

Article

Cooperative Purchasing with General Discount: A Game Theoretical Approach

Jose A. García-Martínez ^{1,*} , Ana Meca ²  and G. Alexander Vergara ^{3,4} ¹ Departamento de. Estudios Económicos y Financieros, Universidad Miguel Hernández de Elche, 03202 Elche, Spain² I. U. Centro de Investigación Operativa, Universidad Miguel Hernández de Elche, 03202 Elche, Spain³ Interuniversity Doctorate in Economics (DEClIDE), I. U. Centro de Investigación Operativa, Universidad Miguel Hernández de Elche, 03202 Elche, Spain⁴ Facultad de Ciencias Económicas, Universidad de San Buenaventura Cali, Cali 760031, Colombia

* Correspondence: jose.garciam@umh.es

Abstract: In some situations, sellers of certain commodities usually provide price discounts for large orders according to a decreasing unit price function. Buyers of such commodities can cooperate and form purchasing groups to benefit from these price discounts. A natural way to allocate the corresponding cost reductions is the equal price rule. We analyze this situation as a cooperative game. We show that when the decreasing unit price function is linear, the equal price rule coincides with the Shapley value and the nucleolus of the cooperative game. However, some buyers may argue that the equal price rule is not acceptable because it favors those who buy just a few units of the product. This can be more problematic when the decreasing unit price function is nonlinear: In that case, the equal price rule loses some of its good properties and it no longer matches the Shapley value or the nucleolus. Unlike the linear case, in this nonlinear case, the Shapley value and nucleolus do not assign the same price to all agents, so there are different price rules. However, they have a computability problem, as both are very laborious to calculate for a large number of agents. To find a suitable alternative, we first study the properties that a different price rule should have in this situation. Second, we propose a family of different price rules that hold those properties and are easy to calculate for a large number of agents. This family of different price rules provides buyers (companies, institutions, consumers, etc.) with an easy-to-implement method which ensures stability in cooperative purchasing.

Keywords: cooperative purchasing; price discounts; equal price rule; different price rules; game theory**MSC:** 91A12, 91A80

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1. Introduction

Since time immemorial, trade has offered buyers the opportunity to pay a lower price for the purchase of large quantities of a product. Today, consumers encounter all kinds of quantity discounts at every turn in both retail shops and online platforms. Beyond consumers, quantity discounts can be seen to permeate business-to-business transactions. Most companies receive a quantity discount on some of their purchases and extend a quantity discount to some of their customers. Mobile operators in Europe receive volume discounts from telecommunications operators based on the number of international calls made [1]. More generally, large retailers and manufacturers such as Lidl and Ikea demand discounts from their suppliers based on a large volume of products purchased. At the same time, many large manufacturers and retailers have programs in place to offer volume discounts to external companies, schools and non-profit organizations. One example of such a program is Apple's Volume Purchase Program, which offers customized volume discounts for purchases.

The omnipresence of quantity discounts for buyers highlights the importance of addressing questions such as how many units should be ordered when dealing with a quantity discount schedule? Should buyers join a group-purchasing organization to try to lower purchase prices? Since the 1950s, quantity discounting has been an important research topic, which has also appeared in economics and operations research literature. In operations, much of the core work for determining optimal order sizes for buyers and for using quantity discounts to coordinate lot sizes in supply chains was carried out by the end of the last century. Several literature review papers address much of this work in various ways. Since the turn of the century, a steady flow of papers has continued to appear. The research area remains vibrant, with quality publications appearing every year in the operations-management field alone. A comprehensive overview of quantity discounts by [2] covers many of the papers published in the last 25 years.

Here, we focus on the questions raised above and demonstrate that it is beneficial for buyers to join a large purchasing group to obtain lower purchase prices and, thus, a significant reduction in their costs. We assume that the buyers (retailers or producers) already know how many units of the product they are going to buy. They may be finished products or raw materials to produce a certain product. That is, they initially know how many units of a commodity they are going to order when they deal with a quantity-discount schedule. Cooperative purchasing initiatives such as purchasing groups, purchasing consortia, and buying offices are becoming increasingly popular due to advances in information technology and the development of online markets. Purchasing groups generate multiple benefits for its participants: buyers can obtain better prices by increasing their purchasing power and reduce costs by consolidating their operations.

In purchasing literature, cooperative purchasing is referred to using many terms. There are certain patterns in those terms, but the terminology is not yet fully stabilized [3]. We define cooperative purchasing as the sharing and/or bundling of purchasing-related information, processes, resources, and/or volumes by two or more organizations in a purchasing group in order to improve their performances [4]. A purchasing group consists of two or more dependent or independent organizations that purchase together, either formally or informally, or through a third party [5]. Cooperative purchasing is a type of cooperative arrangement, often among businesses, to agree to add up demand so as to obtain lower prices from selected suppliers. Retailers' cooperatives are a form of cooperative purchasing. Cooperatives are often used by government agencies to reduce procurement costs [6] and they are also gaining popularity in the private sector [5].

Research on cooperative purchasing has received relatively little attention in the field of operations research. It has so far focused mainly on inductive explanations of practices and deductive qualitative reasoning. There has been little use of game-theory reasoning, to date. One specific issue which has received particularly little research attention is the allocation of costs resulting from purchasing price savings achieved through cooperative purchasing using the so-called equal price (EP) allocation method. This EP method is commonly used, and is defined as all agents paying the same price per item [7]. EP is practically and intuitively appealing, but it may lead to unfair outcomes under certain circumstances. This has been reported previously by [7] and analyzed systematically by [4]. The latter focuses specifically on allocating the total gains resulting from cooperation and formally analyzes how and under what conditions unfairness arises when EP is used. These two issues are important to all types of purchasing groups as they all have to make decisions on how to allocate their gains. They provide an analytical analysis of unfair outcomes of EP, provide recommendations for purchasing groups as to how to deal with them, and contribute to increased awareness and understanding of EP-related problems.

In this paper, we study situations in which a seller of a certain commodity provides price discounts for large orders according to a decreasing unit-price function. Buyers of this commodity can cooperate and form purchasing groups to benefit from these price discounts. It is provided in [8] an analytical and empirical basis for a general quantity-discount function (QDF) which can be used to describe the underlying function of almost

all types of quantity discount. They show that this QDF fits very well with 66 discount schedules found in practice. It is proposed in [8] a QDF with an explicit formula depending on certain parameters, but we propose a general price function which measures the quantity discount buyers encounter when cooperating in purchasing and satisfies properties such as continuity, decreasingness, convexity, and limited growth rate.

On the other hand, as mentioned, it is focused in [4] on allocating the total gains from cooperation by means of a benefit game, and analyze the unfairness resulting from using the commonly used EP method for allocating such gains. They demonstrate that this unfairness is caused by neglecting a particular component of the added value of individual group members. They discuss measures that a purchasing group might consider to mitigate the perception of unfairness, but they do not study in depth the class of cooperative games that they have at hand or propose an alternative to the EP rule. Unlike [4], we analyze these cooperative purchasing situations as cooperative cost games and study them comprehensively. We go beyond the EP rule and other well-known but difficult-to-calculate distributions such as the Shapley value or the compromise value [9]. In particular, we show that when the decreasing unit price function is linear, the EP rule coincides with the Shapley value and the nucleolus. However, some buyers may argue that the EP rule is not acceptable because it favors those who buy just a few units of the product. This can be more problematic when the decreasing unit-price function is nonlinear: In that case, EP loses some of its good properties and no longer matches the Shapley value or the nucleolus. By contrast to the linear case, in this nonlinear case, the Shapley value and nucleolus do not assign the same price to all agents, so they are different price (DP) rules. However, they have a computability problem in that both are very laborious to calculate for a large number of agents. To find an adequate alternative, we first study the properties that a DP rule should have in this situation. Second, we propose a family of DP rules that hold those properties and are easy to calculate for a large number of agents.

Our paper, thus, contributes to the literature on cooperative purchasing models in the following way: First, we extend the study of such models with general discount functions and introduce a new class of cooperative-purchasing cost games with general discounts. Second, we comprehensively analyze cooperative-purchasing cost games with linear discounts and show that the EP rule coincides with the Shapley value and the nucleolus. This equality does not hold for cooperative-purchasing cost games with nonlinear discounts, so we then study such cost games with nonlinear discounts and propose a family of DP rules that are acceptable to all agents and easier to compute than the Shapley value and the nucleolus. They are called α -proportional rules. To make our family of α -proportional rules acceptable to all agents, we distinguish between major agents (who buy large quantities) and non-major agents (who buy small quantities). To the best of our knowledge, there is no formal definition of such agents in the cooperative-purchasing-games literature. The beauty of our α -proportional rules is that, with the proportionality factor α , they reduce the cost of major agents and increase the cost of non-major agents. Fortunately, there is always an α threshold above which any α -proportional rule is acceptable for all agents.

The paper is organized as follows. We begin with a Related Literature section, which describes the literature most closely related to our paper. Next, in Section 3, we develop a formal model of cooperative-purchasing cost with a general discount (CPGD model) and prove that all the buyers included (grand coalition) can obtain significant reductions in costs. The equal price (EP) rule turns out to be an efficient and (coalitionally) stable method for allocating the reduced costs generated by the CPGD model. Then, Section 4 looks at cooperative-purchasing models with decreasing and linear unit-price functions. We demonstrate that the linear nature of the discount price function provides additional information about the corresponding cooperative-purchasing game with linear discount (CPL-game): The marginal contribution of an agent diminishes as a coalition grows. Moreover, the EP rule matches the Shapley value and the nucleolus. In Section 5, we propose a family of allocation rules for cooperative-purchasing games with non-linear discount (CPNL-games). We focus on the different price (DP) method and propose a family of

allocation rules with different prices that are acceptable for all agents: BDP rules. Section 6 focuses on an alternative approach to obtain DP rules for CPNL-games. This consists of allocating the cost of the grand coalition proportionally, with a proportionality factor which combines individual costs and the EP rule. We obtain a highly suitable parametric family of proportional rules, named α -proportional rules, which, notably, is related to the family of BDP rules. Specifically, Section 7 proves that there is always an α threshold above which any α -proportional rule is a BDP rule. Finally, we illustrate our model with a couple of examples in Section 8. Finally, Section 9 draws conclusions and points out further research for scholars in the field.

2. Related Literature

As mentioned above, the use of game theory to study cooperative-purchasing models has so far been limited. However, there are works that have approached the subject from various perspectives. Here, we describe the literature from the past 15 years most closely related to our paper.

