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A Game Theoretic Approach to Cooperation with Pairwise Cost Reduction

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Resumen

El presente trabajo de Tesis Doctoral, se vertebra en el estudio y desarrollo, dentro del campo de la Teoría de Juegos, de los escenarios de Juegos de Costes Cooperativos de Utilidad Transferible, donde la cooperación se produce de manera bilateral entre pares de agentes.

A tal fin se desarrolla en una primera parte un avance en los conceptos de benefactor y beneficiario, a la vez que presenta un nuevo modelo de *Corporate Tax system* descubriendo y analizando las propiedades que aparecen en los juegos de coste coalicionales cuando existen múltiples benefactores y una dualidad de roles, o dicho en otros términos, cuando ambos roles pueden ser desempeñados por un mismo jugador. Unido a ello, se constata el valor de Shapley como criterio o regla de reparto idónea para este tipo de juegos y se presenta una expresión simplificada e intuitiva del mismo que facilita sobremanera su cálculo.

En una segunda parte, y desde la estructura de biform-games , se estudia un escenario híbrido donde los agentes cooperan tras una primera fase competitiva previa donde los jugadores, con el objetivo de reducir sus costes, determinan estratégicamente el nivel de esfuerzo que van a dedicar, o dicho de otro modo, el grado de cooperación con el que van a participar anticipando la reducción de costes que se obtendría, según el nivel de esfuerzo aportado, en el reparto como resultado de la cooperación. Posteriormente a la fase competitiva, se analiza y estudia el modo en el que los jugadores, de manera bilateral, entre pares, cooperan según el nivel de esfuerzo adoptado al objeto de reducir sus costes.

A tal efecto, se presenta un nuevo modelo de juegos cooperativo denominado Pairwise Effort Games (PE Games) desde el que se analiza el impacto de los esfuerzos bilaterales entre pares de jugadores en las reducciones de costes producto de la cooperación y se estudia la existencia de criterios o mecanismos eficientes de asignación de costes que permitan distribuir idóneamente entre la totalidad de jugadores las ganancias obtenidas.

Se demuestra la estabilidad de la gran coalición y la existencia de asignaciones que incentivan a la totalidad de jugadores a cooperar a través de un nivel óptimo de esfuerzo. Se identifica y presenta una familia de repartos con reducciones por pares ponderadas por separado (*Weighted Pairwise Reduction,* WPR) en la que se halla y se constata la generación de dicho nivel óptimo de esfuerzo. Dentro de esta familia, se identifica y se presenta a su vez, la regla que genera el único equilibrio de esfuerzo eficiente.

Por otro lado, se constata que el reparto propuesto por el Valor de Shapley se halla dentro de la familia WPR pero se constata también que los incentivos provocados por dicho reparto conducen a estrategias ineficientes de esfuerzo en la fase competitiva.

Se consigue hallar y demostrar la existencia de equilibrios de esfuerzo en esta fase competitiva. (*Pairwise Effort Equilibria*, PEE).

Una vez presentada y demostrada la existencia de esta familia de valores de reparto WPR, se identifica y se presenta una subfamilia de repartos donde las reducciones por pares no se ponderan por separado sino que, en su lugar, se ponderan de forma agregada. A esta subfamilia se le denomina WPAR *(Weighted Pairwise Aggregate Reduction)*.

Se prueba que el nivel de eficiencia es menor cuando cuando las reducciones entre pares se ponderan de manera agregada para cada agente en lugar de hacerlo separadamente. Se identifica y se propone, tras la comparación entre la familia WPR y la subfamilia WPAR, una regla de reparto dentro de la subfamilia WPAR que, sin alcanzar, tal y como se ha indicado, los valores del nivel óptimo de equilibrio eficiente, si es capaz de generar esfuerzos de equilibrio más cercanos a los esfuerzos de equilibrio óptimamente eficientes.

El trabajo contenido en la presente Tesis doctoral, abre interesantes y prometedoras líneas de estudio e investigación que ahonden tanto en la interdependencia o complementariedad entre los agentes y los diferentes niveles de esfuerzo llevados a cabo, como al desarrollo, entre otras líneas o vías, del estudio de modelos bilaterales con múltiples reducciones de costes y el impacto que dichos esfuerzos realizados provoquen en las mismas.

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Summary

This Doctoral Thesis is based on the study and development, within the field of Game Theory, of the scenarios of Transferable Utility Cooperative Cost Games, where cooperation occurs bilaterally between pairs of agents.

To this end, the first part of this Thesis develops an advance in the concepts of benefactor and beneficiary, and at the same time presents a new model of Corporate Tax system, discovering and analyzing the properties that appear in coalitional cost games when there are multiple benefactors and a duality of roles, or in other words, when both roles can be played by the same player. In addition, the Shapley value as an ideal distribution criterion or rule for this type of games is established and a simplified and intuitive expression of it is presented, which greatly facilitates its calculation.

In a second part, and from the biform-games structure, a hybrid scenario is studied where the agents cooperate after a first competitive phase where the players, with the objective of reducing their costs, strategically determine the level of effort they are going to devote, or in other words, the degree of cooperation with which they are going to participate, anticipating the cost reduction that would be obtained, according to the level of effort contributed, in the distribution as a result of the cooperation.

After the competitive phase, we analyze and study the way in which the players, bilaterally, between pairs, cooperate according to the level of effort adopted in order to reduce their costs.

To this end, a new model of cooperative games called Pairwise Effort Games (PE Games) is presented, from which the impact of bilateral efforts between pairs of players on the cost reductions resulting from cooperation is analyzed, and the existence of efficient cost allocation criteria or mechanisms that allow the gains obtained to be ideally distributed among all the players is studied.

The stability of the grand coalition and the existence of allocations that incentivize all players to cooperate through an optimal level of effort are demonstrated. A family of allocations with Weighted Pairwise Reduction (WPR) is identified and presented in which the generation of such optimal level of effort is found and verified. Within this family, the rule that generates the unique efficient effort equilibrium is identified and presented.

On the other hand, it is found that the distribution proposed by the Shapley value is within the WPR family, but it is also found that the incentives caused by this distribution lead to inefficient effort strategies in the competitive phase.

The existence of Pairwise Effort Equilibria (PEE) in this competitive phase is found and demonstrated.

Once the existence of this family of WPR partitioning values has been presented and demonstrated, a subfamily of partitionings is identified and presented where pairwise reductions are not weighted separately but instead are weighted in aggregate. This subfamily is referred to as WPAR (Weighted Pairwise Aggregate Reduction).

It is proved that the level of efficiency in is lower when the pairwise reductions are weighted in aggregate for each agent instead of separately. It is identified and proposed, after the comparison between the WPR family and the WPAR subfamily, a sharing rule within the WPAR subfamily that, without reaching, as it has been indicated, the values of the optimal efficient equilibrium level, it is able to generate equilibrium efforts closer to the optimally efficient equilibrium efforts.

The work contained in this doctoral thesis opens up interesting and promising lines of study and research that delve into the interdependence or complementarity between the agents and the different levels of effort carried out, as well as the development, among other lines or ways, of the study of bilateral models with multiple cost reductions and the impact that these efforts have on them.

Introduction

Preliminaries: decision making

In the day-to-day reality, the human being as an individual in his personal sphere or the human being as an agent or integral part of a social, economic or professional ecosystem, is continually subjected to the need to make decisions in situations or scenarios that pose more than one possible alternative. Each individual, faced with each decision to be taken, according to his capacity and available information, must determine the best or optimal solution from among all the possible options in order to maximize the benefit or minimize the costs or losses.

Consequently, the need to face decisions is a constant in every aspect of human existence (Bradley, 2017), being these logically more complex as the development achieved is greater, so that decision making, from any field or area whether personal, professional, individual, collective or institutional, directly or indirectly, conditions and shapes with each action our own degree of evolution. In other words, determines and defines our own development and existence.

Economics, as a social discipline intrinsically attached to the events of everyday life, was born, among others, with the aim of analyzing and identifying ways to offer solutions to the continuous questions that arise from a human nature already defined long time ago by Aristotle as "social animal".

Questions and problems that inevitably arise as a consequence of the continuous interactions centred on obtaining means or resources to satisfy needs. In other words, one of the ultimate aims of economics is the study of the different models of organisation and distribution of resources that are not unlimited. The way in which these resources are organised and distributed will be the result of a complex set of conflicting decisions in which the need to act strategically is assumed to be necessary to obtain optimal results.

If resources were not limited, the existence of risk would naturally be less, and therefore decisions would be of significantly less relevance, and certainly economics itself as a discipline might find no reason to continue to exist.

The limitation of resources leads directly or indirectly to the fact that the number of possible decisions or strategies to adopt aimed at satisfying needs is also not unlimited and will force us to continually redefine new strategies and new decisions that will always have the same objective: to optimise the allocation of limited resources (Poveda and Bualó, 2024) or, in other words, to obtain the maximum possible satisfaction with respect to the degree of scarcity of the resources available to us.

As Braund (2005) significatively stated, development or underdevelopment, growth, stagnation, evolution or individual or joint backwardness have depended in the past and will continue to depend in the future on the degree of success and the degree of success of each decision strategy adopted.

Logically, each strategy and each decision is transcendental for the agents who must make them, and it is this importance that makes it necessary to develop a theoretical body with the capacity for real and practical application that allows, through study and analysis, to minimize the probabilities of error and maximize the options for success.

Mathematics and its application to the so-called social sciences allows us to respond to the need to model the situations or decision scenarios that take place in real life, situations or conflict scenarios in which there is a certain degree of opposition in the interests of each player and the achievement of these interests will depend on a decision-making strategy that, to a greater or lesser extent, will be strongly conditioned in turn by the decision-making strategies adopted by the other players.

Game Theory, from its economic and mathematical origin, arises specifically as a response, as Bonome (2010) defines, to the need to analyze and study decision making or, in other words, the strategies that each player can adopt in order to obtain the maximum benefit in any possible scenario where he cooperates or competes against other agents.

Game Theory: origins and brief historical approach

Although there is a more than wide consensus in recognizing the contributions of von Neumann and Morgenstern (1944) as the starting point of game theory as an area or scientific discipline with its own identity and independence, it was not until two centuries ago, in the eighteenth century, when the first references were published that were consolidated as precursors proposing the basis for an imminent formal mathematical development that would determine the theoretical frameworks of the same.

As detailed by Villalón and Caraballo (2015), in their publication "A walk through the history of Game Theory", the works of G.W. Leibniz (1704), P.R.Montmort (1713) or A.N. Caritat (1785) where, respectively, the appearance of a new model of scenarios or logical reasoning games from the premises of probability was noted, the concept of mixed strategy and the proposals of solution through the minimax rule were introduced for the first time, and where, finally, in the work of the Marquis de Condorcet, the Jury Theorem and the well-known Condorcet Paradox were shown for the first time. All of them were contributions of notable relevance that opened up avenues of research hitherto unknown.

The 19th century marked the beginning of a growing production that reflected the incipient interest generated. The first publications on mathematical modeling appeared which, in the end, would be the embryo that would define and lay the future foundations of the discipline. In 1913 the work of F. Zermelo and E. Borel (1913) stands out among them. Borel (1913) laid the early foundations of Game Theory by proposing models between pairs of players who interact moved by opposing or conflicting interests, as bipersonal zero-sum games with imperfect information. Although it is true that this publication focused on the characterization of strategies, it did not, however, show any interest in identifying the existence of an optimal strategy above the rest. It was Borel (1920) who for the first time identified and proposed the existence of a single optimal strategy in certain game scenarios.

During the course of the century, a succession of authors and works, each time with greater projection, developed the bases of a future game theory that already showed a clear potential for its application to more and more problems or situations of everyday life. The works, among others, of Cournot (1838), Bertrand (1883), or Edgeworth (1881) solidly consolidated the lines that in the near future would give rise to the birth and irruption of Game Theory.

The first half of the twentieth century witnessed an exponential increase in academic production that continued to develop and expand both the theoretical bases and new practical scenarios of a future game theory that was about to burst forth in full force. Authors such as Steinhaus (1925) formally delimiting for the first time the concept of strategy or H. Loomis (1946) delving into the proof of the minimax rule proposed by Neumann, are a very succinct example of the unstoppable interest aroused by a discipline that revealed an unusual potential for use and a vast range of applications from its beginnings.

The milestone reached and the repercussion caused by the publication in 1944 of "Theory of games and economic behavior" by Neumann and Morgenstern would mark the moment in which Game Theory would definitively consolidate itself as an independent mathematical branch with the consideration as a scientific discipline and with its own identity. The Nobel Prize awarded to John Nash together with C. Harsanyi and R. Selten in 1994 for their contribution to game theory, introducing the essential concept of equilibrium and analyzing the so-called "market imperfections", recognized for the first time with this award its significance in economic theory, rewarding, in the words of the Royal Swedish Academy of Sciences, "the pioneering analysis of equilibria in the theory of non-cooperative games".

Fifty years had passed since the 1944 publication of Neumann and Morgenstern and Game Theory, as S. Monsalve (2003) rightly defined it, had unanimously reached the status of a fundamental discipline of modern economic analysis.

The Nobel Prize awarded to Nash was not the only one in the field of economics to recognize game theory as a tool for understanding and modeling decision-making processes. Mirrlees and Vickrey in 1996 would obtain it for their work on Game Theory and asymmetric information. Aumann and Schelling would do the same in 2005 for their work on conflict and cooperation scenarios. Hurwicz, Maskin and Myerson were awarded in 2007 for their contributions to the theory of mechanism design and market efficiency. Shapley and Roth in 2012 won the same award for "the theory of stable allocations and the practice of market design", Thaler in 2017 for his work on the possible absence of rationality in decision making, and Wilson and Milgrom were also recognized with the same award in 2020 for their work on the improvement of auction theory.

As can be seen, Game Theory, at present, has reached a notable and consolidated degree of maturity that allows it a relevant status and position, beyond Economics itself, in areas of knowledge a priori as disparate as, among others, Computer Science, Biology, Engineering, Medicine, Political Science, Marketing, Sociology or even within the field of military strategy or even too in Literature, where appears as an significant element within the development of the plot of novels and literary narratives in which the strategic choice of characters is treated (Brams, 1992) or it has been desired to analyze or model the choices of the characters using this discipline (Mehlmann, 2000; Mozaffari and Eghbal, 2020).

With the imminent development of calculation and computation capacities together with the exponential advances in quantification, identification and data management, essential tools are provided which, when applied to Game Theory, will allow it to reach a capacity of faithful reflection of reality that will lead to levels of accuracy in the proposals of strategic solutions not yet reached, projecting for this discipline a long way to go and a future as solid as it is remarkably promising.

Conflict, rationality and interdependence

Game Theory seeks, from the analysis, to propose or identify the optimal solution in scenarios with interactional and decision making problems (Sohrabi and Azgomi, 2020), environments that also could be defined as involving per se a conflict of interests and presenting a marked strategic interdependence among the players. The individual rational behavior of any agent involved will depend, therefore, on his capacity for analysis and also on his capacity to anticipate the actions of the other players in situations, generally, of incomplete or imperfect information.

Logically, we can deduce that the lesser the amount of information available to each player, the greater and deeper the strategic capacity for analysis and anticipation must be in order to define the optimal behavior of each player when the cost or benefit of each action is not given and depends in turn on the actions of the other players.

Strategic interaction, as Hammerstein (1973) indicated, is an intrinsic consequence of the existence of conflicts of interest in scenarios where the behavior of each agent will be conditioned by his own decisions, taking into account that these decisions will in turn be conditioned by the decisions of the other agents.

Consequently, conflict and the strategic way of behaving or acting in the face of it is nothing more than the way in which each player will try to satisfy his own interest to the maximum extent possible, either in a confrontational manner, without any type of transaction or prior agreement, or through agreements for which consensus will be necessary with each agent who, in turn, will also look after his own interest.

By definition, conflicts exist logically and naturally at all social levels, both individually and collectively and from any approach or prism (Bashir et al., 2020), will exist whenever the interests or motivational priorities (Hand, 1998) of one player are different from the interests of another player with whom he interacts, so that this scenario must be rationally managed by all the agents involved if they all seek to obtain the highest possible payoff, benefit or utility value.

It is therefore necessary to identify four pillars or basic elements on which every interaction between players is generally based :

a) Conflict Scenario: Each player pursues the maximum benefit for himself without taking into account the degree of benefit that the rest of the players may finally achieve. Arrow (1951), in a theorization that is still accepted and valid today, and from the point of view of rational utilitarianism, defined conflict as the scenario that occurs when each individual player seeks to satisfy his own interest without taking into account the possible preferences of the other players. He thus established a conceptual framework that is still fully valid today to determine both the rational behavior of each agent and the very structure or foundation of game theory.

b) Strategic behavior: the existence of a conflicnt leads each player, depending on his interests and the information available to him and using the knowledge ot the others players payoffs (Rapoport, 2012), to develop or adopt a certain strategy of action, that will determine the way in which he interrelates respectively with the other players.

c) Rational behavior: This strategy will be focused in order to maximize the satisfaction of individual self-interest. The search for self-interest in turn defines the character of interactions between agents in the framework of game theory, presupposing and taking for granted that every agent will act rationally. To this effect, it is necessary to refer again to the work contributed by Arrow (1951), to understand the relationship between the concept of rationality and utility, defining the former as the behavior focused on obtaining the highest individual utility, rejecting any decision that does not maximize it.

Consequently, players act in an attempt to optimize their utility functions in both competitive and cooperative scenarios, reflecting rational behavior that seeks to maximize their expected allocation of cooperative benefits (Chen, 2024). However, it should not be forgotten that, sometimes, altruistic behavior can also be integrated and contemplated in utility functions.

d) Interdependence: Finally, ties of dependence is the last feature that conditions and determines the behavior of all agents insofar as their decisions will always be conditioned by the decisions of the other players. Therefore, it will always should be necessary to reach points of equilibrium, coalitions or unions that allow the materialization of agreements between two or more players. The non-existence of such agreements would only deprive the purpose or objective of the agents of meaning insofar as the veiled intention to extract, dominate or subordinate some agents to others would be evident, and would also deprive the reason for a game scenario and, logically, the purpose of the game itself of all meaning.

Confrontation or Cooperation

Having arrived at this point and having conveniently identified the structure on which every game is based and takes place, it is equally appropriate to focus on the relationship that causes every interaction to entail by nature the existence of a conflict that must be resolved strategically by each player in a rational way (Schelling, 1958), either competing against another player or cooperating with a group or set of players.

Game Theory, faced with the concept and existence of conflicts, offers unique and remarkable advantages as a mathematical tool capable of analyzing and even modeling such this conflict situations (Rapoport, 2012) totally or partially, allowing to identify possible vectors of strategies between antagonistic agents and finding common objectives between them that move to points of equilibrium or agreement.

To this end, a distinction is made between two possible scenarios or game models: cooperative or non-cooperative or competitive, depending on whether or not decisions are made with or without commitments to the other players, or depending, for example, on whether or not there is transparency in the coalitional costs, the prior possibility for the players involved to share information, or the possibility of reaching prior agreements that allow them to adapt or coordinate their strategies in order to maximize the incomes for all the players involved in a consensual manner. Cooperative games, turn, can also be classified into games with or without transferable utility [1] depending on whether or not the utility that motivates cooperation can be tangible (Peters and Peters, 2015) in the sense of whether or not it can be transferred as currency from one player to another.

