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Decision Support

The generalized range adjusted measure in data envelopment analysis: Properties, computational aspects and duality

Juan Aparicio, Juan F. Monge*

Center of Operations Research, University Miguel Hernandez of Elche. Avda. de la Universidad, s/n, Elche 03202 Alicante, Spain

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ABSTRACT

The measurement of technical efficiency is a topic of great interest in microeconomics and engineering. Data Envelopment Analysis (DEA) is one of the existing techniques for measuring technical efficiency. One of the challenges related to DEA is to introduce a “well-defined” efficiency measure. Overall, it means that the technical efficiency measure should satisfy a list of mathematical and economical properties. Regarding this point, an unresolved question in the DEA literature to date, is whether any measure can satisfy both Indication, also called Pareto-efficiency identification, and uniqueness of the projection point generated by the corresponding efficiency optimization model. With this issue in mind, this paper introduces a new family of measures, inspired on the Range-Adjusted Measure (RAM), which satisfy a list of six properties. This family of measures will be called Generalized Range-Adjusted Measure (GRAM). Additionally, we show in this paper how GRAM can be implemented from a computational point of view and we also provide an economical interpretation of its dual program in terms of (shadow) profit maximization. Finally, an empirical example extracted from the literature serves to illustrate the new methodology.

1. Introduction

Data Envelopment Analysis (DEA) is a non-parametric technique for the estimation of production frontiers in microeconomics and engineering, which is based upon envelopment methods (Banker, Charnes, & Cooper, 1984; Charnes, Cooper, & Rhodes, 1978). DEA gauges the relative efficiency of a set of decision-making units (DMUs) that operate in a multi-input/multi-output framework under a technology; also called production possibility set. In particular, technical inefficiency for a DMU is defined as the distance from the assessed unit to a certain part of the border of the production possibility set, known as the efficient frontier. The efficient frontier represents the reference benchmark. The points located onto this efficient frontier, where the distance from the evaluated unit is achieved, are known in the DEA literature as “projection points”.

In contrast to other existing alternatives for measuring technical efficiency in the literature, as for example Stochastic Frontier Analysis (SFA), DEA does not need to explicitly specify the mathematical formulation of the frontier of the underlying technology to be estimated from a data sample. Additionally, DEA ‘naturally” copes with the multi-output case. Moreover, DEA models, based on mathematical programming, provide benchmarking information through the targets utilized in the evaluation of technically inefficient units. In this regard, the targets are the coordinates of the projection point in the corresponding input-output space. However, in contrast to parametric techniques, Data Envelopment Analysis lacks a goodness of fit indicator (e.g., R^2). In the DEA literature, this weakness has been substituted by the satisfaction of a list of theoretical properties that the efficiency measure must fulfill. In particular, Färe & ll (1978) and Pastor, Ruiz, & Sirvent (1999) mention the following properties: (P1) the measure takes values between zero and one; (P2) monotonicity; (P3) units invariance; (P4) translation invariance; and finally, (P5) the assessed DMU is Pareto-Koopmans efficient if and only if the measure has a value of one (Indication or Pareto-efficiency identification).

The satisfaction of the largest possible number of properties has been one of the reasons why many different technical efficiency measures, i.e., different implementations of the idea of distance from a DMU to the efficient frontier, have been introduced in the DEA literature over the last few decades. In this respect, the measures that satisfy more properties among those previously mentioned, P1-P5, are mainly slacks-based, such as the weighted additive measures (Aparicio, Pastor, & Vidal, 2016; Cooper, Park, & Pastor, 1999; Lovell & Pastor, 1995) and the Slacks-Based Measure (SBM) (Tone, 2001). In particular, the Range-Adjusted Measure (RAM) (Cooper et al. (1999)) satisfies all the aforementioned properties. The other alternatives fail in satisfying at least one of

* Corresponding author.
E-mail addresses: j.aparicio@umh.es (J. Aparicio), monge@umh.es (J.F. Monge).

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the five properties. Radial measures (Banker et al., 1984; Charnes et al., 1978), the directional distance function (Chambers, Chung, & Fare, 1996; 1998) and the hyperbolic measure (Färe, Grosskopf, & Lovell (1985)) do not satisfy Pareto-efficiency since they may project the evaluated unit onto the weakly efficient frontier instead of the strongly efficient subset of the production frontier. As for the Russell input and output measures of technical efficiency and their graph extension (Färe et al. (1985)), translation invariance does not hold (see Pastor & Aparicio (2015)). Additionally, the SBM fulfills the five properties except for translation invariance (see Sharp, Meng, & Liu (2007)). Consequently, the RAM (Cooper et al. (1999)) currently stands out as one of the DEA measures that satisfies a greater number of properties.

More recently, emphasizing the virtues of DEA as a benchmarking tool, Sueyoshi & Sekitani (2009) have introduced a new property for technical efficiency measures, the so-called “unique projection for efficiency comparison” (P6). This property guarantees that the efficiency measure selects a unique projection point onto the efficient frontier, as a benchmark for the assessed DMU. Otherwise, secondary goals should be incorporated into the analysis as a criterion to select a projection point, something that is subjective in practice and may require information from an expert in the sector related to the data. Unfortunately, as Sueyoshi & Sekitani (2009) pointed out, all known DEA efficiency measures do not fulfill this property (see Proposition 9 in Sueyoshi & Sekitani (2009)), not even the RAM.

In order to endow slacks-type measures in DEA with property P6, Aparicio, Monge, & Ramon (2021) have recently introduced a new measure related to RAM, as follows. Whereas RAM maximizes a particular weighted sum of slacks, the new measure maximizes the 'product' of weighted slacks in order to implement the hypervolume maximization. This is why this new proposal is called Multiplicative Range-Adjusted Measure (MRAM). The MRAM fulfills the “unique projection for efficiency comparison” property, although at the expense of failing to fulfill the property of indication. Hence, nowadays in the literature, we have, on the one hand, the RAM measure that satisfies all the properties except P6 and, on the other hand, the MRAM, which meets all the properties except P5.

In this paper, we define a new family of slacks-based technical efficiency measures, based upon the Hölder means, which presents the RAM and the MRAM as extreme elements, in such a way that, in between, all members of the family are efficiency measures satisfying the six properties P1-P6. Nevertheless, from a computational perspective, the new approach is apparently more complex than the RAM, which is based on simple Linear Programming. However, as we will show, this new family of efficiency measures is based on a convex optimization program and can be solved using standard convex optimization software, as is also the case with Linear Programming.

The paper is organized as follows. Section 2 introduces the essential background on Data Envelopment Analysis and, in particular, on the Range-Adjusted Measure (RAM), the Multiplicative Range-Adjusted Measure (MRAM) and the properties that they satisfy. In Section 3, we introduce the new family of technical efficiency measures, which we call the Generalized Range-Adjusted Measure (GRAM), show its main properties and discuss some computational issues. Additionally, in that section, we provide and interpret the dual program associated with the GRAM from an economical perspective. Section 4 illustrates the new approach by reporting to numerical results using a data set taken from the literature. Section 5 concludes.

2. Background

Let us consider that we have observed n Decision Making Units (DMUs) that use m inputs to produce s outputs. These are denoted by (X_j, Y_j), $j = 1, \ldots, n$. It is assumed that $X_j = (x_{1j}, \ldots, x_{mj}) \geq 0_m$, $j = 1, \ldots, n$, and $Y_j = (y_{1j}, \ldots, y_{sj}) \geq 0_s$, $j = 1, \ldots, n$. Additionally, due to the fact that we will resort to the RAM, MRAM and a new approach throughout the paper, the input and output ranges, defined as $R_i^- = \overline{x}_i - \underline{x}_i$, with $\overline{x}_i = \max\{x_{ij}, j = 1, \cdots, n\}$, $\underline{x}_i = \min\{x_{ij}, j = 1, \cdots, n\}$, and $R_r^+ = \overline{y}_r - \underline{y}_r$, with $\overline{y}_r = \max\{y_{rj}, j = 1, \cdots, n\}$, $\underline{y}_r = \min\{y_{rj}, j = 1, \cdots, n\}$ must be assumed to be strictly positive.

The relative efficiency of each DMU_0 in the sample is assessed with reference to the so-called production possibility set $T = \{(X, Y) \geq 0_{m+s} : X \text{ can produce } Y\}$, which can be empirically constructed from the n observations by assuming the postulates that appear in Banker et al. (1984). If, in particular, Variable Returns to Scale (VRS) is assumed, then T can be characterized as follows:

$$T = \{(X, Y) \geq 0_{m+s} : \sum_{j=1}^{n} X_j\lambda_j \leq X, \sum_{j=1}^{n} Y_j\lambda_j \geq Y, \sum_{j=1}^{n} \lambda_j = 1, \lambda_j \geq 0, \forall j\}. \tag{1}$$

Hereinafter, we assume that each DMU is interested in increasing outputs and reducing inputs. This is known as a non-oriented or graph approach in the literature, in contrast to the input and output oriented contexts. The graph version of the model could be trivially particularized to the oriented versions. In order to measure technical inefficiency under this approach, it is useful to introduce the notions of weakly and strongly efficient frontiers. The first one is linked to the so-called Debreu-Farrell definition of technical efficiency (Debreu, 1951; Farrell, 1957), whereas the second one corresponds to the Pareto-Koopmans notion of technical efficiency (Koopmans, 1951). So, the weakly efficient frontier is defined as follows:

$$Fr^w(T) = \{(X, Y) \in T : (U, -V) < (X, -Y) \Rightarrow (U, V) \notin T\}. \tag{2}$$

The strongly efficient frontier is a subset of the weakly efficient frontier and contains all the producible vectors of inputs and outputs that are Pareto-efficient, i.e., they are not Pareto-dominated by any other producible point in T:

$$Fr^s(T) = \{(X, Y) \in T : (U, -V) \leq (X, -Y), (U, V) \neq (X, Y) \Rightarrow (U, V) \notin T\}. \tag{3}$$

Following, Färe & ll (1978), Pastor et al. (1999), Sueyoshi & Sekitani (2009) and Russell & Schworm (2018), we highlight six properties defined for efficiency measurement in DEA:

- (P1): The measure takes values between zero and one.
- (P2): [Monotonicity] An increase in any input, holding other inputs as well as all outputs constant, reduces or keeps the same the value of the measure, and vice versa for the output side.
- (P3): [Units invariance] The values of the efficiency measure do not depend on the units of measurements in the input and output variables.
- (P4): [Translation invariance] The values of the efficiency measure do not depend on any translation in inputs or outputs.
- (P5): [Indication] The measure equals one, if and only if the evaluated unit is Pareto-efficient.
- (P6): [Unique projection for efficiency comparison] The projection point generated by the measure should be the same, regardless of the alternative optimal solution considered from the associated optimization problem.

The first five properties focus on the value of the efficiency measure (the score) and its meaning, while the sixth property is devoted to the benchmarking qualities of the DEA measures.