It is discussed in [10] the problem that arises when a small buying organization uses a contract negotiated by a large buying organization. They show that a relatively small organization would benefit from joining a specific purchasing group, but the inclusion of such an organization might decrease the profits of the bigger organizations in this exchange. In [11], it is noted that it is important to avoid the kind of imbalance of incentives for and contributions by organizations in a purchasing group that can be caused by EP. Finally, reasoning from an equity-theory perspective [12], it can be observed that individuals who feel under-rewarded will try to restore equity. As in purchasing groups, EP may lead to under-rewarded organizations in a group. This may lead to lower commitment on the part of those organizations or result in them leaving the group [13].

In [14], it is considered a distribution system consisting of a set of retailers who face a single-period price-dependent demand for a single product. By taking advantage of the risk-pooling effect and the quantity/volume discount provided by suppliers or third-party carriers, the retailers may place joint orders and keep inventory at central warehouses before demand realization, and allocate inventory among themselves after demand realization to reduce their operating costs. Under certain assumptions, the author shows that there is a stable allocation of profits among the retailers and also shows how to compute it.

In [15], it is introduced a new class of cooperative-purchasing situations: maximum cooperative purchasing (MCP) situations. The allocation of possible cost savings in MCP situations, in which the unit price depends on the quantity of the largest order within a group of players, is analyzed by defining corresponding cooperative MCP-games. The authors show that a decreasing unit price is a sufficient condition for a non-empty core: There is a set of marginal vectors that belong to the core. The nucleolus of an MCP-game can be derived in polynomial time from one of these marginal vectors. Using the decomposition of an MCP-game into unanimity games, they also find an explicit expression for the Shapley value.

It is studied in [16] mechanisms for managing group purchasing by a set of buyers of a given product with a concave purchase-cost function. Cost-sensitive buyers are willing to buy a range of product quantities at different prices. They investigate two types of mechanism that can be used by a group-purchasing organization: ordering and bidding mechanisms. Under the choice of appropriate cost-sharing rules, they introduce a sequential joint-ordering mechanism and a family of ordering strategies under which some buyers' strategic deviations never leave other buyers worse off.

Inventory cost games with discounts are a particular type of cooperative-purchasing model. In an inventory cost game [17] a group of firms dealing with the ordering of a certain commodity decide to cooperate and place their orders jointly. To coordinate the ordering policy of the firms, some revelation of information is needed: the amount of information revealed by each firm to the rest is kept as low as possible, since they may be competitors in the consumer market. In [17], it is focused on proportional division mechanisms for sharing

the joint cost, and introduce and characterize the SOC rule (share the ordering costs). Later, it is analyzed in [18] the class of inventory games that arises from inventory problems with special sale prices. A group of firms trying to minimize their joint inventory costs by cooperating may receive a special discount on set-up cost just by ordering. Reasons for such price reductions range from competitive price wars to attempted inventory reduction by the supplier. This cooperative situation generates the class of inventory games with non-discriminatory temporary discounts. The modified SOC rule, a kind of proportional rule, is proposed as a stable (core-allocation), consistent allocation. More recently, in [19], it is extended inventory-cost games to the situation where the manufacturer provides the retailers with a price discount on purchases in excess of a certain order quantity. The authors define the corresponding inventory game with quantity discount, and show that there is a stable allocation of the total cost, which they call the demand-proportionality rule and which they characterize. At the same time, it is considered in [20] an inventory-cost game involving a single supplier that offers quantity discounts and allows retailers to delay payments. The retailers are tempted to form coalitions in order to minimize their costs. They propose a solution approach which generates stable coalition structures for the retailers taking into account the delay in payments and the amount of the discount offered by the supplier. The approach proposed includes a decision rule that generates preferred coalitions for each retailer and considerably reduces the number of coalition structures explored in order to determine stable solutions.

In [21], it is proposed and studied the family of α -serial cost-sharing rules for cost-sharing problems. Each rule in this family is a parametric combination of the serial cost-sharing rule [22] and the dual serial cost-sharing rule [23]. The parameter α determines how this combination is obtained. The α -serial cost-sharing rule allocates the total production cost, in a cost-sharing problem, in such a way that agents with low demands have to pay cost increments associated with low outputs and cost increments associated with high outputs ($0 < \alpha < 1$). If only one type of cost increase is taken into account, e.g., agents with low demands only have to pay cost increments associated with low outputs, we obtain the serial cost-sharing rule ($\alpha = 1$). On the contrary, if agents with low demands only have to pay cost increments associated with high outputs, we obtain the dual serial cost-sharing rule ($\alpha = 0$). Albizuri's approach and the context are different from ours in this paper. While her α -serial cost-sharing rule is proposed for general cost-sharing problems, our alpha-proportional rule is a very appropriate allocation rule for cooperative-purchasing models with a general discount. It distributes the cost of the grand coalition proportionally, with a proportional factor that combines the maximum cost that each agent has to pay individually (its own individual cost) with the minimum cost that can be achieved through cooperation (the EP rule). The α -proportional rule is natural and intuitive and much easier to calculate than the α -serial cost-sharing rule.

Finally, the book [24] shows that the Shapley value is highly valued by many researchers as a useful and relevant model to analyze, both from a theoretical and applied perspective, allocation problems in the most general sense. It is structured in three parts. They first present some of its very well-known mathematical expressions, starting with those introduced by Lloyd Shapley in 1953. Secondly, they present some of its most important characterizations as an indication of the large number of appealing and interesting properties that this value satisfies. Finally, they select a sample of the Shapley-value extensions to a large number of contexts and their applications to very different fields and sceneries. For a recent survey of Shapley value, nucleolus and other solution concepts in operation management, see [25].

3. Model

We consider a finite set of agents $N = \{1, 2, \dots, n\}$, who want to buy a certain service or good. Each agent $i \in N$ wants to buy a quantity $q_i > 0$ units of the product at a cost $P(q_i)q_i$, where $P(q_i) > 0$ represents variable costs with discount per unit, i.e., the price that agent i pays for quantity q_i . It is, however, independent of player i . Throughout the paper and with no loss of generality, we rank agents according to how much they buy. In other

words, we assume that $q_1 \leq q_2 \leq \dots \leq q_n$. We consider a general discount price function $P : (0, +\infty) \rightarrow \mathbb{R}_+$ with the following properties:

Properties of P
1. Class C^2 at $(0, +\infty)$: there exists $P''(q)$ at all points of $(0, +\infty)$ and it is continuous.
2. Decreasing: for all $q > 0$, $P'(q) < 0$.
3. Convex: for all $q > 0$, $P''(q) \geq 0$.
4. Limited growth rate: for all $q > 0$, $ P'(q) \leq \frac{P(q)}{q}$

Notice that property 2 means that agents who buy large quantities will obtain greater discounts—per unit of product—than agents that buy small quantities. Property 3 means that the biggest discounts occur at the start of the deal. The last property indicates that the cost $P(q)q$ is increasing in q , as the opposite makes no economic sense. This means that the average cost per unit is greater than the marginal cost (this property comes from the fact that $x \geq y \implies P(x)x \geq P(y)y$ is equivalent to $P'(x) \geq -\frac{P(x)}{x}$).

We refer to our model as a cooperative-purchasing model with general discount (N, q, P) (henceforth, CPGD-model), where N is the total number of agents in a purchasing group (i.e., the grand coalition); q is the vector of quantities that each agent $i \in N$ wants to buy, i.e., $q = (q_i)_{i \in N}$; and P is a discount function satisfying properties 1–4.

Given a CPGD-model (N, q, P) , we define the corresponding cooperative purchasing cost game with general discount (N, c) . For any coalition of agents $S \subseteq N$, the cost function $c(S)$ is defined as the total cost that the coalition has to pay on buying the product together: $c(S) = P(q_S)q_S$, with $q_S = \sum_{i \in S} q_i$. From now on, we refer to this as a CPGD-game.

The first question that comes to mind is whether it is profitable for the agents in N to form the grand coalition to obtain a significant reduction in costs. The answer is yes because CPGD-games are always subadditive. A cost game (N, c) is said to be subadditive if $S \cap T = \emptyset$, so $c(S \cup T) \leq c(S) + c(T)$, for all $S, T \subseteq N$. The subadditivity property reveals that the cost of the grand coalition is always less than the sum of the costs of any partition of N in two coalitions S and $N \setminus S$; that is, $c(N) \leq c(S) + c(N \setminus S)$ for all $S \subseteq N$. Consequently, agents have incentives to form the grand coalition in CPGD situations.

The following proposition shows this property for CPGD-games.

Proposition 1. *Every CPGD-game (N, c) is subadditive.*

Proof. Take $S, T \subseteq N$ s.t. $S \cap T = \emptyset$. Then

$$c(S \cup T) = P(q_{S \cup T})q_{S \cup T} = P(q_{S \cup T})q_S + P(q_{S \cup T})q_T \leq P(q_S)q_S + P(q_T)q_T = c(S) + c(T),$$

considering that $P(q_{S \cup T}) \leq P(q_S)$, and $P(q_{S \cup T}) \leq P(q_T)$. $\square$

We have, thus, proved that the grand coalition can obtain significant reductions in costs. In that case, the reduced total cost is given by $c(N) = \sum_{i \in N} P(q_N)q_i$, where $P(q_N)$ is the minimum price with discount per unit that coalition N can obtain.

The second question is whether a method can be found for allocating the costs generated by the CPGD model that is efficient, coalitionally stable and easy to compute. The answer is again yes, but it is not as straightforward as the previous answer. More elaborate work is required, as set out in the following sections.

We start by defining an allocation rule for CPGD-games. This is a map ψ which assigns a vector $\psi(c) \in \mathbb{R}^n$ to every (N, c) , satisfying efficiency, that is, $\sum_{i \in N} \psi_i(c) = c(N)$. Each component $\psi_i(c)$ indicates the cost allocated to $i \in N$, so an allocation rule for CPGD-

games is a method for allocating the reduced total cost among the agents in N when they cooperate.

A very natural, commonly used method is the equal price (EP) rule. Given a CPGD-game (N, c) , the EP rule is given by $\epsilon(c) = (\epsilon_i(c))_{i \in N}$ with $\epsilon_i(c) = P(q_N)q_i$. Each agent $i \in N$ obtains the quantity q_i at the minimum cost $P(q_N)$, and pays $P(q_N)q_i$. The EP rule has good properties for CPGD-games, at least with respect to computability (it is easily computable) and coalitional stability, in the sense of the core. The core of a cost game (N, c) is defined as follows:

$$C(c) := \left\{ x \in \mathbf{R}^n / \sum_{i \in N} x_i = c(N), \sum_{i \in S} x_i \leq c(S), \forall S \subseteq N \right\}.$$

Coalitional stable allocations in the core sense are called core-allocations. A game (N, c) is balanced if and only if $C(c) \neq \emptyset$. We interpret a non-empty core for cost games as indicating a setting where all included cooperation is feasible, in the sense that there are possible cost reductions that leave all agents better off (or, at least, not worse off).

