and random arrival rules with withdrawal limits

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ABSTRACT

This paper deals with extensions of the family of sequential priority rules and the random arrival rule for bankruptcy problems when withdrawal limits are fixed by an arbitrator. We assume two approaches to limit withdrawals. The first is based on the principle that agents can only obtain at most a fixed part of the endowment each time they qualify for an award, and the second that agents can only receive at most a proportion of their claims each time they are attended. For each approach we introduce two pairs of families, each consisting of a sequential priority rule-like family and a random arrival rule-like family. Applying the first approach, we show that the constrained equal awards rule belongs to each of the families. In the second, we prove the same for the proportional rule. We study in detail one pair of families for each approach, and the others are examined in relation to them.

1. Introduction

How to divide a resource between a group of agents who have claims on it, has been a problem of relevance in diverse contexts since ancient times. These problems can be divided into four groups. *Bankruptcy problems* (O'Neill, 1982; Aumann and Maschler, 1985) describe situations in which there is a scarce and perfectly divisible resource (money, water, time) and a finite set of agents who have claims or rights on it, but the resource is not sufficient to fully honor the agents' demands. *Surplus-sharing problems* (Moulin, 1987) describe situations of cooperative ventures that produce a non-negative surplus that must be distributed among the participating agents, taking only into account their opportunity costs. *Surplus-or-loss-sharing problems* (Chun, 1988; Herrero et al., 1999) describe situations in which there is a perfectly divisible budget that has to be divided among a finite set of agents who have entitlements, but in this case the budget can be greater or less than the aggregate entitlement. *Loss-sharing problems* (Bergantiños and Vidal-Puga, 2004) describe situations mathematically similar to bankruptcy problems, but negative allocations are allowed. In this paper, we focus on bankruptcy problems. These problems and their extensions have been widely studied in the literature (see Thomson, 2003, 2015, 2019) and applied to disparate situations (see Pulido et al., 2002, 2008; Niyato and Hossain, 2006; Bergantiños et al., 2012; Casas-Méndez et al., 2011; Gozalvez et al., 2012; Hu et al., 2012; Lucas-Estañ et al., 2012; Giménez-Gómez et al., 2016; Sanchez-Soriano et al., 2016; among others).

Consider the situation where there are indications that a bank may fail. This makes account holders rush to the bank to withdraw their deposits (claims), but the funds available in the bank are not sufficient to satisfy all the customers' demands which, usually, leads to the first to reach the bank withdrawing everything possible. However, recent events in several countries, when account holders attempting to recover their money, were not able to do so, because of the constraints of governmental rules concerning how much they could withdraw. The dynamics described above are behind the well-known sequential priority (SP) rules and their average, the random arrival (RA) rule (O'Neill, 1982) also known as the run-to-the-bank rule (Young, 1994). Thus, to study how these rules can be defined so as to take into account the limits on withdrawals makes sense.

To this end, two alternatives are assumed to limit withdrawals. In both, when the process is repeated to the point that the resource is exhausted, claimants always arrive in the same order. First, we assume that claimants can request at most a fixed proportion $\theta \in\ ]0, 1]$ of the endowment, which is determined beforehand by the arbitrator, and no more than their claims on each occasion. Second, we assume that claimants can ask for at most a proportion of their claims, $\phi \in\ ]0, 1]$, determined by the arbitrator, not exceeding the endowment at each time. Moreover, we consider two ways in which to apply the parameter (θ or ϕ). First, when it is applied just once to the initial endowment or claims and the limit is kept fixed throughout the procedure. Second, when it is applied recursively to the remaining endowment or claims. Thus, we obtain four pairs of families, each pair consisting of a family of sequential priority rules and a family of random arrival rules. We call them *the families of sequential priority and*

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random arrival rules with withdrawals (1) *limited on the basis of the endowment* (SPE-family and RAE-family), (2) *recursively limited on the basis of the endowment* (SPRE-family and RARE-family), (3) *limited on the basis of the claims* (SPC-family and RAC-family), and (4) *recursively limited on the basis of the claims* (SPRC-family and RARC-family), respectively.

We show that the iterative procedure to obtain the rules in each family always converges in a finite number of rounds until the endowment is exhausted. For the SPE, RAE, SPRE and RARE families, we prove that the constrained equal awards (CEA) rule belongs to them all when θ goes to zero, and when θ is 1, the SP-rules belong to the SPE and SPRE families and RA belongs to the RAE and RARE families. Therefore, in some sense, θ can be seen as a tool to achieve a balance between egalitarianism and priority, in the following sense. Lower values of θ imply a more egalitarian distribution, whereas higher values of θ favor agents with higher priority or claims. Thus, when the endowment is sufficiently small in relation to claims, then SP rules or RA rule arise, and when claims are sufficiently small in relation to the endowment, CEA arises. However, we show that a general Lorenz dominance relationship between rules of the same family cannot be established. On the other hand, for the SPC, RAC, SPRC and RARC families, we prove that the proportional (PROP) rule belongs to them all when ϕ goes to zero, and when ϕ is 1, we obtain the same as for the previous rule families. Therefore, in this case, ϕ can be seen as a tool to balance proportionality and priority. In this case, lower values of ϕ give a more proportional distribution, while higher values of ϕ favor again those agents with higher priority. Thereby, when the endowment is sufficiently small in relation to claims, then SP rules or RA arise, and when claims are sufficiently small in relation to the endowment, PROP arises.

Regarding the properties satisfied by the rules in these families, they all satisfy efficiency, homogeneity, continuity, endowment monotonicity and respect of minimal rights. The rules in the RAE, RARE, RAC and RARC families also satisfy anonymity, equal treatment of equals, and order preservation. Moreover, the rules in the SPE, RAE, SPRE and RARE families satisfy invariance under claims truncation, and the rules in the SPC and SPRC families satisfy consistency. Finally, none of the families are closed under duality.

The rest of the paper is organized as follows. Section 2 presents the basic concepts of bankruptcy problems. In Section 3 two families of sequential priority rules and the corresponding two families of random arrival rules with limited withdrawals are studied. Other families are also introduced by changing the way in which withdrawal restrictions are made. Section 4 concludes with some pertinent remarks.

2. Preliminaries

A bankruptcy problem is defined by a 3-tuple (N, c, E), where N is the finite set of claimants with number of agents $|N|$, $c = (c_1, c_2, \ldots, c_{|N|}) \in \mathbb{R}^N_+$ is the vector of their claims, $C = \sum_{i\in N} c_i$ is the total claim, and $E \in \mathbb{R}_+$ is the endowment (assumed perfectly divisible), such that $C > E$,[1] i.e., there is not enough to fully satisfy the claimants' claims. The set of all these problems with set of agents N is denoted by BP^N.

A rule R is a function that associates with each $(c, E) \in BP^N$ a unique point $R(c, E) \in \mathbb{R}^N_+$ such that $\sum_{i\in N} R_i(c, E) = E$, and $0 \le R_i(c, E) \le c_i$, $\forall i \in N$.

Let $(c, E) \in BP^N$, we consider the following classical rules:

- The proportional rule (PROP) (Aristotle, 4th Century BD) is defined by setting $PROP_i(c, E) = \frac{c_i}{C}E,\ \forall i \in N$.
- The constrained equal awards rule (CEA) (Maimonides, 12th Century AD) is defined by setting $CEA_i(c, E) = min\{\alpha, c_i\}$, $\forall i \in N$, where $\alpha \in \mathbb{R}_+$ is chosen so that $\sum_{i\in N} min\{\alpha, c_i\} = E$.
- The constrained equal losses rule (CEL) (Maimonides, 12th Century AD) is defined by setting $CEL_i(c, E) = max\{0, c_i - \lambda\}$, $\forall i \in N$, where $\lambda \in \mathbb{R}_+$ is chosen so that $\sum_{i\in N} max\{0, c_i - \lambda\} = E$.
- The Talmud rule (TAL) (Aumann and Maschler, 1985) is defined in terms of CEA and CEL as follows:

$$TAL_i(c, E) = \begin{cases} CEA_i(\frac{1}{2}c, E) & \text{if } C \ge 2E \\ \frac{c_i}{2} + CEL_i(\frac{1}{2}c, E - \frac{1}{2}C) & \text{if } C \le 2E, \end{cases} \quad \forall i \in N.$$

- The sequential priority rule associated with $\sigma \in \Sigma(N)$ (SP^σ) is defined by setting

$$SP^\sigma_i(c, E) = min\left\{c_i,\ max\{0,\ E - \sum_{j\in N:\sigma(j)<\sigma(i)} c_j\}\right\},\ \forall i \in N,$$

where $\Sigma(N)$ is the set of all possible orders of N. The family of sequential priority rules, SP-family, consists of all these rules.
- The random arrival rule (RA) (O'Neill, 1982) is defined by setting

$$RA_i(c, E) = \frac{1}{n!}\sum_{\sigma\in\Sigma(N)} SP^\sigma_i(c, E),\ \forall i \in N.$$

There are many properties that are very natural and reasonable for a rule. We present some of the most common below. The first five (efficiency, homogeneity, continuity, respect of minimal rights and endowment monotonicity) are basic properties of division rules which are satisfied by all rules in the families introduced in this paper. The following three properties (equal treatment of equals, anonymity and order-preservation) are basic properties related to symmetry which are satisfied by all rules in the random arrival rule-like families. The next property (invariance under claim truncation) is another common property of division rules which is satisfied by all rules in the families of sequential priority and random arrival rules with withdrawals limited on the basis of the endowment. The penultimate property (consistency) is related to possible changes in the claimant set, this property is satisfied by all the rules in the families of sequential priority rules with withdrawals limited on the basis of the claims. Finally, we present a property (self-duality) related to the relationship between the two possible points of view to analyze a bankruptcy problem which is satisfied by the random arrival rule but not by any other rule in the families, and a property for family of rules (closed under duality) which is satisfied by the family of sequential priority rules, but not by any of the families introduced in this paper.