As we mentioned above, the Range-Adjusted Measure (RAM) fulfills P1-P5 (see Cooper et al., 1999) but fails P6 (see Sueyoshi &

Sekitani (2009)). This measure is defined from a Linear Programming optimization model, as follows.

$$\begin{aligned} \min_{\mathbf{s},\boldsymbol{\lambda}} \quad & 1-\left(\sum_{i=1}^{m}\frac{s_{i0}^{-}}{(m+s)R_i^{-}}+\sum_{r=1}^{s}\frac{s_{r0}^{+}}{(m+s)R_r^{+}}\right) \\ s.t. \quad & \\ & \sum_{j=1}^{n} x_{ij}\lambda_j + s_{i0}^{-} = x_{i0} \qquad \forall i=1,\cdots,m \\ & \sum_{j=1}^{n} y_{rj}\lambda_j - s_{r0}^{+} = y_{r0} \qquad \forall r=1,\cdots,s \\ & \sum_{j=1}^{n}\lambda_j = 1 \\ & s_{i0}^{-}, s_{r0}^{+}, \lambda_j \geq 0 \qquad \forall i,r,j \end{aligned} \tag{4}$$

If $(s_{10}^{-*},\ldots,s_{m0}^{-*},s_{10}^{+*},\ldots,s_{s0}^{+*},\lambda_1^*,\ldots,\lambda_n^*)$ is an optimal solution of the above program, then the RAM for the DMU_0 is defined as

$$\Gamma_{RAM}(X_0,Y_0) = 1-\left(\sum_{i=1}^{m}\frac{s_{i0}^{-*}}{(m+s)R_i^{-}}+\sum_{r=1}^{s}\frac{s_{r0}^{+*}}{(m+s)R_r^{+}}\right) = 1-\frac{1}{m+s}\left(\sum_{i=1}^{m}\frac{s_{i0}^{-*}}{R_i^{-}}+\sum_{r=1}^{s}\frac{s_{r0}^{+*}}{R_r^{+}}\right). \tag{5}$$

If the arithmetic average of input and output inefficiencies is substituted by their geometric average in the objective function in model (4), the Multiplicative Range-Adjusted Measure (MRAM) is obtained (see Aparicio et al. (2021)):

$$\begin{aligned} \min_{\mathbf{s},\boldsymbol{\lambda}} \quad & 1-\left(\prod_{i=1}^{m}\frac{s_{i0}^{-}}{R_i^{-}}\prod_{r=1}^{s}\frac{s_{r0}^{+}}{R_r^{+}}\right)^{1/(m+s)} \\ s.t. \quad & \\ & \text{constraints of (4).} \end{aligned} \tag{6}$$

If now $(s_{10}^{-*},\ldots,s_{m0}^{-*},s_{10}^{+*},\ldots,s_{s0}^{+*},\lambda_1^*,\ldots,\lambda_n^*)$ is an optimal solution of (6), then the MRAM for the DMU_0 can be defined as follows.

$$\Gamma_{MRAM}(X_0,Y_0) = 1-\left(\prod_{i=1}^{m}\frac{s_{i0}^{-*}}{R_i^{-}}\prod_{r=1}^{s}\frac{s_{r0}^{+*}}{R_r^{+}}\right)^{1/(m+s)}$$

As Aparicio et al. (2021) have shown, the MRAM satisfies that values between zero and one are always taken (P1), monotonicity (P2), units invariance (P3), translation invariance (P4) and the "unique projection for efficiency comparison" property (P6). However, it fails indication (P5), i.e., MRAM is not able to distinguish between weakly efficient units and strongly efficient units (Pareto-efficient units), as happens with other measures (radial, directional distance function and hyperbolic).

3. The generalized range-Adjusted measure

3.1. Definition of the GRAM

In this section, we introduce a new family of efficiency measures, which is related to the Range-Adjusted Measure (RAM) and the Multiplicative Range-Adjusted Measure (MRAM). This family depends on a parameter p, whose values determine the different members of this family. Hereinafter, this "general" measure will be called the Generalized Range-Adjusted Measure (GRAM).

Next, we define the new technical efficiency measure. Before doing so, let us introduce some notions on the generalization of the arithmetic mean as different ways of summarizing several results. If p is a non zero real number and $\mathbf{z} = (z_1,\cdots,z_k)$ are positive real numbers, then the Generalized Mean with exponent p of these positive real numbers is

$$GM_p(z_1,\cdots,z_k) = \left(\frac{1}{k}\sum_{h=1}^{k} z_h^p\right)^{1/p}.$$

For $p=0$, we set it equal to the Geometric Mean (which is the limit of means with exponents approaching zero):

$$GM_0(z_1,\ldots,z_k) = \sqrt[k]{\prod_{h=1}^{k} z_h}.$$

The Generalized Mean is also known as the Hölder mean. A set of particular values of p are: $M_{-\infty}$, the minimum value of $\mathbf{z}$; M_{-1}, the Harmonic mean; M_0, the geometric mean; M_1, the arithmetic mean; M_2, the quadratic mean; and $M_{+\infty}$, the maximum value of $\mathbf{z}$. The Generalized Mean is also known as the Minkowski distance of order p. For $p \geq 1$, the Hölder mean is related to the l_p norms; (Briec, 1999) studied the Hölder distance function as a measure of technical efficiency. Nevertheless, a metric can be easily obtained for $p<1$ by removing the exponent $1/p$.

Fig. 1 shows unit circles (the set of all points that are at the unit distance from the centre) for different values of p. Notice that for $p<1$ the unit circles are not convex sets.

If the objective functions in models (4) and (6) are substituted by the Generalized Mean[1], maintaining the same set of constraints, then a new family of technical efficiency measures, dependent on the parameter p, may be defined as follows:

$$\begin{aligned} \min_{\mathbf{s},\boldsymbol{\lambda}} \quad & 1-\left(\frac{1}{(m+s)}\sum_{i=1}^{m}\left(\frac{s_{i0}^{-}}{R_i^{-}}\right)^p+\frac{1}{(m+s)}\sum_{r=1}^{s}\left(\frac{s_{r0}^{+}}{R_r^{+}}\right)^p\right)^{1/p} \\ s.t. \quad & \\ & \text{constraints of (4).} \end{aligned} \tag{7}$$

Model (7) satisfies the assumptions (A1) and (A2) in Halická & Trnovská (2021). Therefore, by Theorem 1 in Halická & Trnovská (2021), the existence of at least one optimal solution is guaranteed. If $(s_{10}^{-*},\ldots,s_{m0}^{-*},s_{10}^{+*},\ldots,s_{s0}^{+*},\lambda_1^*,\ldots,\lambda_n^*)$ is an optimal solution of model (7), then the value of the Generalized Range-Adjusted Measure (GRAM), $GRAM_p$, for the DMU_0 is defined as follows:

$$\Gamma_{GRAM_p}(X_0,Y_0) = 1-\left(\frac{1}{(m+s)}\sum_{i=1}^{m}\left(\frac{s_{i0}^{-*}}{R_i^{-}}\right)^p+\frac{1}{(m+s)}\sum_{r=1}^{s}\left(\frac{s_{r0}^{+*}}{R_r^{+}}\right)^p\right)^{1/p}. \tag{8}$$

The projection point generated by the above model for the assessed DMU_0 is defined as (X_0^*,Y_0^*), with components $x_{i0}^* = \sum_{j=1}^{n} x_{ij}\lambda_j^*$, for all i, and $y_{r0}^* = \sum_{j=1}^{n} y_{rj}\lambda_j^*$, for all r, from an optimal solution $(s_{10}^{-*},\ldots,s_{m0}^{-*},s_{10}^{+*},\ldots,s_{s0}^{+*},\lambda_1^*,\ldots,\lambda_n^*)$ of model (7).

RAM and MRAM are members of this family of technical efficiency measures. In particular, $\Gamma_{GRAM_0}(X_0,Y_0) = \Gamma_{MRAM}(X_0,Y_0)$ and $\Gamma_{GRAM_1}(X_0,Y_0) = \Gamma_{RAM}(X_0,Y_0)$. This is one of the objectives of our approach: proposing a general formulation that encompasses these two previously published efficiency measures (RAM and MRAM).

From a computational point of view, we resort to program (9), which has a strictly concave objective function for $0<p<1$ and the same linear constraints as model (7), instead of directly solving program (7). Both models are not exactly equivalent but, as we will go on to prove, any optimal solution of model (7) is an optimal solution of model (9) and vice-versa. This relationship allows us to

[1] In our context, the Generalized Mean and the Geometric Mean are defined on non-negative vectors of slacks. For each input or output variable, a slack equal to zero is related to the non-detection of technical inefficiency in such dimension.

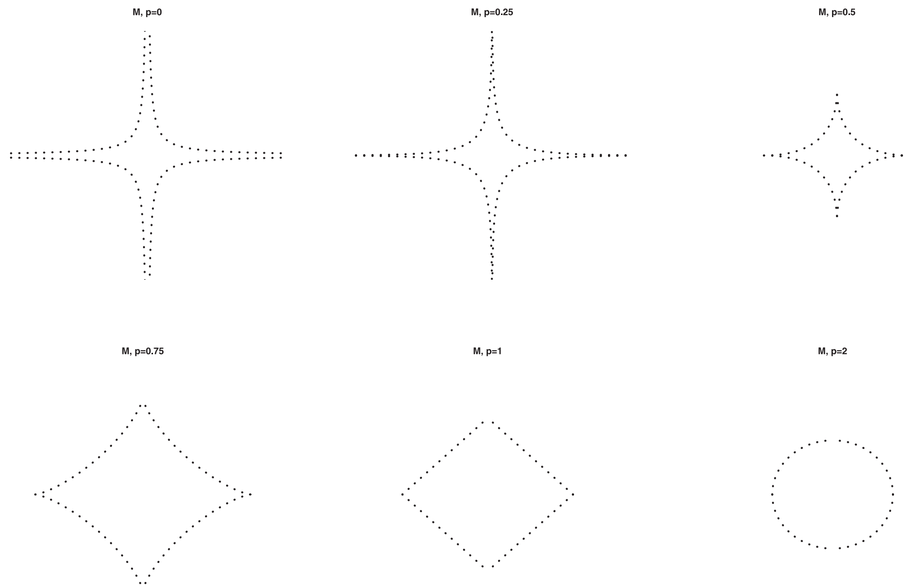

Fig. 1. Unit circles for different values of p.

use the optimal solutions of model (9) to determine the optimal value of model (7) as well as their projection points.

$$\left(1-\Gamma_{GRAM_p}(X_0,Y_0)\right)^p = \max_{\mathbf{s},\boldsymbol{\lambda}} \quad \frac{1}{(m+s)}\sum_{i=1}^{m}\left(\frac{s_{i0}^-}{R_i^-}\right)^p + \frac{1}{(m+s)}\sum_{r=1}^{s}\left(\frac{s_{r0}^+}{R_r^+}\right)^p$$
$$s.t.$$
$$\text{constraints of (4).} \tag{9}$$

Regarding this formulation, we are first going to prove that the objective function of model (9) is strictly concave for $0 < p < 1$.

Proposition 3.1. *Let* $0 < p < 1$. *The function* $h\left(s_{10}^-,\ldots,s_{m0}^-,s_{10}^+,\ldots,s_{s0}^+\right) = \frac{1}{(m+s)}\sum_{i=1}^{m}\left(\frac{s_{i0}^-}{R_i^-}\right)^p + \frac{1}{(m+s)}\sum_{r=1}^{s}\left(\frac{s_{r0}^+}{R_r^+}\right)^p$ *is strictly concave in* $\left(s_{10}^-,\ldots,s_{m0}^-,s_{10}^+,\ldots,s_{s0}^+\right) \geq 0_{m+s}$.