Proposition 2. *Every CPGD-game (N, c) is balanced.*

Proof. The idea is to prove that $\epsilon(c)$ is a core allocation. That is, $\sum_{i \in S} \epsilon_i(c) \leq c(S)$ for every $S \subseteq N$.

Take a coalition $S \subseteq N$. Thus,

$$\sum_{i \in S} \epsilon_i(c) = \sum_{i \in S} P(q_N)q_i \leq \sum_{i \in S} P(q_S)q_i = c(S).$$

Hence, $C(c) \neq \emptyset$ and (N, c) is balanced. $\square$

As shown in Section 4, when P is linear, the corresponding cooperative-purchasing game with linear discount (henceforth, CPL-game) is concave and the EP rule matches the Shapley value and the nucleolus. In such cases, there is no better way to allocate the reduced total cost. However, some agents may argue that the EP is questionable because it favors those agents who buy just few units of the product. Note that these agents pay the same price as the major buyers. The concerns of these agents can be really problematic when P is not linear. In that case, the equal price rule loses some of its good properties and no longer matches the Shapley value or the nucleolus.

In addition, the EP rule takes only two elements into account for a particular agent $i \in N$: the quantity demanded by this agent (q_i) and the aggregate of all quantities ($q_N = \sum_{i \in N} q_i$). It does not take into account the distribution of the individual quantities demanded by agents, i.e., q_j for all $j \in N \setminus \{i\}$. This can also be a problem when there are large asymmetries between large and small buyers.

To solve this problem, Section 5 proposes a family of allocation rules for cooperative-purchasing games with non-linear discount (henceforth, CPNL-games). We focus there on the different price (DP) method and propose a family of allocation rules with different prices that are acceptable for all agents.

4. Equal Price Rule for CPL-Games

We begin by studying CPL-games. The linear nature of the discount price function provides additional information about these games: The marginal contribution of an agent diminishes as a coalition grows. This is well-known as the snowball effect or concavity property. Cooperative game theory provides allocation rules for concave games with good properties (coalitional stability and acceptability). We highlight the Shapley value, first introduced in [26], and the nucleolus, presented in [27].

Here we prove that CPL-games with linear discounts are always concave and the EP rule matches the Shapley value and the nucleolus.

Let (N, c) be a CPL-game with P being a linear discount function. That is, $P : \mathbb{R}_{++} \rightarrow \mathbb{R}_+$, such that, for all $q \in \mathbb{R}_+$, $P(q) := a - bq$, with $b \geq 0$ and a large enough that $a - bq \geq 0$, for all $0 < q \leq \frac{a}{b}$. Thus, for all $S \subseteq N$, $c(S) = aq_S - bq_S^2$.

The next proposition shows that a CPL-game is always concave, in the sense that for all $i \in N$ and all $S, T \subseteq N$ such that $S \subseteq T \subset N$ with $i \in S$, $c(S) - c(S \setminus \{i\}) \geq c(T) - c(T \setminus \{i\})$.

Proposition 3. *Every CPL-game is concave.*

Proof. We first show that an agent's marginal contribution to any coalition is always smaller than its individual cost. We denote by $M_i c(S)$ the marginal contribution of player $i \in S \subseteq N$; that is, $M_i c(S) = c(S) - c(S \setminus \{i\})$, for all $i \in S \subseteq N$. Thus,

$$\begin{aligned} M_i c(S) &= aq_S - bq_S^2 - aq_{S \setminus \{i\}} + bq_{S \setminus \{i\}}^2 = aq_i - b(q_S^2 - q_{S \setminus \{i\}}^2) \\ &= aq_i - b(q_S + q_{S \setminus \{i\}})(q_S - q_{S \setminus \{i\}}) = aq_i - bq_i^2 - 2bq_i q_{S \setminus \{i\}} \\ &= c(\{i\}) - 2bq_i q_{S \setminus \{i\}}. \end{aligned} \quad (1)$$

Finally, take $i \in N$ and $S \subseteq T \subset N$ with $i \in S$; thus

$$M_i c(S) = c(i) - 2bq_i q_{S \setminus \{i\}} \geq c(i) - 2bq_i q_{T \setminus \{i\}} = M_i c(T),$$

where $q_{S \setminus \{i\}} \leq q_{T \setminus \{i\}}$. $\square$

Now we focus on the Shapley value and the nucleolus. The Shapley value assigns a unique allocation (among the agents) of the total surplus generated by the grand coalition. It measures how important each agent is to overall cooperation, and what cost it can reasonably expect. The Shapley value of a concave game is the center of gravity of its core (see [28]). This allocation is, in general, hard to compute when the number of agents is large. Given a CPL-game (N, c) , we denote by $\phi(c)$ the Shapley value, where for each agent $i \in N$, the corresponding cost allocation is

$$\phi_i(c) = \sum_{i \in T \subseteq N} \frac{(n-t)!(t-1)!}{n!} [c(T) - c(T \setminus \{i\})], \text{ with } |T| = t. \quad (2)$$

The nucleolus maximizes the “welfare” of the worst treated coalitions, i.e. those with the smallest excess. We denote by $\nu(c)$ the nucleolus of the CPL-game (N, c) . First, we define the excess of coalition S in (N, c) with respect to allocation $x \in \mathbb{R}^n$ as $e(S, x) = c(S) - \sum_{i \in S} x_i$. This number can be considered as an index of the “welfare” of coalition S at x : The greater $e(S, x)$, the better coalition S is at x . Let $e^*(q)$ be the vector of the 2^n excesses arranged in (weakly) increasing order, i.e., $e_i^*(x) \leq e_j^*(x)$ for all $i < j$. Second, we define the lexicographical order $\succ_l$. For any $x, y \in \mathbb{R}^n$, $x \succ_l y$ if and only if there exists an index k such that for any $i < k$, $x_i = y_i$ and $x_k > y_k$. The nucleolus of the CPL-game (N, c) is the set

$$\nu(c) = \{x \in X : e^*(x) \succ_l e^*(y) \text{ for all } y \in X\}$$

with $X = \{x \in \mathbb{R}^n : \sum_{i \in N} x_i = c(N), x_i \geq c(\{i\}) \text{ for all } i \in N\}$.

It is well-known that the nucleolus is a singleton for balanced games and that it is always a core allocation.

This last Proposition shows that the EP coincides with the Shapley value and the nucleolus for CPL-games.

Proposition 4. *Let (N, c) be a CPL-game. Thus, $\epsilon(c) = \phi(c) = \nu(c)$.*

Proof. To prove this, we first need to describe the class of PS-games introduced by [29]. Thus, a cost game (N, c) satisfies the PS property if for all $i \in N$, there exists $k_i \in \mathbb{R}$ such

that $M_i c(S \cup \{i\}) + M_i c(N \setminus S) = k_i$, for all $i \in N$ and $S \subseteq N \setminus \{i\}$. In [29], it is shown that for PS games, the Shapley value coincides with the nucleolus; that is, $\phi_i(c) = v_i(c) = \frac{k_i}{2}$, for all $i \in N$.

Take a CPL-game (N, c) , and take $i \in N$. First we prove that it is a PS-game with $k_i = 2\epsilon_i(c)$. Take $S \subseteq N \setminus \{i\}$. By (1), it holds that

$$\begin{aligned} M_i c(S \cup \{i\}) + M_i c(N \setminus S) &= c(\{i\}) - 2bq_i q_S + c(\{i\}) - 2bq_i q_{(N \setminus S) \setminus \{i\}} \\ &= 2c(\{i\}) - 2bq_i q_S - 2bq_i (q_{(N \setminus S)} - q_i) \\ &= 2aq_i - 2bq_i q_S - 2bq_{(N \setminus S)} = 2aq_i - 2bq_i q_N \\ &= 2aq_i P(q_N) = 2\epsilon_i(c). \end{aligned}$$

Second, we know that $\phi_i(c) = v_i(c) = \frac{k_i}{2}$, and thus $\phi_i(c) = v_i(c) = \epsilon_i(c)$, for all $i \in N$. $\square$

It can be concluded that, for CPL-games, the EP rule is an appropriate way to allocate the reduced total cost.

5. Balanced Different Price Rule for CPNL-Games

This section studies CPNL-games, where P function is not linear. We focus on DP rules, where a DP rule is defined as $DP(c) = (DP_i(c))_{i \in N}$ with $\frac{DP_i(c)}{q_i} \neq \frac{DP_j(c)}{q_j}$ for all $q_i \neq q_j$ with $i, j \in N$. Note that $\frac{DP_i(c)}{q_i}$ is the price that agent i will pay per unit under this DP rule. We analyze the properties that a different price allocation rule (DP rule) should have in order to be acceptable to all agents.

As mentioned above, the agents could argue that the equal price rule is not acceptable because it favors those agents who buy just few units of the product. This becomes more problematic when the discount function is non-linear. In that case, the equal price rule loses some of its good properties. Unlike the linear case, other acceptable rules, such as the Shapley value and the nucleolus, do not assign the same price to all agents, i.e., they are DP rules but they have a computability problem in that both are very laborious to calculate for a large number of agents.

Therefore, those who buy large quantities of the product (major agents) may not accept the EP rule and they would prefer to pay a lower price than the EP. An easily computable DP rule should be proposed in which the price depends on the quantity demanded (with this new rule, the benefits of cooperation are not distributed as uniformly as in the equal price rule). First, we need to define “major agents”. We propose the following approach: ranking agents according to how much they buy ($q_1 \leq q_2 \leq \dots \leq q_n$), the major agents are all $i \in N$ such that $\sum_{j \in N} P(q_i)q_j < \sum_{j \in N} P(q_j)q_j$, i.e., if all agents pay the individual price of agent i , the total cost is smaller than if agents pay their individual costs. Thus, the set of agents can be split into two subsets: the major agents

$$A_m = \{i \in N, \sum_{j \in N} P(q_i)q_j < \sum_{j \in N} P(q_j)q_j\} \quad (3)$$

and the non-major agents

$$A_{nm} = \{i \in N, \sum_{j \in N} P(q_i)q_j \geq \sum_{j \in N} P(q_j)q_j\}. \quad (4)$$

The DP rule should assign a price lower than the EP to the major agents, and one higher than or equal to the EP to the non-major agents. Therefore, each agent $i \in N$ should pay the price $\sigma(q_i, q_{-i})P(q_N)$ for the profile of quantities $q = (q_1, q_2, \dots, q_i, \dots, q_n) \in \mathbb{R}_+^n$ where $\sigma : [0, +\infty) \times [0, +\infty)^{n-1} \rightarrow \mathbb{R}_{++}$ is a function which determines the different prices. Thus, the price $\sigma(q_i, q_{-i})P(q_N)$ not only depends of the quantity demanded by agent i , but

possibly also on the quantities demanded by other agents and their distribution. This is another significant difference with regard to the equal price rule.