In cooperative environments, where the aim is to maximize the benefits derived from cooperation, perhaps the analysis or identification of the strategies of each player will not have excessive relevance insofar as each agent seeks to cooperate to increase his profits, so that the coalitional payments resulting from cooperation will be the key element or aspect that determines the incentive so that, effectively, all the players decide and accept

[1] Games with Transferable Utility (TU Games) , as Myerson (1992) stated , could be considered as an special case of NTU- Games (Games with nontransferable utility)

to cooperate.

Given that, as rightly synthesized by R. Sexton (1986), for any game cooperation between agents is absolutely voluntary, no player would decide to cooperate if he does not perceive that he can obtain a benefit from it, so that the existence of stable coalitions of players will only occur insofar as such stability is a consequence of the absence of any other possible coalitional alternative that improves the results of the first.

We are faced with two possible scenarios: cooperative or competitive, two scenarios in which there is a conflict of interests (Aumann and Schelling, 2005) that is resolved through negotiation in different ways and in which, indistinctly, each player will always act rationally trying or pursuing to maximize or optimize his payment function , either by confronting another player and trying to win or obtain a benefit to the detriment of his partner, or by using the group as a way to achieve what would not be achievable or feasible individually or through confrontation or absence of cooperation. The way in which each player acts in pursuit of his own interest will make agreements possible or, on the contrary, will lead the players to an impasse with no coalitional value.

From the competitive approach, the interaction under only and exclusively non-cooperative models, entails, in certain scenarios, certain difficulty for its projection or real application beyond the theoretical modeling. An example is the approach of the minimax theoretical body (Neumann, 1928) where, given the mathematical nature of game theory, the existence of an "optimal" strategy in zero-sum games has been a standard in non-cooperative models that has practically been unanimously consolidated as an instrument of analysis in decision making. This theory, that although from the theoretical approach the optimal or ideal way of proceeding in a game can be clearly reflected, from the empirical or practical point of view and applied to the reality of human interactions, it only hints at certain limitations since it is hardly feasible to comply with the conditions required for its application (Riker, 1992), being perhaps one of the most evident limitations the conditional requirement of zero sum, in which there are two entirely confronted interests, whereby everything that a player wins must correspond to what his opponent loses, a scenario that can hardly be observed in a reality not only because of the forcefulness of the restrictions but also because of the significant risk and the absence of incentives or stimuli to conform a game that would not mean anything other than the complete elimination or defeat for one of the two participants.

On the other hand, Cooperative Games do not contemplate a scenario of loss for some players in favor of others, in them, the players pursuing the so-called common good (as a way of expressing the maximum possible benefit for each and every one of the agents involved in the game), will be predisposed to reach certain agreements that entail sharing and even making certain sacrifices aimed at obtaining a benefit that optimally compensates them.

Consequently, Game Theory as a tool, and especially from a social science such as Economics, cannot and should not disassociate itself from reality , in spite of the unbounded complexity and multitude of variables that determine it. In other words, game theory must be able to reflect any possible scenario of interaction, integrating the possible variables and determining factors that can shape and define any game situation in today's interdependent environment, (Zhang et al., 2023) made up of complex dynamics that is permanently changing.

Riker (1992) aptly describes the complexity of human action and the counterproductive nature of trying to establish normative elements common to all types of interaction instead of generalizing descriptive aspects, leading to the presentation of a biased interaction that could hardly offer a rigorous analysis and, therefore, could hardly be presented as an adequate tool to identify possible suitable strategies of behavior if it has not previously been able to contemplate without any type of distortion and as accurately as possible any game or conflict scenario.

In 1992, Ray Noorda, former CEO of Novell, broke with the classical theoretical conception, prevailing since Adam Smith (Dowling, 2020) that only contemplated the competitive path, and polularized the term "Coopetition" [2] to describe the need to compete and coop-

[2] Although R. Noorda is credited with having popularised the concept , as Marlon et al (2024) states the term Coopetition was, decades ago, first introduced by Cherington (1913) in his work entitled "Advertising

erate at the same time as a formula to achieve a return that could not be achieved only by competing. Five years later, Nalebuff and Brandenbruger (1997) consolidated the term coined by Noorda and took a step forward to propose, from the field of game theory, collaboration between competitors as a way to maximize profits or minimize costs and improving from the way in which, until this moment, interfirms strategic alliances and competitive dynamics were understood (da Silva and Cardoso, 2024).

The search for utility from rationality, which can be considered as an interactive rationality (Colombatto et al., 1996), moves each player to act solely interested in obtaining the best payoff or reward, either acting independently or taking into account the actions of other players. This implies that each player will carry out any behavior that leads to maximizing the profit pursued, which leads us on a wide range of occasions to collaboration between agents, collaboration that moves agents to adopt "hybrid" or mixed interaction strategies, in which competitive phases coexist with cooperative phases.

Brandenburger and Stuart, in 2007 modeled, presenting a new class of transferable utility cooperative games (Biform Games), the way in which competition and cooperation hybridize each other to conform payoffs that depend on the previous competitive strategies of each player.

In this type of strategy, prior to cooperation or coalition building, each agent will "prepare" by competing or negotiating, trying to place himself in an advantageous position that allows him to obtain the highest utility or payoff by using or investing the minimum amount or value of resources in the collaborative phase.

Objectives, methodology and contributions

The present thesis work is based on three central objectives, objectives that all start from a single motivation that, from the necessary perseverance and humility, determines and configures as a premise the strong will to provide an added value that contributes to reduce

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as a business force", in which he includes the statements of K.S. Pickett defining the relations among oyster dealers.

or shorten the distance between the theoretical formulation and the practical application in the modeling and allocation criteria, within the scope of Game Theory, of the conflict scenarios of Cooperative Cost Games of transferable utility, where the interaction and cost reduction occurs bilaterally between pairs of players, such bilaterality being coalitionally independent.

The first objective is focused on the development and extension of the concept of roles of benefactor (the player who will always reduce the cost of the rest of the players who are in the same coalition) and beneficiary (the player who will always see his costs reduced as a consequence of joining coalitions in which there is a benefactor) studying the possibility of their performance in a dual manner, that is, studying the effects that happen when the same agent acts with duality of roles, both as benefactor and beneficiary. This objective arises in turn, and inherently to the above, from the analysis of models in which there is a single beneficiary or beneficiaries coexist with benefactors without any player playing both roles simultaneously, i.e., acting as both benefactor and beneficiary in any coalition.

Once the effects and nature of role duality in cooperative environments have been determined, the next step is to analyze the duality of cooperative and competitive behaviors in bilateral cooperation models between pairs of players, which, depending on their degree of cooperation, can also be identified according to the dual or non-dual performance of the roles indicated in the first chapter.

The second objective is focused by one hand on the analysis and study of the effects, under the structure of biform games, of a specific type of asymmetric cooperation that takes place bilaterally between pairs of agents with the common objective of reducing their individual costs between pairs. By other hand is focused on the study and analysis about the impact of pairwise efforts on cost reductions and the result cost structure for this framework while also considering a family of cost allocation rules with pairwise reduction wighted separately. A family of cost allocation rules with weighted pairwise reductions (WPR) is studied, which includes the Shapley value (when all players exert half of the possible effort) and which is a subset of the core of this type of games. Going deeper into the study of the sharing family (WPR), we finally consider the existence of effort equilibria in this competitive phase. (PEE)

The third and last objective is focused on finding, within the WPR family of allocations, a sub-family where pairwise reductions are not weighted separately but, instead, are weighted in aggregate. This sub-family is referred to as the WPAR. On the other hand, the aim is to compare the properties of the two identified allocation families.

The methodology used in general in this thesis is based on the analysis and application of Transferable Utility Cooperative Cost games, using for this purpose the modeling in the form of biform-games (Brandenburger and Stuart, 2007), in order to present and define a new class of games that study the costs and challenges associated with the establishment of a situation of paired effort.

In addition to the above, the methodology applied is structured on the basis of the use of the mathematical framework as a vehicle tool that allows us to observe and analyse the interactions between agents and the results of such cooperation by applying optimal and accurate sharing rules.

As contributions of the work developed and reflected in this thesis document, it must be indicated:

As a first contribution, a new class of cooperative cost games is presented, associated to the coalitional cost models with multiple benefactors, which imply both a less restrictive definition of the concept of beneficiary and a generalisation of the concept of benefactor insofar as it does not contemplate a single benefactor but a multiplicity of them as well as their irreplaceability and the performance of a double role, or what is the same, the possibility that a benefactor can act simultaneously as such and as a beneficiary in any possible coalition in which he/she is integrated.

In addition to this, we formulate a simplified expression of the Shapley value for this new class of games, once the concavity of the new class of games has been proved, to demonstrate the validity and suitability of the Shapley Value as a sharing rule capable of guaranteeing coalitional stability, adequately recognizing the effects of give and take and rewarding especially the dual role played by the benefactors, while presenting a remarkably intuitive and simplified expression of the same that greatly facilitates its calculation.

As second contribution, the introduction of PE-Games (Pairwise Effort Games) a new class of biform games. In PE-Games, a doubly robust cost sharing mechanism is presented. That mechanism not only has good properties regarding the cooperative game in the second stage but also creates appropriate incentives in the non-cooperative game in the first stage that enable efficiency to be achieved.

As third contribution, on one hand the finding of a loss of efficiency when cooperation is restricted or limited only to an aggregate cost reduction by pairs. On the other hand, as a culmination of the work done, to identify, within the family of WPR allocations, a sub-family where pairwise reductions are not weighted separately but instead are weighted in aggregate. This sub-family is called WPAR. We find that this sub-family generates in the competitive stage the unique efficient effort equilibrium.

It is demonstrated that the solution of the cooperative game determines the incentives of agents to make an effort in the first stage, and consequently the efficiency of the final outcome.

Thesis Structure

This thesis document is organized in three chapters. As a due presentation and facilitating the course of the totality of the content, the present work opens with an introduction that serves as a guided itinerary through a brief historical approach of the discipline of Game Theory, to, subsequently, go through the spirit, essence and framework concepts that, within this discipline, are constituted as the necessary foundation that has allowed, on the same, to raise and carry out this doctoral thesis.

The first chapter, titled as "Corporation Tax Games with dual benefactors", corresponds to the book chapter ***"The Shapley Value of corporation tax games with dual benefactors"*** published in Handbook of the Shapley value in 2019 (see Appendix E). It starts with a brief introduction in Section 1.2. Then, the cost-coalitional problems with multiple dual and irreplaceable benefactors and some of their properties are described. After that, in Section 1.3, the class of cooperative cost games associated to cost-coalitional problems with multiple dual and irreplaceable benefactors, the so called multiple corporation tax games, is introduced. Section 1.4 presents a simple and easily computable expression for the Shapley value of multiple corporation tax games. An example illustrating the model and the role played by dual and irreplaceable benefactors is given in Section 1.5. Finally some concluding remarks and highlights for further research are collected in Section 1.6.

The second and third chapters correspond to the article ***"Efficient effort equilibrium in cooperation with pairwise cost reduction"*** published in Omega in 2023 (see Appendix F) . The second chapter titled "Efficient Effort Equilibrium in cooperation with pairwise cost reduction" , after a brief motivation and introduction is organized as follows. Section 2.2 introduces the biformal game and describes in detail the two stages in which the model is developed. Section 2.3 is devoted to analyze the second stage which is a cooperative game. In this cooperative game, agents reduce each other's costs as a result of cooperation, so that the total cost reduction of each agent in a coalition is the sum of the reductions generated by the rest of the members of that coalition. Section 2.4 studies the first stage, i.e. the non-cooperative game that precedes the cooperative game of the second stage. Here, agents anticipate the cost allocation that results from the cooperative game in the second stage by incorporating the effect of exerted effort in their cost functions. It is considered a family of cost allocation rules (in the second state) with separately weighted pairwise reductions (WPR family) and obtain the corresponding effort equilibria in the first state. Then, we develop a general and complete analysis of the efficient effort equilibria. Finally, in this section is found the kernel assignment rule in this WPR family that generates the unique efficient stress equilibria.

Finally, the third chapter is titled "Measuring Efficiency for Pairwise Aggregate Reduction" and is organized similarly to the previous two chapters. After a brief initial introduction, Section 3.2 focuses on a subfamily of the WPR family in which the pairwise reductions are not weighted separately, but are weighted as an aggregate reduction, the WPAR family. It is found that the level of efficiency is lower than that achieved when the pairwise reductions are weighted separately for each agent. Next in Section 3.3 the rule is identified, within this WPAR family, that generates the equilibrium efforts closest to the efficient ones. Finally, Section 3.4 completes the study of our model by comparing the two families of kernel allocation analyzed.

The thesis structure ends with a general conclusion and future researchs, a bibliography part and a last part dedicated to the appendices of the previous chapters, which contain the proofs of the results and summary tables (notation and optimization problems).

Chapter 1

Corporation Tax Games with dual benefactors

1.1 Introduction

In recent years, as a result of an eminently globalized environment, the debate on the necessary cooperation among states and firms has been intensified. The absence of this cooperation among countries can cause both a race to the bottom tax competition in fiscal policies and opacity or financial secrecy. On the part of firms or individuals, it can cause underground economy, tax evasion or fiscal fraud. All of them are inefficient behaviors.

In particular, the underground economy is a significant problem and difficult to deal with. The causes and negative effects of the underground economy have been debated by authors as Feige (2016), Sandmo (2012), Schneider (2000), and Bajada & Schneider (2018), among other authors. The solutions to be adopted to detect and reduce the underground economy have been studied, for example, by Slemrod & Yitzhaki (2002),Torgler (2011), Keen & Slemrod (2017),Williams (2017), or Dell'Anno (2009). Three solutions of particular relevance are the design of optimal tax systems, the increase in transparency and information, and a greater severity of the punishments. These elements allow to increase the capability to detect and discourage the infringing behaviors. These efforts not only benefit the states themselves by allowing an increase in tax collection, but also benefit all the firms that act in accordance with the law, since it eliminates the competitors that acted in a submerged manner.

However, carry out effective policies focused at combating the underground economy, requires a high economic cost in human and material resources that must be faced by the countries governments. Cooperation among countries and firms could reduce these costs. For example, cooperation among countries could be based on the desire for transparency and the transfer of information in order to facilitate the detection of fraudulent behavior, allowing a reduction of costs. In addition, beyond the mandatory legal requirement, a firm can make an effort to improve the transparency of its financial practice. The firm can also just share any kind of relevant information with the tax authorities. This cooperation could be rewarded by a tax reduction.

Inspired by the Spanish tax system, Meca & Varela-Peña (2018) introduce a cooperative model, where the Government is considered the only benefactor, as it keeps costs at the same level, zero cost, while reduce the costs of those investors who act legally (beneficiaries). Investors may decide to cooperate or not cooperate with the Government. If they decide to cooperate, the Government will provide a framework of legal certainty, which is in their benefit. On the contrary, if investors decide not to cooperate with the Government and try to defraud the system by tax evasion, they can be detected and charged with unlawful behav-ior. Once this irregular behavior is demonstrated, they will be punished and required to return all amount defrauded plus a penalty. This means that the costs of not cooperating with the Government would be higher than cooperate, and so all investors are willing to pay the lowest taxes under legal protection of the Government. The authors present the class of corporation tax games as an application of linear cost games to the corporate tax reduction system.

Linear cost games were introduced by Meca & Sosic (2014) as a particular case of k-norm cost games with benefactor and beneficiaries, when k = 1. The authors introduce a class of cost-coalitional problems, which are based on a priori information about the cost faced by each agent in each set that it could belong to. Then, they focus on problems with decreasingly monotonic coalitional costs. In their work they study the effects of giving and receiving, on cost-coalitional problems, when there exist players whose participation in an alliance always contributes to the savings of all alliance members (benefactors), and there also exist players whose cost decreases in such an alliance (beneficiaries). Meca & Sosic also show that when there are multiple benefactors, an agent sees the same individual costs in any coalition that contains at least one benefactor and is not all-inclusive. Thus, with a single benefactor all the members of a coalition may see their cost increase if he leaves the group; they say that he is irreplaceable.

On the other hand, when there are several benefactors, the cost of a member of the coalition remains the same as long as there is another benefactor in the coalition; they say then that each benefactor in this case is replaceable. They study separately the two cases, and use linear and quadratic norm cost games to analyze the role played by benefactors and beneficiaries in achieving stability of different cooperating alliances. Different notions of stability, the core and the bargaining set, are considered there and provided conditions for stability of the grand coalition which leads to minimum value of total cost incurred by all agents.

In this chapter, it is presented a new model of corporate tax system with several firms and countries (multiple dual benefactors). Countries are dual in the sense they are benefactors (they reduce the cost of both firms and other countries) and beneficiaries (the information provided by others countries reduce its cost). They are also irreplaceable benefactors because all the members of a coalition may see their cost increase if one of them leaves the group. It differs from the corporate tax system given by Meca & Varela-Peña (2018) in the following three points. First, there is a single benefactor there. Moreover, the definition of benefactor given by them is a particular case of the definition of dual and irreplaceable benefactor given here. It is noted that dual benefactors here generalize benefactors there. Second, the concept of beneficiary in their paper is less restrictive than the one considered here. It is also noted that a beneficiary here is a beneficiary in the corporate tax system given there (see Section 1.2 for more details). And third, it is proposed here the Shapley value (Shapley, 1953) a as stable allocation rule for sharing the reduced total costs. Meca and Sosic, (2014) and Meca and Varela-Peña (2018) proved that the grand coalition is stable in the sense of the core, but they didn't study the Shapley Value. Here we present a simple expression for the Shapley value of multiple corporation tax games that benefits all agents and, in particular, compensates the benefactors for their dual role and irreplaceable character. As a representative and recent survey of this allocation rule, it may be cited the work done by Moretti & Patrone (2008). More recently, Li et al. (2024) show the remarkable application and usefulness of this allocation rule with respect to the undeniable success of Machine Learning.

1.2 Cost-coalitional problems with multiple dual and irreplaceable benefactors

Let $E = \{1, 2, .., e\}$ be a set of firms, and $P = \{1, 2, .., p\}$ be a set of countries, with $S_j^i \geq 0$ and $\bar{S}_j^i \geq 0$ be respectively a tax and a reduced tax that firm j pays in country i, with $S_j^i > \bar{S}_j^i$. Let $N = E \cup P$ denote the set of all agents (firms and countries), with $|N| = n = e + p$, where $e \geq 1$ and $p \geq 2$. We define $T \subseteq N$ as an arbitrary set of agents in N. If two given countries are in a coalition T, then they cooperate and share information, which implies that they can reduce their levels of tax evasion and underground economy. The size of the reduction depends on how much information a country has and how relevant it is for the other country. Note that, for a country i, the more countries are in a coalition with it, the more relevant information this country gathers, and consequently, the smaller the degree of tax evasion and underground economy it has. Formally, let w_i^T be a measure of the underground economy and tax evasion of country i when it is in a coalition T, thus, given two sets $T \subseteq T' \subseteq N$, we assume that always $w_i^T > w_i^{T'}$ if $(T' \backslash T) \cap P \neq \emptyset$, and $w_i^T = w_i^{T'}$ otherwise. Therefore, always $w_i^T \geq w_i^{T'}$. We denote by w_i a countrys'stand alone measure of tax evasion, i.e., $w_i = w_i^{\{i\}}$.