Proof of Proposition. It is enough to prove the strict concavity of the function $\sum_{i=1}^{m}\left(\frac{s_{i0}^-}{R_i^-}\right)^p + \sum_{r=1}^{s}\left(\frac{s_{r0}^+}{R_r^+}\right)^p$ since $\frac{1}{(m+s)}$ is a strictly positive constant. To prove that, let us analyze each term $\left(\frac{s_{i0}^-}{R_i^-}\right)^p$ and $\left(\frac{s_{r0}^+}{R_r^+}\right)^p$. First of all, we are going to prove that the function $f(z) = z^p$ with $0 < p < 1$ for $z \geq 0$ is strictly concave. It is very easy to prove this for $z > 0$ since $f''(z) = p(p-1)z^{p-2} < 0$. Note also that $f(0) = 0$. We are going to prove that if we have $\hat{z} \geq 0$ and $\check{z} \geq 0$, with $\hat{z} \neq \check{z}$, then $f(\delta\hat{z} + (1-\delta)\check{z}) > \delta f(\hat{z}) + (1-\delta)f(\check{z})$ for all $0 < \delta < 1$, which is the definition of strict concavity on $z \geq 0$. If $\hat{z} > 0$ and $\check{z} > 0$, then the property trivially holds. Let us assume, without loss of generality, that $\hat{z} = 0$. Then, $f(\delta\hat{z} + (1-\delta)\check{z}) = f((1-\delta)\check{z}) = (1-\delta)^p f(\check{z})$, by the definition of the function $f(z)$. By applying logarithms, it is not hard to see that $(1-\delta)^p > (1-\delta)$ for all $0 < \delta < 1$ and $0 < p < 1$. Consequently, $f(\delta\hat{z} + (1-\delta)\check{z}) > (1-\delta)f(\check{z}) = \delta f(\hat{z}) + (1-\delta)f(\check{z})$ because $f(\hat{z}) = 0$. Therefore, $f(z) = z^p$ with $0 < p < 1$ for $z \geq 0$ is a strictly concave function. In this way, we know that each term $\left(\frac{s_{i0}^-}{R_i^-}\right)^p$ and $\left(\frac{s_{r0}^+}{R_r^+}\right)^p$ defines a strictly concave function since $f(z) = kz^p$ is a strictly concave function for $k > 0$. Finally, the sum of strictly concave functions is a strictly concave function, which implies that $h\left(s_{10}^-,\ldots,s_{m0}^-,s_{10}^+,\ldots,s_{s0}^+\right)$ satisfies strict concavity. □

Next, we are going to prove the relationship between the optimal solutions of models (7) and (9). Before doing that, notice that model (9) also satisfies the assumptions (A1) and (A2) in Halická & Trnovská (2021), which implies that the existence of optimal solutions is assured (see Theorem 1 in Halická & Trnovská (2021)).

Proposition 3.2. *Let* $p > 0$. $(s_{10}^{-*},\ldots,s_{m0}^{-*},s_{10}^{+*},\ldots,s_{s0}^{+*},\lambda_1^*,\ldots,\lambda_n^*)$ *is an optimal solution of model (7) if and only if it is an optimal solution of model (9).*

Proof of Proposition.
We will prove part if $(s_{10}^{-*},\ldots,s_{m0}^{-*},s_{10}^{+*},\ldots,s_{s0}^{+*},\lambda_1^*,\ldots,\lambda_n^*)$ is an optimal solution of model (7), then $(s_{10}^{-*},\ldots,s_{m0}^{-*},s_{10}^{+*},\ldots,s_{s0}^{+*},\lambda_1^*,\ldots,\lambda_n^*)$ is an optimal solution of model (9). The other part of the proof is similar. Let us assume that $(s_{10}^{-*},\ldots,s_{m0}^{-*},s_{10}^{+*},\ldots,s_{s0}^{+*},\lambda_1^*,\ldots,\lambda_n^*)$ is not an optimal solution of model (9). Given our supposition, there exists $(s_{10}^{-'},\ldots,s_{m0}^{-'},s_{10}^{+'},\ldots,s_{s0}^{+'},\lambda_1^{'},\ldots,\lambda_n^{'})$ a feasible solution of model (9) such that $\frac{1}{(m+s)}\sum_{i=1}^{m}\left(\frac{s_{i0}^{-'}}{R_i^-}\right)^p + \frac{1}{(m+s)}\sum_{r=1}^{s}\left(\frac{s_{r0}^{+'}}{R_r^+}\right)^p > \frac{1}{(m+s)}\sum_{i=1}^{m}\left(\frac{s_{i0}^{-*}}{R_i^-}\right)^p + \frac{1}{(m+s)}\sum_{r=1}^{s}\left(\frac{s_{r0}^{+*}}{R_r^+}\right)^p$. Now, given that the function $h(z) = z^q$ is a strictly increasing function for $z \geq 0$ and $q > 0$, we have that $\left(\frac{1}{(m+s)}\sum_{i=1}^{m}\left(\frac{s_{i0}^{-'}}{R_i^-}\right)^p + \frac{1}{(m+s)}\sum_{r=1}^{s}\left(\frac{s_{r0}^{+'}}{R_r^+}\right)^p\right)^{1/p} > \left(\frac{1}{(m+s)}\sum_{i=1}^{m}\left(\frac{s_{i0}^{-*}}{R_i^-}\right)^p + \frac{1}{(m+s)}\sum_{r=1}^{s}\left(\frac{s_{r0}^{+*}}{R_r^+}\right)^p\right)^{1/p}$, which is equivalent to

$$1-\left(\frac{1}{(m+s)}\sum_{i=1}^{m}\left(\frac{s_{i0}^{-'}}{R_i^-}\right)^p + \frac{1}{(m+s)}\sum_{r=1}^{s}\left(\frac{s_{r0}^{+'}}{R_r^+}\right)^p\right)^{1/p} <$$
$$1-\left(\frac{1}{(m+s)}\sum_{i=1}^{m}\left(\frac{s_{i0}^{-*}}{R_i^-}\right)^p + \frac{1}{(m+s)}\sum_{r=1}^{s}\left(\frac{s_{r0}^{+*}}{R_r^+}\right)^p\right)^{1/p}.$$

However, this last inequality implies a contradiction with the fact that $(s_{10}^{-*},\ldots,s_{m0}^{-*},s_{10}^{+*},\ldots,s_{s0}^{+*},\lambda_1^*,\ldots,\lambda_n^*)$ is an optimal solution of model (7) since $(s_{10}^{-'},\ldots,s_{m0}^{-'},s_{10}^{+'},\ldots,s_{s0}^{+'},\lambda_1^{'},\ldots,\lambda_n^{'})$ is a trivial feasible solution of model (7). □

Before showing the main properties satisfied by the Generalized Range-Adjusted Measure, let us prove that the projection point generated from an optimal solution of model (9) is unique.

Proposition 3.3. *Let* $0 < p < 1$. *The projection point generated by model (9) is the same, regardless of the alternative optimal solution considered.*

Proof of Proposition. Let us assume that $(s_0^{-'}, s_0^{+'}, \lambda') = (s_{10}^{-'}, \ldots, s_{m0}^{-'}, s_{10}^{+'}, \ldots, s_{s0}^{+'}, \lambda_1', \ldots, \lambda_n')$ and $(s_0^{-''}, s_0^{+''}, \lambda'') = (s_{10}^{-''}, \ldots, s_{m0}^{-''}, s_{10}^{+''}, \ldots, s_{s0}^{+''}, \lambda_1'', \ldots, \lambda_n'')$, such that $(s_0^{-'}, s_0^{+'}) \neq (s_0^{-''}, s_0^{+''})$, are optimal solutions of model (9). Then, $(\bar{s}_0^-, \bar{s}_0^+, \bar{\lambda}) = (\frac{1}{2}s_{10}^{-'} + \frac{1}{2}s_{10}^{-''}, \ldots, \frac{1}{2}s_{s0}^{+'} + \frac{1}{2}s_{s0}^{+''}, \frac{1}{2}\lambda_1' + \frac{1}{2}\lambda_1'', \ldots, \frac{1}{2}\lambda_n' + \frac{1}{2}\lambda_n'')$ is a feasible solution of model (9) since the constraints are linear. Then,

$$\frac{1}{(m+s)}\sum_{i=1}^{m}\left(\frac{\bar{s}_{i0}^-}{R_i^-}\right)^p + \frac{1}{(m+s)}\sum_{r=1}^{s}\left(\frac{\bar{s}_{r0}^+}{R_r^+}\right)^p >$$

$$\frac{1}{2}\left(\frac{1}{(m+s)}\sum_{i=1}^{m}\left(\frac{s_{i0}^{-'}}{R_i^-}\right)^p + \frac{1}{(m+s)}\sum_{r=1}^{s}\left(\frac{s_{r0}^{+'}}{R_r^+}\right)^p\right)$$

$$+\frac{1}{2}\left(\frac{1}{(m+s)}\sum_{i=1}^{m}\left(\frac{s_{i0}^{-''}}{R_i^-}\right)^p + \frac{1}{(m+s)}\sum_{r=1}^{s}\left(\frac{s_{r0}^{+''}}{R_r^+}\right)^p\right),$$

since the objective function of model (9) is strictly concave. However, notice that the sum above coincides with the optimal value of model (9). This is a contradiction with the fact that $(s_0^{-'}, s_0^{+'}, \lambda')$ and $(s_0^{-''}, s_0^{+''}, \lambda'')$ are optimal solutions of model (9), which is maximizing the objective function. Therefore, only two scenarios are possible: (i) Model (9) has a unique optimal solution $(s_0^{-*}, s_0^{+*}, \lambda^*)$, which implies that the projection point is also unique; or (ii) all the optimal solutions of model (9) have the same values for the slacks (s_0^{-*}, s_0^{+*}). By the definition of the projection point, and the constraints of (9), we have that $x_{i0}^* = x_{i0} - s_{i0}^{-*}$, for all i, and $y_{r0}^* = y_{r0} + s_{r0}^{+*}$, for all r, for any optimal solution.

Cooper et al. (1999) proved that the RAM satisfies properties P1-P5, while Sueyoshi & Sekitani (2009) showed that the RAM does not meet P6. In contrast, the MRAM fulfills P1-P4 and P6 but it does not satisfy P5, i.e., the identification of Pareto-Koopmans efficiency (see Aparicio et al. (2021)). Next, we prove that the Generalized Range-Adjusted Measure satisfies all these six properties, as long as the parameter p takes values strictly between 0 and 1.

Proposition 3.4. *If* $0 < p < 1$, *then* $GRAM_p$ *satisfies P1-P6.*

Proof of Proposition.

(P1): It is immediate as a consequence of the definition of Γ_{GRAM_p} since, under Variable Returns to Scale, $s_{i0}^{-*} \leq R_i^-$, for all $i = 1, \ldots, m$ and $s_{r0}^{+*} \leq R_r^+$, for all $r = 1, \ldots, s$.