We consider that if for agents $i, j \in N$ $q_i \leq q_j$, then the function takes a smaller/equal value, i.e., $\sigma(q_i, q_{-i}) \geq \sigma(q_j, q_{-j})$; thus, the different price is greater/equal for agent i . We also consider that this property of σ is limited. This means that the larger one's purchase is, the higher one's cost; i.e., if $q_i \leq q_j$ for $i, j \in N$, then $\sigma(q_i, q_{-i})P(q_N)q_i \leq \sigma(q_j, q_{-j})P(q_N)q_j$ which is equivalent to $\sigma(q_i, q_{-i})q_i \leq \sigma(q_j, q_{-j})q_j$. Both are quite reasonable assumptions.

We also assume that the major agents will pay a price lower than $P(q_N)$ and the non-major agents a price of $P(q_N)$ or higher. We name this property major agent acceptability (henceforth, MA). Formally, for all $i \in A_m$, $\sigma(q_i, q_{-i}) < 1$ and for all $i \in A_{nm}$, $\sigma(q_i, q_{-i}) \geq 1$.

To make the allocation rule also acceptable to the non-major agents, we assume an upper bound for the price that they have to pay. Thus, the price payable by any $i \in A_{nm}$ is assumed to be greater than or equal to the price that all non-major agents can obtain together, i.e., $\sigma(q_i, q_{-i})P(q_N) \leq P(q_{A_{nm}})$ with $q_{A_{nm}} = \sum_{i \in A_{nm}} q_i$, where this condition is equivalent to $\sigma(q_1, q_{-1})P(q_N) \leq P(q_{A_{nm}})$ because $\sigma(q_1, q_{-1}) \geq \sigma(q_j, q_{-j})$ for any $j \in N \setminus \{1\}$. This means that the σ function sets an upper bound for the different price of the non-major agents: $\sigma(q_1, q_{-1}) \leq \frac{P(q_{A_{nm}})}{P(q_N)}$. We call this property non-major agent acceptability (henceforth, NMA).

Another desirable property for the DP rule is efficiency, that is, $\sum_{i \in N} \sigma(q_i, q_{-i})P(q_N)q_i = \sum_{i \in N} P(q_N)q_i$. This is equivalent to $\sum_{i \in N} \sigma(q_i, q_{-i})q_i = \sum_{i \in N} q_i \Leftrightarrow \sum_{i \in N} (\sigma(q_i, q_{-i}) - 1)q_i = 0 \Leftrightarrow \sum_{i \in A_{nm}} (\sigma(q_i, q_{-i}) - 1)q_i = \sum_{i \in A_m} (1 - \sigma(q_i, q_{-i}))q_i$ (Note that $N = A_m \cup A_{nm}$, $\sigma(q_i, q_{-i}) < 1$ if $i \in A_m$, and $\sigma(q_i, q_{-i}) \geq 1$ if $i \in A_{nm}$). Thus, a different price rule is efficient if and only if $\sum_{i \in A_m} (1 - \sigma(q_i, q_{-i}))q_i = \sum_{i \in A_{nm}} (\sigma(q_i, q_{-i}) - 1)q_i$. This means that the function σ weighs the quantities of the major and non-major agents in such a way that the sets A_m and A_{nm} are balanced. We call this property balanced weighting (henceforth, BW).

Finally, we consider that there is a limit to how much the function σ can decrease if the quantity demanded by one agent $i \in N$ increases. This limit is set by the ratio $\frac{P(q'_N)q'_i}{P(q_N)q_i}$ in the following way. Let $q = (q_i, q_{-i})$ and $q' = (q'_i, q_{-i})$, if $q_i \geq q'_i$, then, $\sigma(q_i, q_{-i}) \geq \frac{P(q'_N)q'_i}{P(q_N)q_i} \sigma(q'_i, q_{-i})$. First, note that the ratio $\frac{P(q'_N)q'_i}{P(q_N)q_i} < 1$ (in (11)), it is shown that the equal price $\epsilon_i(c) = P(q_N)q_i$ is increasing in q_i . Therefore, $\frac{P(q'_N)q'_i}{P(q_N)q_i} < 1$. Thus, $\sigma(q_i, q_{-i})$ has to be at least greater than a portion of $\sigma(q'_i, q_{-i})$, that is $\frac{P(q'_N)q'_i}{P(q_N)q_i} \sigma(q'_i, q_{-i})$.

We thus define a balanced different price rule (BDP rule) as $\beta(c) = (\beta_i(c))_{i \in N}$ with $\beta_i(c) = \sigma(q_i, q_{-i})P(q_N)q_i$ for all $i \in N$, and the function $\sigma : [0, +\infty) \times [0, +\infty)^{n-1} \rightarrow \mathbb{R}_{++}$ satisfying the following properties:

1. **Monotonically decreasing through players (MDP).** Given a $q \in \mathbb{R}_+^n$, if $q_i \leq q_j$, then $\sigma(q_i, q_{-i}) \geq \sigma(q_j, q_{-j})$ for all $i, j \in N$.
2. **Limited decrease through players (LDP).** Given a $q \in \mathbb{R}_+^n$, if $q_i \leq q_j$, then $\sigma(q_i, q_{-i})q_i \leq \sigma(q_j, q_{-j})q_j$ for all $i, j \in N$.
3. **Major-agents acceptability (MA).** For all $i \in A_m$, $\sigma(q_i, q_{-i}) < 1$, and for all $i \in A_{nm}$, $\sigma(q_i, q_{-i}) \geq 1$.
4. **Balanced weighting (BW).** $\sum_{i \in A_m} (1 - \sigma(q_i, q_{-i}))q_i = \sum_{i \in A_{nm}} (\sigma(q_i, q_{-i}) - 1)q_i$.
5. **Non-major agents acceptability (NMA).** $\sigma(q_1, q_{-1}) \leq \frac{P(q_{A_{nm}})}{P(q_N)}$.
6. **Limited decrease in a player quantity (LDQ).** Let (N, q, P) and (N, q', P) be two CPGD-models with $q = (q_i, q_{-i})$ and $q' = (q'_i, q_{-i})$. If $q_i \geq q'_i$, then $\sigma(q_i, q_{-i}) \geq \frac{P(q'_N)q'_i}{P(q_N)q_i} \sigma(q'_i, q_{-i})$.

Note that a BDP rule always satisfies the following desirable properties:

1. **Symmetry (SYM).** If two agents i and j in a group are interchanged in the sense that $c(S \cup \{i\}) = c(S \cup \{j\})$ for every $S \subset N \setminus \{i, j\}$, then $\beta_i(c) = \beta_j(c)$. It means that equal

agents in a group should pay equal costs. Indeed, $c(S \cup \{i\}) = c(S \cup \{j\}) \Leftrightarrow q_i = q_j$. Thus, $\sigma(q_i, q_{-i})P(q_N)q_i = \sigma(q_j, q_{-j})P(q_N)q_j$.

2. **Player monotonicity (PMON).** For all $i, j \in N$ s.t. $q_i \leq q_j$, it holds that $\beta_i(c) \leq \beta_j(c)$. This holds by property 2 of function σ .
3. **Cost monotonicity (CMON).** For all $i \in N$ s.t. $q_i \geq q'_i$, it holds that $\beta_i(c) \geq \beta_i(c')$, with $(N, c), (N, c')$ being the CPGD-games corresponding to CPGD-models (N, q, P) and (N, q', P) where $q = (q_{-i}, q_i)$ and $q' = (q_{-i}, q'_i)$. Satisfying this property means that if the number of units of the product to be purchased by one agent in a purchasing group remains the same or increases in comparison to a previous situation, then that agent should pay an equal or higher cost. This holds by property 6 of function σ .
4. **Fair ranking added cost (FRAC).** If for two agents i and j in a group $c(N) - c(N \setminus \{i\}) \geq c(N) - c(N \setminus \{j\})$, then $\beta_i(c) \geq \beta_j(c)$. Satisfying this FRAC property means that an agent with an equal or larger added cost (this is also called marginal costs) should pay an equal or larger cost. Indeed, $c(N) - c(N \setminus \{i\}) \geq c(N) - c(N \setminus \{j\}) \Leftrightarrow P(q_{N \setminus \{i\}})q_{N \setminus \{i\}} \leq P(q_{N \setminus \{j\}})q_{N \setminus \{j\}}$, and by property 4 (limited growth rate) of function P , $q_{N \setminus \{i\}} \leq q_{N \setminus \{j\}} \Leftrightarrow q_i \geq q_j$. Thus, by property 2 of function σ , FRAC holds.

The Dummy player property (DUM) is meaningless in this context. For a dummy player to exist, the function P must be constant, and in that case, all players are dummies.

The following proposition shows that the BDP rule is always efficient and coalitionally stable. In cooperative game theory, efficiency and coalition stability is equivalent to being a core allocation. This means that no coalition has incentives to break the grand coalition to obtain a lower cost.

Theorem 1. *Every BDP rule for CPNL-games is efficient and coalitionally stable.*

Proof. Let (N, c) be a CPNL-game. Take a BDP rule $\beta(c) = (\sigma(q_i, q_{-i})P(q_N)q_i)_{i \in N}$ with the function σ satisfying 1–6. First, the BDP rule is efficient if $\sum_{i \in N} \sigma(q_i, q_{-i})P(q_N)q_i = c(N)$, which, as mentioned above, is equivalent to property 4 (BW) of the function σ . Second, for it to be coalitionally stable it must be proved that $\sum_{i \in S} \sigma(q_i, q_{-i})P(q_N)q_i \leq c(S)$, for all $S \subseteq N$. Three cases can be distinguished.