Any agent $k \in T$ incurs certain non-negative cost, which depends on the subset T. We denote this cost by c_k^T, and by c_k an agents'stand alone cost, i.e., $c_k = c_k^{\{k\}}$. For any coalition $T \subseteq N$, the cost of agents are:

1. $c_j^T = \sum_{i \in P \cap T} \bar{S}_j^i + \sum_{i \in P \setminus (P \cap T)} S_j^i$ for all $j \in T \cap E$, where firm $j \in T$ must pay a tax $\bar{S}_j^i$ to country i if $i \in T$, and S_j^i if $i \notin T$.

2. $c_i^T = g_i\left(w_i^T\right)$ for all $i \in T \cap P$, with g_i being an increasing function such that for all $i, i' \in P$ and for all $T \subseteq N$, where $i, i' \in P \cap T$ it always holds that $g_i\left(w_i^{T \setminus \{i'\}}\right) - g_i\left(w_i^T\right) = z_{ii'}$, with $z_{ii'} > 0$ being how much the country i' reduces the cost of i with the information i' shares with i. [1]

The function g_i measures the cost incurred by country i with a given level of underground economy. Behind the g_i function lies the efectiveness of the resources of country i for a given level of underground economy.

Next, it is identified two special roles that all the agents can play in the model, being benefactors and beneficiaries.

Definition 1.1 *A benefactor is an agent $\bar{k} \in N$ such that for any set $T \subseteq N \setminus \bar{k}$ and for all $k \in T, c_k^T \geq c_k^{T \cup \{\bar{k}\}}$, in addition, for at least one agent $k \in T, c_k^T > c_k^{T \cup \{\bar{k}\}}$. The agents whose cost decreases in an alliance with a benefactor are denoted by beneficiaries.*

The following lemma characterizes the agents of the game as benefactors and beneficiaries.

Lemma 1.1 *An agent k is a benefactor if and only if it is a country. However, both firms and countries can be beneficiaries.*

There are agents that are dual in the sense that they are benefactors and beneficiaries, these are the countries. However, the firms are exclusively beneficiaries. The proof of Lemma 1.1 and all the results of this chapter can be found and consulted in Appendix A.

The following definition is a relevant property of a benefactor.

Definition 1.2 *A benefactor $\bar{k} \in T \subseteq N$ is irreplaceable if $c_k^T \neq c_k^{T \setminus \bar{k}}$ for at least a $k \in T \setminus \bar{k}$.*

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[1] We assume $zii\prime > 0$, thus, countries are always benefactors. However, $zii\prime$ could be as close to zero as we want, i.e., the information that a country shares with other country can be negligible. Therefore, in the limit case in which $zii\prime = 0$, the results should hold. In any case, a wider generalization of this model will be consider in future research.

Lemma 1.2 *Countries are irreplaceable benefactors.*

It is noted the vector of individual agents' costs in all possible subsets by

$c^N = \left(c_k^T\right)_{k \in T, \emptyset \neq T \subseteq N}$.

Thus, the set of agents N and the cost coalitional vector c^N define a cost-coalitional problem with multiple dual and irreplaceable benefactors $\left(N, c^N\right)$.

A desirable property is that cooperation is beneficial. It can be guaranteed, if the cost in large subsets do not exceed their cost in smaller ones. The following definition formalize this idea.

Definition 1.3 *A cost-coalitional vector c^N satisfies cost monotonicity if $c_k^T \geq c_k^{T'}$ for all $k \in T$, with $T \subset T' \subseteq N$.*

The following lemma shows that the cost-coalitional problem with multiple dual benefactors has this property.

Lemma 1.3 *The cost coalitional problem $\left(N, c^N\right)$ has the property of cost monotonicity.*

Next step is to define cost games related to the cost-coalitional problem with multiple dual benefactors and prove the cooperation in beneficial for all the agents in the model, benefactors and beneficiaries.

1.3 Multiple corporation tax games

For a given cost-coalitional problem with multiple dual and irreplaceable benefactors $\left(N, c^N\right)$ we define the multiple corporation tax game (N, c), where $c(T) = \sum_{k \in T} c_k^T$ for all $T \subseteq N$, and $c(\emptyset) = 0$.

The following issue is considered. Is it profitable for the agents in N to form the gran coalition to pay lower taxes and so reduce the degree of tax evasion? Here, it is proved that the answer to this question is positive because (N, c) is a subadditive game, in the sense that $c\left(T \cup T'\right) \leq c(T) + c\left(T'\right)$, for any $T, T' \subseteq N$, and $T \cap T' = \emptyset$. Notice that the subadditivity condition implies that if N is partitioned into disjoint coalitions (whose integrants reduce the degree of tax evasion) the corresponding cost will not decrease.

In fact it is demonstrated that (N, c) is not only subadditive but also concave, in the sense that for all $k \in N$ and all $T, T' \subset N$ such that $T \subset T' \subset N$ with $k \in T$, then $c(T) - c(T\backslash\{k\}) \geq c(T') - c(T'\backslash\{k\})$. It is a well-known result in cooperative game theory that every concave game is subadditive. Moreover, the concavity property provides with additional information about the game: the marginal contribution of an agent diminishes as a coalition grows. It is wellknown as the snow ball effect. For more details on cooperative game theory see, for example, González-Díaz et al. (2023).

First, in the Lemma 1.4, it is found out which are the cost marginal contributions of the agents (firms and countries).

Lemma 1.4 *Let $\left(N, c^N\right)$ be a cost-coalitional problem with multiple dual and irreplaceable benefactors and (N, c) the associated multiple corporation tax game. Then, for any $T \subseteq N$,*

1. *if $j \in E \cap T$, $c(T) - c(T\backslash\{j\}) = c_j^T$;*
2. *if $i \in P \cap T$,*

$$c(T) - c(T\backslash\{i\}) = c_i^T - \sum_{j \in E \cap T} \left(S_j^i - \bar{S}_j^i\right) - \sum_{i' \in P \cap T\backslash\{i\}} \left(g_{i'}\left(w_{i'}^{T\backslash\{i\}}\right) - g_{i'}\left(w_{i'}^T\right)\right).$$

In point 1, this proposition states that a firm j always contributes to a coalition $T\backslash\{j\}$ exactly with its cost in coalition T, which is c_j^T. As a firm is always and exclusively a beneficiary in this model, it has not effect in the cost of others agents: either countries or firms. However, a country is a benefactor to both firms and others countries, therefore, its marginal contribution is smaller than its cost in coalition T. If country i is withdrawn from a coalition T, the individual cost of firms and others countries in coalition T increases.

The following Theorem states that our class of games are concave.

Theorem 1.1 *The multiple corporation tax games (N, c) are concave.*

So it is proved that in a cost-coalitional problem with multiple dual and irreplaceable benefactors $\left(N, c^N\right)$ it is efficient that all firms pay lower taxes and all countries manage to jointly reduce their degrees of tax evasion. In that case, the reduced total cost is given by $c(N) = \sum_{i \in P} c_i^N + \sum_{j \in E} c_j^N$.

An allocation rule for multiple corporation tax games is a map ψ which assigns a vector $\psi(N,c) \in R^N$ to every (N,c), satisfying that $\sum_{k\in N} \psi_k(N,c) = c(N)$. Each component $\psi_k(N,c)$ indicates the cost allocated to $k \in N$, so an allocation rule for multiple corporation tax games is a procedure to allocate the reduced total cost among the agents in N when they cooperate. An allocation rule should have good properties from the following points of view.

1. Computability. For a particular game the rule should be computable in a reasonable CPU time, even when the number of agents is large.

2. Coaltional Stability. It is very convenient that the rule proposes an allocation which belongs to the core of the cost game. This means that, for every multiple corporation tax game $(N,c), \phi$ should satisfy the following:

$\sum_{k\in T} \psi_k(N,c) \leq c(T)$, for every $T \subseteq N$.

This condition assures that no group of agents T is disappointed with the proposal of the rule, because the cost allocated to it is less than or equal to the cost it would support if its members formed a coalition to pay lower taxes, and reduce the levels of tax evasion, independently of the agents in $N \backslash T$.

3. Acceptability. The rule must be understandable and acceptable by the agents.

A very natural allocation rule for multiple corporation tax games is $\psi_k(N,c) = c_k^N$, for all $k \in N$. It has good properties at least with respect to computability and coalitional stability. Notice that, for every $T \subseteq N, \sum_{k\in T} \psi_k(N,c) = \sum_{k\in T} c_k^N \leq \sum_{k\in T} c_k^T = c(T)$

Nevertheless, the benefactors will have serious difficulties accepting the above allocation rule that rewards the beneficiaries excessively while they do not receive enough compensation for their dual role of giving and receiving.

Since the multiple corporation tax games are concave, cooperative game theory provides allocation rules for them with good properties at least with respect to items coalitional stability and acceptability. It is highlighted the Shapley value and the nucleolus, which always provide core allocations in this context (see González-Díaz et al., 2010 for details on them). Both are, in general, hard to compute when the number of agents increases.

Next, it is presented a simple and easily calculated expression for the Shapley value of multiple corporation tax games that compensates the benefactors for their dual role and irreplaceable character.

1.4 The Shapley value

One of the most important allocation rules for cost games is the Shapley value (see Shapley, 1953). As we already mentioned, the Shapley value is specially convenient for concave games: it is the barycenter of its core (see Shapley, 1971).

We denote by $\phi(N, c)$ the shapley value of multiple corporation tax game (N, c), where for each agent $k \in N$,

$\phi_k(N, c) = \sum_{T \subseteq N; k \in T} \gamma(T)[c(T) - c(T \backslash \{k\})]$, with $\gamma(t) = \frac{(n-t)!(t-1)!}{n!}$, $|T| = t$.

The following Theorem states that the Shapley value can be easily computed in the class of multiple corporation tax games. Moreover, it shows that the Shapley value provides an acceptable allocation for multiple corporation tax games: it increases the cost of a beneficiary in a half of the benefits it obtains from benefactors, and it decreases the cost of a benefactor in a half of the benefits it provided to the beneficiaries.

Theorem 1.2 *For any multiple corporation tax game* (N, c)*, the Shapley value is*

1. For all $j \in E, \phi_j(N, c) = c_j^N + \frac{1}{2} \sum_{i \in P} \left(S_j^i - \bar{S}_j^i \right)$
2. For all $i \in P, \phi_i(N, c) = c_i^N - \frac{1}{2} \sum_{j \in E} \left(S_j^i - \bar{S}_j^i \right) + \frac{1}{2} \sum_{i' \in P \backslash \{i\}} \left(z_{ii'} - z_{i'i} \right)$

From Theorem 1.2 it can be derived that the Shapley value compensates benefactors. Note first that, the cost of a firm j in the grand coalition is c_j^N. This firm j is benefited from a country i in an amount which is $S_j^i - \bar{S}_j^i$. The Shapley value reduces this benefit exactly in a half, and consequently this is the amount in which the cost of firm j is increased, see point 1 of Theorem. In addition, the country i is compensated exactly in this amount, and consequently its costs is reduced, see point 2 of Theorem. However, a country in its relation with others countries is simultaneously benefactor and beneficiary. Let's first look at the role as beneficiary of i, in any coalition, the country i is benefited from country i' in a cost reduction of $z_{ii'}$, in this case, country i plays the role of beneficiary and i' of benefactor. Thus, the Shapley value reduces the benefit $z_{ii'}$ of country i in a half, in others words, it increases its cost in this amount. Nevertheless, at the same time, the country i benefits country i' in an amount equal to $z_{i'i}$. Now, country i is the benefactor and i' the beneficiary.

In this case, the Shapley value works in the same way, it compensates the benefactor and increasing the cost of the beneficiary in a half of $z_{i'i}$. Therefore, in the relation between two countries both are simultaneously benefactors and beneficiaries, however, if $z_{ii'} - z_{i'i} > 0$, then country i could be seen as a "net" beneficiary and i' as a "net" benefactor, on the contrary if $z_{ii'} - z_{i'i} < 0$. Thus, country i can be a "net" benefactor with some countries and a "net" beneficiary with others.

In conclusion, regarding to the individual cost in the grand coalition, the Shapley values increases the cost of a beneficiary in a half of the benefits it obtains from benefactors, and it decreases the cost of a benefactor in a half of the benefits it provided to the beneficiaries. As in this model there are dual agents (benefactors and beneficiaries), the final effect on these agents depends on which role is stronger.

1.5 An example

In this example, a simple situation with two countries " A " and " B ", and two firms "1" and "2" with activity in both countries is proposed. These countries are very concerned about their own levels of underground economy, tax evasion, and fraud. To fight against this illegal behavior, these countries must to face a high economic cost in human and material resources. However, this cost can be reduced if both countries decide to cooperate and, for example, they share resources and/or information in its fight.

On the other hand, firms have to pay in each country a certain amount of taxes. Nevertheless, these firms can choose to cooperate with a particular country. For example, beyond the mandatory legal requirement, a firm can make an effort to improve the transparency of its financial practice. The firm can also just share any kind of relevant information with the tax authorities. This cooperation is rewarded by a tax reduction. In particular, country A will fix a reduction of 10%, and B will do it of 15%. Thus, each firm must pay either a tax $\left(S_j^i\right)$ or a reduce tax $\left(\bar{S}_j^i\right)$ as it is given in Table 1.1

$S_1^A = 2$	$S_1^B = 4$	$S_2^A = 5$	$S_2^B = 8$
$\bar{S}_1^A = 1,8$	$\bar{S}_1^B = 3,4$	$\bar{S}_2^A = 4,5$	$\bar{S}_2^B = 6,8$

Table 1: Tax and reduced tax of each firm (in millions of euros)

It is considered that the cost function of any country $c_i^T = g_i\left(w_i^T\right)$ has two terms. The first term does not depend on the type of coalition the country belongs to. In other words, it does not depend on the information other countries could provide. This is a kind of fixed cost. The second term does depend on which coalition the country is. In particular, $g_A\left(w_A\right) = 4 + w_A^T$ and $g_B\left(w_B\right) = 8 + 2w_B^T$. In addition, the level of underground economy or tax evasion are normalized to 1 in any coalition with only one country, i.e., without the help of others countries. Thus, $w_i^T = 1$ for any $i \in P, T \subset N$ such that $P \cap T \backslash \{i\} = \emptyset$. However, in any coalition $T' \subset N$ such that $A, B \in T', w_A^{T'} = 0.5$ and $w_B^{T'} = 0.6$.

Table 2 shows the cost-coalitional vector and corresponding cost game (last column); i.e. for any coalition $T \subseteq N$, the cost of each agent c_k^T, and the cost of this coalition $c(T)$

$Coalition\backslash^{Agent}$	A	B	1	2	$c(T)$
{A}	5				5
{B}		10			10
{1}			6		6
{2}				13	13
{A,B}	4.5	9.2			13.70
{A,1}	5		5.80		10.80
{A,2}	5			12.50	17.50
{B,1}		10	5.40		15.40
{B,2}		10		11.80	21.80
{1,2}			6	13	19
{A,B,1}	4.50	9.20	5.20		18.90
{A,B,2}	4.5	9.20		11.30	25
{A,1,2}	5		5.80	12.50	23.30
{B,1,2}		10	5.40	11.80	27.20
{A,B,1,2}	4.50	9.20	5.20	11.30	30.20

Table 2: Cost coalitional vector and cost game.

From the previous table, it is straightforward to obtain $z_{ii'}$, where $z_{ii'} = c_i^{T\setminus\{i'\}} - c_i^T$ for all $T \subseteq N$ such that $i, i' \in P \cap T$. Therefore, $z_{AB} = 0.5$ and $z_{BA} = 0.8$, i.e., country B reduces the cost of country A in 0,5 and country A reduces the cost of country B in 0.8. Consequently, country A is a net-benefactor with country B, and country B a net-beneficiary with country A.

We can calculate now the Shapley value by using the expressions from Theorem 4. Note that, in this case, we only need the values of Table 1, the last row of Table 2 $\left(c_A^N, c_B^N, c_1^N \text{ and } c_2^N\right)$, and both values z_{AB} and z_{BA}. Therefore, Theorem 9 allows to reduces significantly the amount of information and time to compute Shapley value.

In Table 3, it is shown for any agent its individual cost, the cost in the grand coalition, the Shapley value, and the difference between the last two values.

$Agent \setminus^{Value}$	$c(\{k\})$	$\psi_k(N,c)$	$\phi_k(N,c)$	$\psi_k(N,c) - \phi_k(N,c)$
A	5	4.5	4	0,5
B	10	9.2	8,5	0,8
1	6	5,2	5,6	$-0,4$
2	13	11,3	12,2	$-0,9$

Table 3: Comparsison individual costs, cost in the grand coalition and the Shapley value

Notice that costs in the gran coalition reduce the indivudual costs of each player. Regarding to the cost in the grand coalition, Shapley value decreases the cost of benefactors in a half of the benefits that it provided to the beneficiaries. Additionally, it increases the cost beneficiaries in a half of the benefits that they obtain from benefactors. For example, for country $A, \phi_A(N,c) = c_A^N - \frac{1}{2}\left(\left(S_1^A - \bar{S}_1^A\right) + \left(S_2^A - \bar{S}_2^A\right)\right) + \frac{1}{2}\left(z_{AB} - z_{BA}\right)$. As $z_{AB} - z_{BA} = -0.3$, country A is a net-benefactor. Thus, Shapley value decreases its cost in a half of this difference. However, for country B, the cost is increased in the same amount because it is a net-beneficiary. In this example, there are only two countries, however, if there was more countries, a given country could be a net benefactor with some countries and a net beneficiary with others, this depends on the sign of $z_{ii'} - z_{i'i}$.

1.6 Concluding remarks

In this chapter, a new model of Corporate tax games with benefactors and beneficiaries as an application of linear cost games to the corporate tax reduction system introducing the figure of multiple, dual and irreplaceable benefactors has been presented. It has been used the Shapley value as a rule of stable allocation to sharing costs reduced. Moreover, its properties are studied, it has been verified the snowball effect derived from the concavity of the model proving that the larger the coalition the lower the costs for its members and we proved that, these games are concave, i.e., the marginal contribution of a firm and a country diminishes as a coalition grows (snowball effect). Hence, the grand coalition is stable in the sense of the core. This means that firms have strong incentives to cooperate with the countries instead of being fraudsters. Then, we propose the Shapley value as an easily computable core-allocation that benefits all agents and, in particular, compensates the benefactors for their dual and irreplaceable role.

The model here, distinguishes two groups of agents: dual benefactors (countries) and beneficiaries (firms), while the original model presented by Meca and Sosic (2014), considered two disjoint groups of agents, benefactors and beneficiaries. A natural extension would be to consider that all agents can be dual (benefactors and beneficiaries). We believe that similar results to those obtained here could be achieved.

Chapter 2

Efficient Effort Equilibrium in cooperation with pairwise cost reduction

2.1 Introduction

In real life, the amount of information that a player possesses or knows about the other players is significantly limited, so that the degree of certainty about the possible behaviors or strategies of third parties is logically constrained and, therefore, the payoff matrices will depend on the possible strategies aimed at minimizing as far as possible the effects of such uncertainty with which each player enters a game. Cooperation will be nothing more than a way of obtaining a profit that would otherwise be unattainable, inferior, very costly or even inaccessible, by making use of the resources or values of third parties while sharing or offering one's own resources or values. The greater the benefit obtained from the other players and the lower the amount and cost of sharing or offering one's own resources, the greater the total profit obtained.