(P2): Let us consider DMU_0 and DMU_a such that (X_a, Y_a) is Pareto dominated by (X_0, Y_0). This means that, without loss of generality, there exists at least an input i', with $i' = 1, \ldots, m$ such that $x_{i'a} > x_{i'0}$. We have to show that the value of $GRAM_p$ for DMU_a is smaller than $GRAM_p$ for the DMU_0. Let $k_a = x_{i'a} - x_{i'0} > 0$. Let $(s_0^{-*}, s_0^{+*}, \lambda_0^*) = (s_{10}^{-*}, \ldots, s_{m0}^{-*}, s_{10}^{+*}, \ldots, s_{s0}^{+*}, \lambda_{10}^*, \ldots, \lambda_{n0}^*)$ be an optimal solution of model (7) when DMU_0 is evaluated. Then, $(\bar{s}_a^-, \bar{s}_a^+, \bar{\lambda}_a)$ with $\bar{s}_a^+ = s_0^{+*}$, $\bar{\lambda}_a = \lambda_0^*$, $\bar{s}_{ia}^- = s_{i0}^{-*}$ for $i \neq i'$ and $\bar{s}_{i'a}^- = s_{i'0}^{-*} + k_a$ is a feasible solution of model (7) when DMU_a is evaluated. Regarding the objective function of this last model, we know that

$$\Gamma_{GRAM_p}(X_a, Y_a) \leq 1 - \left(\frac{1}{(m+s)}\sum_{i=1}^{m}\left(\frac{\bar{s}_{ia}^-}{R_i^-}\right)^p + \frac{1}{(m+s)}\sum_{r=1}^{s}\left(\frac{\bar{s}_{ra}^+}{R_r^+}\right)^p\right)^{1/p} \leq$$

$$1 - \left(\frac{1}{(m+s)}\sum_{i=1}^{m}\left(\frac{s_{i0}^{-*}}{R_i^-}\right)^p + \frac{1}{(m+s)}\sum_{r=1}^{s}\left(\frac{s_{r0}^{+*}}{R_r^+}\right)^p\right)^{1/p} = \Gamma_{GRAM_p}(X_0, Y_0),$$

which is the relationship between the efficiency scores of DMU_0 and DMU_a that we wanted to find.

(P3): It is a consequence of the definition of Γ_{GRAM_p}, since the objective function is one minus an average of dimensionless ratios.

(P4): Thanks to constraint $\sum_j \lambda_j = 1$, any feasible solution of model (7) without translating the data is a feasible solution of model (7) when inputs and outputs are translated and vice versa. The value of the objective function does not change because the input and output ranges also remain unchanged under a translation of the data.

(P5): The GRAM model fits the general slacks-based scheme proposed in Halická & Trnovská (2021) since it satisfies the assumptions (A1) and (A2) in Halická & Trnovská (2021). This implies that (P5) easily follows from Theorem 3 in Halická & Trnovská (2021).

(P6): Let us assume that $(s_0^{-'}, s_0^{+'}, \lambda') = (s_{10}^{-'}, \ldots, s_{m0}^{-'}, s_{10}^{+'}, \ldots, s_{s0}^{+'}, \lambda_1', \ldots, \lambda_n')$ and $(s_0^{-''}, s_0^{+''}, \lambda'') = (s_{10}^{-''}, \ldots, s_{m0}^{-''}, s_{10}^{+''}, \ldots, s_{s0}^{+''}, \lambda_1'', \ldots, \lambda_n'')$ are optimal solutions of model (7) such that each solution generates a different projection point. By the definition of projection point, this is only possible if $(s_0^{-'}, s_0^{+'}) \neq (s_0^{-''}, s_0^{+''})$. Additionally, by Proposition 3.2, any optimal solution of model (7) is an optimal solution of model (9). This implies that both solutions, i.e., $(s_0^{-'}, s_0^{+'}, \lambda')$ and $(s_0^{-''}, s_0^{+''}, \lambda'')$, are also optimal solutions of model (9). However, they yield different projection points from model (9) since $(s_0^{-'}, s_0^{+'}) \neq (s_0^{-''}, s_0^{+''})$. This fact contradicts Proposition 3.3.

□

The GRAM model fits the so-called general slacks-based scheme since it satisfies the assumptions (A1) and (A2) in Halická & Trnovská (2021). This implies that monotonicity (P2 in this paper) can be extended to strict monotonicity by Theorem 5 by Halická & Trnovská (2021). Additionally, the GRAM also satisfies the so-called super efficiency property (see Halická & Trnovská (2021)).

Next, we show a simple numerical example in order to compare the three technical efficiency measures: RAM, MRAM and GRAM.

Example 1. Let us consider the following seven observations (DMUs), which use one input for producing one output. Table 1

Table 1
Example 1 .

DMU		I1	O1
DMU 1		1.0	1.0
DMU 2		2.0	4.0
DMU 3		5.0	7.0
DMU 4		2.0	1.0
RAM	0.750	2.00	4.00
MRAM	0.856	1.50	2.50
GRAM$_{0.5}$	0.833	1.75	3.25
DMU 5		3.0	2.0
RAM	0.750	2.00	4.00
MRAM	0.764	2.00	4.00
GRAM$_{0.5}$	0.757	2.00	4.00
DMU 6		5.0	4.0
RAM	0.750	2.00	4.00
MRAM	0.750	3.50	5.50
GRAM$_{0.5}$	0.750	3.50	5.50
DMU 7		7.0	7.0
RAM	0.833	5.00	7.00
MRAM	1.000	7.00	7.00
GRAM$_{0.5}$	0.917	5.00	7.00

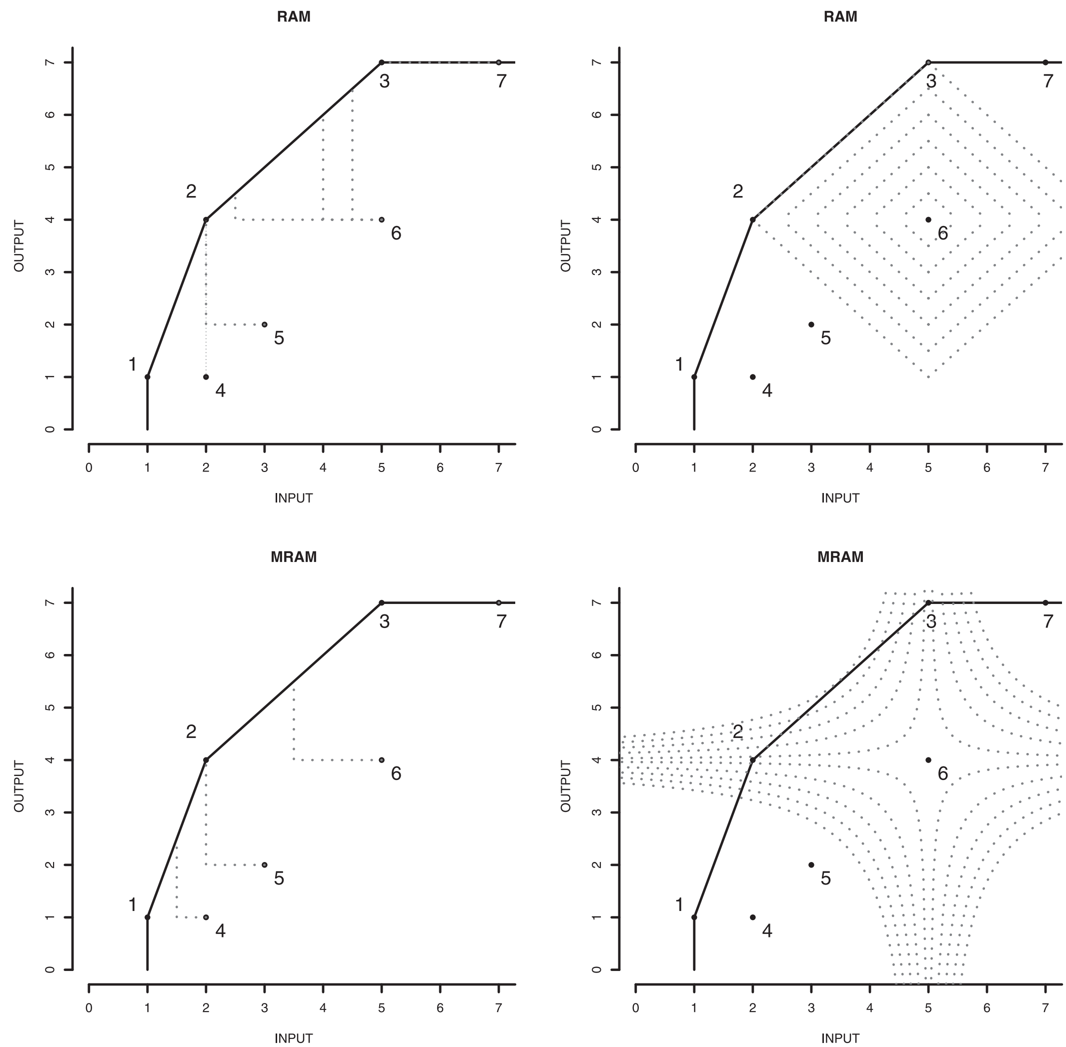

Fig. 2. Example 1. RAM and MRAM measures.

shows the data and the results associated with the implementation of the RAM, MRAM and GRAM for $p = .5$. See also Fig. 2 and Fig. 3 for a graphical representation where the exact level curves have been included for DMU 6. Specifically, Table 1 reports the efficiency score and the input and output corresponding to the projection point. Notice that RAM presents multiple projection points in the case of certain units. In particular, for DMU 6, Table 1 shows only one of the infinite alternative solutions (any convex combination of DMUs 2 and 3). They are very different. One extreme possible solution indicates that the unit should not change its output and should decrease its input by 60%, while another extreme alternative solution points out the opposite, i.e., fixing the input and focus the effort on increasing the output by 75%. However, this situation does not happen with the new GRAM. For DMU 6, GRAM with $p = .5$ determines the mid-point between DMUs 2 and 3 as a unique projection point. Regarding MRAM, this measure is not always able to identify Pareto-efficiency. For example, in the case of DMU 7, note that efficiency scores are strictly less than one for RAM and GRAM, indicating that this unit is not Pareto-efficient (indeed, it is dominated by DMU 3), but the score is one for MRAM. Additionally, it is worth mentioning that the efficiency score corresponding to GRAM for $p = .5$ can coincide with the arithmetic mean of the efficiency scores of RAM and MRAM in a natural way for this numerical example. This is not by chance. For the context of only one input and only one output, the following result is always true:

$$\left(\frac{1}{2}(s^-)^{\frac{1}{2}} + \frac{1}{2}(s^+)^{\frac{1}{2}}\right)^2 = \frac{s^- + s^+ + 2(s^-)^{\frac{1}{2}}(s^+)^{\frac{1}{2}}}{4} = \frac{1}{2}\left(\frac{s^- + s^+}{2}\right) + \frac{1}{2}\left(s^- s^+\right)^{\frac{1}{2}}.$$

This result requires that the optimal slacks are the same for the three measures. In general, this result is not true but it sheds some light on the relationship between the new approach and the RAM and MRAM.

As a final observation, we would like to highlight that this new family of efficiency measures, the Generalized Range-Adjusted Measure, for its non-extreme members, i.e., $0 < p < 1$, inherits the best properties of each of the extreme members of the family, i.e., $p = 0$ (the MRAM) and $p = 1$ (the RAM). See Fig. 4 for a graphical representation of the GRAM measure when p goes from 0 to 1.