- (1) $S \subseteq A_m$. Here, for all $i \in S$, $\sigma(q_i, q_{-i}) < 1$ and so $\sum_{i \in S} \sigma(q_i, q_{-i})P(q_N)q_i \leq \sum_{i \in S} P(q_S)q_i = c(S)$.
- (2) $S \subseteq A_{nm}$. We now prove that $\sum_{i \in S} (\sigma(q_i, q_{-i})P(q_N) - P(q_S))q_i \leq 0$. By P.5 (NMA) we know that $\forall i \in A_{nm}$, $\sigma(q_i, q_{-i}) \leq \frac{P(q_{A_{nm}})}{P(q_N)}$. Then, $\sigma(q_i, q_{-i})P(q_N) \leq P(q_{A_{nm}})$. Take into account that $P(q_{A_{nm}}) \leq P(q_S)$, for all $S \subseteq A_{nm}$, it is found that $\sigma(q_i, q_{-i})P(q_N) - P(q_S) \leq 0$ for any $i \in S \subseteq A_{nm}$. Hence, $\sum_{i \in S} (\sigma(q_i, q_{-i})P(q_N) - P(q_S))q_i \leq 0$.
- (3) $S \cap A_{nm} \neq S$. By an argument similar to that above

$$\sum_{i \in S} \sigma(q_i, q_{-i})P(q_N)q_i = \sum_{i \in S \cap A_m} \sigma(q_i, q_{-i})P(q_N)q_i + \sum_{i \in S \cap A_{nm}} \sigma(q_i, q_{-i})P(q_N)q_i \leq \sum_{i \in S \cap A_m} P(q_N)q_i + \sum_{i \in S \cap A_{nm}} P(q_N)q_i$$

Hence, by the subadditive property,

$$\sum_{i \in S \cap A_m} P(q_N)q_i + \sum_{i \in S \cap A_{nm}} P(q_N)q_i \leq \sum_{i \in S} P(q_N)q_i = c(S),$$

and so, $\sum_{i \in S} \sigma(q_i, q_{-i})P(q_N)q_i \leq c(S)$. $\square$

Summarizing, any BDP rule always satisfies the properties SYM, PMON, CMON, FRAC, efficiency and coalitional stability. Moreover, it is acceptable for both major and non-major agents. It can be concluded that a BDP rule is a good DP rule for CPNL-games.

Below, we focus on an alternative approach for obtaining DP rules for CPNL-games. This consists of allocating the cost of the grand coalition proportionally, with a proportionality factor that combines the individual costs (faced when each agent buys the product on its own) and the EP rule (available when agents face cooperative purchasing). We obtain a highly suitable parametric family of proportional rules which, notably, is related to the family of BDP rules.

6. The Family of α -Proportional Rules

Consider a CPNL-game (N, c) and a parameter $0 \leq \alpha \leq 1$. We define an α -proportional rule as $\Theta(c, \alpha) = (\Theta_i(c, \alpha))_{i \in N}$ where

$$\Theta_i(c, \alpha) := \theta_i(\alpha) P(q_N) q_N \text{ for all } i \in N, \quad (5)$$

with

$$\theta_i(\alpha) := \frac{\alpha P(q_N) q_i + (1 - \alpha) P(q_i) q_i}{\sum_{j \in N} [\alpha P(q_N) q_j + (1 - \alpha) P(q_j) q_j]}. \quad (6)$$

Note that $\theta_i(\alpha)$ is a convex combination of the EP rule $P(q_N) q_i$ and the individual cost $P(q_i) q_i$, which is normalized to one. Hence, $0 \leq \theta_i(\alpha) \leq 1$ and $\sum_{i \in N} \theta_i(\alpha) = 1$. Thus, $\Theta_i(c, \alpha)$ allocates the cost of the grand coalition $P(q_N) q_N$ proportionally to $\theta_i(\alpha)$.

Note first the 1-proportional rule matches the EP rule, that is, for all $i \in N$

$$\Theta_i(c, 1) = \theta_i(1) P(q_N) q_N = \frac{P(q_N) q_i}{\sum_{j \in N} P(q_N) q_j} P(q_N) q_N = \frac{P(q_N) q_i}{P(q_N) q_N} P(q_N) q_N = P(q_N) q_i = \epsilon_i(c).$$

The 0-proportional rule is the rule proportional to the individual cost, and it is related to the EP rule: for all $i \in N$

$$\Theta_i(c, 0) = \frac{P(q_i) q_i}{\sum_{j \in N} P(q_j) q_j} P(q_N) q_N = \frac{P(q_i) q_N}{\sum_{j \in N} P(q_j) q_j} P(q_N) q_i = \frac{P(q_i) q_N}{\sum_{j \in N} P(q_j) q_j} \epsilon_i(c) = \frac{P(q_i) q_N}{\sum_{j \in N} P(q_j) q_j} \Theta_i(c, 1)$$

Therefore, $\Theta_i(c, 0)$ is greater or less than $\Theta_i(c, 1)$ depending on the ratio $\frac{P(q_i) q_N}{\sum_{j \in N} P(q_j) q_j}$ is greater or less than 1. We assume w.l.o.g. that $q_1 \leq q_2 \leq \dots \leq q_n$. If at least one of these inequalities is strict, clearly, $\frac{P(q_1) q_N}{\sum_{j \in N} P(q_j) q_j} > 1$ and $\frac{P(q_n) q_N}{\sum_{j \in N} P(q_j) q_j} < 1$. As P is a continuously decreasing discount function, there is a unique threshold $\bar{q} \in (q_1, q_n)$, such that,

$$\frac{P(\bar{q}) q_N}{\sum_{j \in N} P(q_j) q_j} = 1 \Leftrightarrow P(\bar{q}) = \frac{\sum_{j \in N} P(q_j) q_j}{q_N} \quad (7)$$

This threshold $\bar{q}$ makes it possible to define two sets of agents that are independent of the parameter α : those who buy small quantities $\bar{S} = \{i \in N, q_i < \bar{q}\}$, and those who buy large quantities $\bar{L} = \{i \in N, q_i > \bar{q}\}$.

The agents $i \in \bar{S}$, who buy small quantities, are harmed by the 0-proportional rule in comparison to the EP because $\frac{P(q_1) q_N}{\sum_{j \in N} P(q_j) q_j} > 1$ for all $q_i < \bar{q}$, which implies that $\Theta_i(c, 0) > \Theta_i(c, 1) = \epsilon_i(c)$. However, the agents in $i \in \bar{L}$ who buy large quantities benefit because $\frac{P(q_1) q_N}{\sum_{j \in N} P(q_j) q_j} < 1$, for all $q_i > \bar{q}$. If there is an agent $i \in$ such that $q_i = \bar{q}$, that agent is neutral to the rule, i.e., $\Theta_i(c, 0) = \Theta_i(c, 1)$. That agent will pay the equal price for any $\alpha \in (0, 1)$.

The proposition below summarizes the above reasoning by relating the α -proportional rules $\Theta(c, \alpha)$ to the EP rule $\epsilon(c)$. It also shows the increasing or decreasing character of the α -proportional rule with respect to the parameter α .

Proposition 5. Let (N, c) be a CPNL-game. The α -proportional allocation rule holds:

1. For $\alpha = 1$, all agents pay the equal price: $\Theta_i(c, 1) = \epsilon_i(c)$, for all $i \in N$.
2. For any $\alpha < 1$,
 - (a) For all $i \in \bar{L}$, $\Theta_i(c, \alpha) < \epsilon_i(c)$ and $\Theta_i(c, \alpha)$ decreases in α .
 - (b) For all $i \in \bar{S}$, $\Theta_i(c, \alpha) > \epsilon_i(c)$ and $\Theta_i(c, \alpha)$ increases in α .
 - (c) If there is $i \notin \bar{L} \cup \bar{S}$, then $\Theta_i(c, \alpha) = \epsilon_i(c)$.

Proof. It is straightforward to prove point 1, i.e., $\Theta_i(c, 1) = \epsilon_i(c)$ for all $i \in N$.

To prove point 2, it is necessary to assess the derivative of function $\Theta_i(c, \alpha)$ with respect to α .

Indeed, $\frac{d(\Theta_i(c, \alpha))}{d\alpha} = \frac{d(\theta_i(\alpha))}{d\alpha} P(q_N) q_N$. We now calculate $\frac{d(\theta_i(\alpha))}{d\alpha}$ by writing it as a function of $\epsilon_i(c) = P(q_N) q_i$ and $c(\{i\}) = P(q_i) q_i$. Thus,

$$\begin{aligned} \frac{d(\theta_i(\alpha))}{d\alpha} &= \frac{d\left(\frac{\alpha \epsilon_i(c) + (1-\alpha) c(\{i\})}{\sum_{j \in N} [\alpha \epsilon_j(c) + (1-\alpha) c(\{j\})]}\right)}{d\alpha} \\ &= \frac{\sum_{j \in N} (\epsilon_i(c) - c(\{i\})) [\alpha \epsilon_j(c) + (1-\alpha) c(\{j\})] - \sum_{j \in N} (\alpha \epsilon_j(c) + (1-\alpha) c(\{j\})) [\epsilon_j(c) - c(\{j\})]}{\left(\sum_{j \in N} [\alpha \epsilon_j(c) + (1-\alpha) c(\{j\})]\right)^2} \\ &= \frac{\sum_{j \in N} (\epsilon_i(c) - c(\{i\})) [\alpha \epsilon_j(c) + (1-\alpha) c(\{j\})] - (\alpha \epsilon_i(c) + (1-\alpha) c(\{i\})) [\epsilon_j(c) - c(\{j\})]}{\left(\sum_{j \in N} [\alpha \epsilon_j(c) + (1-\alpha) c(\{j\})]\right)^2} \\ &= \frac{\sum_{j \in N} \epsilon_i(c) c(\{j\}) - \epsilon_j(c) c(\{i\})}{\left(\sum_{j \in N} [\alpha \epsilon_j(c) + (1-\alpha) c(\{j\})]\right)^2}. \end{aligned}$$

Note that the sign of $\frac{d(\Theta_i(c, \alpha))}{d\alpha}$ depends only on the sign of $\sum_{j \in N} \epsilon_i(c) c(\{j\}) - \epsilon_j(c) c(\{i\})$. Let us look at the latter:

$$\begin{aligned} \sum_{j \in N} \epsilon_i(c) c(\{j\}) - \epsilon_j(c) c(\{i\}) &= \epsilon_i(c) \sum_{j \in N} c(\{j\}) - c(\{i\}) \sum_{j \in N} \epsilon_j(c) \\ &= P(q_N) q_i \sum_{j \in N} P(q_j) q_j - P(q_i) q_i \sum_{j \in N} P(q_N) q_j \\ &= P(q_N) q_i \sum_{j \in N} P(q_j) q_j - P(q_i) q_i P(q_N) q_N \\ &= P(q_N) q_i \left(\sum_{j \in N} P(q_j) q_j - P(q_i) q_N \right). \end{aligned}$$

Clearly, $\sum_{j \in N} \epsilon_i(c) c(\{j\}) - \epsilon_j(c) c(\{i\}) \leq 0 \Leftrightarrow \sum_{j \in N} P(q_j) q_j - P(q_i) q_N \leq 0$. The last inequality holds if and only if $q_i > \bar{q}$ (see expression (7)). Hence, if $i \in \bar{L}$, then $\Theta_i(c, \alpha) < \epsilon_i(c)$ and $\Theta_i(c, \alpha)$ decreases in α .