Once the knowledge is established that not only competitive models occur under conflict scenarios, understanding as such any interaction between players in which each of them pursues the satisfaction of their own individual interest, but also that any cooperative model takes place under the premise of the existence of a conflict and on the basis of a rationality that will be the feature that guides and determines the behavior and therefore the decisions of any player, it becomes absolutely necessary to understand that any agent, acting from the most absolute rationality, behaves in a selfish way, that is, he will seek to obtain the maximum benefit by using all the ways and tools available or that he may have at his disposal.

To conceive, therefore, that in cooperative scenarios the players always participate by contributing the maximum effort (contributing all the resources at their disposal) would imply, in the first place, erroneously assuming that in cooperative environments the respective degrees of cooperation are symmetrical and would imply, equally erroneously, in the second place, admitting that the players prioritize cooperation over the individual interest that motivates each one of them to interact.

Reality, therefore, is much more complex and dynamic, marked to a large extent by a rationality that determines and shapes the behavior of each agent, who will do everything possible to try to obtain the maximum return or benefit while devoting the least cost to achieve it. In other words, and increasingly so in a social, economic and business environment perceived by the agents as deeply competitive, the players will try, whenever there is an opportunity, to cooperate in positions of advantage that will allow them to redirect the conflict of cooperation towards more favorable positions for themselves in order, as far as possible, to obtain the maximum from the others while giving the minimum in return.

Logically, when faced with the proposal or possibility of cooperation, the positions with which each player decides to do so will be marked by a previous preparation or strategic behavior in which, given the assumption that the rest of the players will try to receive the maximum payment offering the lowest value, individualism will be increased and maximized (Freeman, 2013), an individualism marked by a sense of emergency that will materialize in attempts to reach an agreement between pairs of agents with the aim of limiting the maximum cost or effort involved in cooperating for both, while guaranteeing certain minimums with respect to the possible returns on these costs. In other words, an attempt will be made to reach and close a previous or preparatory phase of negotiation under a competitive scenario in which the best possible conditions will be sought bilaterally for each player in order to reach the cooperation phase in the most solid way, ensuring the best or most optimal ratio of expenditure faced and payment received.

Let us look at the above from the operating modes and dynamics applied by essential economic agents such as companies, entities or corporations. Taking as an example the behavior of companies or corporations in their continuous search for growth and the pursuit of market expansion and reduction of operating costs while maintaining or increasing competitiveness, we can clearly appreciate the existence of different forms of cooperation depending on the degree of integration or interdependence of the partners and the intended objectives of the agreements. These forms have been widely studied in the economic literature (see, for example, Todeva and Knoke (2005) for a review). There is a specific type of cooperation in transferable utility models whose properties and characteristics differentiate it from the rest. It can occur between agents that share, for example, resources, knowledge or infrastructure. The common purpose is to obtain individual advantages such as the reduction of their respective individual costs. The particularity of this form of cooperation lies in the fact that the cost reduction is based on bilateral interactions.

We consider that form of cooperation here in which, given any pair of cooperating agents, one agent reduces the cost of the other by a certain amount which is independent of cooperation with other agents. This means that if there are more agents in the coalition the amount of the cost reduction does not change. This pairwise cost reduction is independent of the coalition to which the pair of agents may belong. Therefore, for any agent, the total cost reduction in any coalition can easily be calculated as the sum of the reductions obtained from each bilateral interaction with the other members of the coalition.

There are several situations where this kind of cooperation with pairwise cost reduction occurs and is profitable, e.g. strategic collaboration agreements between firms to reduce logistical operational costs. The need to increase market share requires logistics firms to expand their radius of action as far as possible. This means major investments in expensive infrastructures at new sites, which increase operational costs. Agreements are established between companies to reduce those costs while maintaining control of their respective markets and hindering access by new competitors. They offer the resources held by each firm in its respective area of influence under advantageous conditions. This enables them to expand their operating ranges with significant cost savings. Interactions occur bilaterally, with each company using the resources of the other. These cost reductions are independent of any cost reductions that can also be obtained by interacting with other agents in larger coalitions.

Among other frequent examples where these modes of cooperation are commonplace, we can also cite bilateral free trade agreements between countries where free trade agreements are quite common in a globalized economy such as today's. They facilitate trade in goods and services between countries, reducing trade barriers and, consequently, the cost of trade. These cost reductions are specific to each pair of countries, and are independent of any other agreements that either country may decide to enter into with other countries.

Another example of common scenarios is the exchange of market data. Today, information about customers and their buying patterns is of vital importance to companies. It enables them to maximize the cost-effectiveness of advertising and to focus on their ideal target markets. Cooperation between companies (usually in complementary industries) consists of sharing information about their respective customers. This reduces costs for each of the companies involved. The information that a particular firm provides is specific to it, so the value of the information that it receives from another specific firm is independent of information from other firms. Even if two firms provide information about the same customer, the information itself is different because it describes the purchase of a different good or service. This can increase the value of that particular customer as a target, which again boosts the value of this particular kind of cooperation.

The latest example, among an endless number of examples, may well be that of cooperation agreements between companies to reduce costs by increasing the reach of their respective telecommunications networks. In highly competitive sectors such as mobile telephony and online services, cooperation between operators has become quite common. For example, they can share the location of their respective antennas, enabling them to extend the reach of their networks. This results in greater benefits by offering a broader service, while avoiding the costs that would be incurred if each company were to install its own structures. Again, the cost reduction is bilateral when two players decide to share and use each other's antennas. These cost savings are independent of any collaboration agreements that each firm may have with other agents to share antennas in larger coalitions.

In this kind of cooperation, the cost reduction between agents may be highly asymmetric when they cooperate in pairs. For example, if two agents A and B decide to cooperate, agent A could provide a major reduction for agent B, while the reduction provided in the opposite direction could be more modest. These asymmetries can induce imbalances or discriminations that could jeopardize cooperation. A fair distribution mechanism for the costs generated by cooperation is undoubtedly needed to ensure the stability of any strategic partnership, as Thomson (2010) points out.

In addition, it is quite common for this kind of cooperation to require the agents involved to make a set level of effort. It is natural to think that the amount by which one agent can reduce the costs of the other (if they decide to cooperate) could depend on the effort that the agent exerts. For example, if one country can obtain information relevant to another (e.g. information on tax evasion and the flight of capital involving its citizens), the amount and quality of the specific information may depend on the effort exerted by the first country in gathering it. This extends the situation beyond a cooperative model.

The first works presenting bilateral cooperation models between pairs of agents when these, in two different phases, show different degrees or levels of cooperation as a result of a first strategic competitive encounter, were carried out in 2007 by Brandenburger and Stuart, introducing the concept of biform-games, a class of games in which the coalitional value will be given according to the previous strategies adopted in the negotiation phase between players.

Although there is currently a gradual growth in the volume of work on this type of game, unfortunately at present there has been no significant volume of production aimed at going deeper and opening up new avenues beyond the initial concept. Recently, among others, it is fair to highlight the work carried out by Manuel and D. Martin (2020) studying the interdependence in the formation of coalitions with respect to the different wills to cooperate on the part of the players and also studying certain properties and adaptation of the Shapley value to act as a valid and suitable sharing rule.

In this thesis work, we model the sequence of decisions as a bi-form game (Brandenburger and Stuart, 2007), a new class of games that allows, as it has been indicated, to model and analyze this type of widespread and increasingly frequent scenarios where there is a phase of non-cooperation prior to the cooperative phase while allowing to integrate and incorporate the different asymmetries under which the players decide to interact with different degrees of cooperation, degrees that are agreed or fixed in a previous non-cooperative or negotiation phase.

In the model introduced in this thesis work, in the first stage of the bi-form game, agents decide how much (costly) effort they are willing to exert. This has a direct impact on their pairwise cost reductions. This first stage is modeled as a non-cooperative game in which agents determine the level of pairwise effort to reduce the costs of their partners. In the second stage, agents engage in bilateral interactions with multiple independent partners where the cost reduction brought by each agent to another agent is independent of any possible coalition. We study this bilateral cooperation in the second stage as a cooperative game in which cooperation leads agents to reduce their respective costs, so that the total reduction in costs for each agent in a coalition is the sum of the reductions generated by the rest of the members of that coalition. In the non-cooperative game of the first stage, the agents anticipate the cost allocation that will result from the cooperative game in the second stage by incorporating the effect of the effort made into their cost functions. Based on this model, we explore costs, benefits, and challenges associated with setting up a pairwise effort situation.

We investigate the impact of pairwise efforts on cost reductions and the resulting cost structure for this framework. In particular, we explore the design of a cost-allocation mechanism that efficiently allocates the gains from pairwise effort to all parties. To that end, we first compute the optimal level of cost reduction, taking into account the pairwise cost reductions collectively accrued by all agents. An ideal allocation scheme should encourage agents to participate in it and, at the same time, establish proper incentives to make efforts prior to cooperation. Specifically, we first show that it is profitable for all agents to participate in a pairwise effort situation. Then we study how the total reduction in costs should be allocated to the participants in such a situation. We do this by modeling the pairwise cost reduction between agents that takes place in the second stage as a cooperative game, which we refer to as the pairwise effort game or "PE-game".

We prove that the marginal contribution of an agent diminishes as a coalition grows in PE-games (i.e. they are concave games) and thus all-included cooperation is feasible, in the sense that there are possible cost reductions that make all agents better off or, at least, not worse off (i.e. PE-games are balanced, which means that the core is not empty). This all-included cooperation is also consistent (i.e. PE-games are totally balanced, which means the core of every subgame is non-empty). We identify various allocation mechanisms that enable all-included cooperation to be feasible (i.e. allocation mechanisms that belong to the core of PE-games). In particular, we discuss a family of cost allocations with weighted pairwise reduction which is always a subset of the core of PE-games. This is a broad family of core-allocations which includes the Shapley value, which is obtained when all the weights work out to a half. We provide a highly intuitive, simple expression for the Shapley value, which matches the Nucleolus in our model. To select one of these core-allocations in the second stage, we take into account the incentives that it generates in the efforts made by agents, and consequently in the aggregate cost of a coalition. We show that the Shapley value can induce inefficient effort strategies in equilibrium in the non-cooperative model. However, it is always possible to find core-allocations with weighted pairwise reductions that create appropriate incentives for agents to make optimal efforts that minimize aggregate costs, i.e. core-allocations that generate an efficient level of effort in equilibrium.

This thesis work contributes to the literature by presenting a doubly robust cost sharing mechanism. This mechanism not only has good properties in the second-stage cooperative game, but also creates suitable incentives in the first-stage non-cooperative game that allow efficiency to be achieved.

Cooperative game theory has developed a substantial mathematical framework for identifying and providing suitable cost sharing allocations (see, e.g., Fiestras-Janeiro et al. 2011; Guajardo and Rönnqvist 2016, for a survey). Multiple solutions have been proposed from a wide range of approaches (see, e.g., Moulin 1987; Slikker and Van den Nouweland 2012; Lozano et al. 2013). The Shapley Value (Shapley 1953) is considered one of the most outstanding of them, and a suitable solution concept (see, e.g., Moretti and Patrone 2008; Serrano 2009 for a survey). As an allocation rule it has very good properties, such as efficiency, proportionality, and individual and coalitional rationality. However, it has the disadvantage of posing computational difficulties, which increase as the number of players increases. Nonetheless, there is a large body of literature in which the Shapley value is proposed as a simple, easy-to-apply solution in different economic scenarios (see, e.g., Littlechild and Owen 1973; Bilbao et al. 2008; Li and Zhang 2009; Kimms and Kozeletskyi 2016; Le et al. 2018; Meca et al. 2019). These papers give simplified solutions for different classes of games. They take the cost structure as given and do not consider the system externalities that arise when agents make efforts to give and receive cost reductions. Consequently, the present work incorporates both the non-cooperative aspects of making efficient efforts (modeling decisions related to pairwise cost reductions) and the cooperative nature of giving and receiving cost reductions in pairwise effort situations.

As in principal-agent literature, we refer to action by agents as "effort". In this setting, the concept of "effort" is widely used in analyzing different kinds of problem. One of the first was the moral hazard problems: See for example Holmstrom (1982). Other examples are Holmstrom (1999) and Dewatripont et al. (1999), who identify conditions under which more information can induce an agent to make less effort. The approach in our model is quite different, in that we do not consider any kind of principal. As far as we know, our model is novel in that it analyzes the incentive for agents to make efforts in a bi-form game: A non-cooperative stage where agents choose how much effort to make and a cooperative second stage. As mentioned, we show that the solution of the cooperative game determines the incentives of agents to make an effort in the first stage, and consequently the efficiency of the final outcome.

Bernstein et al. (2015) also use a bi-form model to analyze the role of process improvement in a decentralized assembly system in which an assembler lays in components from several suppliers. The assembler faces a deterministic demand and suppliers incur variable inventory costs and fixed production setup costs. In the first stage of the game suppliers invest in process improvement activities to reduce their fixed production costs. Upon establishing a relationship with suppliers, the assembler sets up a knowledge sharing network which is modeled as a cooperative game between suppliers in which all suppliers obtain reductions in their fixed costs. They compare two classes of allocation mechanism: Altruistic allocation enables non-efficient suppliers to keep the full benefits of the cost reductions achieved due to learning from the efficient supplier. The Tute allocation mechanism benefits a supplier by transferring the incremental benefit generated by including an efficient supplier in the network. They find that the system-optimal level of cost reduction is achieved under the Tute allocation rule.

The hybridization linked to biform-games, integrating both cooperative and competitive phases and relating the profits obtained with the strategy profiles of each player, allows a great versatility to model and adapt to a wide range of situations, which is why the volume of publications and the increasing breadth in the range of possible fields or areas of application of biform-games is showing an obvious and undeniable evolution and exponential growth in recent years (e.g. Summerfield and Dror (2013), Ray and Vohra (2015), Fox et al (2021) for some survey references). Our bi-form game is novel in terms of incentive for efforts by agents and is also richer in results: We find the allocation rule that generates the unique efficient effort in equilibrium in cooperation with pairwise cost reduction.

2.2 Model

We consider a model with a finite set of agents $N = \{1, 2, ..., n\}$, where each agent has a good (for example resources, knowledge or infrastructure) and has to perform a certain activity. The total cost of an agent's activity can be reduced if it cooperates with another agent, which means that the two agents share their goods. These cost reductions obtained by sharing goods in pairs depend on the effort made previously by each agent. Our model consists of two different stages. In the first stage, agents choose their effort levels as in a non-cooperative game. In the second stage, agents cooperate to reduce their costs, and allocate the minimum cost they achieve by pairwise cost reductions as in a cooperative game. The proposed cost allocation for the cooperative game in the second stage determines the payoff function of the non-cooperative game in the first stage. Therefore, we model the sequence of decisions as a bi-form game (Brandenburger and Stuart, 2007). The two stages of the model are described in detail below.

First Stage (non-cooperative game): Each agent $i \in N$ chooses in this state an effort level $e_i = (e_{i1}, ..., e_{i(i-1)}, e_{i(i+1)}, ...e_{in}) \in [0, 1]^{n-1}$, where $e_{ij} \in [0, 1]$ stands for the level of effort by agent i to reduce the cost of agent j if they cooperate in the second stage. These efforts have a cost $c_i(e_i) \in \mathbb{R}_+$ for any $i \in N$. We assume that $c_i(.) : [0, 1]^{n-1} \to \mathbb{R}_+$ is a scalar field of class $C^2([0, 1]^{n-1})$.[1] Moreover, for all $e_{ij} \in [0, 1]$ with $j \in N \backslash \{i\}$, it is assumed that $\frac{\partial c_i(e_i)}{\partial e_{ij}} > 0$, $\frac{\partial^2 c_i(e_i)}{\partial e_{ij}^2} > 0$, and $\frac{\partial^2 c_i(e_i)}{\partial e_{ij} \partial e_{ih}} = 0$ for all $h \neq i, j$, which implies that the marginal cost $\frac{\partial c_i(e_i)}{\partial e_{ij}}$ is independent of the effort that i exerts with agents other than j.[2]

Second Stage (cooperative game): Given the effort made in the first stage, agents cooperate, so for any pair of cooperating agents $i, j \in N$ and a given effort e_{ij}, agent i reduces the total cost of agent j by an amount $r_{ji}(e_{ij}) \in \mathbb{R}_+$, and vice versa. These

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[1] A scalar field is said to be class C^2 at $[0, 1]^{n-1}$ if its 2-partial derivatives exist at all points of $[0, 1]^{n-1}$ and are continuous.

[2] This last assumption implies that the Hessian matrix is a diagonal matrix. In addition, note that, given our assumptions about c_i, w.l.o.g. we could consider that $c_i(e_i) = \sum_{j \in N \backslash \{i\}} c_{ij}(e_{ij})$ where $c_{ij}(.) : [0, 1] \to \mathbb{R}_+$. We omit it from the paper so as not to introduce more notation into the model.

particular reductions between agents $i, j \in N$ are independent of cooperation with other agents. We also assume for all $j \in N \backslash \{i\}$ that function $r_{ij}(.) : [0,1] \to R_+$ is class C^2, increasing and concave[3] at $[0,1]$. Thus, these agents participate in bilateral interactions with multiple independent partners whose cost reductions are coalitionally independent, i.e. the cost reduction given by each agent to another agent is independent of any possible coalition. This means that the total reduction in cost for each agent in a coalition $S \subseteq N$ is the sum of the pairwise cost reductions given to that agent by the rest of the members of the coalition, i.e. for agent i, it is $\sum_{j \in S \backslash \{i\}} r_{ij}(e_{ji})$. We assume perfect information regarding agents' costs and cost reductions depending on efforts.

Given an effort profile $e = (e_1, e_2, ..., e_n) \in [0,1]^{n(n-1)}$ in the first stage, the second stage can be seen as a cooperative game, more specifically a transferable utility cost game (N, e, c), where N is the finite set of players, and $c : 2^N \to R$ is the so-called characteristic function of the game, which assigns to each subset $S \subseteq N$ the cost $c(S)$ that is incurred if agents in S cooperate. By convention, $c(\emptyset) = 0$. The cost of agent i in coalition $S \subseteq N$ is given by $c^S(i) := c_i(e_i) - \sum_{j \in S \backslash \{i\}} r_{ij}(e_{ji})$. This cost can be interpreted as the reduced cost of agent i in coalition S. Note that the larger the coalition, the greater the cost reduction it achieves, i.e. for all $i \in S \subseteq T \subseteq N$, $c^T(\{i\}) \leq c^S(\{i\})$. The total reduced cost for coalition S is given by

$$c(S) := \sum_{i \in S} c^S(\{i\}) = \sum_{i \in S} [c_i(e_i) - \sum_{j \in S \backslash \{i\}} r_{ij}(e_{ji})]. \tag{2.1}$$

When all agents cooperate, they form what is called the grand coalition, which is denoted by N. Thus, $c(N)$ is the aggregate cost of the grand coalition. The allocation of the grand coalition cost achieved through cooperation, in the second stage, assigns a reduced final cost to each agent, that is, $\psi_i(e)$, for all $i \in N$, where $\psi_i : E \to R$ with $E := \prod_{i \in N} E_i$ and $E_i := [0,1]^{n-1}$. Then, we define the cost allocation rule $\psi : E \to \mathbb{R}^n$ s.t. $\psi(e) = (\psi_i(e))_{i \in N}$ and $\sum_{i \in N} \psi_i(e) = c(N)$.