3.2. Computational aspects

This section is devoted to dealing with the computational aspects of the new measure since, in contrast to the RAM, GRAM

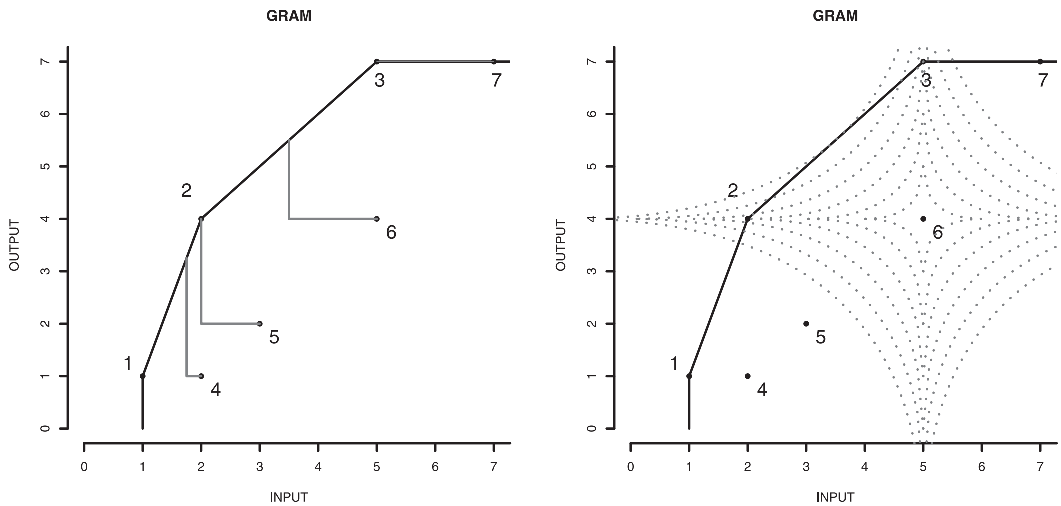

Fig. 3. Example 1. GRAM measure.

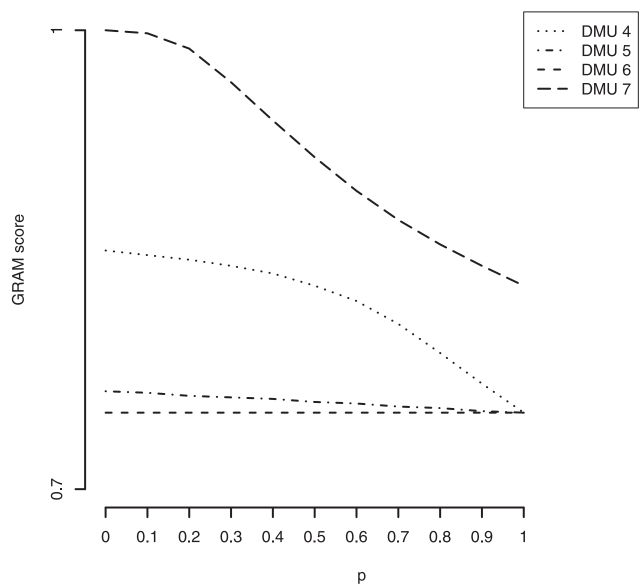

Fig. 4. Example 1. GRAM measure.

is associated with a non-linear optimization program. The GRAM problem can be solved directly using a specialized non-linear optimization software, such as BARON, LINGO and MOSEK, for example. In convex solvers, see (MOSEK, 2021), the concave optimization problem (9) can be expressed as

$$
\begin{array}{lll}
\max_{\mathbf{s},\boldsymbol{\lambda},\mathbf{t}} & t & \\
\quad s.t. & & \\
& t_i\, t \le \frac{1}{(m+s)}\left(\frac{s_{i0}^-}{R_i^-}\right)^p & \forall i=1,\cdots,m \\
& t_r\, t \le \frac{1}{(m+s)}\left(\frac{s_{r0}^+}{R_r^+}\right)^p & \forall r=1,\cdots,s \\
& \sum_{i=1}^m t_i + \sum_{r=1}^s t_r = 1 & \\
& \text{constraints of (4).} &
\end{array}
\tag{10}
$$

If $0 < p < 1$ is a rational number (see Appendix A), then we can resort to Second-Order Cone Programming (SOCP) for determining global optimal solutions of the non-linear program (9).

3.3. Duality

In this subsection, the duality and some properties of the equivalent GRAM problem (9) are studied. Given that different formulations of the same problem yield different dual problems and given the complexity of the problem (7) when obtaining its dual is the goal, this section will present the dual formulation of the auxiliary problem (9).

Proposition 3.5. *The dual problem of (9) is*

$$
\begin{array}{lll}
\min_{\mathbf{u},\mathbf{v},\mathbf{z},\boldsymbol{\theta}} & \sum_{i=1}^m v_i x_{i0} - \sum_{r=1}^s u_r y_{r0} + \theta + \frac{1-p}{(m+s)}\sum_{i=1}^m z_i^- + \frac{1-p}{(m+s)}\sum_{r=1}^s z_r^+ & \\
\quad s.t. & & \\
& \sum_{i=1}^m v_i x_{ij} - \sum_{r=1}^s u_r y_{rj} + \theta \ge 0 & \forall j=1,\cdots,n \\
& v_i \ge \left(\frac{p}{(m+s)(R_i^-)}\right)(z_i^-)^{(p-1)/p} & \forall i=1,\cdots,m \\
& u_r \ge \left(\frac{p}{(m+s)(R_r^+)}\right)(z_r^+)^{(p-1)/p} & \forall r=1,\cdots,s \\
& \theta \quad free. &
\end{array}
\tag{11}
$$

Proof of Proposition.

The Lagrange problem associated with the problem (9) is given by

$$
\begin{aligned}
L(\mathbf{s},\boldsymbol{\lambda},\mathbf{u},\mathbf{v},\boldsymbol{\theta}) = & \sum_{i=1}^m \left(\frac{1}{(m+s)(R_i^-)^p}(s_{i0}^-)^p - v_i s_{i0}^-\right) \\
& + \sum_{r=1}^s \left(\frac{1}{(m+s)(R_r^+)^p}(s_{r0}^+)^p - u_r s_{r0}^+\right) \\
& + \boldsymbol{\lambda}^T(-\mathbf{X}^T\mathbf{v} + \mathbf{Y}^T\mathbf{u} + \theta) - \theta + diag(\mathbf{X_o}^T)\mathbf{v} - diag(\mathbf{Y_0})^T\mathbf{u}
\end{aligned}
$$

with the domain $\mathbb{R}_+^m \times \mathbb{R}_+^s \times \mathbb{R}_+^n$ for the primal variables $\mathbf{s_0^-}, \mathbf{s_0^+}, \boldsymbol{\lambda}$ and $\mathbb{R}^m \times \mathbb{R}^s \times \mathbb{R}$ for the Lagrange multipliers (dual variables) $\mathbf{v}, \mathbf{u}, \boldsymbol{\theta}$. In this way, the dual problem is

$\min\{_{\mathbf{s}\ge\mathbf{0},\boldsymbol{\lambda}\ge\mathbf{0}} \sup L(\mathbf{s},\boldsymbol{\lambda},\mathbf{u},\mathbf{v},\boldsymbol{\theta})\}$.

Since the dual function $L(\mathbf{s},\boldsymbol{\lambda},\mathbf{u},\mathbf{v},\boldsymbol{\theta})$ is separable, the "sup" problem is divided into separate "sup" problems for each primal variable $\mathbf{s}^-$, $\mathbf{s}^+$ and $\boldsymbol{\lambda}$. The slack variables $\mathbf{s}$ appear in concave functions of the same type ($\phi(s) = \frac{1}{(m+s)R^p}s^p - ws$, where $\mathbf{w} := \{$'$\mathbf{v}$', '$\mathbf{u}$'$\}$), and L is affine in $\boldsymbol{\lambda}$ ($\boldsymbol{\lambda}^T(-\mathbf{X}^T\mathbf{v} + \mathbf{Y}^T\mathbf{u} + \theta)$).

For each slack variable $\mathbf{s}$, if $w > 0$, the supreme of $\phi(s)$ is reached when $s = \left(\frac{(m+s)R^p w}{p}\right)^{1/(p-1)}$. Moreover, L is linear in $\boldsymbol{\lambda}$, hence

$$\sup_{\lambda\geq 0} \boldsymbol{\lambda}^T(-\mathbf{X}^T\mathbf{v}+\mathbf{Y}^T\mathbf{u}+\theta) = \begin{cases} 0 & -\mathbf{X}^T\mathbf{v}+\mathbf{Y}^T\mathbf{u}+\theta \leq 0 \\ +\infty & otherwise \end{cases}$$

By substituting the optimal value of $\mathbf{s}$ into the dual function, the dual function can be re-expressed as:

$$\begin{aligned} L(\mathbf{s},\boldsymbol{\lambda},\mathbf{u},\mathbf{v},\boldsymbol{\theta}) = \sum_{i=1}^{m} & \left(\frac{1}{(m+s)(R_i^-)^p}\left(\frac{(m+s)(R_i^-)^p v_i}{p}\right)^{p/(p-1)}\right. \\ & \left. -v_i\left(\frac{(m+s)(R_i^-)^p v_i}{p}\right)^{1/(p-1)}\right) \\ & +\sum_{r=1}^{s}\left(\frac{1}{(m+s)(R_r^+)^p}\left(\frac{(m+s)(R_r^+)^p u_r}{p}\right)^{p/(p-1)}\right. \\ & \left. -u_r\left(\frac{(m+s)(R_r^+)^p u_r}{p}\right)^{1/(p-1)}\right) \\ & +\boldsymbol{\lambda}^T(-\mathbf{X}^T\mathbf{v}+\mathbf{Y}^T\mathbf{u}+\theta)-\theta+diag(\mathbf{X_o}^T)\mathbf{v}-diag(\mathbf{Y_0})^T\mathbf{u} \end{aligned}$$

Additionally, introducing additional positive variables $z_i^- = (v_i(R_i^-)(m+s)/p)^{p/(p-1)}$ and $z_r^+ = (u_r(R_r^+)(m+s)/p)^{p/(p-1)}$ to eliminate the nonlinear terms in the objective function, and considering the upper bound $-\mathbf{X}^T\mathbf{v}+\mathbf{Y}^T\mathbf{u}+\theta \leq 0$, the dual problem can be expressed as:

$$\begin{aligned} \min_{\mathbf{u},\mathbf{v},\mathbf{z},\boldsymbol{\theta}} \quad & \sum_{i=1}^{m} v_i x_{i0} - \sum_{r=1}^{s} u_r y_{r0} + \theta + \frac{1-p}{(m+s)}\sum_{i=1}^{m} z_i^- + \frac{1-p}{(m+s)}\sum_{r=1}^{s} z_r^+ \\ s.t. \quad & \\ & \sum_{i=1}^{m} v_i x_{ij} - \sum_{r=1}^{s} u_r y_{rj} + \theta \geq 0 \qquad \forall j = 1,\cdots,n \\ & v_i \geq \left(\frac{p}{(m+s)(R_i^-)}\right)(z_i^-)^{(p-1)/p} \qquad \forall i = 1,\cdots,m \\ & u_r \geq \left(\frac{p}{(m+s)(R_r^+)}\right)(z_r^+)^{(p-1)/p} \qquad \forall r = 1,\cdots,s \\ & \theta \quad free \end{aligned}$$

Next, we show that the dual problem of the equivalent GRAM model (9) can be economically interpreted in terms of (shadow) input and output prices and (shadow) profit. To do so, we first prove that one of the decision variables of model (11) represents optimal profit at optimum. Before proving that result, let us introduce the notion of actual profit and optimal profit given a technology T and input and output price vectors, K and Q, respectively. On the one hand, given a DMU_0, the actual profit is defined as revenue minus cost, i.e., $\pi_0 = \sum_r q_r y_{r0} - \sum_i k_i x_{i0}$. On the other hand, given a technology and input and output price vectors, optimal profit is defined as the maximum profit that can be achieved for any feasible unit in T at the fixed prices, i.e., $\pi(K,Q) = \max\{\sum_r q_r y_r - \sum_i k_i x_i : (X,Y) \in T\}$. Note that, by definition, $\pi(K,Q) \geq \pi_0$ since DMU_0 belongs to T. The next proposition states that, at optimum, the value of the decision variable θ in (11) coincides with the optimal profit calculated at (shadow) input and output prices $\mathbf{v}$ and $\mathbf{u}$ (see also Halická & Trnovská (2021)).