Given a rule R, it satisfies:

- Efficiency (EFF): $\sum_{i\in N} R_i(c, E) = E$.
- Homogeneity (HOM): If $\forall\lambda > 0$, $R(\lambda c, \lambda E) = \lambda R(c, E)$.
- Continuity (CT): If for each sequence of problems $\{(c^m, E^m)\}_{m\in\mathbb{N}} \subset BP^N$, such that

$$\lim_{m\to+\infty}(c^m, E^m) = (c, E),\ \text{then}\ \lim_{m\to+\infty} R(c^m, E^m) = R(c, E).$$

- Respect of minimal rights (RMR): $R_i(c, E) \ge max\{0, E - \sum_{j\in N-i} c_j\}$.
- Endowment monotonicity (EM): If $\sum_{i\in N} c_i \ge E > F$, then $R(c, E) \ge R(c, F)$.
- Equal treatment of equals (ETE): If $c_i = c_j$, then $R_i(c, E) = R_j(c, E)$.

[1] If $C = E$, then there is no problem, the claimants are awarded their claims.

- Anonymity (AN): If $\forall \pi \in \Pi(N)$, $R(\pi(c), E) = \pi(R(c, E))$, where Π is the set of all one-to-one maps from N to N.
- Order-preservation (OP): If $c_i \geq c_j$, then $R_i(c, E) \geq R_j(c, E)$ and $c_i - R_i(c, E) \geq c_j - R_j(c, E)$.
- Invariance under claims truncation (IT): If we take $c'_i = min\{c_i, E\}, \forall i \in N$, then $R_i(c, E) = R_i(c', E)$, for all $i \in N$.
- Consistency (CS): If for all $N' \subset N$, $R_i(c, E) = R_i\left((c_j)_{j\in N'}, E - \sum_{j\in N\setminus N'} R_j(c, E)\right)$, for all $i \in N'$.
- Self-duality (SD): If for all $(c, E) \in BP^N$, $R(c, E) = c - R(c, C - E)$.

In the case of families of rules, we can consider closedness under duality in the following sense: Let $\mathcal{F}$ be a family of rules. We say that it is closed under duality (CD) if for all $R \in \mathcal{F}$, there is $R' \in \mathcal{F}$ such that $R(c, E) = c - R'(c, C - E)$, for all $(c, E) \in BP^N$.

EFF simply says that the whole endowment must be distributed in its entirety. HOM and CT are properties of non-arbitrariness (Moreno-Ternero and Villar, 2006a). HOM not only states that a change of the units of measurements of the data of the problem does only affect the final allocation with the same change of units, but also that large amounts are treated in the same way as small quantities (Marchant, 2008). CT states that small changes in the inputs of the problem do not cause large changes in the final allocation. RMR states that each claimant is guaranteed at least what is left over after the claims of the rest of the claimants are fully honored, provided this amount is non-negative. EM is related to solidarity (Moreno-Ternero and Roemer, 2006) and states that when the endowment increases then no claimants end up worse off. ETE and AN are related to impartiality and equity. AN states that the rule does not depend on the name of the claimants and ETE says that claimants with the same claims must be treated equally. OP requires that, given two claimants, the one with the larger claim receives at least as much as the one with the smaller claim and that his loss be at least as large as the loss incurred by the one with the larger claim. IT says that if the claims are truncated at the endowment, then the final allocation does not change. CS can be seen as a principle of robustness (Thomson, 2011), and it states that if some agents leave with their awards, then we do not need to recalculate the awards for those agents that remain. SD means that the rule divides the endowment in the awards problem in the same way as it divides the losses in the corresponding deficit problem in which the claims are the same and the endowment is equal to the deficit in the other. Excellent surveys on axiomatic analysis of rules are given by Thomson (2003, 2015, 2019).

Finally, an alternative to compare allocations in terms of distributional equality is by means of the Lorenz order. This criterion has been commonly used to evaluate or compare rules in the context of claims adjudication (see, Moreno-Ternero and Villar, 2006b; Bosmans and Lauwers, 2011; Thomson, 2012). Given two vectors $x, y \in \mathbb{R}^N_+$ such that $\sum_{i\in N} x_i = \sum_{i\in N} y_i$, we say that x Lorenz dominates y, $x \succeq_L y$, if $\sum_{j=1}^{k} x_{(j)} \geq \sum_{j=1}^{k} y_{(j)}$, for all $k = 1, 2, \ldots, |N|$, where for a vector $z \in \mathbb{R}^N_+$, $z_{(1)}, \ldots, z_{(|N|)}$ are its coordinates rewritten in increasing order.

3. Sequential priority rules with withdrawal limits

The dynamic rationale behind sequential priority rules is based on the order of arrival of the claimants before the arbitrator who has to solve the problem of conflicting claims. When a claimant comes before the arbitrator, it is assumed that she can request that her claim should be satisfied as far as possible. However, we will now assume that she cannot request her claim in its totality, but only up to a limit fixed by the arbitrator based on different criteria related to the problem at hand. This not only keeps to the idea of sequential priority rules, but also introduces a new element which should be taken into account. When a limit on the withdrawals is set, sequential priority rules associated with an order might not exhaust the endowment, and then following Dominguez (2013) they would be lower bound rules.[2] Dominguez and Thomson (2006) introduced a recursive definition of a rule to find the only rule satisfying invariance under reasonable awards first.[3],[4] Following the recursion idea, Dominguez (2013) introduced the recursive extension of lower bound rules. Given a lower bound rule and a problem, this procedure consists in assigning to each agent her lower bound; then revising each agent's claim and the endowment downwards accordingly, and applying again the lower bound rule, and so on and so forth. Dominguez (2013) proves that a lower bound rule satisfies positivity[5] if and only if there is a unique rule being its recursive extension. This result is important because it is simple to check whether a recursive extension of a lower bound rule converges to a rule and, moreover, is unique.

In order to limit withdrawals, we assume that the limit is set either on the part of the endowment or on the part of the claim that can be withdrawn. Each approach defines different families of lower bound rules when sequential priority rules are applied, whose recursive extensions lead to different families of rules.

3.1. The SPE and RAE families

Let us assume that the arbitrator sets a limit on the part of the endowment E that claimants can request, but never exceeding their claims. This limit is given in terms of a parameter θ that determines the proportion of the endowment that can be claimed. Therefore, θ lies in the interval $]0, 1]$, i.e., from almost nothing can be requested by claimants, to all E, if and when it is below their claims. For intermediate values of θ, claimants can receive up to θE, but again not exceeding their claims. The procedure is the same as in sequential priority rules, but it may occur that after one round of claims the endowment has not been fully distributed. In this case, the endowment and the claims are revised downwards by the awards and a new round in the same order of arrival will take place with exactly the same limit on the part of the endowment that can be claimed. Thus, successive rounds of requests will take place until the endowment is exhausted.

Definition 1. For each problem $(c, E) \in BP^N$, given $\sigma \in \Sigma(N)$ an order of arrival of the claimants, and $\theta \in]0, 1]$ the proportion of the endowment E that the claimants can claim in a round, i.e., the maximum claim to be satisfied in a round is θE, *the sequential priority rule associated with* σ *and withdrawals limited to* θE, $SP^{\sigma,\theta E}$, is given by

$$SP_i^{\sigma,\theta E}(c, E) = \sum_{k=1}^{+\infty} SP_i^{\sigma,\theta E,k}(c^k, E^k),\ \forall i \in N,$$

2 A lower bound rule is defined as a rule but relaxing the requirement of efficiency to feasibility, i.e., $\sum_{i\in N} R_i(c, E) \leq E$. Some examples are the minimal rights lower bound rule (Curiel et al., 1987), the securement lower bound rule (Moreno-Ternero and Villar, 2004), or the conditional equal division lower bound rule (Moulin, 2002).

3 A rule satisfies reasonable awards if $R(c, E) \geq \frac{1}{|N|}(\min\{c_i, E\})_{i\in N}$. This property is called securement by Moreno-Ternero and Villar (2004).

4 A rule is invariant under reasonable awards first if $R(c, E) = \frac{1}{|N|}(\min\{c_i, E\})_{i\in N} + R\left(c - \frac{1}{|N|}(\min\{c_i, E\})_{i\in N}, E - \frac{1}{|N|}\sum_{i\in N}\min\{c_i, E\}\right)$.

5 A lower bound rule satisfies positivity if for each problem with $C > E > 0$, the lower bound rule provides an allocation different from the vector with all its coordinates equal to 0.

where $SP_i^{\sigma,\theta E,k}(c^k,E^k)$, $i\in N$ and $k=1,2,\ldots$, are recursively defined as follows

$$c_i^k = c_i^{k-1} - SP_i^{\sigma,\theta E^1,k-1}(c^{k-1},E^{k-1}),$$
$$E^k = E^{k-1} - \sum_{i\in N} SP_i^{\sigma,\theta E^1,k-1}(c^{k-1},E^{k-1}), \text{ and}$$

$$SP_i^{\sigma,\theta E^1,k}(c^k,E^k) = \min\left\{c_i^k, \theta E^1, \max\left\{0,\ E^k - \sum_{j\in N:\sigma(j)<\sigma(i)} \min\{c_j^k,\theta E^1\}\right\}\right\},\ \forall i\in N,$$

where $c^1=c$ and $E^1=E$.

The SPE-family consists of all rules given by Definition 1.

First note that in each round the claims are less than or equal to the previous one, and the endowment to be distributed diminishes, but the maximum part of the endowment that can be claimed remains constant and exactly equal to θE. The idea of setting an upper bound on the amount that can be withdrawn each time is the simplest, not leading to new recalculations during the distribution process. This upper bound can easily be calculated as a proportion of the initial endowment. The approach in which the bound is calculated as a function of the endowment of that step is presented in Section 3.3.

$SP^{\sigma,\theta E}$ has been obtained as a recursive extension of the lower bound rule resulting from applying the sequential priority rule associated with an order when the limit described above is imposed on the withdrawals. It is easy to check that this lower bound rule satisfies positivity, thus by Dominguez (2013, Th.1), we know that $SP^{\sigma,\theta E}$ is the only recursive extension of that lower bound rule. Therefore, the procedure for defining the rule converges, but it is unknown how many rounds are necessary. The next result establishes that the number of rounds depends only on the parameters c, E, and θ of the problem and is finite. Therefore, the order of arrival of the claimants does not influence in the number of rounds, and the calculation of the $SP^{\sigma,\theta E}$ is reduced to a finite sum. This is important because it can simplify algorithms to compute these rules.