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[3] $\partial r_{ji}(e_{ij})/\partial e_{ij} > 0$ (increasing) and $\partial^2 r_{ji}(e_{ij})/\partial e_{ij}^2 < 0$ (concave).

The non-cooperative cost game in the first stage is defined through that cost allocation rule ψ by $(N, \{E_i\}_{i\in N}, \{\psi_i\}_{i\in N})$, where E_i is the strategy space of agent $i \in N$ (its effort level space), and ψ_i is the payoff function of agent i, but in this case is a cost function. Hence, for an effort profile $e \in E$, the corresponding cost function is $\psi(e)$. That effort is made in anticipation of the result of the cooperative cost game that follows in the second stage. Therefore, we first analyze the second stage (see Section 2.3), and focus on different ways of allocating the grand coalition cost. We determine cost allocation rules with good computability properties and coalitional stability for this cooperative cost game. Notice that a given cost allocation rule will generate precise incentives in the first state and consequently particular equilibrium effort strategies [4]. In turn, these particular effort strategies will have an associate cost of the grand coalition. At this point, a question about efficiency arises. The lower the cost of the grand coalition generated in equilibrium is, the more efficient the equilibrium effort strategies and the allocation rule considered will be.

Therefore, there are two dimensions to be considered. First, the cost allocation rule for the cooperative game should have good properties (computability and coalitional stability). Second, the allocation rule must induce the right incentives in the non-cooperative game to obtain the lowest cost of the grand coalition. This interesting, relevant question is analyzed in the section dedicated to analyze efforts and optimal rules.

Therefore, we consider the following assumptions:

(CA) Cost assumptions: $c_i \in C^2$, and $\frac{\partial c_i(e_i)}{\partial e_{ij}} > 0$ (increasing), $\frac{\partial^2 c_i(e_i)}{\partial e_{ij}^2} > 0$ (convex), and $\frac{\partial^2 c_i(e_i)}{\partial e_{ij}\partial e_{iK}} = 0$, if $k \neq j$ (additively separable).

(RA) Reduction assumptions: $r_{ji} \in C^2$, and $\partial r_{ji}(e_{ij})/\partial e_{ij} > 0$ (increasing), $\partial^2 r_{ji}(e_{ij})/\partial e_{ij}^2 < 0$ (concave).

A summary of the notation and the main optimization problems (Table 1 and 2) can be found in Appendix D.

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[4] An effort strategy profile is said to be in equilibrium when each agent has nothing to gain by changing only their own effort strategy given the strategies of all the other agents (Nash equilibrium).

2.3 Cooperation with Pairwise Cost Reduction

This section presents the analysis of cooperation with pairwise cost reduction in the second stage. Agents make their efforts in pairwise sharing in the first stage, and initiate cooperation with efforts $e = (e_1, ..., e_i, ..., e_n)$. We model a Pairwise Effort Game (henceforth, PE-game) as a multiple-agent cooperative game where each agent i incurs an initial cost of $c_i(e_i)$. All agents in a pairwise effort group (coalition) give cost reductions to and receive such reductions from other agents. As a result, all agents in the coalition reduce their initial costs to levels that depend on the efforts made in the first stage by the others. No agent outside the pairwise effort situation benefits from this pairwise cost reduction opportunity. We introduce all the game-theoretic concepts used in this paper, but readers are referred to González-Díaz et al. (2010) for more details on cooperative and non-cooperative games.

We refer to the pairwise effort situation as a PE-situation and denote it by the tuple $(N, e, \{c_i(e_i), \{r_{ji}(e_{ij})\}_{j\in N\setminus\{i\}}\}_{i\in N})$. We associate a cost game (N, e, c) with each PE-situation as defined by (2.1).

The class of PE-games has some similarities with the class of linear cost games introduced in Meca and Sosic (2014). They define the concept of cost-coalitional vectors as a collection of certain a priori information, available in the cooperative model, represented by the costs of the agents in all possible coalitions to which they could belong. The cost of a coalition in their study is thus the sum of the costs of all agents in that coalition. However, the PE-games considered here are significantly different from their linear cost games. Linear cost games focus on the role played by benefactors (giving) and beneficiaries (receiving) as two groups of disjoint agents, but PE-games consider that all agents could be dual benefactors, in the sense that they be benefactors and beneficiaries at the same time. In addition, PE-games are based on a bilateral cooperation between agents that enables both to reduce their costs but is coalitionally independent.

We now consider a PE-situation $(N, e, \{c_i(e_i), \{r_{ij}(e_{ij})\}_{j\in N\setminus\{i\}}\}_{i\in N})$ and consider whether it is profitable for the agents in N to form the grand coalition to obtain a significant reduction in costs. We find that the answer is yes, and show that the associated PE-game

(N, e, c) is concave, in the sense that for all $i \in N$ and all $S, T \subseteq N$ such that $S \subseteq T \subset N$ with $i \in S$, so $c(S) - c(S \setminus \{i\}) \geq c(T) - c(T \setminus \{i\})$. This concavity property provides additional information about the game: the marginal contribution of an agent diminishes as a coalition grows. This is well-known and is called the "snowball effect".

The first result in this section shows that PE-games are always concave. This means that the grand coalition can obtain a significant reduction in costs. In that case, the reduced total cost is given by $c(N) = \sum_{i \in N} c_i(e_i) - R(N)$, where $R(N) = \sum_{i \in N} \sum_{j \in N \setminus \{i\}} r_{ij}(e_{ji})$ is the total reduction produced by bilateral reductions between all agents in the situation, which turns out to be the total cost savings for all agents. The proof of next Proposition , together with all our other proofs for this chapter, is shown in Appendix B.

Proposition 2.1 *Every PE-game is concave.*

An allocation rule for PE-games is a map ψ which assigns a vector $\psi(e) \in R^n$ to every (N, e, c), satisfying efficiency, that is, $\sum_{i \in N} \psi_i(e) = c(N)$. Each component $\psi_i(e)$ indicates the cost allocated to $i \in N$, so an allocation rule for PE-games is a procedure for allocating the reduced total to all the agents in N when they cooperate. It is a well-known result in cooperative game theory that concave games are totally balanced: The core of a concave game is non-empty, and since any subgame of a concave game is concave, the core of any subgame is also non-empty. That means that coalitionally stable allocation rules can always be found for PE-games. We interpret a non-empty core for PE-games as indicating a setting where all included cooperation is feasible, in the sense that there are possible cost reductions that make all agents better off (or, at least, not worse off). The totally balanced property suggests that this all-included cooperation is consistent, i.e. for every group of agents whole-group cooperation is also feasible.

A highly natural allocation rule for PE-games is $\varphi_i(e) = c^N(\{i\}) = c_i(e_i) - R_i(N)$, for all $i \in N$, with $R_i(N) = \sum_{j \in N \setminus \{i\}} r_{ij}(e_{ji})$ being the total reduction received by agent $i \in N$ from the rest of the agents $j \in N \setminus \{i\}$. It has good properties at least with respect to computability and coalitional stability in the sense of the core. Formally, the core of a PE-game (N, c) is defined as follows

$$Core\,(N,c) = \{x \in \mathbb{R}^n / \sum_{i\in N} x_i = c(N), \sum_{i\in S} x_i \leq c(S) \forall S \subseteq N\}. \tag{2.2}$$

Notice that $\varphi\,(e) \in Core\,(N,c)$. Indeed, $\sum_{i\in N} \varphi_i\,(e) = c(N)$ and for every $S \subseteq N$, $\sum_{i\in S} \varphi_i\,(e) = \sum_{i\in S} c^N(i) \leq \sum_{i\in S} c^S(i) = c(S)$. Nevertheless, the agents could argue that this allocation does not provide sufficient compensation for their dual role of giving and receiving. Note that the allocation only considers their role as receivers.

PE-games are concave, so cooperative game theory provides allocation rules for them with good properties, at least with respect to coalitional stability and acceptability of items. We highlight the Shapley value (see Shapley 1953), which assigns a unique allocation (among the agents) of a total surplus generated by the grand coalition. It measures how important each agent is to the overall cooperation, and what cost can it reasonably expect. The Shapley value of a concave game is the center of gravity of its core (see Shapley 1971). In general, this allocation becomes harder to compute as the number of agents increases. We present a simple expression here for the Shapley value of PE-games that takes into account all bilateral relations between agents and compensates them for their dual role of giving and receiving.

Given a general cost game (N,c), we denote the Shapley value by $\phi(c)$, where the corresponding cost allocation for each agent $i \in N$, is

$$\phi_i(c) = \sum_{i\in T\subseteq N} \frac{(n-t)!(t-1)!}{n!} \left[(c(T) - c(T \backslash \{i\})\right], \text{ with } \mid T \mid = t. \tag{2.3}$$

The Shapley value has many desirable properties, and it is also the only allocation rule that satisfies a certain subset of those properties (see Moulin, 2004). For example, it is the only allocation rule that satisfies the four properties of Efficiency, Equal treatment of equals, Linearity and Null player (Shapley, 1953).

Given a PE-game (N,e,c), we denote by $\phi(e)$ the Shapley value of the cost game. The following Theorem shows that the Shapley value provides an acceptable allocation for

PE-games. Indeed, it reduces the individual cost of an agent by the average of the total reduction that it obtains from the others ($R_i(N)$) plus half of the total reduction that it provides to the rest of the agents, i.e. $G_i(N) = \sum_{j \in N \setminus \{i\}} r_{ji}(e_{ij})$.

Theorem 2.1 Let (N, e, c) be a PE-game. For each agent $k \in N, \phi_k(e) = c_k(e_k) - \frac{1}{2}[R_k(N) + G_k(N)]$.

From Theorem 2.1 it can be derived that the Shapley value, $\phi(e)$, considers the dual role of giving and receiving of all agents, and the final effect on those agents depends on which role is stronger. As mentioned above, if an allocation does not compensate them for their dual role of giving and receiving, and it only considers their role as receivers, as the individual cost in the grand coalition, $\varphi(e)$, does, the cooperation would not be desirable for those dual agents. This non-acceptability can be avoided by using the Shapley value, which also coincides with the Nucleolus (Schmeiler 1989) for PE-games.

The nucleolus selects the allocation in which the coalition with the smallest excess (the worst treated) has the highest possible excess. The nucleolus maximizes the "welfare" of the worst treated coalitions. Denote by $\nu(e) \in R^n$ the *Nucleolus* of the PE-game (N, e, c), associated with a PE-situation $(N, e, \{c_i(e_i), \{r_{ij}(e_{ij})\}_{j \in N \setminus \{i\}}\}_{i \in N})$. First, we define the excess of coalition S in (N, e, c) with respect to allocation x as $d(S, x) = c(S) - \sum_{i \in S} x_i$. This excess can be considered as an index of the "welfare" of coalition S at x: The greater $d(S, x)$, the better coalition S is at x. Let $d^*(x)$ be the vector of the 2^n excesses arranged in (weakly) increasing order, i.e., $d_i^*(x) \leq d_j^*(x)$ for all $i < j$. Second, we define the lexicographical order $\succ_l$. For any $x, y \in R^n$, $x \succ_l y$ if and only if there is an index k such that for any $i < k$, $x_i = y_i$ and $x_k > y_k$. The nucleolus of the PE-game (N, e, c) is the set

$$\nu(e) = \{x \in X : d^*(x) \succ_l d^*(y) \text{ for all } y \in X\} \tag{2.4}$$

with $X = \{x \in R^n : \sum_{i \in N} x_i = c(N),\ x_i \geq c(\{i\}) \text{ for all } i \in N\}$.

It is well known that the Nucleolus is a singleton for balanced games and that it is always a core-allocation.

The Proposition proves that for PE-games the Shapley value matches the Nucleolus.

This is a very good property that few cost games satisfy.

Proposition 2.2 *Let (N, e, c) be a PE-game. For each agent $k \in N$, $\nu_k(e) = \phi_k(e)$.*

Therefore, given an effort profile, the Shapley value is a very suitable way of allocating the reduced cost due to cooperation. Note that, the cost reduction as a result of cooperation between any pair of agents $i, j \in N$ is $r_{ij}(e_{ji}) + r_{ji}(e_{ij})$, and the Shapley value assigns one half of this amount to i and the other half to j. This seems a reasonable way to split this aggregate cost reduction. However, if agents knew before choosing their levels of efforts that the cost reductions resulting from their efforts were going to be allocated according to the Shapley value, the incentives created could generate inefficiencies. Some agents could find it optimal to exert too little effort and in some situations this could be inefficient.

For example, consider a PE-situation in which one agent has the ability to produce a substantial reduction in costs for other agents with a low effort cost and the rest of the agents have almost no ability to reduce costs for others even with a high effort cost. If the Shapley value is used as the allocation rule for this game, agents may not have incentives to make any level of effort. Note that in the first step agents have to decide how much effort to make. However, if the Shapley value is modified to give a greater portion of the pairwise cost reduction to the especially productive agent, it might make more effort and thus produce a greater reduction in cost for other agents. This change in the Shapley value generates new allocation rules, which can reduce the cost of the grand coalition regarding the Shapley allocation. The following example with three agents illustrates these ideas.

Example 2.1 Consider a pairwise inter-organizational situation with three firms, i.e. $N = \{1, 2, 3\}$. For any effort profile $e \in [0, 1]^6$, the PE-situation is given by the following initial costs,

$c_1(e_{12}, e_{13}) = 100 + 100e_{12} + 4e_{12}^2 + 100e_{13} + 4e_{13}^2$
$c_2(e_{21}, e_{23}) = 100 + 100e_{21} + 4e_{21}^2 + 100e_{23} + 4e_{23}^2$
$c_3(e_{31}, e_{32}) = 100 + 100e_{31} + 4e_{31}^2 + 100e_{32} + 4e_{32}^2$

and the following pairwise reduced costs, all of them in thousands of Euros,

$r_{i1}(e_{1i}) = 2 + 200e_{1i} - 3e_{1i}^2$ with $i = 2, 3$
$r_{i2}(e_{2i}) = 2 + 3e_{2i} - e_{2i}^2$ with $i = 1, 3$
$r_{i3}(e_{3i}) = 2 + 3e_{3i} - e_{3i}^2$ with $i = 1, 2$

If the allocation rule in the second stage is the Shapley value, the firms choose their levels of effort according to this cost allocation function. It is straight forward to show that in this case the unique effort equilibrium e^*, is one in which the three firms make no effort, i.e. $e^*_{ij} = 0$ for $i, j \in N$.[5] Thus, the Shapley value distributes the cost of the grand coalition $c^*(N) = 288$ equally, i.e. for each firm $i = 1, 2, 3$, $\phi_i(e^*) = c_i(e^*_i) - \frac{1}{2}\sum_{j \in N\setminus\{i\}}[r_{ij}(e^*_{ji}) + r_{ji}(e^*_{ij})] = 100 - \frac{1}{2}((2 + 2) + (2 + 2)) = 96$.

Note that, for example, in the relationship between firm 1 and 2, the pairwise cost reduction is $r_{12}(e_{21}) + r_{21}(e_{12})$, and the Shapley value gives $\frac{1}{2}$ of this amount to firm 1 and the other $\frac{1}{2}$ to firm 2. However, if the proportion that firm 1 obtains is increased, e.g. from $\frac{1}{2}$ to $\frac{3}{4}$, and the part for firm 2 is thus reduced to $\frac{1}{4}$, the incentive of firm 1 to make an effort can be increased. The same goes for firms 1 and 3 so that the incentive of firm 1 to make an effort for firm 3 is also increased. These changes in the Shapley value lead to a new allocation rule which we denote by $\Omega(e) = (\Omega_1(e), \Omega_2(e), \Omega_3(e))$ for any effort profile $e \in [0, 1]^6$. With this new allocation rule, the equilibrium efforts are zero for firms 2 and 3, and one for firm 1. That is, $e^{**}_{1j} = 1$, for $j = 2, 3$, $e^{**}_{2j} = 0$, for $j = 1, 3$, and $e^{**}_{3j} = 0$, for $j = 1, 2$. In this case,

[5] Theorem 2.3 in Section 2.4 shows the efforts of equilibrium in the non-cooperative game in the general case.

the grand coalition cost $c^{**}(N) = 102$ is allocated equally between firms 2 and 3, and the rest to firm 1. That is, $\Omega_i(e^{**}) = 100 - \frac{1}{4}[(2+200-3)+2] - \frac{1}{2}(2+2) = 47,75$ for $i = 2,3$, and $\Omega_1(e^{**}) = 100+100+4+100+4 - \frac{3}{4}[(2+(2+200-3))+(2+(2+200-3))] = 6,5$.

Hence, the new allocation rule $\Omega(e^{**})$ greatly reduces the grand coalition cost (by 136.000 Euros) as well as the costs of each firm; i.e. a reduction of 89.500 Euros for firm 1 and 23.250 Euros for firms 2 and 3. In relative terms, with the Shapley value each company pays 33.33% of the total cost. However, with the modified Shapley value agent 1 only pays 4.4% of the total cost, while agents 2 and 3 pay 47.8% each. Therefore, the modified Shapley value generates a more efficient outcome in the sense that it creates more appropriate incentives for firms.

To reach efficient effort strategies in equilibrium (henceforth EEE) in the first stage, we consider a new family of allocation rules, for PE-games (second stage), based on the Shapley value. This family consists of the rules $\Omega(e) \in R^n$, where for all $i \in N$,

$$\Omega_i(e) = c_i(e_i) - \sum_{j \in N\backslash\{i\}} [\omega^i_{ij} r_{ij}(e_{ji}) + \omega^i_{ji} r_{ji}(e_{ij})],$$

with $\omega^i_{ij}, \omega^i_{ji} \in [0,1]$, for all $j \in N\backslash\{i\}$, such that $\omega^i_{ij} = 1 - \omega^j_{ij}$ and $\omega^i_{ji} = 1 - \omega^j_{ji}$. The Shapley value is a particular case of this family of rules in which $\omega^i_{ij} = \omega^i_{ji} = \frac{1}{2}$, for all $i \in N$ and all $j \in N\backslash\{i\}$. This family of cost allocation for PE-games is referred to as *cost allocation with weighted pairwise reduction*.

The Theorem below shows that the family of cost allocations with weighted pairwise reduction is always a subset of the core of PE-games. This property identifies a wide subset of the large core of PE-games, including the Shapley value (and thus the Nucleolus).

Theorem 2.2 *Let (N, e, c) be a PE-game. For every family of weights $\omega^i_{ij}, \omega^i_{ji} \in [0,1]$, i, $j \in N, i \neq j$, such that $\omega^i_{ij} = 1 - \omega^j_{ij}$ and $\omega^i_{ji} = 1 - \omega^j_{ji}$, $\Omega(e)$ belongs to the core of (N, e, c).*

Now a complete analysis of the EEE for cooperation in pairwise cost reduction can be conducted.

2.4 Efficiency, Equilibrium Strategies, and Optimal Rule

We first define an *efficient effort profile* as the effort profile that minimizes the cost of the grand coalition, $c(N) = \sum_{i\in N}[c_i(e_i) - \sum_{j\in N\setminus\{i\}} r_{ij}(e_{ji})]$.