Proposition 3.6. *Let $(\mathbf{v}^*,\mathbf{u}^*,\mathbf{z}^*,\boldsymbol{\theta}^*)$ be an optimal solution of model (11). Then, $\theta^* = \pi(\mathbf{v}^*,\mathbf{u}^*)$.*

Proof of Proposition.

Let us first suppose that $\theta^* > \pi(\mathbf{v}^*,\mathbf{u}^*)$. Then, $(\mathbf{v}^*,\mathbf{u}^*,\mathbf{z}^*,\pi(\mathbf{v}^*,\mathbf{u}^*))$ is a feasible solution of model (11) because $\pi(\mathbf{v}^*,\mathbf{u}^*) = \max\left\{\sum_r u_r^* y_r - \sum_i v_i^* x_i : (X,Y) \in T\right\} \geq \pi_j$ for all $j = 1,\ldots,n$. The value of the objective function of (11) evaluated at $(\mathbf{v}^*,\mathbf{u}^*,\mathbf{z}^*,\pi(\mathbf{v}^*,\mathbf{u}^*))$, equals $\sum_i v_i^* x_{i0} - \sum_r u_r^* y_{r0} + \pi(\mathbf{v}^*,\mathbf{u}^*) + \sum_i p(z_i^-)^*/(m+s)(R_i^-)^p + (k-l)\sum_r p(z_r^+)^*/(m+s)(R_r^+)^p < \sum_i v_i^* x_{i0} - \sum_r u_r^* y_{r0} + \theta^* + \sum_i p(z_i^-)^*/(m+s)(R_i^-)^p + \sum_r p(z_r^+)^*/(m+s)(R_r^+)^p$, which is a contradiction with the optimality of $(\mathbf{v}^*,\mathbf{u}^*,\mathbf{z}^*,\boldsymbol{\theta}^*)$. Let us now assume that $\theta^* < \pi(\mathbf{v}^*,\mathbf{u}^*)$. By the first set of constraints in model (11), we have that $\sum_r u_r^* y_{rj} - \sum_i v_i^* x_{ij} \leq \theta^* < \pi(\mathbf{v}^*,\mathbf{u}^*)$ for all $j = 1,\ldots,n$, which is a contradiction, since the optimal profit is always achieved at least on a DMU_j (see Ray, 2007).

□

Thanks to the above proposition, we can rewrite model (11) as

$$\begin{aligned} \min_{\mathbf{u},\mathbf{v},\mathbf{z}} \quad & \left[\pi(\mathbf{v},\mathbf{u}) + \sum_{i=1}^{m} v_i x_{i0} - \sum_{r=1}^{s} u_r y_{r0} + \frac{1-p}{(m+s)}\sum_{i=1}^{m} z_i^- + \frac{1-p}{(m+s)}\sum_{r=1}^{s} z_r^+\right] \\ s.t. \quad & \\ \pi(\mathbf{v},\mathbf{u}) \geq & \sum_r u_r y_{rj} - \sum_i v_i x_{ij} \qquad \forall j = 1,\cdots,n \\ v_i \geq & \left(\frac{p}{(m+s)(R_i^-)}\right)(z_i^-)^{(p-1)/p} \qquad \forall i = 1,\cdots,m \\ u_r \geq & \left(\frac{p}{(m+s)(R_r^+)}\right)(z_r^+)^{(p-1)/p} \qquad \forall r = 1,\cdots,s \end{aligned} \tag{12}$$

This relates the GRAM to the so-called loss function, as introduced by Pastor, Lovell, & Aparicio (2012). The objective function is composed of two parts. First, the difference between optimal profit and actual profit at prices $\mathbf{v}$ for inputs and $\mathbf{u}$ for outputs, which must be minimized. Second, a penalty term related to the satisfaction of so-called normalization conditions (see Halická & Trnovská (2018)). In our case, the set of normalization conditions is given by the second and third sets of constraints in model (12).

4. An empirical illustration of the new measure

4.1. Illustrative example

In this section, we briefly illustrate the use of the new approach with a database taken from Ramón, Ruiz, & Sirvent (2010). In this way, we consider a sample consisting of 20 DMUs, each one using 4 inputs to produce 2 outputs. The data are shown in Table 2. In Tables 3 and 4, we focus our attention on non-Pareto efficient DMUs. In particular, for this set of units, we report the slacks and efficiency score for different values of the parameter p (0, 0.1, 0.25, 0.5, 0.75, 0.9 and 1). As we mentioned above, $p = 0$ corresponds to the MRAM, whereas $p = 1$ corresponds to the RAM.

Table 2
Data.

DMU	x_1	x_2	x_3	x_4	y_1	y_2
1	1.7	1.7	2.3	1	2	1
2	5.9	3.3	3.3	1.3	1	2
3	5.6	5	3.3	1.4	1	2
4	2.9	2	1.7	2	2	2
5	2.6	3.3	3.3	1.7	1	2
6	1.7	2.5	1.4	2	1.5	1.5
7	3	5	2.5	2	2	1
8	1.6	4	2	1.5	2	1
9	5.3	2.5	3.3	2	1	2
10	3.3	2	2	2	2	1
11	4	2.5	5	3.3	2	2
12	8	5	1.7	1.1	2	3
13	8	4	3.5	1.7	2	2
14	4.3	3.2	3.3	2	1.5	1
15	1.9	2.5	3.5	1.7	2	1
16	3	5.5	2.4	1.3	1	1.5
17	5.9	4.2	4.9	2	2	2
18	2.8	4.6	2.3	1.9	1	2
19	1.7	2.2	1.4	1.3	1	1
20	3.3	1.7	3.3	3.3	2	1

Table 3
The results for non-Pareto efficient units.

DMU	p	s_1^-	s_2^-	s_3^-	s_4^-	s_1^+	s_2^+	Score
DMU 2	0.00	1.240 (21.0%)	0.112 (3.4%)	1.373 (41.6%)	0.068 (5.2%)	1.000 (100.0%)	0.056 (2.8%)	0.8897
	0.10	1.245 (21.1%)	0.114 (3.5%)	1.373 (41.6%)	0.068 (5.2%)	1.000 (100.0%)	0.055 (2.8%)	0.8778
	0.25	1.255 (21.3%)	0.120 (3.6%)	1.373 (41.6%)	0.066 (5.1%)	1.000 (100.0%)	0.053 (2.7%)	0.8573
	0.50	1.298 (22.0%)	0.146 (4.4%)	1.374 (41.6%)	0.057 (4.4%)	1.000 (100.0%)	0.046 (2.3%)	0.8161
	0.75	1.439 (24.4%)	0.231 (7.0%)	1.377 (41.7%)	0.026 (2.0%)	1.000 (100.0%)	0.022 (1.1%)	0.7683
	0.90	1.558 (26.4%)	0.302 (9.2%)	1.379 (41.8%)	0.001 (0.1%)	1.000 (100.0%)	0.001 (0.1%)	0.7365
	1.00	1.563 (26.5%)	0.305 (9.2%)	1.379 (41.8%)	0.000 (0.0%)	1.000 (100.0%)	0.000 (0.0%)	0.7154
DMU 3	0.00	0.557 (9.9%)	1.595 (31.9%)	1.387 (42.0%)	0.208 (14.9%)	1.000 (100.0%)	0.149 (7.5%)	0.7865
	0.10	0.573 (10.2%)	1.605 (32.1%)	1.387 (42.0%)	0.205 (14.6%)	1.000 (100.0%)	0.146 (7.3%)	0.7757
	0.25	0.609 (10.9%)	1.627 (32.5%)	1.389 (42.1%)	0.196 (14.0%)	1.000 (100.0%)	0.141 (7.1%)	0.7581
	0.50	0.745 (13.3%)	1.710 (34.2%)	1.394 (42.2%)	0.162 (11.6%)	1.000 (100.0%)	0.120 (6.0%)	0.7253
	0.75	1.142 (20.4%)	1.952 (39.0%)	1.405 (42.6%)	0.069 (4.9%)	1.000 (100.0%)	0.057 (2.9%)	0.6860
	0.90	1.456 (26.0%)	2.140 (42.8%)	1.410 (42.7%)	0.002 (0.1%)	1.000 (100.0%)	0.002 (0.1%)	0.6564
	1.00	1.468 (26.2%)	2.147 (42.9%)	1.411 (42.8%)	0.000 (0.0%)	1.000 (100.0%)	0.000 (0.0%)	0.6356
DMU 7	0.00	0.701 (23.4%)	3.035 (60.7%)	0.436 (17.4%)	0.532 (26.6%)	0.000 (0.0%)	0.436 (43.6%)	1.0000
	0.10	0.755 (25.2%)	3.159 (63.2%)	0.469 (18.8%)	0.548 (27.4%)	0.000 (0.0%)	0.450 (45.0%)	0.9614
	0.25	0.757 (25.2%)	3.163 (63.3%)	0.470 (18.8%)	0.548 (27.4%)	0.000 (0.0%)	0.451 (45.1%)	0.8801
	0.50	0.762 (25.4%)	3.165 (63.3%)	0.469 (18.8%)	0.552 (27.6%)	0.000 (0.0%)	0.448 (44.8%)	0.8148
	0.75	0.787 (26.2%)	3.172 (63.4%)	0.456 (18.2%)	0.573 (28.7%)	0.000 (0.0%)	0.427 (42.7%)	0.7750
	0.90	0.880 (29.3%)	3.195 (63.9%)	0.410 (16.4%)	0.650 (32.5%)	0.000 (0.0%)	0.350 (35.0%)	0.7547
	1.00	1.300 (43.3%)	3.300 (66.0%)	0.200 (8.0%)	1.000 (50.0%)	0.000 (0.0%)	0.000 (0.0%)	0.7397
DMU 9	0.00	1.874 (35.4%)	0.184 (7.4%)	1.591 (48.2%)	0.111 (5.6%)	1.000 (100.0%)	0.092 (4.6%)	0.8449
	0.10	1.881 (35.5%)	0.189 (7.6%)	1.591 (48.2%)	0.110 (5.5%)	1.000 (100.0%)	0.090 (4.5%)	0.8325
	0.25	1.899 (35.8%)	0.200 (8.0%)	1.591 (48.2%)	0.106 (5.3%)	1.000 (100.0%)	0.087 (4.4%)	0.8117
	0.50	1.971 (37.2%)	0.243 (9.7%)	1.593 (48.3%)	0.091 (4.6%)	1.000 (100.0%)	0.075 (3.8%)	0.7719
	0.75	2.201 (41.5%)	0.381 (15.2%)	1.597 (48.4%)	0.041 (2.1%)	1.000 (100.0%)	0.035 (1.8%)	0.7267
	0.90	2.392 (45.1%)	0.495 (19.8%)	1.600 (48.5%)	0.001 (0.1%)	1.000 (100.0%)	0.001 (0.1%)	0.6958
	1.00	2.400 (45.3%)	0.500 (20.0%)	1.600 (48.5%)	0.000 (0.0%)	1.000 (100.0%)	0.000 (0.0%)	0.6748
DMU 10	0.00	0.782 (23.7%)	0.088 (4.4%)	0.092 (4.6%)	0.326 (16.3%)	0.000 (0.0%)	0.663 (66.3%)	1.0000
	0.10	0.806 (24.4%)	0.101 (5.1%)	0.096 (4.8%)	0.339 (17.0%)	0.000 (0.0%)	0.660 (66.0%)	0.9856
	0.25	0.820 (24.8%)	0.105 (5.3%)	0.089 (4.5%)	0.350 (17.5%)	0.000 (0.0%)	0.649 (64.9%)	0.9538
	0.50	0.850 (25.8%)	0.112 (5.6%)	0.075 (3.8%)	0.375 (18.8%)	0.000 (0.0%)	0.625 (62.5%)	0.9250
	0.75	0.896 (27.2%)	0.124 (6.2%)	0.052 (2.6%)	0.413 (20.7%)	0.000 (0.0%)	0.587 (58.7%)	0.9056
	0.90	0.955 (28.9%)	0.139 (7.0%)	0.022 (1.1%)	0.463 (23.2%)	0.000 (0.0%)	0.537 (53.7%)	0.8958
	1.00	1.000 (30.3%)	0.150 (7.5%)	0.000 (0.0%)	0.500 (25.0%)	0.000 (0.0%)	0.500 (50.0%)	0.8895
DMU 11	0.00	0.725 (18.1%)	0.273 (10.9%)	3.260 (65.2%)	1.357 (41.1%)	0.000 (0.0%)	0.060 (3.0%)	1.0000
	0.10	0.794 (19.9%)	0.320 (12.8%)	3.298 (66.0%)	1.354 (41.0%)	0.000 (0.0%)	0.059 (3.0%)	0.9693
	0.25	0.838 (21.0%)	0.346 (13.8%)	3.299 (66.0%)	1.346 (40.8%)	0.000 (0.0%)	0.051 (2.6%)	0.8965
	0.50	0.939 (23.5%)	0.405 (16.2%)	3.300 (66.0%)	1.328 (40.2%)	0.000 (0.0%)	0.032 (1.6%)	0.8189
	0.75	1.068 (26.7%)	0.481 (19.2%)	3.300 (66.0%)	1.306 (39.6%)	0.000 (0.0%)	0.006 (0.3%)	0.7569
	0.90	1.100 (27.5%)	0.500 (20.0%)	3.300 (66.0%)	1.300 (39.4%)	0.000 (0.0%)	0.000 (0.0%)	0.7231
	1.00	1.100 (27.5%)	0.500 (20.0%)	3.300 (66.0%)	1.300 (39.4%)	0.000 (0.0%)	0.000 (0.0%)	0.7024