Proposition 1. *Let $(c,E)\in BP^N$, and $\theta\in\,]0,1]$. When applying the sequential priority rule, the maximal number of rounds needed to fully distribute the endowment is given by the first $m\in\mathbb{Z}_+$, such that*

$$\sum_{i\in N^m} c_i + |N\backslash N^m| m\theta E \geq E,$$

where $N^l=\{i\in N|c_i\leq l\theta E\}$.

Proof. In the first round the maximum amount that can be claimed is θE. If $\sum_{i\in N^1} c_i + |N\backslash N^1|\theta E \geq E$, then the procedure stops in just one round. If not, a second round is necessary. Before starting the second round, $\sum_{i\in N^1} c_i + |N\backslash N^1|\theta E$ has been already distributed, and the new claims will be 0 for claimants in N^1 and $c_i-\theta E$ for the remaining claimants.

The aggregate claim until round 2 will be given by $\sum_{i\in N^2} c_i + |N\backslash N^2|2\theta E$. if this amount is larger than E, the procedure will stop. If not, then it continues. Since E is finite and $E<C$, this procedure will end in a finite number of rounds. Thus, the procedure will continue until $\sum_{i\in N^m} c_i + |N\backslash N^m|m\theta E\geq E$, for some $m\in\mathbb{Z}_+$. Note that after the first round m that meets the latter condition, it makes no sense to continue, because the endowment is exhausted. □

We can visualize the rule $SP^{\sigma,\theta E}$ for two players as follows. First, it gives the first claimant in the order σ, claimant i_1, up to $\min\{c_{i_1},\theta E\}$. Then, that agent stops receiving additional units and the second claimant i_2 receives up to $\min\{c_{i_2},\theta E, E-\min\{c_{i_1},\theta E\}\}$. If the endowment is exhausted, then the procedure stops. Otherwise, the claims and the endowment are revised downwards accordingly, and the procedure continues until the endowment is fully distributed (see Fig. 1).

Some remarks on this family are the following. First, when $\theta=1$ we have the sequential priority rule associated with order σ. Second, $SP^{\sigma,\theta E}(c,E)$ is a continuous function of the parameter θ in the interval $]0,1[$. Third, the case $\theta=0$ is not included in Definition 1, however $SP^{\sigma,0E}$ could be defined by taking limits when θ goes to 0 on the right,

$$SP^{\sigma,0E}(c,E) = \lim_{\theta\to 0^+} SP^{\sigma,\theta E}(c,E),\ \forall (c,E)\in BP^N.$$

Finally, the computational complexity of algorithms to compute these rules is $O(m|N|)$ while the computational complexity of sequential priority rules is $O(|N|)$. Nevertheless, since many rounds will be identical when m is large, the computational complexity of algorithms to compute the SPE-rules is at most $O(|N|^2)$.

Proposition 2. *For all $\sigma\in\Sigma(N)$, (i) $SP^{\sigma,1E}=SP^\sigma$ and (ii) $SP^{\sigma,0E}=CEA$.*

Proof. (i) By definition, $SP^{\sigma,1E}=SP^\sigma$, for all $\sigma\in\Sigma(N)$.
(ii) For each $(c,E)\in B^N$, let $\alpha^*\in\mathbb{R}_+$ such that $\sum_{i\in N} min\{c_i,\alpha^*\}=E$. Since $\sum_{i\in N} c_i\geq E$, $\alpha^*<E$, and then $\theta=\frac{\alpha^*}{E}<1$. For each $\sigma\in\Sigma(N)$, we take $\theta(h)=\frac{\alpha^*}{hE}$, $h=1,2,\ldots$ It is obvious that $\lim_{h\to+\infty}\theta(h)=0$.

For each h, a claimant obtains at most $\frac{\alpha^*}{h}$ in each round. Therefore, given any order $\sigma\in\Sigma(N)$, after h rounds claimants with claims larger than α^* have obtained precisely α^*, and the rest of claimants their claims. Now, by the definition of CEA, we can conclude that after h rounds claimants have exactly obtained what CEA prescribes for them. Therefore, $SP^{\sigma,\theta(h)E}(c,E)=CEA(c,E)$, for all $\sigma\in\Sigma(N)$.[6]

Now, since $SP^{\sigma,\theta E}(c,E)$ is a continuous function in the parameter θ in the interval $]0,1[$, $\forall\sigma\in\Sigma(N)$; $\lim_{h\to+\infty}\theta(h)=0$; and $SP^{\sigma,\theta(h)E}(c,E)=CEA(c,E)$, $\forall\sigma\in\Sigma(N)$,

$$SP^{\sigma,0E}(c,E) = \lim_{\theta\to 0^+} SP^{\sigma,\theta E}(c,E) = CEA(c,E),\ \forall\sigma\in\Sigma(N). \quad \square$$

Therefore, *CEA* and SP^σ belong to the SPE-family. Moreover, we observe that *CEA* corresponds to the case $\theta=0$, whereas SP^σ corresponds to the other extreme value, $\theta=1$.[7] The extreme of the interval associated with *CEA* would be that where a "more" egalitarian treatment would be given to the claimants, and at the opposite extreme of the interval "less" egalitarian treatment is given. This comment raises the question: can a Lorenz domination comparison be made between two rules on the basis of the values of parameters indexing them? The answer is negative. Therefore, we cannot establish a general result on the Lorenz domination comparison between two rules as it is established, for example, for the TAL-family[8] in Moreno-Ternero and Villar (2006a). This is shown in the following example.

[6] Note that given a problem there are infinite possible values of the parameter θ for which sequential priority rules with withdrawals limited on the endowment side coincide with CEA.

[7] Note that we include 0 by taking the limit when θ goes to 0 on the right.

[8] For all $(c,E)\in BP^N$, the TAL-family is the set of all rules given by:

$$T_i^\theta(c,E) = \begin{cases} \min\{\theta c_i,\lambda\} & \mathit{if} E\leq\theta C \\ \max\{\theta c_i, c_i-\mu\} & \mathit{if} E\geq\theta C, \end{cases}\quad \forall i\in N,$$

where $\theta\in[0,1]$, and λ and μ are chosen so that $\sum_{i\in N} T_i^\theta(c,E)=E$.

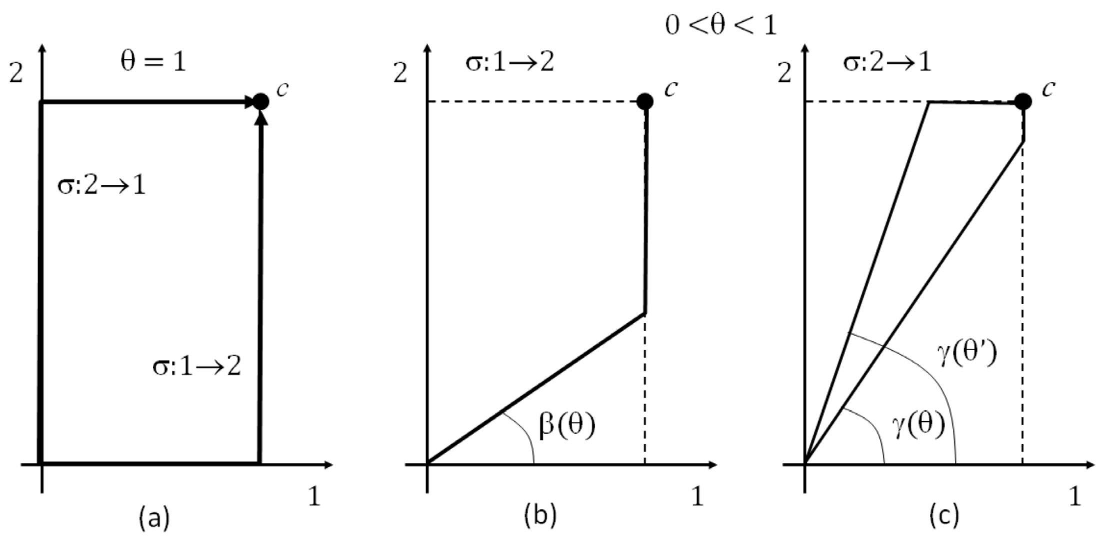

Fig. 1. Rules in the two-claimant case. This figure illustrates the path of awards of the rules within the SPE-family for $N = \{1, 2\}$ and $c_1 < c_2$. Fig. 1(a) shows the path of awards when $\theta = 1$, in both possible orders. Fig. 1(b) shows the shape of the path of awards when $0 < \theta < 1$ and $\sigma : 1 \to 2$. And Fig. 1(c) shows two possible shapes of the path of awards when $0 < \theta < 1$ and $\sigma : 2 \to 1$. Finally, with regard to the angles $\beta(\theta)$ and $\gamma(\theta)$ we have the following. $\beta(\theta) \le 45°$ and $\gamma(\theta) \ge 45°$, for all $\theta \in]0, 1[$, and $\beta(\theta)$ and $\gamma(\theta)$ depend on θ. Indeed, if $\lceil \frac{1}{\theta} \rceil$ is even, then $\tan \beta(\theta) = \frac{2-\lceil \frac{1}{\theta} \rceil}{\lceil \frac{1}{\theta} \rceil \theta}$, and if $\lceil \frac{1}{\theta} \rceil$ is odd, then $\tan \beta(\theta) = \frac{\left(\lceil \frac{1}{\theta} \rceil - 1\right)\theta}{2-\left(\lceil \frac{1}{\theta} \rceil - 1\right)\theta}$. However, in some cases it is simple to determine $\beta(\theta)$, for example, if $0.5 \le \theta < 1$, then $\tan \beta(\theta) = \frac{1-\theta}{\theta}$. Regarding $\gamma(\theta)$, $\beta(\theta) + \gamma(\theta) = 90°$.

Example 1. Let $(c, E) \in B^N$, such that $N = \{1, 2, 3\}$, $c = (3, 4, 5)$ and $E = 10$. Consider the order $\sigma_1 : 1 \to 2 \to 3$. In Fig. 2 we observe that given any two parameters $\theta_1, \theta_2 \in]0, 1]$, such that $\theta_1 < \theta_2$, we can establish in general neither $SP^{\sigma_1,\theta_1 E}(c, E) \succeq_L SP^{\sigma_1,\theta_2 E}(c, E)$ nor $SP^{\sigma_1,\theta_2 E}(c, E) \succeq_L SP^{\sigma_1,\theta_1 E}(c, E)$.