Definition 2.1 *An effort profile* $\tilde{e} = (\tilde{e}_1, ..., \tilde{e}_i, ..., \tilde{e}_n)$ *with* $\tilde{e}_i = (\tilde{e}_{i1}, ..., \tilde{e}_{i(i-1)}, \tilde{e}_{i(i+1)}, ...\tilde{e}_{in}) \in [0,1]^{n-1}$ *is efficient if* $\tilde{e} = \arg \underset{e\in[0,1]^{n(n-1)}}{min} \sum_{i\in N}[c_i(e_i) - \sum_{j\in N\setminus\{i\}} r_{ij}(e_{ji})]$

An efficient effort profile $\tilde{e}$ is well defined because $c(N)$ as a function of e is strictly convex in e_{ij} for all $i,\, j \in N, i \neq j$. [6]

The following proposition shows that the effort e_{ij} is efficient if the marginal cost of that effort equals the marginal reduction that this effort generates; otherwise, the effort is zero or one. The proof of Proposition appears in Appendix B, together, as indicated, with those of all the other proofs in this section.

Proposition 2.3 *There exists a unique efficient effort profile* $\tilde{e} = (\tilde{e}_1, ..., \tilde{e}_i, ..., \tilde{e}_n)$ *with* $\tilde{e}_i = (\tilde{e}_{i1}, ..., \tilde{e}_{i(i-1)}, \tilde{e}_{i(i+1)}, ...\tilde{e}_{in}) \in [0,1]^{n-1}$*, such that*

- $\tilde{e}_{ij} = 0$ if $\frac{\partial c_i(e_i)}{\partial e_{ij}} > \frac{\partial r_{ji}(e_{ij})}{\partial e_{ij}}$ for all $e_{ij} \in [0,1]$,

- $\tilde{e}_{ij} = 1$ if $\frac{\partial c_i(e_i)}{\partial e_{ij}} < \frac{\partial r_{ji}(e_{ij})}{\partial e_{ij}}$ for all $e_{ij} \in [0,1]$,

- $\tilde{e}_{ij} \in (0,1)$ is the unique solution of $\left.\frac{\partial c_i(e_i)}{\partial e_{ij}}\right|_{e_{ij}=\tilde{e}_{ij}} = \left.\frac{\partial r_{ji}(e_{ij})}{\partial e_{ij}}\right|_{e_{ij}=\tilde{e}_{ij}}$, otherwise.

We now focus on the non-cooperative effort game that arises under the family of *cost allocation with weighted pairwise reduction* (henceforth, WPR family). Then we analyze efficiency in equilibrium.

Consider the WPR family, i.e., $\Omega_i(e) = c_i(e_i) - \sum_{j\in N\setminus\{i\}}[\omega^i_{ij} r_{ij}(e_{ji}) + \omega^i_{ji} r_{ji}(e_{ij})]$ for all $i \in N$ with $\omega^i_{ij}, \omega^i_{ji} \in [0,1]$, $i,\, j \in N, i \neq j$, such that $\omega^i_{ij} = 1 - \omega^j_{ij}$ and $\omega^i_{ji} = 1 - \omega^j_{ji}$. For each specification of these weights, a particular allocation rule can be obtained that induces a certain equilibrium effort strategy in the first stage, which in turn generates the associated

[6] Note that the second derivative in e_{ij} is equal to $\frac{\partial^2 c_i(e_i)}{\partial e_{ij}^2} - \frac{\partial^2 r_{ji}(e_{ij})}{\partial e_{ij}^2}$, which is always positive because $\frac{\partial^2 c_i(e_i)}{\partial e_{ij}^2} > 0$ and $\frac{\partial^2 r_{ji}(e_{ij})}{\partial e_{ij}^2} < 0$.

cost allocation in equilibrium. The aim of this section is twofold. First, we identify the efficient allocation rule within the WPR family, i.e., that which results in the lowest cost of the grand coalition. Second, we show that the effort profile induced in equilibrium by this allocation rule coincides with the efficient effort profile of Proposition 2.3.

The non-cooperative cost game associated with $\Omega = (\Omega_i)_{i\in N}$ in the first stage is defined by $(N, \{E_i\}_{i\in N}, \{\Omega_i\}_{i\in N})$, where for every agent $i \in N$, $E_i := [0,1]^{n-1}$ is the players' i strategy set, and for all effort profiles $e \in E := \prod_{i\in N} E_i$, and Ω_i is the cost function for agent $i \in N$. We call this an effort game.

In this game, we use the following definition of equilibrium.

Definition 2.2 *The effort profile $e^* = (e_1^*, ..., e_n^*) \in E$ is an equilibrium for the game* $(N, \{E_i\}_{i\in N}, \{\Omega_i\}_{i\in N})$ if e_i^* is the optimal effort for agent $i \in N$ given the strategies of all the other agents $j \in N\backslash\{i\}$.

First, note that the optimal effort for agent $i \in N$ given the strategies of all the other agents $j \in N\backslash\{i\}$ is the effort e_i that minimizes $\Omega_i(e_i, e_{-i})$. Note that the function $\Omega_i(e_i, e_{-i})$ is strictly convex in the effort e_{ij} that agent i exerts for any $j \in N\backslash\{i\}$.[7] This means that for agent i there is a unique optimal level of effort $\hat{e}_{ij}$ for each $j \in N\backslash\{i\}$. That optimal level $\hat{e}_{ij}$ depends on the parameter ω_{ji}^i, on the marginal cost of agent i in regard to effort $\hat{e}_{ij}$ (i.e. $\frac{\partial c_i(e_i)}{\partial e_{ij}}$), and on the marginal cost-reduction for agent j in regard to effort $\hat{e}_{ij}$, (i.e. $\frac{\partial r_{ji}(e_{ij})}{\partial e_{ij}}$). Consequently, although the cost function of agent i depends on other agents' efforts (e_{ji} for all $j \in N\backslash\{i\}$), the optimal effort does not.

To obtain the optimal effort, we analyze the derivative of the convex function $\Omega_i(e)$ with respect to e_{ij}, for any $j \in N\backslash\{i\}$. It must be noted that $\frac{\partial \Omega_{ii}(e)}{\partial e_{ij}} \geq 0 \iff \frac{\partial c_i(e_i)}{\partial e_{ij}} \geq \omega_{ji}^i \frac{\partial r_{ji}(e_{ij})}{\partial e_{ij}}$ for all $e_{ij} \in [0,1]$. The following result characterizes the optimal effort level for agent $i \in N$ in the first stage of the game.

[7] Note that $\frac{\partial \Omega_i(e)}{\partial e_{ij}} = \frac{\partial c_i(e_i)}{\partial e_{ij}} - \omega_{ji}^i \frac{\partial r_{ji}(e_{ij})}{\partial e_{ij}}$ and $\frac{\partial_i^2 \Omega(e)}{\partial e_{ij}^2} = \frac{\partial^2 c_i(e_i)}{\partial e_{ij}^2} - \omega_{ji}^i \frac{\partial^2 r_{ji}(e_{ij})}{\partial e_{ij}^2} > 0$ because, as assumed above, $\frac{\partial^2 c_i(e_i)}{\partial e_{ij}^2} > 0$ and $\frac{\partial^2 r_{ji}(e_{ij})}{\partial e_{ij}^2} < 0$

Lemma 2.1 *Let $(N, \{E_i\}_{i\in N}, \{\Omega_i\}_{i\in N})$ be an effort game and $\hat{e}_{ij}$ be the optimal level of effort that agent i exerts to reduce the costs of agent j. Thus,*

- $\hat{e}_{ij} = 0$ if and only if $\frac{\partial c_i(e_i)}{\partial e_{ij}} > \omega^i_{ji}\frac{\partial r_{ji}(e_{ij})}{\partial e_{ij}}$, for all $e_{ij} \in [0,1]$,
- $\hat{e}_{ij} = 1$ if and only if $\frac{\partial c_i(e_i)}{\partial e_{ij}} < \omega^i_{ji}\frac{\partial r_{ji}(e_{ij})}{\partial e_{ij}}$, for all $e_{ij} \in [0,1]$,
- $\hat{e}_{ij} \in (0,1)$ that holds $\left.\frac{\partial c_i(e_i)}{\partial e_{ij}}\right|_{e_{ij}=\hat{e}_{ij}} = \omega^i_{ji} \left.\frac{\partial r_{ji}(e_{ij})}{\partial e_{ij}}\right|_{e_{ij}=\hat{e}_{ij}}$, otherwise.

The following theorem shows the unique allocation rule of the WPR family that induces an efficient effort profile in equilibrium. This allocation rule gives all the reductions to the agent that generates them. Formally, let $H(e) := (H_i(e))_{i\in N}$ be the allocation rule in the WPR family with $\omega^i_{ji} = 1$ for i, $j \in N, i \neq j$, that is $H_i(e) = c_i(e_i) - \sum_{j\in N\setminus\{i\}} r_{ji}(e_{ij})$ for $i \in N$. We consider an allocation rule as efficient if it induces an efficient effort profile in equilibrium.

Theorem 2.3 *Consider the effort game $(N, \{E_i\}_{i\in N}, \{H_i\}_{i\in N})$. Let e^*_{ij} be the level of effort that an agent i exerts to reduce the costs of agent j in the unique equilibrium with i, $j \in N, i \neq j$. Thus,*

- $e^*_{ij} = 0$ if and only if $\left.\frac{\partial c_i(e_i)}{\partial e_{ij}}\right|_{e_{ij}=0} > \left.\frac{\partial r_{ji}(e_{ij})}{\partial e_{ij}}\right|_{e_{ij}=0}$
- $e^*_{ij} = 1$ if and only if $\left.\frac{\partial c_i(e_i)}{\partial e_{ij}}\right|_{e_{ij}=1} < \left.\frac{\partial r_{ji}(e_{ij})}{\partial e_{ij}}\right|_{e_{ij}=1}$
- $e^*_{ij} \in (0,1)$ that holds $\left.\frac{\partial c_i(e_i)}{\partial e_{ij}}\right|_{e_{ij}=e^*_{ij}} = \left.\frac{\partial r_{ji}(e_{ij})}{\partial e_{ij}}\right|_{e_{ij}=e^*_{ij}}$, otherwise.

In addition, $e^*_{ij} = \tilde{e}_{ij}$ for i, $j \in N, i \neq j$ and $H_i(e)$ is the only allocation rule of the WPR family that always induces an efficient effort profile in equilibrium.

Corollary 2.1 Let Θ be the set of all allocation rules for PE-games. There is no $\psi \in \Theta$ such that the effort equilibrium profile induced in the non-cooperative game $(N, \{E_i\}_{i\in N}, \{\psi_i\}_{i\in N})$ generates a lower cost of the grand coalition than allocation rule H.

As mentioned, the effort e_{ij} is efficient when its marginal cost matches the marginal reduction that it generates; otherwise, the effort is zero or one. Allocation rule $H(e)$ aligns the incentives of agents in the first stage game with this idea. It gives all the reduction to the agent that generates it. In that case, the best response of any agent is to make its marginal cost equal to the marginal reduction that its effort generates; otherwise, this agent exerts the minimal or maximal effort depending on which is higher: the marginal cost or the marginal reduction.

We illustrate this analysis with the 3-firm case given in the Example 1 in Section 3.4

In this section we work out the allocation rule (in the second stage) within the WPR family that generates the unique efficient effort equilibrium (in the first stage). However, there are situations in which pairwise reductions cannot be weighted separately, i.e. it is not possible to assign different weights to what an agent gives and what the same agent receives in a pairwise interaction. For example, there may be situations in which there is a unique cost reduction for any pair of agents that depends on the effort exerted by both agents, i.e. an aggregate reduction. In that case they have to decide how to split the whole cost reduction. Such cases require a weight to be assigned to the pairwise aggregate reduction.

The question that arises in this new scenario is whether the level of efficiency maintained is the same as that attained when the pairwise reductions are weighted separately for each agent. Unfortunately, the answer is no: the level of efficiency decreases in this new scenario as shown in next chapter.

2.5 Concluding remarks

In this second chapter, we analyze and study under the form of bi-form games, structured in a first competitive phase and a second cooperative phase, a mode or form of bilateral cooperation that, from the scope of the Transferable Utility Cooperative Cost Games, allows the reduction of costs between pairs of agents being this reduction independent of the behavior or cooperation of each agent of the pair outside the pair. In other words, the cost reduction will remain invariant regardless of the size of the coalition in which each pair of players is integrated.

As a consequence of the study of the costs, benefits and challenges associated with this pairwise effort environment, it is found that in the first competitive phase the agents anticipate to determine the individual level of effort that will shape their cost function in the cooperative game of the second phase. As a result of the modeling of the pairwise cost reduction in the cooperative phase, a new class of PE-Games (Pairwise Effort Games) is introduced where both its concavity and the stability of a large coalition allowing a cost reduction for all players are found. A cost sharing mechanism is presented that generates sufficient incentives to generate the optimal efforts by the players to minimize the aggregate costs through the contribution of optimal efforts. The allocation rule (WPR) that generates the indicated unique efficient effort equilibria is found and presented.

Based on the evidence and certainties observed, there is inevitably a question that makes it necessary to examine whether the same level of efficiency is achieved both by weighting the reductions individually and by weighting them in aggregate. This scenario is developed and analyzed in the next chapter. It is shown that the same level of efficiency is not achieved, and that it is signifficantly lower if the weighting is done on an aggregate basis.

Chapter 3

Measuring Efficiency for Pairwise Aggregate Reduction

3.1 Introduction

The impact and importance that the introduction of the notion of transferable utility (TU) had on the field of game theory is undeniable. This has been so to the extent that it presented a new class of games whose allocations depended on the previous strategic behavior of the players, modifying an initial theoretical rigidity that presupposed full willingness to cooperate for all players, modeling eminently symmetrical scenarios that are not, in reality, at all frequent.

Analyzing the prior strategic element of the players in cooperative environments or scenarios therefore implies analyzing the fact that the coalitional value inevitably depends on the strategic behaviors adopted by each player, so that the greater the competitiveness, the lower the coalitional values and therefore the lower the incentive to cooperate.

Given that, when a scenario of cooperation arises, the agents will act rationally in their decisions, this rationality will move each agent to maximize his reward in terms of cost-benefit through the strategic management of his involvement or degree of cooperation, since this will be the only aspect that he will be able to manage in order to try to maximize the payments resulting from the coalitions formed in the cooperative phase.

The modeling of this type of situation in the form of the biform-games introduced by Brandenburger and Stuart (2007) contemplates the existence of competitive and cooperative behaviors when reaching agreements between players. Not surprisingly, the concept and proposal of coexistence between both a priori opposite terms, is promoted and normalized towards the business environment reaching a great diffusion and notoriety with the publication by the same author ten years earlier of the book Coopetition (Brandenburger, 1997), where it begins to penetrate and popularize the attempt to take the most beneficial of both behaviors to obtain the greatest possible gain or advantage.

The description or applied development of the coexistence of two strategies was even earlier developed by Zhou in 1994 with the so-called investment and cooperation strategies where, in a given situation, although the agents had to cooperate jointly to increase the level of innovation, on the other hand they tried individually to agree on the optimum level of investment effort that would allow a certain degree of innovation but incurring costs that would optimize the profits derived from it.

One of the most significant contributions of the so-called Biform-Games has undoubtedly been the progress made in bridging the lines separating Game Theory as a theoretical body and the reality to which it is applied, in an attempt to become an effective tool capable of providing or proposing optimal solution proposals.

At this point, it is necessary to stress the fact that, in any cooperation scenario, any asymmetry, understanding as such the different degrees of cooperation with which each player decides to join the game, necessarily causes a loss or reduction of the possible total joint gain that could have occurred if such asymmetry had not existed.

Consequently, when one or more players decide to negotiate or compete in order to set levels or degrees of cooperation lower than the maximum degree or full cooperation, the future distributions or allocations obtained as a result of such cooperation will imply a greater loss the lower the degree of cooperation established.

As pointed out in the work of Liu and Xiang (2023) and even earlier in his time also by Stuart (2001), biform-games are a class of games that tends to be competitive, insofar as it is the vector of strategies defined in the non-cooperative phase that defines the subsequent cooperative phase where the distribution of the profits generated on the basis of the different individual strategies that each agent previously adopts is analyzed, and all this taking into account that the profits or distributions generated by biform-games are not generated for the individual players but for the totality of coalitions in the game.

Taking into account the above, the core or main axis of this third and last part of this thesis work, focuses on measuring the level of efficiency of efforts in equilibrium for a particular family of weighted pairwise aggregate reduction. For this purpose, it is necessary, once the non-existence of an empty core has been identified, as well as the family (WPR) and subfamily (WPAR) of allocation or distribution, and Shapley's proposal of distribution within the latter, to determine the efficiencies and scope of the efforts in equilibrium, determine the efficiencies and scope of a competitive or bargaining phase that leads to the identification of a sharing rule in a cooperative phase different from the one proposed according to Shapley's value, demonstrating the existence of a unique sharing vector within the WPAR subfamily that coincides with the optimal vector of efficient effort determined in the first bargaining phase or absence of cooperation and that in turn coincides with the vector of effort values corresponding to the efficient Nash equilibrium.

3.2 Efficiency of Pairwise Aggregate Reduction

Consider the family of cost allocation with weighted pairwise aggregate reduction $A(e) \in R^n$ defined as follows:

$$A_i(e) = c_i(e_i) - \sum_{j \in N \setminus \{i\}} \alpha_{ij}[r_{ij}(e_{ji}) + r_{ji}(e_{ij})], \tag{3.1}$$

with $\alpha_{ij} \in [0, 1]$. The interaction between agents i and j generates an aggregate cost reduction which is $r_{ij}(e_{ji}) + r_{ji}(e_{ij})$. The parameter α_{ij} measures the proportions in which this reduction is shared between agents i and j, i.e. α_{ij} is the proportion for agent i and $\alpha_{ji} = 1 - \alpha_{ij}$ for agent j.

Note that $A(e)$ is a subfamily of the WPR family $\Omega(e)$, where now $\omega^i_{ij} = \omega^j_{ij} = \alpha_{ij}$, for all $i, j \in N$. From now on we refer to this subfamily as the WPAR family. It is important to note that the Shapley value and the Nucleolus belong to the WPAR family with $\alpha_{ij} = \frac{1}{2}$ for all i, $j \in N$, $i \neq j$. We consider whether the allocation rule $H(e)$, which generates the efficient effort in equilibrium, is applicable in this situation. Unfortunately, $H(e)$ does not fit the scheme of pairwise aggregate reduction.

This section analyzes the non-cooperative effort game that arises in the first stage when cost allocations in the WPAR family are considered.

Our goal is to find out, within the WPAR family, a core-allocation in the cooperative game of the second stage that induce the effort equilibrium level in the first stage closest to the efficient one. We consider that an effort profile $e^{'} \in E$ is more efficient than a profile $e^{''} \in E$ if the aggregate cost generated in the second stage by $e^{'}$ is lower than that generated by $e^{''}$.

We therefore first study the non-cooperative effort game that arises under this new cost allocation $A(e)$, that is $(N, \{E_i\}_{i\in N}, \{A_i\}_{i\in N})$.