First, some non-Pareto efficient units present a score that equals one under the MRAM ($p = 0$), which is again related to the fact that this measure does not satisfy property P5 (Indication). This does not happen with the other values of parameter p. In all these cases, the measures identify all the units in the tables as technically inefficient (score strictly less than one). The new approach guarantees, as happens with the RAM, that the efficiency score values one if and only if the evaluated unit is Pareto-Koopmans efficient. Of course, RAM and GRAM identify the same set of DMUs as efficient (score equals one). Second, the efficiency score decreases as p increases. This is due to the existing relationship between the Hölder means. In particular, if $p > h$, then $\Gamma_{GRAM_p}(X_0, Y_0) \leq \Gamma_{GRAM_h}(X_0, Y_0)$. The equality holds if and only if all the elements of (X_0, Y_0) have the same value. Third, the GRAM with $p = .5$ seems to yield the most balanced score and slacks among all the measures considered in this numerical example. The score for GRAM with p=0.5 does not coincide with the arithmetic average of the scores associated with the RAM and MRAM. Nevertheless, the value of the new measure is close to this mean. As we have previously shown, the relationship between the new approach and RAM and MRAM when $p = .5$ occurs naturally when considering only one input and only one output. In this sense, we suggest 0.5 as a possible selection for the parameter p in the case of the GRAM when the analyst needs to opt for a specific option, instead of resorting to a grid which has different values for p.

4.2. Computational experience

In order to computationally evaluate the new GRAM measure, we have tested the GRAM model using randomly generated instances. We have used a uniform distribution in [50,100] to generate the input/output values. The results are shown in Table 5. The first three columns are the number of DMUs, and the number of inputs/outputs. The number of DMUs considered was 100 or 1000 and the number of inputs/outputs varies from 2 to 8. The next block of two columns gives the average RAM measure for the inefficient units (Mean Score) and the related elapsed time (time) in seconds. The next two blocks report the same information for the MRAM and GRAM (with $p = .5$). From the table, we conclude that the use of the non-linear models (MRAM ad GRAM) as opposed to the RAM model requires more computational time to solve all the problems. Nevertheless, we can see in Table 5 (last row) that the computational time required by the GRAM model to solve a thousand non-linear problems is approximately four minutes. This means that, in practice, the computational time will not actually be a problem for GRAM.

Our experiment was conducted on a PC with a 2.5 GHz dual-core Intel Core i5 processor, 8 Gb of RAM and the operating system was OS X 10.15. The optimization software package CPLEX v20.0 was used for the computations, see (CPLEX, 2021).

Table 4
The results for non-Pareto efficient units.

DMU	p	s_1^-	s_2^-	s_3^-	s_4^-	s_1^+	s_2^+	Score
DMU 13	0.00	2.884 (36.1%)	0.597 (14.9%)	1.627 (46.5%)	0.306 (18.0%)	0.000 (0.0%)	0.260 (13.0%)	1.0000
	0.10	2.962 (37.0%)	0.662 (16.6%)	1.681 (48.0%)	0.313 (18.4%)	0.000 (0.0%)	0.269 (13.5%)	0.9617
	0.25	3.044 (38.1%)	0.712 (17.8%)	1.684 (48.1%)	0.295 (17.4%)	0.000 (0.0%)	0.256 (12.8%)	0.8826
	0.50	3.309 (41.4%)	0.872 (21.8%)	1.691 (48.3%)	0.237 (13.9%)	0.000 (0.0%)	0.212 (10.6%)	0.8222
	0.75	3.977 (49.7%)	1.273 (31.8%)	1.701 (48.6%)	0.097 (5.7%)	0.000 (0.0%)	0.094 (4.7%)	0.7835
	0.90	4.465 (55.8%)	1.562 (39.1%)	1.705 (48.7%)	0.003 (0.2%)	0.000 (0.0%)	0.004 (0.2%)	0.7555
	1.00	4.484 (56.1%)	1.574 (39.4%)	1.705 (48.7%)	0.000 (0.0%)	0.000 (0.0%)	0.000 (0.0%)	0.7353
DMU 14	0.00	2.050 (47.7%)	1.362 (42.6%)	1.275 (38.6%)	0.542 (27.1%)	0.500 (33.3%)	0.458 (45.8%)	0.6788
	0.10	2.052 (47.7%)	1.363 (42.6%)	1.274 (38.6%)	0.543 (27.2%)	0.500 (33.3%)	0.457 (45.7%)	0.6777
	0.25	2.056 (47.8%)	1.364 (42.6%)	1.272 (38.5%)	0.547 (27.4%)	0.500 (33.3%)	0.453 (45.3%)	0.6759
	0.50	2.069 (48.1%)	1.367 (42.7%)	1.265 (38.3%)	0.558 (27.9%)	0.500 (33.3%)	0.442 (44.2%)	0.6730
	0.75	2.108 (49.0%)	1.377 (43.0%)	1.246 (37.8%)	0.590 (29.5%)	0.500 (33.3%)	0.410 (41.0%)	0.6699
	0.90	2.216 (51.5%)	1.404 (43.9%)	1.192 (36.1%)	0.680 (34.0%)	0.500 (33.3%)	0.320 (32.0%)	0.6679
	1.00	2.600 (60.5%)	1.500 (46.9%)	1.000 (30.3%)	1.000 (50.0%)	0.500 (33.3%)	0.000 (0.0%)	0.6644
DMU 15	0.00	0.095 (5.0%)	0.736 (29.4%)	1.229 (35.1%)	0.610 (35.9%)	0.000 (0.0%)	0.073 (7.3%)	1.0000
	0.10	0.103 (5.4%)	0.774 (31.0%)	1.247 (35.6%)	0.619 (36.4%)	0.000 (0.0%)	0.080 (8.0%)	0.9820
	0.25	0.098 (5.2%)	0.774 (31.0%)	1.250 (35.7%)	0.615 (36.2%)	0.000 (0.0%)	0.084 (8.4%)	0.9405
	0.50	0.089 (4.7%)	0.772 (30.9%)	1.255 (35.9%)	0.608 (35.8%)	0.000 (0.0%)	0.092 (9.2%)	0.9010
	0.75	0.083 (4.4%)	0.771 (30.8%)	1.258 (35.9%)	0.603 (35.5%)	0.000 (0.0%)	0.097 (9.7%)	0.8745
	0.90	0.100 (5.3%)	0.775 (31.0%)	1.250 (35.7%)	0.617 (36.3%)	0.000 (0.0%)	0.083 (8.3%)	0.8616
	1.00	0.200 (10.5%)	0.800 (32.0%)	1.200 (34.3%)	0.700 (41.2%)	0.000 (0.0%)	0.000 (0.0%)	0.8534
	0.00	0.119 (4.0%)	3.172 (57.7%)	0.236 (9.8%)	0.052 (4.0%)	0.937 (93.7%)	0.037 (2.5%)	0.9142
DMU 16	0.10	0.110 (3.7%)	3.204 (58.3%)	0.275 (11.5%)	0.050 (3.8%)	0.963 (96.3%)	0.034 (2.3%)	0.9000
	0.25	0.099 (3.3%)	3.250 (59.1%)	0.331 (13.8%)	0.045 (3.5%)	1.000 (100.0%)	0.030 (2.0%)	0.8720
	0.50	0.107 (3.6%)	3.255 (59.2%)	0.331 (13.8%)	0.043 (3.3%)	1.000 (100.0%)	0.029 (1.9%)	0.8100
	0.75	0.172 (5.7%)	3.295 (59.9%)	0.334 (13.9%)	0.026 (2.0%)	1.000 (100.0%)	0.020 (1.3%)	0.7377
	0.90	0.283 (9.4%)	3.362 (61.1%)	0.337 (14.0%)	0.001 (0.1%)	1.000 (100.0%)	0.001 (0.1%)	0.6924
	1.00	0.289 (9.6%)	3.366 (61.2%)	0.337 (14.0%)	0.000 (0.0%)	1.000 (100.0%)	0.000 (0.0%)	0.6626
DMU 17	0.00	1.300 (22.0%)	1.135 (27.0%)	3.086 (63.0%)	0.370 (18.5%)	0.000 (0.0%)	0.244 (12.2%)	1.0000
	0.10	1.421 (24.1%)	1.244 (29.6%)	3.160 (64.5%)	0.352 (17.6%)	0.000 (0.0%)	0.261 (13.1%)	0.9564
	0.25	1.523 (25.8%)	1.307 (31.1%)	3.164 (64.6%)	0.330 (16.5%)	0.000 (0.0%)	0.245 (12.3%)	0.8646
	0.50	1.823 (30.9%)	1.490 (35.5%)	3.174 (64.8%)	0.261 (13.1%)	0.000 (0.0%)	0.197 (9.9%)	0.7900
	0.75	2.506 (42.5%)	1.903 (45.3%)	3.191 (65.1%)	0.105 (5.3%)	0.000 (0.0%)	0.086 (4.3%)	0.7380
	0.90	2.982 (50.5%)	2.189 (52.1%)	3.200 (65.3%)	0.004 (0.2%)	0.000 (0.0%)	0.003 (0.2%)	0.7021
	1.00	3.000 (50.8%)	2.200 (52.4%)	3.200 (65.3%)	0.000 (0.0%)	0.000 (0.0%)	0.000 (0.0%)	0.6772
DMU 18	0.00	0.004 (0.1%)	2.123 (46.2%)	0.017 (0.7%)	0.010 (0.5%)	0.636 (63.6%)	0.001 (0.1%)	0.9885
	0.10	0.004 (0.1%)	2.125 (46.2%)	0.019 (0.8%)	0.010 (0.5%)	0.637 (63.7%)	0.001 (0.1%)	0.9819
	0.25	0.004 (0.1%)	2.130 (46.3%)	0.025 (1.1%)	0.009 (0.5%)	0.640 (64.0%)	0.001 (0.1%)	0.9644
	0.50	0.002 (0.1%)	2.148 (46.7%)	0.046 (2.0%)	0.004 (0.2%)	0.653 (65.3%)	0.000 (0.0%)	0.9166
	0.75	0.000 (0.0%)	2.166 (47.1%)	0.066 (2.9%)	0.000 (0.0%)	0.666 (66.6%)	0.000 (0.0%)	0.8504
	0.90	0.000 (0.0%)	2.167 (47.1%)	0.067 (2.9%)	0.000 (0.0%)	0.667 (66.7%)	0.000 (0.0%)	0.8132
	1.00	0.000 (0.0%)	2.167 (47.1%)	0.067 (2.9%)	0.000 (0.0%)	0.667 (66.7%)	0.000 (0.0%)	0.7907
DMU 20	0.00	1.553 (47.1%)	0.000 (0.0%)	0.970 (29.4%)	2.232 (67.6%)	0.000 (0.0%)	0.000 (0.0%)	1.0000
	0.10	1.598 (48.4%)	0.000 (0.0%)	0.999 (30.3%)	2.297 (69.6%)	0.000 (0.0%)	0.000 (0.0%)	0.9996
	0.25	1.599 (48.5%)	0.000 (0.0%)	0.999 (30.3%)	2.298 (69.6%)	0.000 (0.0%)	0.000 (0.0%)	0.9730
	0.50	1.600 (48.5%)	0.000 (0.0%)	1.000 (30.3%)	2.300 (69.7%)	0.000 (0.0%)	0.000 (0.0%)	0.8859
	0.75	1.600 (48.5%)	0.000 (0.0%)	1.000 (30.3%)	2.300 (69.7%)	0.000 (0.0%)	0.000 (0.0%)	0.8086
	0.90	1.600 (48.5%)	0.000 (0.0%)	1.000 (30.3%)	2.300 (69.7%)	0.000 (0.0%)	0.000 (0.0%)	0.7693
	1.00	1.600 (48.5%)	0.000 (0.0%)	1.000 (30.3%)	2.300 (69.7%)	0.000 (0.0%)	0.000 (0.0%)	0.7454