Moreno-Ternero and Villar (2006a) interpret the parameter that generates the TAL-family as an index of the distributive power of the rule between claimants with relatively lower and higher claims. We can give a similar interpretation of the parameter, but between claimants with lower and higher priority in the following sense. Lower values of θ imply that $SP^{\sigma,\theta E}$ is more egalitarian and then favors claimants with lower priority, because they yield allocations closer to CEA, while higher values of θ imply that $SP^{\sigma,\theta E}$ favors claimants with higher priority, since they yield allocations closer or equal to SP^{σ}. For intermediate values of θ the rules oscillate between CEA and SP^{σ} (see Proposition 2 and Fig. 2).

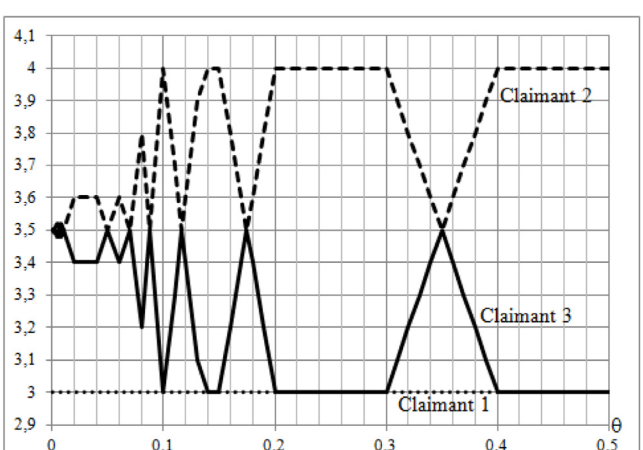

Fig. 2. $SP^{\sigma_1,\theta E}(c, E)$ in function of parameter θ for the problem in Example 1.

The SP-rules satisfy all properties in Section 2 except AN, ETE, OP, and SD. It is easy to check that the SPE-rules satisfy the same properties but CS and CD as the following example shows.

Example 2. Let us consider first the problem given in Example 1 and the order $\sigma_1 : 1 \to 2 \to 3$. For $\theta = 0.16$, $SP^{\sigma_1,0.16E}(c, E) = (3, 3.8, 3.2)$. Its dual problem is given by $((3, 4, 5), 2)$. As in the case of sequential priority rules, we consider the reverse order $\sigma_6 : 3 \to 2 \to 1$ to obtain a dual rule to $SP^{\sigma_1,0.16E}$, i.e., $SP^{\sigma_1,0.16E}(c, E) = c - SP^{\sigma_6,\theta(C-E)}(c, C - E)$. After some calculations we obtain $\theta = 0.9$.

Let us consider now the problem $(c, E') = ((3, 4, 5), 9)$. $SP^{\sigma_1,0.16E'}(c, E') = (3, 3.12, 2.88)$ and $SP^{\sigma_1,0.16E'}(c, E') = c - SP^{\sigma_6,\frac{53}{75}(C-E')}(c, C - E')$. Therefore, there is not a dual rule for $SP^{\sigma_1,0.16E}$ in the family.

If we now consider the order σ_1 and $\theta = 0.34$, $SP^{\sigma_1,0.34E}(c, E) = (3, 3.6, 3.4)$. If claimant 2 leaves the problem with his/her award, then we have the problem $(c', E') = ((4, 5), 7)$, and $\sigma_1' : 2 \to 3$, but $SP^{\sigma_1',0.34E'}(c', E') = (4, 3)$.

To conclude with the analysis of this family, the properties satisfied by the SPE-rules are summarized in Table 1.

Now we go with the corresponding extension of RA. It is well-known that RA is the simple average of all sequential priority rules. Therefore, it seems reasonable that the random arrival rule with withdrawals limited to θE can be defined in the same way.

Definition 2. For each $(c, E) \in BP^N$, *the random arrival rule with withdrawals limited to* θE, RA^{θ}, with $\theta \in [0, 1]$, is given by

$$RA^{\theta}(c, E) = \frac{1}{n!} \sum_{\sigma \in \Sigma(N)} SP^{\sigma,\theta E}(c, E).$$

The RAE-family consists of all rules given by Definition 2.

From the formula it is not easy to visualize RA^{θ}, in general, but for two claimants cases in Fig. 3 the path of awards of some rules within the family are shown. On the other hand, since RA^{θ} is the simple average of all $SP^{\sigma,\theta E}$'s, $\sigma \in \Sigma(N)$, the following results are straightforward.

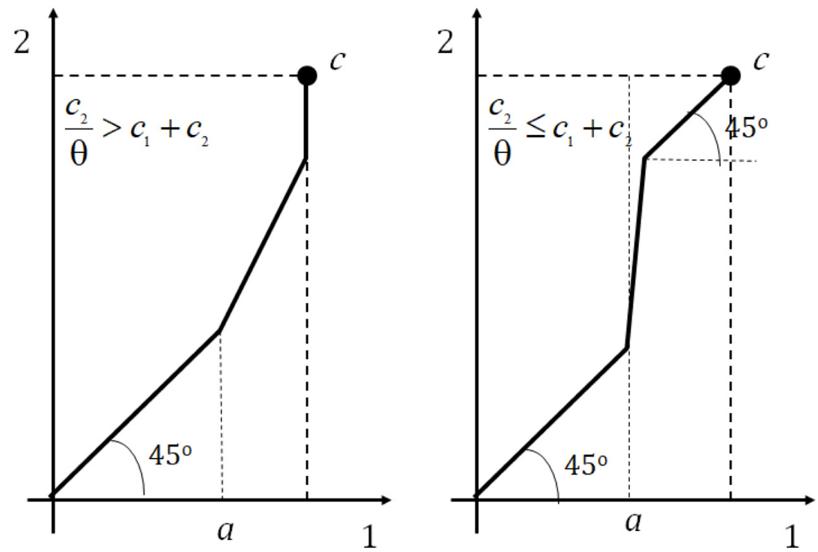

Fig. 3. RA^θ for the two claimants case. This figure illustrates the path of awards depending on the value of the endowment for some rules in the RAE-family for $N = \{1, 2\}$ and $c \in \mathbb{R}^N_+$ with $c_1 < c_2$. The path of awards follows the 45° line until claimant 1 gets a, then either it continues inclined until claimant 1 gets c_1 if $\frac{c_2}{\theta} > c_1 + c_2$, and then vertical until the vector of claims, or it continues inclined until $\theta E = c_2$ if $\frac{c_2}{\theta} \leq c_1 + c_2$, from where it follows the line of slope 1 until it reaches the vector of claims. If $\theta = 1$, then we are in the conditions of the figure on the right and the path of awards follows the 45° line until claimant 1 gets $\frac{c_1}{2}$, then it follows a vertical line until $E = c_2$, and then it follows the line of slope 1 until it reaches the vector of claims, i.e., the path of awards of RA, where $a = \frac{c_1}{\lceil \frac{1}{\theta} \rceil \theta}$, if $\lceil \frac{1}{\theta} \rceil$ is even, and $a = \frac{c_1}{2-\left(\lceil \frac{1}{\theta} \rceil - 1\right)\theta}$, if $\lceil \frac{1}{\theta} \rceil$ is odd.

Table 1
Properties of the four pairs of families when $\theta \in]0, 1[$ and $\phi \in]0, 1[$.

	SP^σ	RA	$SP^{\sigma,\theta E}$	RA^θ	$SP^{\sigma,\phi c}$	$RA^{\phi c}$	$\overline{SP}^{\sigma,\theta E}$	$\overline{RA}^\theta$	$\overline{SP}^{\sigma,\phi c}$	$\overline{RA}^{\phi c}$
EFF	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
HOM	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
CT	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
RMR	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
EM	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
AN	✗	✓	✗	✓	✗	✓	✗	✓	✗	✓
ETE	✗	✓	✗	✓	✗	✓	✗	✓	✗	✓
OP	✗	✓	✗	✓	✗	✓	✗	✓	✗	✓
IT	✓	✓	✓	✓	✗	✗	✓	✓	✗	✗
CS	✓	✗	✗	✗	✓	✗	✗	✗	✓	✗
SD	✗	✓	✗	✗	✗	✗	✗	✗	✗	✗
CD	✓	–	✗	✗	✗	✗	✗	✗	✗	✗

Proposition 3. *The following statements hold:*

1. *For each $(c, E) \in BP^N$, $RA^\theta(c, E)$ is a continuous function of $\theta \in]0, 1[$.*
2. *$RA^0 = CEA$, and $RA^1 = RA$.*

Therefore, CEA and RA also belong to the RAE-family. Moreover, we again observe that both rules correspond to the opposite extreme values of the segment $[0, 1]$ in which the parameter θ lies. Thus, the same comments that were made for the SPE-family can now be made for the RAE-family, including those related to the Lorenz domination (see Fig. 4). Moreover, as in Moreno-Ternero and Villar (2006a), θ can be regarded as a kind of index of the distributive power of the rule between claimants with relatively lower and higher claims. Thus, lower values of θ imply that RA^θ is more egalitarian and then favors claimants with relatively lower claims, while higher values of θ imply that RA^θ is based more on the expected priority on arrivals and then claimants with relatively larger claims are favored.

Finally, it is easy to check that RA^θ inherits the properties in Section 2 satisfied by the SPE-rules with parameter θ. Therefore, when we take the average of all SPE-rules with parameter θ, none of the properties that these rules satisfy are lost. In addition, RA^θ satisfies other properties not satisfied by those, as AN, ETE and OP. These properties are easily derived from the symmetry of the rule obtained when all orders of arrival are taken into account. Moreover, RA satisfies all properties in Section 2 except CS, but the RAE-rules do not satisfy SD and this family is not CD as the following example shows.

Example 3. Consider again the problem in Example 1. $RA^{0.42}(c, E) = (2.6, 3.6, 3.8)$ and $RA^{0.42}(c, C-E) = \left(\frac{2}{3}, \frac{2}{3}, \frac{2}{3}\right)$, therefore $RA^{0.42}$ is not self-dual. In fact, $RA^\theta(c, C-E) = \left(\frac{2}{3}, \frac{2}{3}, \frac{2}{3}\right)$, for all $\theta \in]0, 1]$, therefore the RAE-family is not CD.