To simplify notation and analysis, we consider that for all $i \in N$ and $j \in N \backslash \{i\}$, $c_i'(e_{ij}) := \frac{\partial c_i(e_i)}{\partial e_{ij}}$, $c_i''(e_{ij}) := \frac{\partial^2 c_i(e_i)}{\partial e_{ij}^2}$, $r_{ji}'(e_{ij}) := \frac{\partial r_{ji}(e_{ij})}{\partial e_{ij}}$ and $r_{ji}''(e_{ij}) := \frac{\partial^2 r_{ji}(e_{ij})}{\partial e_{ij}^2}$. Note that, as the WPAR family is a subfamily of WPR, the properties of the latter apply to the former.

Before analyzing the EEE of the above non-cooperative effort game, we define thresholds of alpha parameters that enable them to be reached.

Definition 3.1 *Given an effort game $(N, \{E_i\}_{i\in N}, \{A_i\}_{i\in N})$, we define the following lower and upper thresholds for each pair of agents i and j,*

$\underline{\alpha}_{ij} := \frac{c_i'(0)}{r_{ji}'(0)}$, $\bar{\alpha}_{ij} := \frac{c_i'(1)}{r_{ji}'(1)}$, $\underline{\alpha}_{ji} := \frac{c_j'(0)}{r_{ij}'(0)}$, and $\bar{\alpha}_{ji} := \frac{c_j'(1)}{r_{ij}'(1)}$.

It is clear that $0 < \underline{\alpha}_{ij} < \bar{\alpha}_{ij}$ because c_i' is an increasing function and r_{ji}' decreasing one. Analogously, $0 < \underline{\alpha}_{ji} < \bar{\alpha}_{ji}$.

The first Theorem in this chapter characterizes all possible types of effort equilibrium according to the value of the parameter α_{ij}, for all $i, j \in N, i \neq j$. The proof of Theorem appears in Appendix C, together with all the other proofs in this chapter.

Theorem 3.1 *Let $(N, \{E_i\}_{i\in N}, \{A_i\}_{i\in N})$ be an effort game. The pairwise efforts in any unique equilibrium (e_{ij}^*, e_{ji}^*) are given by*

$$e^*_{ij} = \begin{cases} 0 \text{ if and only if } \alpha_{ij} \leq \underline{\alpha}_{ij} \\ e^I \text{ if and only if } \underline{\alpha}_{ij} < \alpha_{ij} < \bar{\alpha}_{ij} \\ 1 \text{ if and only if } \alpha_{ij} \geq \bar{\alpha}_{ij} \end{cases}$$

$$e^*_{ji} = \begin{cases} 0 \text{ if and only if } \alpha_{ij} \geq 1 - \underline{\alpha}_{ji} \\ e^J \text{ if and only if } 1 - \bar{\alpha}_{ji} < \alpha_{ij} < 1 - \underline{\alpha}_{ji} \\ 1 \text{ if and only if } \alpha_{ij} \leq 1 - \underline{\alpha}_{ji} \end{cases}$$

where $e^I \in (0,1)$ is the unique solution of $c'_i(e_i) - \alpha_{ij} r'_{ji}(e_{ij}) = 0$ and $e^J \in (0,1)$ is the unique solution of $c'_j(e_j) - (1 - \alpha_{ij}) r'_{ij}(e_{ji}) = 0$.

The next corollary shows how the pairwise equilibrium efforts e^*_{ij} depend on α_{ij}, for all $i, j \in N, i \neq j$. As expected, as the proportion of aggregate cost reduction obtained by an agent increases, the effort that agent exerts also increases (or at least stays the same).

Corollary 3.1 Let $(N, \{E_i\}_{i \in N}, \{A_i\}_{i \in N})$ be the effort game and (e^*_{ij}, e^*_{ji}) the pairwise efforts equilibrium. Thus,

- $\frac{\partial e^*_{ij}}{\partial \alpha_{ij}} > 0$, if $\alpha_{ij} \in (\underline{\alpha}_{ij}, \bar{\alpha}_{ij})$; $\frac{\partial e^*_{ij}}{\partial \alpha_{ij}} = 0$, otherwise.
- $\frac{\partial e^*_{ji}}{\partial \alpha_{ij}} < 0$, if $\alpha_{ij} \in (1 - \bar{\alpha}_{ji}, 1 - \underline{\alpha}_{ji})$; $\frac{\partial e^*_{ji}}{\partial \alpha_{ij}} = 0$, otherwise.

The results above are very useful when the goal is to incentivize agents $i, j \in N$ to make more pairwise effort e_{ij} by means of parameter α_{ij}. However, we wish to go further, specifically to achieve the highest level of efficiency within the WPAR family. In other words we wish to find the α^*_{ij}, for all $i, j \in N$ that minimizes the aggregate cost function $\sum_{i \in N} A_i(e^*)$ in equilibrium, where both $A_i\,(e)$ and the effort equilibrium e^* depend on α_{ij}.

3.3 A procedure for finding an Efficient Effort Equilibrium induced by WPAR

The search for alpha parameters which will lead to the EEE can be simplified by taking into account the bilateral independent interactions of agents. Note first that any pair of agents have a particular α_{ij}, and second that the optimal effort made by any agent $i \in N$ in regard to any agent $j \in N\backslash\{i\}$ is independent of the optimal effort that agent i exerts in regard to any other agent $h \in N\backslash\{i,j\}$. Thus, minimizing $\sum_{i\in N} A_i(e^*)$ in terms of α_{ij} is equivalent to minimizing $A_i(e^*) + A_j(e^*)$, since each particular α_{ij} only appears in $A_i(e^*)$ and $A_j(e^*)$. Fortunately, the problem can be further simplified: Note that, $A_i(e^*)$ and $A_j(e^*)$ are the sums of different terms, but α_{ij} only appears in those terms related to the interaction between i and j (see (3.1)). These terms are $c_i(e_i^*) - \alpha_{ij}(r_{ij}(e_{ji}^*) + r_{ji}(e_{ij}^*))$ from $A_i(e^*)$, and $c_j(e_j^*) - (1-\alpha_{ij})(r_{ji}(e_{ij}^*) + r_{ij}(e_{ji}^*))$ from $A_j(e^*)$. Thus, a new function $A_i^*(\alpha_{ij}) := c_i(e_i^*) - \alpha_{ij}(r_{ij}(e_{ji}^*) + r_{ji}(e_{ij}^*))$ can be considered, and analogously $A_j^*(1-\alpha_{ij})$. Note that $\frac{\partial^x(A_i(e^*))}{\partial\alpha_{ij}^x} = \frac{\partial^x(A_i^*(\alpha_{ij}))}{\partial\alpha_{ij}^x}$ and $\frac{\partial^x(A_j(e^*))}{\partial\alpha_{ij}^x} = \frac{\partial^x(A_j^*(1-\alpha_{ij}))}{\partial\alpha_{ij}^x}$for $x = 1, 2, ...$ Therefore, for each pair i and j, it is possible to define the function $L_{ij}^*(\alpha_{ij}) := A_i^*(\alpha_{ij}) + A_j^*(1-\alpha_{ij})$. Hence, minimizing $\sum_{i\in N} A_i(e^*)$ is equivalent to minimizing $L_{ij}^*(\alpha_{ij})$, with

$$\begin{aligned} L_{ij}^*(\alpha_{ij}) &= c_i(e_i^*) + c_j(e_j^*) - \left[\alpha_{ij}(r_{ij}(e_{ji}^*) + r_{ji}(e_{ij}^*)) + (1-\alpha_{ij})(r_{ji}(e_{ij}^*) + r_{ij}(e_{ji}^*))\right] \\ &= c_i(e_i^*) + c_j(e_j^*) - (r_{ij}(e_{ji}^*) + r_{ji}(e_{ij}^*)) \qquad (3.2) \end{aligned}$$

The function $L_{ij}^*(\alpha_{ij})$ depends on α_{ij} through the equilibrium efforts e_{ij}^* and e_{ji}^* because they depend on α_{ij}. We now focus on finding the α_{ij} that minimizes function $L_{ij}^*(\alpha_{ij})$, and provide a procedure for finding the EEE for pairwise aggregate reduction.

We can summarize this reasoning as follows. Let $\alpha = (\alpha_i)_{i\in N}$ and $\alpha_i = (\alpha_{ij})_{j\in N\backslash\{i\}}$, $\alpha^* = \arg \underset{\alpha\in[0,1]^{n(n-1)}}{min} \sum_{i\in N} A_i(e^*) \iff \alpha_{ij}^* = \arg \underset{\alpha_{ij}\in[0,1]}{min} A_i(e^*) + A_j(e^*)$ for all $i \in N \iff$ $\alpha_{ij}^* = \arg \underset{\alpha_{ij}\in[0,1]}{min} c_i(e_i^*) - \alpha_{ij}(r_{ij}(e_{ji}^*) + r_{ji}(e_{ij}^*)) + c_j(e_j^*) - (1-\alpha_{ij})(r_{ji}(e_{ij}^*) + r_{ij}(e_{ji}^*))$ for all $i, j \in N, i \neq j \iff \alpha_{ij}^* = \arg \underset{\alpha_{ij}\in[0,1]}{min} c_i(e_i^*) + c_j(e_j^*) - (r_{ji}(e_{ij}^*) + r_{ij}(e_{ji}^*))$ for all i, $j \in N, i \neq j$. As $L_{ij}^*(\alpha_{ij}) = c_i(e_i^*) + c_j(e_j^*) - (r_{ij}(e_{ji}^*) + r_{ji}(e_{ij}^*))$, then $\alpha_{ij}^* = \arg \underset{\alpha_{ij}\in[0,1]}{min}$ $L_{ij}^*(\alpha_{ij})$ for all i, $j \in N, i \neq j$.

For any effort game considered here, there are only six possible distributions of the lower and upper thresholds of the alpha parameter.[1] These cases are

$$
\begin{array}{ll}
\textbf{Case A} & \underline{\alpha}_{ij} < \bar{\alpha}_{ij} < 1 - \bar{\alpha}_{ji} < 1 - \underline{\alpha}_{ji} \\
\textbf{Case B} & \underline{\alpha}_{ij} < 1 - \bar{\alpha}_{ji} < \bar{\alpha}_{ij} < 1 - \underline{\alpha}_{ji} \\
\textbf{Case C} & \underline{\alpha}_{ij} < 1 - \bar{\alpha}_{ji} < 1 - \underline{\alpha}_{ji} < \bar{\alpha}_{ij} \\
\textbf{Case D} & 1 - \bar{\alpha}_{ji} < \underline{\alpha}_{ij} < \bar{\alpha}_{ij} < 1 - \underline{\alpha}_{ji} \\
\textbf{Case E} & 1 - \bar{\alpha}_{ji} < \underline{\alpha}_{ij} < 1 - \underline{\alpha}_{ji} < \bar{\alpha}_{ij} \\
\textbf{Case F} & 1 - \bar{\alpha}_{ji} < 1 - \underline{\alpha}_{ji} < \underline{\alpha}_{ij} < \bar{\alpha}_{ij}
\end{array}
\tag{3.3}
$$

The last theorem characterizes the optimal α^*_{ij} in cases A-F. Thus, Theorem 3.2 provides the α^*_{ij} that incentivizes an efficient effort equilibrium for WPAR [2]. In Theorem 3.2 we use the following notation:

1. $\check{\alpha}^{[a,b]}_{ij} \in [a, b]$ with $0 \leq a < b \leq 1$ is:
$$
\check{\alpha}^{[a,b]}_{ij} = \begin{cases} a & \text{if } \frac{\partial(L^*_{ij}(\alpha_{ij}))}{\partial \alpha_{ij}} > 0 \text{ for all } \alpha_{ij} \in [a, b] \\ b & \text{if } \frac{\partial(L^*_{ij}(\alpha_{ij}))}{\partial \alpha_{ij}} < 0 \text{ for all } \alpha_{ij} \in [a, b] \\ \text{Solution of } \frac{\partial(L^*_{ij}(\alpha_{ij}))}{\partial \alpha_{ij}} = 0 & \text{otherwise} \end{cases} .
$$

2. $\Lambda(\alpha) = \begin{cases} 0 \text{ if } \alpha < 0 \\ \alpha \text{ if } \alpha \in (0, 1) \\ 1 \text{ if } \alpha > 1 \end{cases}$

Theorem 3.2 *Let* $(N, \{E_i\}_{i \in N}, \{A_i\}_{i \in N})$ *be an effort game, and* $L^*_{ij}(\alpha_{ij}) = c_i(e^*_i) +$

---

[1] Note that $\underline{\alpha}_{ji} < \bar{\alpha}_{ji}$ and $\underline{\alpha}_{ij} < \bar{\alpha}_{ij}$.

[2] The function L^*_{ij} is a piecewise function, and although it is continuous in $\alpha_{ij} \in [0, 1]$ it is not differentiable at all points in its domain. Since it is defined over intervals, it is generally non-differentiable at the endpoints of these intervals. Therefore, to compute the minimum, it is also necessary to evaluate the function at the interval endpoints. In addition, due to its convexity, the minimum can also be an interior point within any of the intervals. However, each interval entails a distinct derivative function, thereby contributing to the complexity of the computation process.

The introduction of Theorem 5 streamlines the evaluation procedure by reducing the number of points to be assessed, presenting them in a case-by-case framework.

$c_j(e_j^*) - (r_{ij}(e_{ji}^*) + r_{ji}(e_{ij}^*))$. *The optimal solution* $\alpha_{ij}^* = \arg \min_{\alpha_{ij} \in [0,1]} L_{ij}^*(\alpha_{ij})$ *is in each case,*

Case A α_{ij}^* is any element of $[\bar{\alpha}_{ij}, 1 - \bar{\alpha}_{ji}]$

Case B $\alpha_{ij}^* = \check{\alpha}_{ij}^{[1-\bar{\alpha}_{ji}, \bar{\alpha}_{ij}]}$

Case C $\alpha_{ij}^* = \begin{cases} \text{any element of } [\bar{\alpha}_{ij}, 1] \text{ if } \alpha^C = \Lambda(\bar{\alpha}_{ij}) \text{ and } \Lambda(\bar{\alpha}_{ij}) < 1 \\ \alpha^C \text{ otherwise} \end{cases},$

where $\alpha^C = \arg min\{L_{ij}^*(\check{\alpha}_{ij}^{[1-\bar{\alpha}_{ji}, 1-\underline{\alpha}_{ji}]}), L_{ij}^*(\Lambda(\bar{\alpha}_{ij}))\}$.

Case D $\alpha_{ij}^* = \begin{cases} \text{any element of } [0, 1-\bar{\alpha}_{ji}] \text{ if } \alpha^D = \Lambda(1-\bar{\alpha}_{ji}) \text{ and } \Lambda(1-\bar{\alpha}_{ji}) > 0 \\ \alpha^D \text{ otherwise} \end{cases},$

where $\alpha^D = \arg min\{L_{ij}^*(\Lambda(1-\bar{\alpha}_{ji})), L_{ij}^*(\check{\alpha}_{ij}^{[\underline{\alpha}_{ij}, \bar{\alpha}_{ij}]})\}$.

Case E $\alpha_{ij}^* = \begin{cases} \text{any element of } [0, 1-\bar{\alpha}_{ji}] \text{ if } \alpha^E = \Lambda(1-\bar{\alpha}_{ji}) \text{ and } \Lambda(1-\bar{\alpha}_{ji}) > 0 \\ \text{any element of } [\bar{\alpha}_{ij}, 1] \text{ if } \alpha^E = \Lambda(\bar{\alpha}_{ij}) \text{ and } \Lambda(\bar{\alpha}_{ij}) < 1 \\ \alpha^E \text{ otherwise} \end{cases},$

where $\alpha^E = \arg min\{L_{ij}^*(\Lambda(1-\bar{\alpha}_{ji})), \check{\alpha}_{ij}^{[\underline{\alpha}_{ij}, 1-\underline{\alpha}_{ji}]}, L_{ij}^*(\Lambda(\bar{\alpha}_{ij}))\}$.

Case F $\alpha_{ij}^* = \begin{cases} \text{any element of } [0, 1-\bar{\alpha}_{ji}] \text{ if } \alpha^F = \Lambda(1-\bar{\alpha}_{ji}) \text{ and } \Lambda(1-\bar{\alpha}_{ji}) > 0 \\ \text{any element of } [\bar{\alpha}_{ij}, 1] \text{ if } \alpha^F = \Lambda(\bar{\alpha}_{ij}) \text{ and } \Lambda(\bar{\alpha}_{ij}) < 1 \\ \alpha^F \text{ otherwise} \end{cases}$

where $\alpha^F = \arg min\{L_{ij}^*(\Lambda(1-\bar{\alpha}_{ji})), L_{ij}^*(\Lambda(\bar{\alpha}_{ij}))\}$.

To conclude the section, we describe a procedure for finding an efficient effort in equilibrium induced by the WPAR family.

EEE Procedure

Given an effort game $(N, \{E_i\}_{i\in N}, \{A_i\}_{i\in N})$,

1. we first calculate the lower and upper thresholds of the bilateral interaction between any pair of agents by using Thresholds Definition ;

2. we then focus on the list (3.3) and determine which case (A-F) applies;

3. Theorem 3.2 provides an optimal α^*_{ij} for all $i, j \in N$, to minimize the centralized (aggregate) cost allocation $\sum_{i \in N} A_i(e^*)$;

4. with this α^*_{ij}, Theorem 3.1 gives the associated efficient effort equilibrium (e^*_{ij}, e^*_{ji}) for every pair of agents, and thus an efficient effort equilibrium e^* for the game;

5. at this point the optimal cost allocation that incentivizes agents $i, j \in N$ to make an efficient effort equilibrium e^*_{ij} and e^*_{ji} is known, i.e.

$$A^*_i(e^*) = c_i(e^*_i) - \sum_{j \in N \setminus \{i\}} \alpha^*_{ij}[r_{ij}(e^*_{ji}) + r_{ji}(e^*_{ij})];$$

We illustrate this procedure with the 3-firm case given in Example 3.1 in section 3.4

3.4 Comparison of WPR and WPAR families

We complete the study of our model of cooperation with pairwise cost reduction by comparing the two families of core-allocations analyzed. We find that there is a loss of efficiency when cooperation is restricted to a pairwise aggregate cost reduction. That loss of efficiency can be measured. In addition, we show that those agents who receive less than the total reduction generated and bear the total cost of this effort always exert less effort than the efficient agent.

As mentioned above, the allocation rule $H(e)$ induces an equilibrium effort e^{*H} that matches the efficient effort of Proposition, i.e. $e^{*H} = \tilde{e}$. This means that there is no rule that generates a lower cost of the grand coalition, see Corollary. However, as also mentioned above, WPAR is a subfamily of WPR, but $H(e)$ is not in WPAR, so e^{*A} is not always equal to e^{*H}.

Let $A^{*}(e)$ be the allocation rule in WPAR that induces the effort profile $e^{*A^{*}}$ that minimizes the cost of the grand coalition, i.e. the efficient allocation in this subfamily. The difference, in terms of efficiency, between the cost of the grand coalition with $e^{*A^{*}}$ and $\tilde{e}$ can be measured. Note that for any particular functions $c_i(e_i)$ and $r_{ij}(e_{ji})$ for i, $j \in N$, $i \neq j$, the associated $e^{*A^{*}}$ and $\tilde{e}$ can be obtained. Let Δ be this difference or loss of efficiency, where

$$\Delta = \sum_{i \in N} [c_i(e_i^{*A^{*}}) - \sum_{j \in N \setminus \{i\}} r_{ij}(e_{ji}^{*A^{*}})] - \sum_{i \in N} [c_i(\tilde{e}_i) - \sum_{j \in N \setminus \{i\}} r_{ij}(\tilde{e}_{ji})]. \tag{3.4}$$

The following proposition shows the relation between efforts $e^{*A^{*}}$ and $\tilde{e}$.