On the other hand, $\sum_{j \in N} \epsilon_i(c) c(\{j\}) - \epsilon_j(c) c(\{i\}) \geq 0 \Leftrightarrow \sum_{j \in N} P(q_j) q_j - P(q_i) q_N \geq 0$. This last inequality holds if and only if $q_i < \bar{q}$ (see expression (7)). Hence, if $i \in \bar{S}$, then $\Theta_i(c, \alpha) > \epsilon_i(c)$ and $\Theta_i(c, \alpha)$ increases in α .

Finally, if there is $i \notin \bar{L} \cup \bar{S}$, then $q_i = \bar{q}$, so $\Theta_i(c, 0) = \Theta_i(c, 1) = \epsilon_i(c)$. As $\Theta_i(c, \alpha)$ is continuous in α , $\Theta_i(c, \alpha) = \epsilon_i(c)$ for all $\alpha \in [0, 1]$. $\square$

Proposition 5 shows that for $\alpha = 1$ all agents pay as per the EP rule. As α decreases, those agents who buy large quantities ($\bar{L}$) start to pay less than under the EP rule and those who buy small quantities ($\bar{S}$) pay more. These differences with respect to the EP rule increase with α and peak when $\alpha = 0$. Therefore, the parameter α quantifies how different the prices are. Thus, for $\alpha = 1$, there are no different prices and for $\alpha = 0$ the difference in prices is maximum.

The next question is whether there is any link between the α -proportional rules and BDP rules. In other words, is any α -proportional rule a BDP rule? The following section shows that there is always an α threshold above which any α -proportional rule is a BDP rule.

7. Condition for an α -Proportional Rule to be a BDP Rule

To compare the α -proportional rule and the BDP rule, it is first necessary to rewrite the former for any agent $i \in N$ as follows:

$$\begin{aligned} \Theta_i(c, \alpha) &= \theta_i(\alpha) P(q_N) q_N = \frac{\alpha P(q_N) q_i + (1-\alpha) P(q_i) q_i}{\sum_{j \in N} [\alpha P(q_N) q_j + (1-\alpha) P(q_j) q_j]} P(q_N) q_N = \\ &= \frac{\alpha P(q_N) + (1-\alpha) P(q_i)}{\sum_{j \in N} [\alpha P(q_N) q_j + (1-\alpha) P(q_j) q_j]} q_N P(q_N) q_i. \end{aligned}$$

Denote

$$\sigma_\alpha(q_i, q_{-i}) := \frac{\alpha P(q_N) + (1-\alpha) P(q_i)}{\sum_{j \in N} [\alpha P(q_N) q_j + (1-\alpha) P(q_j) q_j]} q_N, \quad (8)$$

thus, $\Theta_i(c, \alpha) = \sigma_\alpha(q_i, q_{-i}) P(q_N) q_i$.

First, note that the function σ_α depends on the distribution of the agents' quantities $(q_1, q_2, \dots, q_i, \dots, q_n)$ unlike the equal price rule. This dependence is reflected in the second term of the denominator, which is the sum of the individual cost without cooperation, i.e., $\sum_{j \in N} P(q_j)q_j$. Note that (8) is equal to $\frac{\alpha c(N) + (1-\alpha)P(q_i)q_N}{\alpha c(N) + (1-\alpha)\sum_{j \in N} P(q_j)q_j}$.

Remember that a BDP rule $\beta(c)$ is defined as $\beta_i(c) = \sigma(q_i, q_{-i})P(q_N)q_i$, for all $i \in N$, with the function σ satisfying properties 1–6. The question is whether $\sigma_\alpha(q_i, q_{-i})$ satisfies these six properties for any $q \in R_+^n$ and all $\alpha \in [0, 1]$. Although it does not, we show that a threshold $\alpha < 1$ can be always found above which $\sigma_\alpha(q_i, q_{-i})$ does so.

Returning to the major and non-major agents, note that here major agents are those who buy large quantities, i.e., $A_m = \bar{L}$. This follows comparing the definitions of A_m , $\bar{L}$ and expression (7). Analogously, it can be shown that A_{nm} is equal to set $\bar{S}$ plus any $i \in N$ such that $q_i = \bar{q}$ (if any), i.e., $A_{nm} = \bar{S} \cup (\bar{L} \cup \bar{S})^C$.

The following theorem states an α threshold above which any α -proportional rule is a BDP rule.

Theorem 2. For any CPNL-game, there is always an $\alpha^* < 1$, such that for any $\alpha \in [\alpha^*, 1)$ any α -proportional rule is a BDP rule.

Proof. Let (N, c) be a CPNL-game and $\Theta(c, \alpha)$ be an α -proportional rule for that game, with $\sigma_\alpha(q_i, q_{-i})$ given by (8). To prove that $\Theta(c, \alpha)$ is a BDP rule, it must be shown that $\sigma_\alpha(q_i, q_{-i})$ satisfies properties 1–6 of function σ .

- (MDP) Take $i, j \in N$ s.t. $q_i \leq q_j$ then, by property 2 (Decreasingness) of function P , it follows that $P(q_i) \geq P(q_j)$, and so $\sigma_\alpha(q_i, q_{-i}) \geq \sigma_\alpha(q_j, q_{-j})$.
- (LDP) Take $i, j \in N$ s.t. $q_i \leq q_j$. It can be shown that $\sigma_\alpha(q_i, q_{-i})q_i \leq \sigma_\alpha(q_j, q_{-j})q_j$. Indeed, by property 4 (limited growth rate) of function P , it emerges that $\alpha P(q_N)q_i + (1-\alpha)P(q_i)q_i \leq \alpha P(q_N)q_j + (1-\alpha)P(q_j)q_j$. Hence,

$$\frac{\alpha P(q_N) + (1-\alpha)P(q_i)}{\sum_{j \in N} [\alpha P(q_N)q_j + (1-\alpha)P(q_j)q_j]} q_N q_i < \frac{\alpha P(q_N) + (1-\alpha)P(q_j)}{\sum_{j \in N} [\alpha P(q_N)q_j + (1-\alpha)P(q_j)q_j]} q_N q_j.$$
- (MA) We now prove that for $\alpha < 1$, $\sigma_\alpha(q_i, q_{-i}) < 1$, for all $i \in A_m$, and $\sigma_\alpha(q_i, q_{-i}) > 1$, for all $i \in A_{nm}$.

Indeed, as mentioned above, $A_m = \bar{L}$ and $A_{nm} = \bar{S} \cup (\bar{L} \cup \bar{S})$. Thus, if $\alpha < 1$, from point 2.a. of Proposition 5, we know that, for all $i \in A_m$, $\Theta_i(c, \alpha) < \epsilon_i(c)$, which is equivalent to $\sigma_\alpha(q_i, q_{-i})P(q_N)q_i < P(q_N)q_i \Leftrightarrow \sigma_\alpha(q_i, q_{-i}) < 1$. Analogously, from point 2.b. and 2.c. of Proposition 5, it can be shown that, for all $i \in A_{nm}$, $\sigma_\alpha(q_i, q_{-i}) \geq 1$. Finally, note that if $\alpha = 1$, then $\Theta_i(c, 1) = \epsilon(c)$ and $\sigma_\alpha(q_i, q_{-i}) = 1$ for all $i \in N$.

- (BW) It is straightforward to prove that $\sum_{i \in A_m} (1 - \sigma_\alpha(q_i, q_{-i}))q_i = |\sum_{i \in A_{nm}} (1 - \sigma_\alpha(q_i, q_{-i}))q_i|$.

Indeed,

$$\begin{aligned} \sum_{i \in A_m} (1 - \sigma_\alpha(q_i, q_{-i}))q_i &= |\sum_{i \in A_{nm}} (1 - \sigma_\alpha(q_i, q_{-i}))q_i| \Leftrightarrow \\ \sum_{i \in N} (\sigma_\alpha(q_i, q_{-i}) - 1)q_i &= 0 \Leftrightarrow \sum_{i \in N} \sigma_\alpha(q_i, q_{-i})q_i = \sum_{i \in N} q_i \Leftrightarrow \\ \Leftrightarrow \sum_{i \in N} \sigma_\alpha(q_i, q_{-i})q_i &= q_N \Leftrightarrow \sum_{i \in N} \left(\frac{\alpha P(q_N) + (1-\alpha)P(q_i)}{\sum_{j \in N} [\alpha P(q_N)q_j + (1-\alpha)P(q_j)q_j]} q_N \right) q_i = q_N \\ \Leftrightarrow \sum_{i \in N} \left(\frac{\alpha P(q_N) + (1-\alpha)P(q_i)}{\sum_{j \in N} [\alpha P(q_N)q_j + (1-\alpha)P(q_j)q_j]} \right) q_i &= 1 \Leftrightarrow \frac{\sum_{i \in N} [\alpha P(q_N)q_i + (1-\alpha)P(q_i)q_i]}{\sum_{j \in N} [\alpha P(q_N)q_j + (1-\alpha)P(q_j)q_j]} = 1. \end{aligned}$$

- (NMA) We show that there is always an $\alpha^* < 1$ such that for any $\alpha \in [\alpha^*, 1)$, $\sigma_\alpha(q_1, q_{-1}) \leq \frac{P(q_{A_{nm}})}{P(q_N)}$. Note that, as shown above, $A_{nm} = \bar{S} \cup (\bar{L} \cup \bar{S})^C$.