Finally, Table 1 shows the properties satisfied by the rules in the RAE-family.

3.2. The SPRC and RARC families

Let us now assume that the arbitrator sets a limit on the part of the claims that can be made. This limit is given in terms of a parameter ϕ that determines the proportion of the claims that can be awarded, but never more than the endowment. Therefore, ϕ lies in the interval $]0, 1]$, i.e., from almost nothing can be requested by claimants, to everything they claim. For intermediate values of ϕ, claimants can receive up to ϕ times what remains of their claims. The procedure is analogous to that described in Section 3.1 but applying ϕ to remaining claims in each round that takes place until the endowment is exhausted.

Definition 3. For each problem $(c, E) \in BP^N$, given $\sigma \in \Sigma(N)$ an order of arrival of the claimants, and $\phi \in]0, 1]$ the proportion of the individual remaining claims that claimants can make in a round, *the sequential priority rule associated with σ and withdrawals recursively limited to ϕc*, $SP^{\sigma,\phi c}$, is given by

$$SP_i^{\sigma,\phi c}(c, E) = \sum_{k=1}^{+\infty} SP_i^{\sigma,\phi c^k,k}(c^k, E^k), \ \forall i \in N,$$

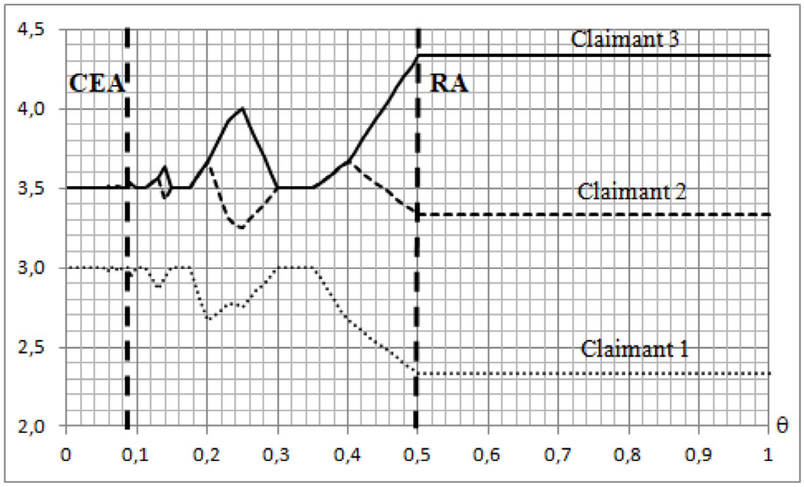

Fig. 4. $RA^{\theta}(c, E)$ in function of θ for Example 1.

where $SP_i^{\sigma,\phi c^k,k}(c^k, E^k)$, $i \in N$ and $k = 1, 2, \ldots$ are recursively defined as follows

$$c_i^k = c_i^{k-1} - SP_i^{\sigma,\phi c^{k-1},k-1}(c^{k-1}, E^{k-1}),$$

$$E^k = E^{k-1} - \sum_{i \in N} SP_i^{\sigma,\phi c^{k-1},k-1}(c^{k-1}, E^{k-1}), \text{ and}$$

$$SP_i^{\sigma,\phi c^k,k}(c^k, E^k) = \min\left\{\phi c_i^k, \max\left\{0, E^k - \sum_{j \in N:\sigma(j)<\sigma(i)} \min\{\phi c_j^k, E^k\}\right\}\right\}, \forall i \in N,$$

where $c^1 = c$ and $E^1 = E$.

The SPRC-family consists of all rules given by Definition 3.

In this case, note that the maximum amount that the claimants could request changes from one round to another and the limits on the withdrawals are different from one claimant to another depending on their claims. As in Section 3.1, it is easy to check that $SP^{\sigma,\phi c}$ also satisfies the conditions in Dominguez (2013, Th.1), therefore the procedure converges to a unique rule. Furthermore, we can establish a similar result regarding the necessary number of rounds needed to reach the limit.

Proposition 4. *Let $(c, E) \in BP^N$, and $\phi \in]0, 1]$. When applying the sequential priority rule, the maximal number of rounds needed to fully distribute the endowment is given by the first $m \in \mathbb{Z}_+$ such that*

$$\sum_{j=1}^{m} (-1)^{j+1} \beta_j^m \phi^j \geq \frac{E}{C},$$

where

$$\beta_1^m = m, \beta_2^m = \beta_2^{m-1} + \beta_1^{m-1}, \ldots, \beta_{m-1}^m = \beta_{m-1}^{m-1} + \beta_{m-2}^{m-1}, \beta_m^m = 1.$$

Proof. First, we note that in the first round each claimant i will claim ϕc_i, and then ϕC in total. If $\phi C \geq E$, then the procedure stops in one round. If not, in a second round, each claimant i will claim $\phi(c_i - \phi c_i)$, and then $\phi(C - \phi C)$ in total. Moreover, the satisfied claim will be the sum of both rounds, i.e., $2\phi C - \phi^2 C$. Therefore, the satisfied claim up to one round will be the sum of the satisfied claim until the previous round plus that satisfied in the current round. Now, if $2\phi C - \phi^2 C \geq E$, then the procedure stops, otherwise it continues. In the third round, the aggregate claim is given by $3\phi C - 3\phi^2 C + \phi^3 C$. Again, if $3\phi C - 3\phi^2 C + \phi^3 C \geq E$, then the procedure stops, otherwise it continues. Now, let us suppose by induction that the total satisfied claim in round m is given by

$$\sum_{j=1}^{m} (-1)^{j+1} \beta_j^m \phi^j C,$$

where

$$\beta_1^m = m, \beta_2^m = \beta_2^{m-1} + \beta_1^{m-1}, \ldots, \beta_{m-1}^m = \beta_{m-1}^{m-1} + \beta_{m-2}^{m-1}, \beta_m^m = 1,$$

and $\sum_{j=1}^{m} (-1)^{j+1} \beta_j^m \phi^j C < E$.

In round $m + 1$, we will have that the aggregate claim is

$$\sum_{j=1}^{m} (-1)^{j+1} \beta_j^m \phi^j C + \phi\left(C - \sum_{j=1}^{m} (-1)^{j+1} \beta_j^m \phi^j C\right).$$

After some calculations we obtain that the aggregate claim in round $m + 1$ has the structure given in the statement of the proposition. Furthermore, since E is finite and $C > E$, this procedure will end after a finite number of steps. □

We can visualize the rule $SP^{\sigma,\phi c}$ for two players as follows. First, it gives the first claimant in the order σ, claimant i_1, up to $\min\{\phi c_{i_1}, E\}$. Then, that agent stops receiving additional units and the second claimant i_2 receives up to $\min\{\phi c_{i_2}, E - \min\{\phi c_{i_1}, E\}\}$. If the endowment is exhausted, then the procedure stops. Otherwise, the claims and the endowment are revised down accordingly, and the procedure continues until the endowment is fully distributed (see Fig. 5).

Note that when $\phi = 1$ we have SP^{σ}, for every $\sigma \in \Sigma(N)$. Furthermore, $SP^{\sigma,\phi c}(c, E)$ is a continuous function of the parameter ϕ in the interval $]0, 1[$. Third, the case $\phi = 0$ is not included in Definition 3, however $SP^{\sigma,0c}$ could be defined by taking limits when ϕ goes to 0 on the right,

$$SP^{\sigma,0c}(c, E) = \lim_{\phi \to 0^+} SP^{\sigma,\phi c}(c, E), \forall (c, E) \in BP^N.$$

Proposition 5. *For all $\sigma \in \Sigma(N)$, (i) $SP^{\sigma,1c} = SP^{\sigma}$ and (ii) $SP^{\sigma,0c} = PROP$.*

Proof. (i) By definition, $SP^{\sigma,1c} = SP^{\sigma}$, for all $\sigma \in \Sigma(N)$.
(ii) First note that given ϕ, for all orders of arrivals, the number of rounds will be the same, i.e., this does not depend on the particular order of arrival.

Let us suppose that $m+1$ rounds are necessary. Until round m all claimants have obtained a part of the endowment proportional

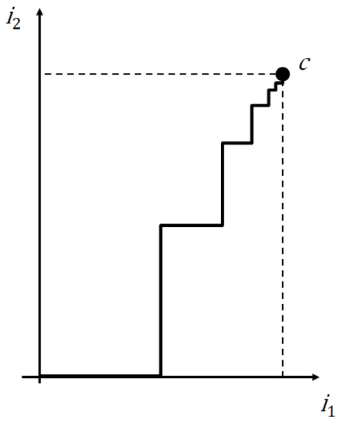

Fig. 5. Illustration of $SP^{\sigma,\phi c}(c,E)$ in function of E for two claimants.

to their claims, it is just in the last round, $m+1$, that the proportionality is broken. Indeed, the quotient between the allocated endowment to each claimant and the total allocated endowment until round m is given by

$$\frac{\sum_{j=1}^{m}(-1)^{j+1}\beta_j^m\phi^j c_i}{\sum_{j=1}^{m}(-1)^{j+1}\beta_j^m\phi^j C}=\frac{c_i}{C},\ \forall i\in N.$$

We now call $p(m,\phi)=\sum_{j=1}^{m}(-1)^{j+1}\beta_j^m\phi^j$.

Second, m depends on parameter ϕ, and $m(\phi)$ is a decreasing function such that when ϕ goes to 0, m goes to infinity.

Third, the polynomial $p(m,\phi)$ is an increasing function in both terms, m and ϕ. In addition,

$$p(m,1)=1 \text{ and } \lim_{\phi\to^{-}1}p(m,\phi)=1 \text{ and } \lim_{m\to+\infty}p(m,\phi)=1.$$

Finally, the part of the endowment to be distributed in round $m+1$ is given by $E-p(m,\phi)C$. For each ϕ we will have $m(\phi)+1$ rounds for each order of arrival.