Table 5
Results for simulated data.

			RAM		*MRAM*		$GRAM_{p=.5}$	
DMU	Inputs	Outputs	Mean Score	time	Mean Score	time	Mean Score	time
100	2	2	0.632	0.007	0.726	1.5	0.639	3.0
1000	2	2	0.615	0.141	0.700	26.1	0.616	28.6
100	4	4	0.744	0.017	0.839	0.8	0.757	7.6
1000	4	4	0.712	0.325	0.803	25.1	0.713	73.4
100	8	8	0.777	0.023	0.876	0.2	0.804	14.5
1000	8	8	0.762	0.609	0.861	10.1	0.765	254.3

5. Conclusions

This paper is concerned with the measurement of technical efficiency from a Data Envelopment Analysis (DEA) perspective. In a DEA, the literature usually considers the following properties for technical efficiency measures: (P1) the measure takes values between zero and one; (P2) monotonicity; (P3) units invariance; (P4) translation invariance; (P5) the assessed DMU is Pareto-Koopmans efficient if and only if the measure has a value of one (Indication); and, finally, (P6) unique projection for efficiency comparison (see Färe & ll (1978), Pastor et al. (1999) and Sueyoshi & Sekitani (2009), to name just a few). So far, no known efficiency measure satisfies all the aforementioned properties. One of the DEA measures that meets more properties is the Range-Adjusted Measure (RAM) by Cooper et al. (1999). It satisfies P1-P5, but fails P6. Indeed, following Sueyoshi & Sekitani (2009), none of the traditional measures in DEA fulfills P6. A recent version of the RAM, the so-called Multiplicative Range-Adjusted Measure (MRAM) by Aparicio et al. (2020) has endowed the traditional RAM with property P6 (unique projection for efficiency comparison), but at the expense of failing to fulfill property P5 (Indication or Pareto-efficiency detection).

In this paper, we have defined a new family of non-radial non-oriented technical efficiency measures inspired by the RAM and the Hölder means. Because of the analogy to the RAM, we have called it the Generalized Range-Adjusted Measure (GRAM), which depends on the value of a parameter p, with $0 \le p \le 1$. When $p = 0$, the GRAM collapses to the MRAM, while when $p = 1$, it is equivalent to the RAM. Regarding this new family, we have proved that if $0 < p < 1$, then the GRAM satisfies all the mentioned properties, i.e., P1-P6. In this way, we have introduced, for the first time, a "well-defined" DEA efficiency measure in the line recently pinpointed by some authors as, for example, Aparicio and Pastor (2013) and Sueyoshi & Sekitani (2009). Additionally, the interpretation of the value of the measure is straightforward since it represents a certain average of dimensionless input- and output-specific technical inefficiencies. From a computational perspective, the GRAM for $0 < p < 1$ is, by definition, associated with a non-linear optimization program. However, in these respects, another goal has been achieved: the measure may be computed by solving a Second-Order Cone Programming (SOCP) model when the parameter p is a rational number. Finally, we have also provided the dual model of GRAM, relating it to the notion of profit maximization.

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Appendix A. SOCP formulation when $0 < p < 1$ is a rational number

If $0 < p < 1$ is a rational number $p = l/k$, GRAM problem (9) is equivalent, regarding the set of optimal solutions, to the following optimization problem

$$\begin{array}{lll} \max\limits_{\mathbf{s},\mathbf{t},\boldsymbol{\lambda}} & \left(\sum\limits_{i=1}^{m} \dfrac{t_{i0}^{-}}{(m+s)R_i^p} + \sum\limits_{r=1}^{s} \dfrac{t_{r0}^{+}}{(m+s)(R_r^p)} \right) & \\ s.t. & & \\ & (t_{i0}^{-})^k \le (s_{i0}^{-})^l & \forall i = 1, \cdots, m \\ & (t_{r0}^{+})^k \le (s_{r0}^{+})^l & \forall r = 1, \cdots, s \\ & \text{constraints of (4).} & \end{array} \tag{A.1}$$

Any inequality of the form $t^{2^{\hat{n}}} \le t_1 t_2 \cdots t_{2^{\hat{n}}}$ can be expressed by $2^{\hat{n}} - 1$ inequalities (hyperbolic constraints) of the form $s_{ab}^2 \le s_a s_b$, where all the new variables introduced are required to be nonnegative. If $\hat{n} = 1$, then this is already a standard single constraint $t_{12}^2 \le t_1 t_2$. Otherwise, we can break it into a rotated cone constraint, plus two similar constraints of the same form but with an exponent that is half of the original exponent. For example, if $\hat{n} = 2$ then $t^4 \le t_1 t_2 t_3 t_4$ becomes $t^2 \le t_{12} t_{34}$, $t_{12}^2 \le t_1 t_2$ and $t_{34}^2 \le t_3 t_4$. The same transformation can be applied recursively until all the constraints are conic. If we have the inequality $t^k \le s^l$, with $k + h = 2^{\hat{n}}$, we can use the equivalent inequality $t^{k+h} \le s^l\, t^h\, 1^{k-l}$. See Fig. A.5 for a tree decomposition representation of the constraint $t^5 \le s^3$ into 7 hyperbolic constraints, where $\mathbf{T_{ab}}$ represents the hyperbolic constraint related with to geometric mean of variables from a to b in $s^l\, t^h\, 1^{k-l}$. Notice that the second-order cone representation is not unique.

Moreover, each hyperbolic constraint $t_{ab}^2 \le t_a t_b$ can be transformed into second-order cone constraints considering

$$\left\| \begin{pmatrix} 2\, t_{ab} \\ t_a - t_b \end{pmatrix} \right\| \le t_a + t_b, \tag{A.2}$$

as a consequence of rotating the second order cone Q through an angle of forty-five degrees in the t_a, t_b-plane. See (Alizadeh & Goldfarb, 2003) for more details regarding Second Order Cone Programming. Second Order Cone Programming has been applied in very different fields, but also in DEA; (Halická & Trnovská, 2019; Hasannasab, Margaritis, Roshdi, & Rouse, 2019) and (Sueyoshi & Sekitani, 2007) present SOCP formulations to solve the classical nonlinear Hyperbolic and Russell graph problems, respectively. Recently, Alcaraz, Anton-Sanchez, Aparicio, Monge, & Ramón (2021) have introduced an algorithm based on Second Order Cone Programming to solve the Russell graph problems, aggregating input and output specific inefficiencies through the geometric mean.

Let us now return to our equivalent GRAM problem (9). Model (A.1) can be expressed as a Second Order Cone Programming problem as

$$\begin{array}{lll} \max_{\mathbf{s},\mathbf{t},\boldsymbol{\lambda}} & \left(\sum_{i=1}^{m} \frac{t_{i0}^{-}}{(m+s)R_i^p} + \sum_{r=1}^{s} \frac{t_{r0}^{+}}{(m+s)(R_r^p)} \right) & \\ & s.t. \quad \boldsymbol{T}_{\boldsymbol{i}ab} \succcurlyeq_Q \mathbf{0} & \forall (a,b) \in \text{Recursive Tree of} (t_{i0}^{-})^k \le (s_{i0}^{-})^l,\ \forall i \\ & s.t. \quad \boldsymbol{T}_{\boldsymbol{r}ab} \succcurlyeq_Q \mathbf{0} & \forall (a,b) \in \text{Recursive Tree of} (t_{r0}^{+})^k \le (s_{r0}^{+})^l,\ \forall r \\ & \text{constraints of (4).} & \end{array} \tag{A.3}$$

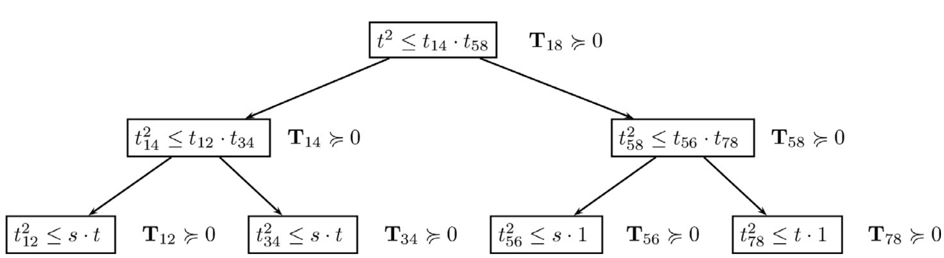

Fig. A1. Decomposition Tree of $t^5 \le s^3$ into hyperbolic constraints.

There are different types of software in the literature to solve Second-Order Cone problems, such as CPLEX, Gurobi, MOSEL, BARON, LINGO, among others. SOCP problems are a subclass of convex problems and can be solved using any algorithm for convex optimization, such as interior-point methods (Nesterov & Nemirovski, 1994) or first-order methods (Boyd, Parikh, Chu, Peleato, & Eckstein, 2011).

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