We first prove that for all $i \in A_{nm}$, $\sigma_\alpha(q_i, q_{-i})$ is decreasing in α . Indeed, as $\Theta_i(c, \alpha) = \sigma_\alpha(q_i, q_{-i})P(q_N)q_i$, thus $\frac{d(\Theta_i(c, \alpha))}{d\alpha} = \frac{d(\sigma_\alpha(q_i, q_{-i}))}{d\alpha} P(q_N)q_i$. Therefore, $\frac{d(\sigma_\alpha(q_i, q_{-i}))}{d\alpha} < 0$ if and only if $\frac{d(\Theta_i(c, \alpha))}{d\alpha} < 0$, since $P(q_N)q_i > 0$. The sign of the last derivative always holds for all $i \in \bar{S}$ (see point 2.b of Proposition 5). In addition, if there exists any $i \notin \bar{L} \cup \bar{S}$ then, by point 2.c of Proposition 5, $\frac{d(\Theta_i(c, \alpha))}{d\alpha} = 0$.

Now note that $\sigma_\alpha(q_1, q_{-1}) \leq \frac{P(q_{A_{nm}})}{P(q_N)}$ is equivalent to

$$\frac{\alpha P(q_N) + (1-\alpha)P(q_1)}{\sum_{j \in N} [\alpha P(q_N)q_j + (1-\alpha)P(q_j)q_j]} q_N \leq \frac{P(q_{Anm})}{P(q_N)}. \quad (9)$$

This last inequality always holds for $\alpha = 1$. Indeed, $\frac{P(q_N)}{\sum_{j \in N} P(q_N)q_j} q_N = 1 < \frac{P(q_{Anm})}{P(q_N)}$, because of $q_{Anm} < q_N$. Thus, $\sigma_{\alpha=1}(q_1, q_{-1}) = 1$ and $\sigma_{\alpha}(q_1, q_{-1})$ is decreasing in α . Thus, only two different situations can occur: First, there is a root $\bar{\alpha} \in (0, 1)$ such that $\sigma_{\bar{\alpha}}(q_1, q_{-1}) = \frac{P(q_{Anm})}{P(q_N)}$, i.e., (9) holds with equality, thus, $\sigma_{\alpha}(q_1, q_{-1}) \leq \frac{P(q_{Anm})}{P(q_N)}$ for all $\alpha \in (\bar{\alpha}, 1)$. Second, there is no such $\bar{\alpha}$ that $\sigma_{\bar{\alpha}}(q_1, q_{-1}) = \frac{P(q_{Anm})}{P(q_N)}$. In that case, $\sigma_{\alpha}(q_1, q_{-1}) \leq \frac{P(q_{Anm})}{P(q_N)}$, for all $\alpha \in (0, 1)$.

Assume that $\alpha^* = \bar{\alpha}$ if $\bar{\alpha} \in (0, 1)$ and $\alpha^* = 0$ otherwise. We conclude that there is always an $\alpha^* < 1$, such that for any $\alpha \in [\alpha^*, 1]$, $\sigma_{\alpha}(q_1, q_{-1}) \leq \frac{P(q_{Anm})}{P(q_N)}$.

6. (LDQ) Take (N, q, P) and (N, q', P) two CPGD-models with $q = (q_i, q_{-i})$, $q' = (q'_i, q_{-i})$, and $q_i \geq q'_i$. It must be shown that $\sigma_{\alpha}(q_i, q_{-i})P(q_N)q_i \geq \sigma_{\alpha}(q'_i, q_{-i})P(q'_N)q'_i$. In fact, it must be proven that the function $\sigma_{\alpha}(q_i, q_{-i})P(q_N)q_i$ is increasing in q_i , i.e.

$$\frac{\partial(\sigma_{\alpha}(q_i, q_{-i})P(q_N)q_i)}{\partial q_i} \geq 0.$$

To simplify the proof, $\sigma_{\alpha}(q_i, q_{-i})P(q_N)q_i$ can be rewritten as a function of $\epsilon_i(c) = P(q_N)q_i$, $c(\{i\}) = P(q_i)q_i$ and $c(N) = P(q_N)q_N$. In addition, to simplify the notation, we do not explicitly indicate that all the following derivatives are in regard to q_i ; we denote them by $\epsilon'_i(c)$, $c'(\{i\})$, and $c'(N)$.

First we rewrite the function $\sigma_{\alpha}(q_i, q_{-i})P(q_N)q_i$ as follows:

$$\sigma_{\alpha}(q_i, q_{-i})P(q_N)q_i = \frac{\alpha P(q_N)q_i + (1-\alpha)P(q_i)q_i}{\sum_{j \in N} [\alpha P(q_N)q_j + (1-\alpha)P(q_j)q_j]} P(q_N)q_N = \frac{(\alpha \epsilon_i(c) + (1-\alpha)c(\{i\}))c(N)}{\alpha c(N) + (1-\alpha)\sum_{j \in N} c(\{j\})}.$$

Denote by $f(q_i) = (\alpha \epsilon_i(c) + (1-\alpha)c(\{i\}))c(N)$ and $g(q_i) = \alpha c(N) + (1-\alpha)\sum_{j \in N} c(\{j\})$, thus

$$\sigma_{\alpha}(q_i, q_{-i})P(q_N)q_i = \frac{f(q_i)}{g(q_i)} \text{ and } \frac{\partial(\sigma_{\alpha}(q_i, q_{-i})P(q_N)q_i)}{\partial q_i} = \frac{f'(q_i)g(q_i) - f(q_i)g'(q_i)}{(g(q_i))^2}, \text{ with}$$

$$\frac{\partial(\sigma_{\alpha}(q_i, q_{-i})P(q_N)q_i)}{\partial q_i} > 0 \Leftrightarrow f'(q_i)g(q_i) - f(q_i)g'(q_i) > 0$$

In addition, it is known that

$$f'(q_i) = (\alpha \epsilon'_i(c) + (1-\alpha)c'(\{i\}))c(N) + (\alpha \epsilon_i(c) + (1-\alpha)c(\{i\}))c'(N)$$

$$g'(q_i) = \alpha c'(N) + (1-\alpha)\sum_{j \in N} c'(\{j\})$$

After some calculations, it can be shown that

$$f'(q_i)g(q_i) - f(q_i)g'(q_i) > 0$$

$\Leftrightarrow$

$$(\alpha \epsilon'_i(c) + (1-\alpha)c'(\{i\}))c(N)(\alpha c(N) + (1-\alpha)\sum_{j \in N} c(\{j\}))$$

$>$

$$(\alpha \epsilon_i(c) + (1-\alpha)c(\{i\}))(1-\alpha)(c(N)c'(\{i\}) - c'(N)\sum_{j \in N} c(\{j\}))$$

Clearly, $(\alpha c(N) + (1-\alpha)\sum_{j \in N} c(\{j\})) > \alpha \epsilon_i(c) + (1-\alpha)c(\{i\})(1-\alpha)$, because $c(N) > \epsilon_i(c)$ and $\sum_{j \in N} c(\{j\}) > c(\{i\})$.

To end the proof, we prove that

$$(\alpha \epsilon'_i(c) + (1-\alpha)c'(\{i\}))c(N) > (1-\alpha)\left(c(N)c'(\{i\}) - c'(N)\sum_{j \in N} c(\{j\})\right) \quad (10)$$

It is straightforward to show that (10) is equivalent to

$$\alpha \epsilon'_i(c)c(N) > -(1-\alpha)c'(N)\sum_{j \in N} c(\{j\}),$$

which always holds because $\epsilon'_i(c) > 0$ and $c'(N) > 0$. Note that, by property 4 (limited growth rate) of function P , it is straightforward to prove that $c'(N) > 0$. Next, we show that $\epsilon'_i(c) > 0$,

$$\epsilon'_i(c) = P'(q_N)q_i + P(q_N) > P'(q_N)q_N + P(q_N) = c'(N) > 0, \quad (11)$$

because of $q_i < q_N$, $P'(q_N) < 0$ (by property 2 (decreasingness) of function P) and $P(q_N) \geq 0$ (by definition). This completes the proof of property 6 (LDQ)

We thus conclude that there is always an $\alpha^* < 1$, such that for any $\alpha \in [\alpha^*, 1)$, the function $\sigma_\alpha(q_i, q_{-i})$ satisfies properties 1–6 of function σ . Hence, for all $\alpha \in [\alpha^*, 1)$, any α -proportional rule is a BDP rule. $\square$

It can be seen from the above demonstration that the function $\sigma_\alpha(q_i, q_{-i})$ satisfies property 3 (MA) for $\alpha < 1$, property 5 (NMA) for $\alpha > \alpha^*$. However, the other four properties are satisfied for any $\alpha \in [0, 1]$.

Summarizing, the 1-proportional rule is the EP rule. If major agents do not find this allocation rule acceptable, the parameter α may decrease (to the threshold α^*) and the allocation rule thus becomes a BDP rule. It will be then acceptable to major agents. The smaller the parameter α is, the greater the price differences are in regard to the quantity demanded. Moreover, the lower the major-agent prices are, the higher the non-major-agent prices are. For this reason, a threshold for parameter α is needed. Beyond that threshold, non-major agents do not find the α -proportional rule acceptable, so it becomes coalitionally instable, i.e., it is no longer a core allocation.

Finally, if $\alpha^* = 0$, the α -proportional rule is always a BDP rule, for all $\alpha \in [0, 1)$. The following corollary shows a necessary and sufficient condition on the price function P for this to happen.

Corollary 1. *The α -proportional rule is always a BDP rule for all $\alpha \in [0, 1)$ if and only if $\sum_{j \in N} P(q_j)q_j \geq \frac{P(q_1)}{P(q_{Ann})} P(q_N)q_N$.*

Proof. The threshold α^* is obtained in the proof of Theorem 2 from the inequality $\sigma_\alpha(q_1, q_{-1}) \leq \frac{P(q_{Ann})}{P(q_N)}$. As $\sigma_\alpha(q_1, q_{-1})$ is decreasing in α , if the above inequality holds for $\alpha = 0$, it also holds for all $\alpha \in [0, 1)$. Thus,

$$\begin{aligned} \sigma_{\alpha=0}(q_1, q_{-1}) &\leq \frac{P(q_{Ann})}{P(q_N)} \Leftrightarrow \frac{P(q_1)}{\sum_{j \in N} [P(q_j)q_j]} q_N \leq \frac{P(q_{Ann})}{P(q_N)} \\ &\Leftrightarrow \sum_{j \in N} [P(q_j)q_j] \geq \frac{P(q_1)}{P(q_{Ann})} P(q_N)q_N. \quad \square \end{aligned}$$

8. Numerical Illustration

In this section, we give a numerical illustration. We use a discount price function $P(q_i)$ which has the properties 1–4 described in the Model section. As we mentioned, it is provided in [8] an analytical and empirical basis for a general quantity-discount function (QDF). They show that this QDF fits very well with 66 discount schedules found in practice. They propose a QDF with an explicit formula depending on certain parameter: $P(q) = p_m + \frac{S}{q^\eta}$, where p_m is the theoretical minimum price, S scales the function $P(q)$ for quantity q , and η represent the steepness of a quantity-discount function. As it is carried out in [4], we also assume that $\eta = 0.5$ because, as they show in [8], the schedules with a positive steepness have a mean steepness of 0.58. In the first example, Table 1 shows the price and cost for each agent in three cases for $P(q_i) = 80 + \frac{7000}{\sqrt{q_i}}$.