Now, for every $\varepsilon>0$, since

$$\lim_{m\to+\infty}p(m,\phi)=1 \text{ and } \lim_{\phi\to^{+}0}m(\phi)=+\infty,$$

there is ϕ_ε small enough that $E-p(m(\phi_\varepsilon),\phi_\varepsilon)C=\delta(\phi_\varepsilon)<\varepsilon$, with $\delta(\phi_\varepsilon)\geq 0$.

Since until round $m(\phi_\varepsilon)$ the distributed endowment has been proportionally allocated and it is independent of the order of arrival of the claimants, for each σ and for all $i\in N$,

$$SP^{\sigma,\phi_\varepsilon c}=\frac{c_i}{C}(E-\delta(\phi_\varepsilon))+\varepsilon_i(\sigma),$$

where $\sum_{i\in N}\varepsilon_i(\sigma)=\delta(\phi_\varepsilon)$ and $\varepsilon_i(\sigma)\geq 0$, $\forall i\in N$. Therefore,

$$\left|\frac{c_i}{C}(E-\delta(\phi_\varepsilon))+\varepsilon_i(\sigma)-\frac{c_i}{C}E\right|=\left|\varepsilon_i(\sigma)-\frac{c_i}{C}\delta(\phi_\varepsilon)\right| \leq \delta(\phi_\varepsilon)<\varepsilon.\quad\square$$

Therefore, the SP-rules and PROP belong to the SPRC-family. Moreover, we observe that both correspond to the opposite extremes of the segment $[0,1]$ in which the parameter ϕ lies.[9] Therefore, the extreme of the interval associated with PROP would be that where a "greater" proportionality to the claims would be given to the claimants, and at the opposite extreme of the interval a "lower" proportionality is given. Taking this into account, we can regard the parameter ϕ as a kind of index of proportionality vs priority, in the following sense. Lower values of ϕ imply that the $SP^{\sigma,\phi c}$ is more proportional to the claims, while higher values of ϕ imply that $SP^{\sigma,\phi c}$ favors claimants with higher priority.

Regarding the properties satisfied by the SPRC-rules, it can be immediately proved that they satisfy EFF, HOM, CT, EM, and RMR by definition of the rules. In the next proposition we prove that they also satisfy CS.

Proposition 6. *The SPRC-rules satisfy CS.*

Proof. Let $(c,E)\in BP^N$, $\sigma\in\Sigma(N)$ and $\phi\in\ ]0,1]$. For each $N'\subset N$, we consider the problem $(c_{N'},E_{N'})=\Big((c_i)_{i\in N'},\sum_{i\in N'}SP_i^{\sigma,\phi c}(c,E)\Big)$, and the induced order of arrival $\sigma'=\sigma|_{N'}$ that keeps the same relative order among the claimants in N' as they had in σ. We have to prove that $SP_i^{\sigma',\phi c_{N'}}(c_{N'},E_{N'})=SP_i^{\sigma,\phi c}(c,E)$, $\forall i\in N'$. To do this, we proceed on the basis of the necessary number of rounds to calculate $SP^{\sigma,\phi c}(c,E)$.

First note that there are only two alternatives regarding the structure of awards in a round k. If the endowment is large enough, then all claimants receive their claims for that round, i.e., ϕc^k. Otherwise, at most one agent receives some positive amount strictly less than her claim for that round, and the other claimants receive either all their claims for the round or nothing. Specifically, those who precede that claimant receives everything and those who follow nothing.

$m=1$. This case is analogous to the sequential priority rule associated with σ, therefore by CS of that rule

$$SP_i^{\sigma',\phi c_{N'}}(c_{N'},E_{N'})=SP_i^{\sigma,\phi c}(c,E),\ \forall i\in N'.$$

$m=2$. In this case, $\sum_{i\in N}\phi c_i^1<E^1$ and $\sum_{i\in N}\phi c_i^2\geq E^2$. Then

$$SP_i^{\sigma,\phi c^1,1}(c^1,E^1)=\phi c_i^1,\ \forall i\in N'.$$

By definition of $SP^{\sigma,\phi c}$,

$$\sum_{i\in N'}\phi c_i^1\leq\sum_{i\in N'}SP_i^{\sigma,\phi c}(c,E).$$

If equality holds, then the result follows immediately. Otherwise,

$$SP_i^{\sigma',\phi c_{N'}^1,1}(c_{N'}^1,E_{N'}^1)=\phi c_i^1,\ \forall i\in N'.$$

We now consider the problem (c^2,E^2), which is in the conditions of the case $m=1$, then by CS

$$SP_i^{\sigma',\phi c_{N'}^2,2}(c_{N'}^2,E_{N'}^2)=SP_i^{\sigma,\phi c^2,2}(c^2,E^2),\ \forall i\in N'.$$

Therefore,

$$SP_i^{\sigma',\phi c_{N'}}(c_{N'},E_{N'})=SP_i^{\sigma,\phi c}(c,E),\ \forall i\in N'.$$

$m>2$. In this case, $\sum_{i\in N}\phi c_i^k<E^k$, $\forall k=1,2,\ldots,m-1$, and $\sum_{i\in N}\phi c_i^m\geq E^m$. Then

$$SP_i^{\sigma,\phi c^k,k}(c^k,E^k)=\phi c_i^k,\ \forall i\in N',\ k=1,2,\ldots,m-1.$$

By definition of $SP^{\sigma,\phi c}$,

$$\sum_{i\in N'}\phi c_i^1<\sum_{i\in N'}SP_i^{\sigma,\phi c}(c,E).$$

In general, for $k=1,2,\ldots,m-2$,

$$\sum_{i\in N'}\phi c_i^k<\sum_{i\in N'}SP_i^{\sigma,\phi c}(c,E)-\sum_{j=1}^{k-1}\sum_{i\in N'}\phi c_i^k,$$

[9] Note again that we include 0 by taking the limit when ϕ goes to 0 on the right.

and for $k = m - 1$

$$\sum_{i\in N'} \phi c_i^{m-1} \le \sum_{i\in N'} SP_i^{\sigma,\phi c}(c,E) - \sum_{j=1}^{m-2}\sum_{i\in N'} \phi c_i^k.$$

We observe that the right hand sides of the previous inequalities are precisely $E_{N'}^k$, $k = 1, 2, \ldots, m-1$. Again, if the equality holds for $k = m-1$, then the result immediately follows. Otherwise,

$$\begin{aligned} SP_i^{\sigma',\phi c_{N'}^k,k}(c_{N'}^k, E_{N'}^k) &= SP_i^{\sigma,\phi c^k,k}(c^k, E^k) \\ &= \phi c_i^k, \\ &\quad \forall i \in N',\ \forall k = 1, 2, \ldots, m-1, \end{aligned}$$

and we proceed as in the case $m = 2$. $\square$

However, the SPRC-rules do not satisfy IT, since it is obvious that $\phi c_i \neq \phi \min\{c_i, E\}$, so the awards to claimant i in the successive rounds are diminished and, then the final award of this claimant will be different. Moreover, the SPRC-rules do not satisfy SD and the SPRC-family is not CD as the following example shows.

Example 4. Consider once again the problem in Example 1, and the order σ_1. $SP^{\sigma_1,0.7c}(c,E) = (2.73, 3.64, 3.63)$, but it is not possible to find any order and any ϕ such that $SP^{\sigma_1,0.7c}(c,E) = c - SP^{\sigma,\phi c}(c, C-E)$. Indeed,

1. $\sigma_1 : 1 \to 2 \to 3$. In this case, we have that $0.5 \le SP_1^{\sigma,\phi c}(c, C-E) \le 2$, therefore we cannot get 2.73.
2. $\sigma_2 : 1 \to 3 \to 2$. As in previous case.
3. $\sigma_3 : 2 \to 1 \to 3$. In this case, $\frac{2}{3} \le SP_2^{\sigma,\phi c}(c, C-E) \le 2$, therefore we cannot get 3.64.
4. $\sigma_4 : 2 \to 3 \to 1$. We are in the same situation as in σ_3.
5. $\sigma_5 : 3 \to 1 \to 2$. In this case, when $\phi = 0.274$, we obtain $c - SP^{\sigma,0.274c}(c, C-E) = (2.37, 4, 3.63)$, but it is the only value for which we obtain $SP_3^{\sigma,\phi c}(c, C-E) = 1.37$.
6. $\sigma_6 : 3 \to 2 \to 1$. We have a similar situation as the previous case. For $\phi = 0.274$, we obtain $c - SP^{\sigma,0.274c}(c, C-E) = (3, 3.37, 3.63)$, but it is the only value for which we obtain $SP_3^{\sigma,\phi c}(c, C-E) = 1.37$.

We finish this analysis by referring to Table 1 to see the properties satisfied by the SPRC-rules.

Regarding the corresponding extension of the RA, the random arrival rule with withdrawals recursively limited to ϕc is the simple average of these rules with parameter ϕ.

Definition 4. For each $(c, E) \in BP^N$, *the random arrival rule with withdrawals recursively limited to* ϕc, $RA^{\phi c}$, with $\phi \in [0, 1]$, is given by

$$RA^{\phi c}(c,E) = \frac{1}{n!}\sum_{\sigma\in\Sigma(N)} SP^{\sigma,\phi c}(c,E).$$

The RARC-family consists of all rules given by Definition 4.

The shape of the path of awards of $RA^{\phi c}$ for the two claimants case is shown in Fig. 6. We observe that it is a succession of paths of awards of RA starting one after the previous one along the main diagonal, with successive vectors of claims $(\phi c_1, \phi c_2)$, $(\phi(c_1 - \phi c_1), \phi(c_2 - \phi c_2))$, $\big(\phi(c_1 - \phi c_1 + \phi^2 c_1),$ $\phi(c_2 - \phi c_2 + \phi^2 c_2)\big)$, etc.

Since $RA^{\phi c}$ is obtained by the simple average of all $SP^{\sigma,\phi c}$'s for all $\sigma \in \Sigma(N)$, the following results are straightforwardly obtained:

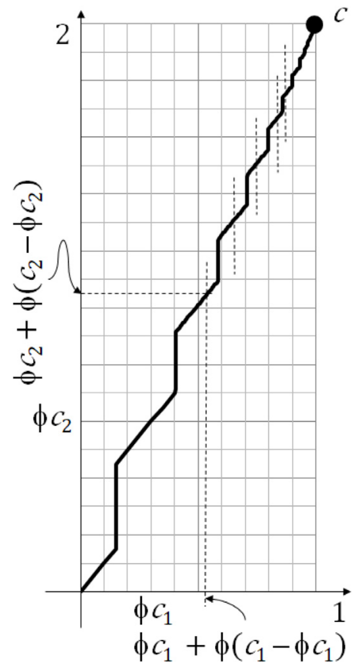

Fig. 6. Path of awards of $RA^{\phi c}$ for the two claimants case.

Proposition 7. *The following statements hold:*

1. *For each* $(c, E) \in BP^N$, $RA^{\phi c}(c, E)$ *is a continuous function of* $\phi \in]0, 1[$.
2. $RA^{0c} = PROP$, *and* $RA^{1c} = RA$.

Therefore, PROP and RA also belong to the RARC-family. Furthermore, we observe that RA and PROP correspond to opposite ends of the interval in which ϕ lies. At this point, the analysis and interpretation of ϕ are analogous to those made for the rules in the SPRC-family, simply changing priority for expected priority.

Moreover, Chun and Thomson (2005) proved that *RA* of the replicated problem converges to *PROP* when the number of replicas goes to infinity. A k-replica problem is a new problem in which there are $k-1$ clones of each claimant with exactly the same claim and the endowment is multiplied by k. Thus, we can say that in the replica problem there are k claimants of each "type". This convergence result is possible because *RA* satisfies ETE. In our case, we also obtain a convergence result (Proposition 7) in which a parametric[10] extension of *RA* converges to *PROP* when the parameter goes to 0. However, we do not use ETE, but the fact that the same result of convergence is obtained for the parametric extension of the SP-family which gives rise to the parametric extension of RA by means of the average of its members (see Proposition 5). In that case, the proof is based on the own recursive structure of those sequential priority rules. On the relationship between these two ways to obtain PROP from RA, we will make additional comments in Section 4.

Regarding the properties that the rules in the family satisfy. EFF, HOM, CT, EM, and RMR are preserved under averaging. AN, ETE and OP are satisfied because the symmetry induced when all possible orders of arrival are considered. CS is not satisfied for the same reasons as RA does not satisfy CS. IT is not satisfied because the limits imposed to the claimants are affected when claims are changed. Finally, the following example shows that neither the rules satisfy SD nor the family is CD.

[10] In the sense that the rules are indexed by parameter ϕ.

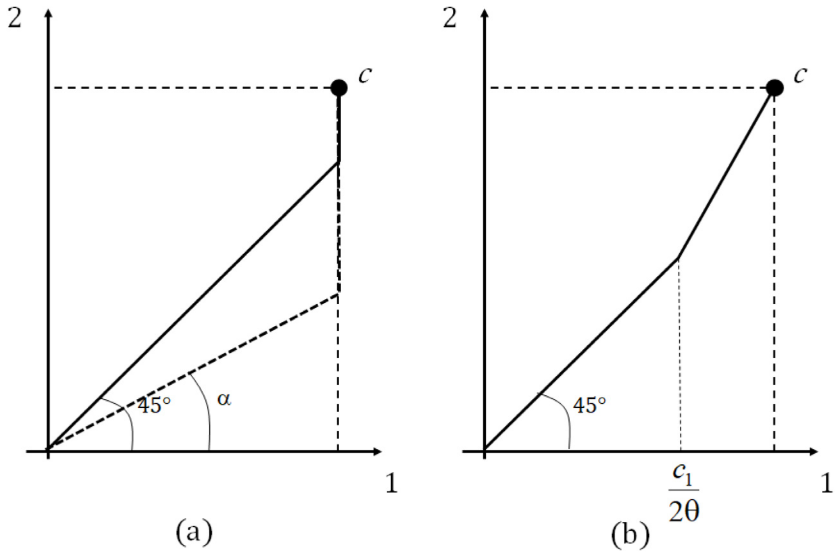

Fig. 7. $\overline{SP}^{\sigma,\theta E}$ and $\overline{RA}^{\theta}$ for the two claimants case with $c_1 < c_2$. (a) $\overline{SP}^{\sigma,\theta E}(c, E)$ with $\sigma : 1 \to 2$, the solid line corresponds to the case $\theta \leq \frac{1}{2}$ and the dashed line corresponds to the case $\theta > \frac{1}{2}$, $\alpha = \arctan\left(\frac{1-\theta}{\theta}\right)$. (b) $\overline{RA}^{\theta}(c, E)$. If $\theta \leq \frac{1}{2}$, then the 45° line continues until (c_1, c_1), and then it follows vertically until c.

Example 5. Consider the problems $(c, E) = ((3, 4), 5)$ and $(c', E) = ((3, 5), 5)$, and $\phi = 0.3$.

$RA^{0.3c}(c, E) = (2.12535, 2.87465) = (3, 4) - RA^{0.5831c}(c, C - E)$

$RA^{0.3c}(c', E) = (1.843, 3.157) \neq (3, 5) - RA^{0.5831c}(c', C' - E)$.

Finally, in Table 1 the list of properties satisfied by the members of the family is shown.

3.3. The other two pairs of families

If we look carefully at the procedures in Sections 3.1 and 3.2, we observe that in the first case the withdrawal limit is constant in each round while in the second it is cumulatively affected by the parameter. Now this is exchanged.

Definition 5. For each problem $(c, E) \in BP^N$, given $\sigma \in \Sigma(N)$ an order of arrival of the claimants, and $\theta \in]0, 1]$, *(i) the sequential priority rule associated with σ and withdrawals recursively limited to θE*, $\overline{SP}^{\sigma,\theta E}$, is given by

$$\overline{SP}_i^{\sigma,\theta E}(c, E) = \sum_{k=1}^{+\infty} SP_i^{\sigma,\theta E^k,k}(c^k, E^k), \ \forall i \in N,$$

where $SP_i^{\sigma,\theta E^k,k}(c^k, E^k)$, $i \in N$ and $k = 1, 2, \ldots$, are recursively defined as follows

$$c_i^k = c_i^{k-1} - SP_i^{\sigma,\theta E^{k-1},k-1}(c^{k-1}, E^{k-1}),$$

$$E^k = E^{k-1} - \sum_{i \in N} SP_i^{\sigma,\theta E^{k-1},k-1}(c^{k-1}, E^{k-1}), \text{ and}$$

$$SP_i^{\sigma,\theta E^k,k}(c^k, E^k) = \min\left\{c_i^k, \theta E^k, \max\left\{0, \ E^k - \sum_{j \in N:\sigma(j)<\sigma(i)} \min\{c_j^k, \theta E^k\}\right\}\right\}, \forall i \in N,$$

where $c^1 = c$ and $E^1 = E$.

(ii) The random arrival rule with withdrawals recursively limited to θE, $\overline{RA}^{\theta}$, with $\theta \in [0, 1]$, is given by

$$\overline{RA}^{\theta}(c, E) = \frac{1}{n!} \sum_{\sigma \in \Sigma(N)} \overline{SP}^{\sigma,\theta E}(c, E).$$

The SPRE and RARE families are the sets of rules given by Definition 5.i and Definition 5.ii, respectively.

Note that the only difference with the recursive procedure in Section 3.1 is that now the parameter is applied to each remaining endowment and not only to the initial one. This implies that the limit on the part of the endowment that can be claimed is different from one round to the next. In Fig. 7, for the two claimants case, the paths of awards of $\overline{SP}^{\sigma,\theta E}$ and $\overline{RA}^{\theta}$ are shown. Furthermore, we have the following result.

Proposition 8. *The following statements hold:*

1. *$\overline{SP}^{\sigma,\theta E}(c, E)$, $\forall \sigma \in \Sigma(N)$, and $\overline{RA}^{\theta}(c, E)$ are continuous functions of $\theta \in]0, 1[$.*
2. *For each $(c, E) \in BP^N$, and for each $\theta \in]0, 1]$, when applying the sequential priority rule, the maximal number of rounds needed to fully distribute the endowment is given by the first $m \in \mathbb{Z}_+$, such that*

$$\sum_{i \in N^m} c_i + |N \backslash N^m| \sum_{k=1}^{m} \theta E^k \geq E,$$

where $N^h = \{i \in N | c_i \leq \sum_{k=1}^{h} \theta E^k\}$, $E^0 = 0$, $E^1 = E$, and

$$E^k = E^{k-1} - \left(\sum_{i \in (N^{k-1} \backslash N^{k-2})} \left(c_i - \sum_{j=1}^{k-2} \theta E^j \right) + |N \backslash N^{k-1}| \theta E^{k-1} \right).$$

3. *$\overline{SP}^{\sigma,0E} = \overline{RA}^{0} = CEA$.*
4. *$\overline{RA}^{1} = RA$, and $\overline{SP}^{\sigma,1E} = SP^{\sigma}$, $\forall \sigma \in \Sigma(N)$.*

Proof. (1) and (4) follow from the definitions of the rules. (2) is proved similarly to Proposition 1. (3) is obtained using similar arguments to those used in Proposition 5, but changing the argument of proportional distribution for the argument of constrained egalitarian distribution in each round. □

Regarding the properties satisfied by the members of these two rule families, it is not difficult to prove which properties they satisfy (see Table 1).

The last pair of rule families discussed in this paper are given in the following definition.