Table 1. Example 1 with $\alpha^* > 0$.

		$P(q_i) = 80 + \frac{7000}{\sqrt{q_i}}$						
Agent	q_i	No-Cooperate		Cooperate				
		Individual		Equal Price		BDP(α -Proportional) $_{\alpha=\alpha^*=0.368}$		
		Price	Cost	Price	Cost	Price	Cost	σ_α
1	50	1069.9	53,497.5	192.7	9633.4	448.9	22,446.6	2.330
2	60	983.7	59,021.8	192.7	11,560.1	416.2	24,970.9	2.160
3	70	916.7	64,166.2	192.7	13,486.8	390.7	27,350.9	2.028
4	80	862.6	69,009.9	192.7	15,413.5	370.2	29,616.8	1.921
5	100	780.0	78,000.0	192.7	19,266.9	338.8	33,883.8	1.759
6	500	393.0	196,524.8	192.7	96,334.5	191.9	95,955.5	0.996
7	600	365.8	219,464.3	192.7	115,601.4	181.6	108,932.6	0.942
8	700	344.6	241,202.6	192.7	134,868.3	173.5	121,453.6	0.901
9	800	327.5	261,989.9	192.7	154,135.2	167.0	133,613.4	0.867
10	900	313.3	282,000.0	192.7	173,402.1	161.6	145,478.2	0.839

First, there is no cooperation and each agent buys individually; thus, the price of agent i is $P(q_i)$ and its cost $P(q_i)q_i$. Second, agents cooperate with an equal price allocation rule. In this case, the price is $P(q_N)$ and the cost $P(q_N)q_i$. Third, agents cooperate but with a BDP allocation rule. In particular, we consider an α -proportional rule, where the parameter α considered is the threshold α^* given by Theorem 2. The price is, in this case, $\sigma_{\alpha^*}(q_i, q_{-i})P(q_N)$ and the cost $\sigma_{\alpha^*}(q_i, q_{-i})P(q_N)q_i$. In this third case, we also show the value of function σ_{α^*} , i.e., $\sigma_{\alpha^*}(q_i, q_{-i}) = \frac{\alpha^*P(q_N) + (1-\alpha^*)P(q_i)}{\sum_{j \in N} [\alpha^*P(q_N)q_j + (1-\alpha^*)P(q_j)q_j]} q_N$. Note that function σ is always greater than one for all non-major agents and less than one for major agents. Thus, in this case, the non-major agents are players 1–5 and 6–10 the major ones.

It is also known that the α -proportional rule is a BDP rule for all $\alpha \in [\alpha^*, 1)$, so as α increases the price and cost of non-major agents will decrease and those of major agents will increase. Hence, agents 1–5 will prefer the highest possible α , and agents 6–10 the lowest. Note that, in the limit case, that is $\alpha = 1$, the value of $\sigma_1(q_i, q_{-i})$ is one for all agents and the price matches the equal price, which is the most favorable situation for non-major agents. The opposite situation is $\alpha^* = 0.368$, which is the most favorable situation possible for major agents. If α were strictly lower than 0.368, this would not be acceptable to non-major agents, and in that case, the α -proportional rule would not be a BDP rule.

Notice that, here, the threshold between major and non-major agents is $\bar{q} = 493,75$. This can be easily obtained from Equation (7). Indeed,

$$\bar{q} = \left(\frac{7.000}{\frac{\sum_{j \in N} c(j)}{q_N} - 80} \right)^2 = \left(\frac{7.000}{395,02 - 80} \right)^2 = \left(\frac{7.000}{315,02} \right)^2 = 493,75.$$

Although, in the first example, $\alpha^* > 0$, it could be zero if the condition from Corollary 1 holds, i.e., if the discount-price function holds that $\sum_{j \in N} P(q_j)q_j \geq \frac{P(q_1)}{P(q_{A_{nm}})}P(q_N)q_N$. In that case, the α -proportional rule is a BDP rule for all $\alpha \in [0, 1)$.

Table 2 shows a second example with $P(q_i) = 10 + \frac{7000}{\sqrt{q_i}}$ where $\alpha^* = 0$, since $\sum_{j \in N} P(q_j)q_j = 1.254.586 > 1.249.760, 2 = (\frac{999,9}{378,93})473.622 = \frac{P(q_1)}{P(q_5)}P(q_N)q_N$. We present the α -proportional for $\alpha = 0$. Notice that the former is acceptable to all agents for all $\alpha \in [0, 1)$ and again $\bar{q} = 493,75 = \left(\frac{7.000}{325,02 - 10} \right)^2$.

Table 2. Example 2 with $\alpha^* = 0$.

$P(q_i) = 10 + \frac{7000}{\sqrt{q_i}}$								
Agent	q_i	No-Cooperate		Cooperate				
		Individual		Equal Price		BDP(α -Proportional) $_{\alpha=\alpha^*=0}$		
		Price	Cost	Price	Cost	Price	Cost	σ_α
1	50	999.9	49,997.5	122.7	6133.4	377.4	18,868.5	3.076
2	60	913.7	54,821.8	122.7	7360.1	344.8	20,689.2	2.811
3	70	846.7	59,266.2	122.7	8586.8	319.5	22,366.5	2.605
4	80	792.6	63,409.9	122.7	9813.5	299.1	23,930.3	2.438
5	100	710.0	71,000.0	122.7	12,266.9	267.9	26,794.7	2.184
6	500	323.0	161,524.8	122.7	61,334.5	121.9	60,957.8	0.994
7	600	295.8	177,464.3	122.7	73,601.4	111.6	66,973.2	0.910
8	700	274.6	192,202.6	122.7	85,868.3	103.6	72,535.3	0.845
9	800	257.5	205,989.9	122.7	98,135.2	97.2	77,738.5	0.792
10	900	243.3	219,000.0	122.7	110,402.1	91.8	82,648.4	0.749

9. Conclusions, Limitations and Implications

We study situations in which a seller provides general price discounts for large orders according to a decreasing unit-price function. In these situations, buyers can cooperate and form purchasing groups to benefit from these price discounts. In this paper, we analyze these cooperative-purchasing situations as cooperative cost games and call them CPGD-games. We prove that the grand coalition can obtain significant reductions in costs (i.e., CPGD-games are subadditive). Then, we show that CPGD-games are balanced; that is, there is always a method for allocating the reduced costs generated by the CPGD model that is efficient and (coalitionally) stable: the equal price (EP) rule.

Next, we focus on cooperative-purchasing models with decreasing and linear unit-price functions. We demonstrate that the linear nature of the discount-price function provides additional information about the corresponding cooperative purchasing game with linear discount (CPL-game): the marginal contribution of an agent diminishes as a coalition grows. This is well-known as the snowball effect or concavity property. We also prove that the EP rule matches the Shapley value and the nucleolus. In such cases, it seems that there is no better way to allocate the reduced total cost. However, some agents may argue that the EP is questionable because it favors those agents who produce and sell just a few units of the product. Note that these agents pay the same price as major buyers. These agent concerns can be really problematic when the unit price function is not linear. To solve this problem, we propose a family of allocation rules for cooperative purchasing games with non-linear discount (CPNL-games). We focus on the different price (DP) method and propose a family of allocation rules with different prices that are acceptable to all agents: balanced different price rules (BDP rules).

Finally, we concentrate on an alternative approach to obtain DP rules for CPNL-games. This consists of allocating the cost of the grand coalition proportionally, with a proportionality factor that combines the individual costs (faced when each agent buys the product on its own) and the EP rule (available when agents face cooperative purchasing). We obtain a highly suitable parametric family of proportional rules, named α -proportional rules which, notably, are related to the family of BDP rules. Specifically, we prove that there is always an α threshold above which any α -proportional rule is a BDP rule. There is, thus, a range of acceptable alpha parameters for all agents, both major (agents who buy large quantities) and non-major (agents who buy small quantities).

This family of α -proportional rules provides a cost-sharing method, for the cooperative-purchasing model with general discount, that is easy to calculate, and guarantees stability

in the cooperation because it is acceptable to both major and non-major buyers. All of them are fully satisfied with this cost-sharing method. We believe that our rule can be a useful tool for cooperative-purchasing organizations with any kind of buyers (firms, institutions, consumers, etc.).

Future research can look first for real situations in which our model could be applied and, based on the properties required in each situation, determine what alpha parameter(s) within the interval would be most suitable, i.e., which α -proportional rule is most suitable for each situation (an α -proportional rule that favors majors or one that is more favorable to non-majors). Secondly, researchers could look for certain properties of the α -proportional rules that are only satisfied by the family of α -proportional rules with a view to obtaining a characterization of the family of α -proportional rules. Third, the analysis can be extended to two-stage situations in which the quantity demanded by agents can be chosen strategically in the first stage, i.e., two-stage models where agents play a non-cooperative game in the first stage to choose the quantity demanded and play our GPGD-game (as a cooperative game) in the second stage. It is, therefore, of great interest to study the characteristics of the equilibrium profile of quantities demanded by agents induced by our family of α -proportional rules.

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Abbreviations

The following abbreviations are used in this manuscript:

EP	Equal price
QDF	Quantity-discount function
DP	Different price
CPGD-model	Cooperative-purchasing model with general discount
CPGD-game	Cooperative-purchasing cost game with general discount
CPL-game	Cooperative-purchasing game with linear discount
CPNL-game	Cooperative-purchasing game with non-linear discount
MCP-situations	Maximum cooperative-purchasing situations
MCP-games	Maximum cooperative-purchasing games
MDP	Monotonically decreasing through players
LDP	Limited decrease through players
MA	Major-agents' acceptability
BW	Balanced weighting
NMA	Non-major agents' acceptability
LDQ	Limited decrease in a player quantity
SYM	Symmetry
PMON	Player monotonicity
CMON	Cost monotonicity
FRAC	Fair ranking added cost

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