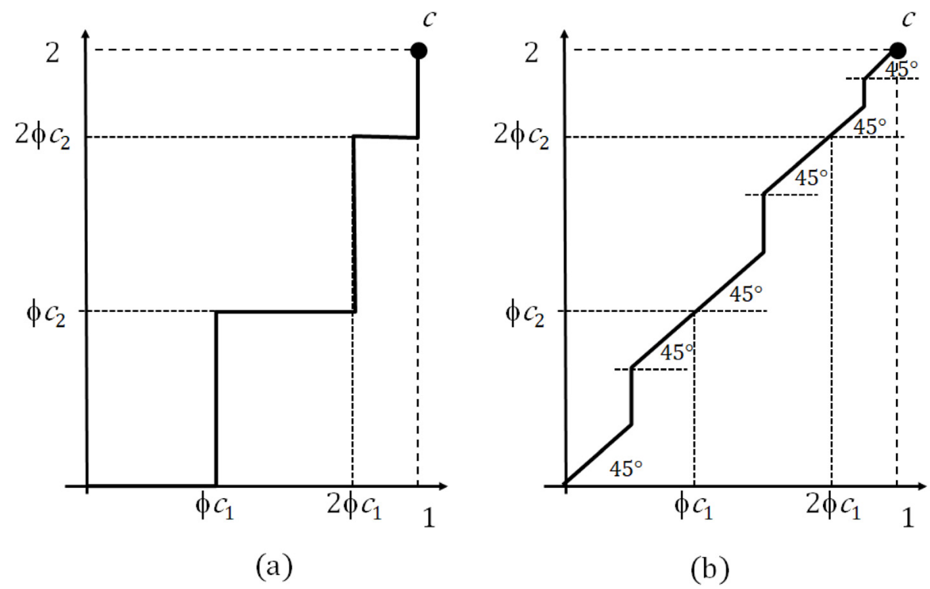

Fig. 8. $\overline{SP}^{\sigma,\phi c}$ and $\overline{RA}^{\phi c}$ for the two claimants case with $c_1 < c_2$. (a) $\overline{SP}^{\sigma,\phi c}(c, E)$ with $\sigma : 1 \to 2$. (b) $\overline{RA}^{\phi c}(c, E)$.

Definition 6. For each problem $(c, E) \in BP^N$, given $\sigma \in \Sigma(N)$ an order of arrival of the claimants, and $\phi \in \,]0, 1]$, *(i) the sequential priority rule associated with σ and withdrawals limited to ϕc, $\overline{SP}^{\sigma,\phi c}$*, is given by

$$\overline{SP}_i^{\sigma,\phi c}(c, E) = \sum_{k=1}^{+\infty} SP_i^{\sigma,\phi c,k}(c^k, E^k),\ \forall i \in N,$$

where $SP_i^{\sigma,\phi c,k}(c^k, E^k)$, $i \in N$ and $k = 1, 2, \ldots$, are recursively defined as follows

$$c_i^k = c_i^{k-1} - SP_i^{\sigma,\phi c^1,k-1}(c^{k-1},$$
$$E^{k-1}), E^k = E^{k-1} - \sum_{i\in N} SP_i^{\sigma,\phi c^1,k-1}(c^{k-1}, E^{k-1}),\ \text{and}$$

$$SP_i^{\sigma,\phi c^1,k}(c^k, E^k)$$
$$= \min\left\{\phi c_i^1, c_i^k, \max\left\{0,\ E^k - \sum_{j\in N:\sigma(j)<\sigma(i)} \min\{\phi c_j^1, c_j^k, E^k\}\right\}\right\}, \forall i \in N,$$

where $c^1 = c$ and $E^1 = E$.

(ii) The random arrival rule with withdrawals limited to ϕc, $\overline{RA}^{\phi c}$, with $\phi \in [0, 1]$, is given by

$$\overline{RA}^{\phi c}(c, E) = \frac{1}{n!} \sum_{\sigma\in\Sigma(N)} \overline{SP}^{\sigma,\phi c}(c, E).$$

The SPC and RAC families are the sets of rules given by Definition 6.i and Definition 6.ii, respectively.

Note that the only difference with the recursive procedure in Section 3.2 is that now the parameter is applied just once and not repeatedly during the procedure. This implies that the limit on the part of the claims that can be made is always the same throughout the procedure. In Fig. 8, for the two claimants case, the paths of awards of $\overline{SP}^{\sigma,\phi c}$ and $\overline{RA}^{\phi c}$ are shown. Moreover, we have the following result.

Proposition 9. *The following statements hold:*

1. *$\overline{SP}^{\sigma,\phi c}(c, E)$, $\forall \sigma \in \Sigma(N)$, and $\overline{RA}^{\phi c}(c, E)$ are continuous functions of $\phi \in \,]0, 1[$.*
2. *For each $(c, E) \in BP^N$, and for each $\phi \in \,]0, 1]$, when applying the sequential priority rule, the maximal number of rounds needed to fully distribute the endowment is given by the first $m \in \mathbb{Z}_+$ such that*

$$\sum_{i\in N^m} c_i + m \sum_{i\in N\setminus N^m} \phi c_i \geq E,$$

where $N^m = \{i \in N | c_i \leq m\phi c_i\}$.

3. *$\overline{SP}^{\sigma,0c} = \overline{RA}^{0c} = PROP$.*
4. *$\overline{SP}^{\sigma,1c} = SP^{\sigma}$, $\forall \sigma \in \Sigma(N)$, and $\overline{RA}^{1c} = RA$.*

Proof. (1) and (4) follow from the definitions of the rules. (2) is proved similarly to Proposition 4. (3) is obtained using similar arguments to those used in Proposition 2, but changing CEA for PROP. □

Finally, it is not difficult to prove which properties the rules in this pair of families satisfy by using similar arguments to those in Section 3.2 (see Table 1).

4. Final remarks

We have addressed the problem of distribution in a bankruptcy problem using the dynamic rationale behind the SP-family and RA, but assuming that claimants can only make limited withdrawals. Depending on how that limitation is set, four pairs of families of rules have been defined. Each pair of families consists of one family extending the SP-family and the other extending RA.

On the one hand, two pairs of families are based on the limitation of withdrawals on the endowment side (Sections 3.1 and 3.3). When the parameter θ is applied just once to the initial endowment and then the withdrawal limit is the same in all rounds, the SPE and RAE families are obtained, and when θ is applied in each round to the remaining endowment, we obtain the SPRE and RARE families. CEA belongs to them all, so θ can be used to obtain more egalitarian allocations. In particular, when the part of the endowment that can be withdrawn is sufficiently small in relation to claims, then CEA arises.

On the other hand, the other two pairs of families are based on the limitation of withdrawals on the claims side (Sections 3.2 and 3.3). When the parameter ϕ is applied just once to the initial claims and then the limitation on the claims is the same in all rounds, we obtain the SPC and RAC families and, when ϕ is applied at each round to the remaining claims, we obtain the SPRC and RARC families. PROP belongs to all these families, so ϕ can be used to obtain more proportional allocations. Specifically, PROP arises when the part of the claims that can be withdrawn is sufficiently small in relation to the endowment.

Therefore, this means that if an arbitrator is interested in egalitarianism, then limitations should be imposed on the part of the endowment that can be withdrawn, while if she is worried about proportionality, then it should be applied to the claims. The reason for this is when a limitation is imposed on the amount that can be withdrawn from the endowment, then somehow all agents are treated as if they were equal, and only in the event of their being able to receive more than what they initially requested, their claims would be taken into account. However, when the limitation is imposed on their claims, these would be taken into account at all times, and only if they were to receive more than the endowment, this would then be considered.

Moreover, Table 1 shows the properties that the rules of these families satisfy. It also includes the properties satisfied by the rules in the SP-family and RA to see which properties are inherited.

Some additional remarks are as follows: First, these families are added to other well-known families of rules for bankruptcy problems as the Young's family (Young, 1987), the ICI and CIC families (Thomson, 2000, 2008), and their subfamilies the TAL-family (Moreno-Ternero and Villar, 2006a) and the reverse TAL-family (van den Brink and Moreno-Ternero, 2017). These last two families have in common with our families that depend on a single parameter. Likewise, while none of these families have RA as one of their members, we introduce four new families of rules that do.

Second, all these rules families have their counterpart when we approach the problem from the perspective of sharing losses. In this case, we would obtain the analogous families of rules, and their analysis would be similar, in every way, since given a bankruptcy problem (c, E) what we would have to solve is the problem of losses $(c, C - E)$, whose analysis is identical to that carried out in this paper, and the final allocation would be given by $c - R(c, C - E)$, where R would be the rule used to distribute the losses among the claimants. Furthermore, it is well-known that $RA(c, E) = c - RA(c, C - E)$ and $PROP(c, E) = c - PROP(c, C - E)$, i.e., *RA* and *PROP* are self-dual; and $CEL(c, E) = c - CEA(c, C - E)$, i.e., *CEA* and *CEL* are dual (see Thomson, 2015). Therefore, the results obtained for the introduced families of rules of awards can easily be extended to the context of sharing losses by taking into account the previous relationships. Obviously, instead of obtaining *CEA*, we would obtain *CEL*. For *PROP* and *RA* we would in essence have the same results.

Another alternative is to consider that the order were chosen randomly at each round. What would happen is not obvious, although perhaps we could use similar arguments to those presented in this paper to provide an answer. We would have to consider all arrival possibilities for each of the rounds, so we would have $(|N|!)^m$ cases, where m is the number of rounds that we know is independent of the order of arrival. Of course, all those cases could be grouped into categories and maybe the analysis could be simplified. In this sense, apparently, when taking the average of all the corresponding allocations, a parametric random arrival rule would be obtained. Therefore, this is an interesting approach for further research.

Third, Chun and Thomson (2005) prove that under replication RA behaves like PROP. In some way this means that when there are a large number of claimants with small claims in relation to the endowment RA behaves like PROP. In this paper we obtain a similar result but from a different perspective. In our case, we obtain the result by limiting the part of the claims that claimants can withdraw at each round, and when this part is sufficiently small in relation to the endowment we obtain that RA behaves like PROP. As we observe the relation between claims and the endowment in both approaches is completely analogous. In addition, the following analogy can be established between both approaches. Somehow the k-replica of a problem would be *"equivalent"* to considering $\phi = \frac{1}{k}$, i.e., while in Chun and Thomson (2005) the approach is by means of multiplication of the problem, in our approach is by means of division of the claims, but the final effect is the same.

Fourth, maybe the most important problem of RA is its calculation when the number of claimants is large. Aziz (2013) designed an algorithm in polynomial time for computing RA when the endowment and the claims are integers which are polynomial in n. However, in the general case, RA is NP-complete to implement (Aziz, 2013). For obtaining algorithms in polynomial time for computing the rules of the families introduced in this paper, even more restrictive conditions than in Aziz (2013) should be considered due to the more complex structure of these rules.

Finally, this approach could also be used to extend the Shapley value (Shapley, 1953) to the case of limited withdrawals. This could contribute to filling a gap in the vast literature on the Shapley value (see Roth, 1988; Algaba et al., 2019).

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