Proposition 3.1 *Let $e_{ij}^{*A^{*}}$ for i, $j \in N$, $i \neq j$ be the equilibrium efforts of $A^{*}(e)$, that minimize the cost of the grand coalition in the family WPAR. Thus, the efficient effort*

$$\tilde{e}_{ij} \geq e_{ij}^{*A^{*}} \textit{for all } i,\ j \in N,\ i \neq j.$$

As mentioned above, when an agent receives less than the total reduction that it generates and bears the total cost of that effort, then that agent always exerts less effort than the efficient one

Finally, readers may think that the rationale behind the efficient rule, $H(e)$, in the WPR family, could also apply to the WPAR family. However, this is not the case. To reach an efficient effort equilibrium in the WPR family, for each pair of agents $i, j \in N, i \neq j$, the weight ω^i_{ji} must be 1, because $\frac{\partial \Omega_i(e)}{\partial e_{ij}} = \frac{\partial c_i(e_i)}{\partial e_{ij}} - \omega^i_{ji}\frac{\partial r_{ji}(e_{ij})}{\partial e_{ij}}$, and ω^j_{ij} must also be 1, because $\frac{\partial \Omega_j(e)}{\partial e_{ji}} = \frac{\partial c_j(e_j)}{\partial e_{ji}} - \omega^j_{ij}\frac{\partial r_{ij}(e_{ji})}{\partial e_{ji}}$. However, this is no longer true for the WPAR family.[3]

The following example with three agents illustrates the comparison of the two core allocation families and completes the demonstration.

Example 3.1 Consider a pairwise inter-organizational situation with three firms, i.e. $N = \{1, 2, 3\}$. For any effort profile $e \in [0, 1]^6$, the PE-situation is given by the following initial costs,

$c_1(e_{12}, e_{13}) = 100 + 100e_{12} + 4e_{12}^2 + 100e_{13} + 4e_{13}^2$
$c_2(e_{21}, e_{23}) = 100 + 100e_{21} + 4e_{21}^2 + 100e_{23} + 4e_{23}^2$
$c_3(e_{31}, e_{32}) = 100 + 100e_{31} + 4e_{31}^2 + 100e_{32} + 4e_{32}^2$

and the following pairwise reduced costs, all of them in thousands of Euros,

$r_{i1}(e_{1i}) = 2 + 110e_{1i} - 2e_{1i}^2$ with $i = 2, 3$
$r_{i2}(e_{2i}) = 2 + 105e_{2i} - 3e_{2i}^2$ with $i = 1, 3$
$r_{i3}(e_{3i}) = 2 + 105e_{3i} - 3e_{3i}^2$ with $i = 1, 2$

by Definition, the pair of firms $\{1, 2\}$ has the thresholds $\underline{\alpha}_{12} = 0.91$, $\bar{\alpha}_{12} = 1.02$, $\underline{\alpha}_{21} = 0.95$, and $\bar{\alpha}_{21} = 1.09$, which correspond to Case F in the Table 3.3. By using Theorem 3.2, it can easily be checked that $\alpha^F = \Lambda(\bar{\alpha}_{12}) < 1$ and $\alpha^*_{12} = 1$. Thus, by Theorem 3.1,

[3] In WPAR, for each pair of agents $i, j \in N, i \neq j$, the weight α_{ij} is not always 1, because $\frac{\partial A_i(e)}{\partial e_{ij}} = \frac{\partial c_i(e_i)}{\partial e_{ij}} - \alpha_{ij}\frac{\partial r_{ji}(e_{ij})}{\partial e_{ij}}$ and $\frac{\partial A_j(e)}{\partial e_{ji}} = \frac{\partial c_j(e_j)}{\partial e_{ji}} - \alpha_{ji}\frac{\partial r_{ij}(e_{ji})}{\partial e_{ji}}$ but $\alpha_{ij} = 1 - \alpha_{ji}$. Note that if $\alpha_{ij} = 1$, then $\alpha_{ji} = 0$ and the derivative conditions for efficiency in Proposition ?? would be violated. Bear in mind that the weights ω^i_{ji} that appear in each derivative $\frac{\partial \Omega_i(e)}{\partial e_{ij}}$ for $i, j \in N, i \neq j$ are independent of one another. However, the weights α_{ij} that appear in the each derivative $\frac{\partial A_i(e)}{\partial e_{ij}}$ for $i, j \in N, i \neq j$ are not, because $\alpha_{ij} = 1 - \alpha_{ji}$. In addition, it is known that $\omega^i_{ij} = \omega^i_{ji} = \alpha_{ij}$ in WPAR for all $i, j \in N, i \neq j$, where $\omega^i_{ij} = 1 - \omega^j_{ij}$ and $\omega^i_{ji} = 1 - \omega^j_{ji}$. The fact that pairwise cost reduction is aggregated by α_{ij} in the subfamily WPAR means that it is not possible to apply the efficient argument used for the WPR family.

$e^*_{12} = 0.833$, $e^*_{21} = 0$. As firms 2 and 3 are identical, $\alpha^*_{13} = 1$, $e^*_{13} = 0.833$ and $e^*_{31} = 0$. Finally, for the pair $\{2, 3\}$, $\underline{\alpha}_{23} = 0.95$, $\bar{\alpha}_{23} = 1.09$, $\underline{\alpha}_{32} = 0.95$, and $\bar{\alpha}_{32} = 1.09$. This is again Case F. Note that in case F, $\alpha^F = \arg min\{L^*_{23}(\Lambda(1-\bar{\alpha}_{32})), L^*_{23}(\Lambda(\bar{\alpha}_{23}))\}$, where in this particular case $L^*_{23}(\Lambda(1-\bar{\alpha}_{32})) = L^*_{23}(\Lambda(\bar{\alpha}_{23}))$ with $\Lambda(1-\bar{\alpha}_{32}) = 0$ and $\Lambda(\bar{\alpha}_{23}) = 1$ Thus, two solutions emerge: (i) $e^*_{23} = 0.357$, $e^*_{32} = 0$, and $\alpha^*_{23} = 1$, and (ii) $e^*_{23} = 0$, $e^*_{32} = 0.357$, and $\alpha^*_{23} = 0$.

Therefore, there are two EEE in WPAR.

(i) $e^*_{12} = e^*_{13} = 0.833$, $e^*_{21} = 0$, $e^*_{23} = 0.357$, $e^*_{31} = e^*_{32} = 0$

(ii) $e^*_{12} = e^*_{13} = 0.833$, $e^*_{21} = e^*_{23} = 0, e^*_{31} = 0$, $e^*_{32} = 0.357$

We now calculate the efficient efforts in this example by Proposition 3.3. They are the solutions of $c'_i(e_{ij}) - r'_{ji}(e_{ij}) = 0$, thus, $\tilde{e}_{12} = \tilde{e}_{13} = 0.833$, and $\tilde{e}_{21} = \tilde{e}_{23} = \tilde{e}_{31} = \tilde{e}_{32} = 0.357$. Note that by Theorem 3.3 these efforts are also the effort equilibrium obtained by the allocation rule $H(e)$.

Example 1 *This example is a particular subcase of Case F. This implies that α^*_{ij} is zero or one, which in turn implies that one of the agents makes no effort and the other makes the efficient value. However, they are never able to make the efficient effort simultaneously under WPAR. The loss of efficiency in WPAR with regard to WPR can be calculated with the help of (3.4).*

$\Delta = \sum_{i\in N}[c_i(e^{*A^*}_i) - \sum_{j\in N\setminus\{i\}} r_{ij}(e^{*A^*}_{ji})] - \sum_{i\in N}[c_i(\tilde{e}_i) - \sum_{j\in N\setminus\{i\}} r_{ij}(\tilde{e}_{ji})] = 278.776 - 276.104 = 2.67$.

3.5 Concluding remarks

In this chapter, the analysis focuses on a sub-family (WPAR) within the WPR family, where it is found that bilateral reductions between pairs of players are not weighted separately, and that the weighting occurs as the sum of all the reductions obtained, i.e. as an aggregate reduction.

Consequently, in this third and last chapter, we have focused firstly on the analysis of the non-cooperative phase in which the non -cooperative effort game takes place when considering the cost allocations belonging to the WPAR family.

The relationship between equilibrium efforts and weights is studied. It is found that the pairwise equilibrium efforts depend on the parameter alpha , that is, on the weight or measure of the proportion in which the cost reduction is shared between each pair of players. The totality of possible equilibria are characterized.

Consequently, it is shown that the increase in the proportion of aggregate cost reduction achieved by each agent leads in turn to that agent's willingness to exert more effort or, in the worst case, to maintain the level of effort unchanged, but never to be able to exert a lower level of effort.

A procedure to calculate and find an efficient effort in equilibrium induced by the WPAR allocation family is identified and presented.

Finally, both allocation families ,WPR and WPAR, are compared by analyzing the behavior when cooperation is limited to a cost reduction in an aggregated pairwise basis as well as when cooperation is individually weighted.

After analyzing the results obtained, on the one hand, the loss of efficiency is proven when cooperation is limited to aggregate cost reductions by pairs and, on the other hand, it is confirmed that this loss or variation in efficiency can be effectively measured, confirming the stability of the grand coalition and demonstrating that those players who see their costs reduced to a lesser extent with respect to the cost reduction they have caused to the rest will always cooperate with a lower level of effort than the efficient one.

Conclusions and future research

In the first chapter it has been presented a new model of Corporate tax games with benefactors and beneficiaries as an application of linear cost games to the corporate tax reduction system introducing the figure of multiple, dual and irreplaceable benefactors. It has been used the Shapley value as a rule of stable allocation to sharing costs reduced. Moreover, its properties are studied, it has been verified the snowball effect derived from the concavity of the model proving that the larger the coalition the lower the costs for its members and it has been proved that, these games are concave, i.e., the marginal contribution of a firm and a country diminishes as a coalition grows (snowball effect). Hence, the grand coalition is stable in the sense of the core. This means that firms have strong incentives to cooperate with the countries instead of being fraudsters. Then, it is proposed the Shapley value as an easily computable core-allocation that benefits all agents and, in particular, compensates the benefactors for their dual and irreplaceable role.

The model presented, distinguishes two groups of agents: dual benefactors (countries) and beneficiaries (firms), while the original model presented by [6], considered two disjoint groups of agents, benefactors and beneficiaries. A natural extension would be to consider that all agents can be dual (benefactors and beneficiaries). Consequently, it is certain that similar results to those obtained here could be achieved.

The second and third chapter, that are directly interrelated, presents a model of cooperation with pairwise cost reduction. The direct impact of pairwise effort on cost reductions is investigated by means of a bi-form game. First, the agents determine the level of pairwise effort to be made to reduce the costs of their partners. Second, they participate in a bilateral interaction with multiple independent partners where the cost reduction that each agent gives to another agent is independent of any possible coalition. As a result of cooperation, agents reduce each other's costs. In the non-cooperative game that precedes cooperation, the agents anticipate the cost allocation that will result from the cooperative game by incorporating the effect of the effort made into their cost functions. It is noted that all-included cooperation is feasible, in the sense that there are possible cost reductions that make all agents better off (or, at least, not worse off), and consistent. It then identifies a family of feasible cost allocations with weighted pairwise reduction. One of these cost allocations is selected by taking into account the incentives generated in the efforts that agents make, and consequently in the total cost of coalitions. Surprisingly, it is found that the Shapley value, which coincides with the Nucleolus in this model, can induce inefficient effort strategies in equilibrium in the non-cooperative model. However, it is always possible to select a core-allocation with appropriate pairwise weights that can generate an efficient effort.

Future research could take any of several directions. First, this thesis assumes that the individual effort cost function $c_i(e_i)$ is independent of the effort of other agents, and that the marginal cost $\frac{\partial c_i(e_i)}{\partial e_{ij}}$ is independent of the effort that i makes in regard to agents other than j, i.e. $\frac{\partial c_i^2(e_i)}{\partial e_{ij}\partial e_{ih}} = 0$. A similar assumption is made with the cost reduction function $r_{ij}(e_{ji}^*)$. There is some degree of independence between efforts. This is a reasonable assumption in many contexts, but in some settings different assumptions might be needed. For example, there are situations with strategic complementarity in which the efforts of agents reinforce each other. In such cases the cost function is supermodular. In other cases there is strategic substitutability, so that efforts offset each other and the function is submodular. Focusing on the effort cost function of one agent, if $\frac{\partial c_i^2(e_i)}{\partial e_{ij}\partial e_{ih}} > 0$ then there is complementarity between the efforts, and if $\frac{\partial c_i^2(e_i)}{\partial e_{ij}\partial e_{ih}} < 0$, then there is substitutability. This is a very interesting future extension. It could also be worth considering this complementarity/substitutability not only between the different efforts that one agent makes in regard to other agents but also between the efforts made by different agents. This assumption can be made on both the effort cost functions and the cost reduction function. Obviously, complementarity on the effort cost function has the opposite effect to that on the cost reduction function.

The second direction is close to the first. The pairwise total cost reduction could be considered as a general function which is increasing in the efforts e_{ij} and e_{ji}, that is $R_{ij}(e_{ij}, e_{ji})$. In our model, this function is additively separable, i.e. $R_{ij}(e_{ij}, e_{ji}) = r_{ij}(e_{ji}) + r_{ji}(e_{ij})$. However, as mentioned above, there could be situations with strategic complementarity or substitutability in which the efforts of agents reinforce or offset each other. In that case, the function $R_{ij}(e_{ij}, e_{ji})$ would not be separable. This is also an interesting question for analysis.

Another direction is related to the assumption of bilateral interaction between agents. This has the advantage of being analytically more tractable and is widely applied in practice (e.g., Fang and Wang 2019; Amin et al. 2020, Park et al. 2010), but overall interaction between agents, dependent on groups, is an important factor that we believe does not affect the success of cooperation. One possible future extension would be to investigate the cooperative model with multiple cost reduction and the impact of the efforts made on those cost reductions.

Finally, we identify a large family of core-allocations with weighted pairwise reduction which contains the Shapley value and the Nucleolus and always provides a level of efficient effort in equilibrium. This family is very rich in itself, as a set solution concept for our cooperative model. Research into this core-allocation family can be furthered through an in-depth analysis of its structure and its geometric relationship to the core.

Conclusiones y vias futuras de investigación

En el primer capítulo se ha presentado un nuevo modelo de Corporate Tax Games con benefactores y beneficiarios como aplicación de los juegos de costes lineales al sistema de reducción del impuesto de sociedades introduciendo la figura de benefactores múltiples, duales e insustituibles. Se ha utilizado el valor de Shapley como regla de asignación estable al reparto de costes reducidos. Además, se estudian sus propiedades, se ha comprobado el efecto bola de nieve derivado de la concavidad del modelo demostrando que cuanto mayor es la coalición menores son los costes para sus miembros y se ha comprobado que, estos juegos son cóncavos, es decir, la contribución marginal de una empresa y un país disminuye a medida que crece una coalición (efecto bola de nieve). Por lo tanto, la gran coalición es estable en el sentido del núcleo. Esto significa que las empresas tienen fuertes incentivos para cooperar con los países en lugar de ser defraudadores. A continuación, se propone el valor de Shapley como una asignación del núcleo fácilmente computable que beneficia a todos los agentes y, en particular, compensa a los benefactores por su doble e insustituible papel.

El modelo presentado, distingue dos grupos de agentes: benefactores duales (países) y beneficiarios (empresas), mientras que el modelo original presentado por [6], consideraba dos grupos disjuntos de agentes, benefactores y beneficiarios. Una extensión natural sería considerar que todos los agentes pueden ser duales (benefactores y beneficiarios). En consecuencia, es seguro que podrían alcanzarse resultados similares a los aquí obtenidos.

Los capítulos segundo y tercero, que están directamente interrelacionados, presentan un modelo de cooperación con reducción de costes por parejas. El impacto directo del esfuerzo por parejas en la reducción de costes se investiga mediante un juego biforme. En primer lugar, los agentes determinan el nivel de esfuerzo por parejas que deben realizar para reducir los costes de sus socios. En segundo lugar, participan en una interacción bilateral con múltiples socios independientes en la que la reducción de costes que cada agente ofrece a otro es independiente de cualquier posible coalición. Como resultado de la cooperación, los agentes reducen los costes de los demás. En el juego no cooperativo que precede a la cooperación, los agentes anticipan la asignación de costes que resultará del juego cooperativo incorporando a sus funciones de costes el efecto del esfuerzo realizado. Se observa que la cooperación todo incluido es factible, en el sentido de que existen posibles reducciones de costes que hacen que todos los agentes estén mejor (o, al menos, no peor), y consistente. A continuación, se identifica una familia de asignaciones de costes factibles con reducción ponderada por pares. Se selecciona una de estas asignaciones de costes teniendo en cuenta los incentivos generados en los esfuerzos que realizan los agentes y, en consecuencia, en el coste total de las coaliciones. Sorprendentemente, se encuentra que el valor de Shapley, que coincide con el Núcleo en este modelo, puede inducir estrategias de esfuerzo ineficientes en equilibrio en el modelo no cooperativo.

Siempre será posible, sin embargo, seleccionar una asignación de núcleos con ponderaciones por pares adecuadas que puedan generar un esfuerzo eficiente.

La investigación futura podría tomar varias direcciones. En primer lugar, en esta tesis se asume que la función de coste del esfuerzo individual $c_i(e_i)$ es independiente del esfuerzo de otros agentes, y que el coste marginal $\frac{\partial c_i(e_i)}{\partial e_{ij}}$ es independiente del esfuerzo que i realiza respecto a otros agentes distintos de j, *i.e.* $\frac{\partial c_i^2(e_i)}{\partial e_{ij} \partial e_{ih}} = 0$. Se hace una suposición similar con la función de reducción de costes $r_{ij}(e^*_{ji})$. Existe cierto grado de independencia entre los esfuerzos. Se trata de una suposición razonable en muchos contextos, pero para otros contextos, no obstante, será necesario partir de supuestos diferentes. Por ejemplo, hay situaciones con complementariedad estratégica en las que los esfuerzos de los agentes se refuerzan mutuamente. En estos casos, la función de costes es supermodular. En otros casos hay sustituibilidad estratégica, de modo que los esfuerzos se compensan entre sí y la función es submodular. Centrándonos en la función de coste del esfuerzo de un agente, si $\frac{\partial c_i^2(e_i)}{\partial e_{ij}\partial e_{ih}} > 0$ entonces hay complementariedad entre los esfuerzos, y si $\frac{\partial c_i^2(e_i)}{\partial e_{ij}\partial e_{ih}} < 0$, entonces hay sustituibilidad.

Se trata de una ampliación futura muy interesante. También podría ser interesante considerar esta complementariedad/sustituibilidad no sólo entre los diferentes esfuerzos que realiza un agente respecto a otros agentes, sino también entre los esfuerzos realizados por diferentes agentes. Ello puede hacerse tanto sobre las funciones de coste del esfuerzo como sobre la función de reducción de costes. Obviamente, la complementariedad en la función de coste del esfuerzo tiene el efecto contrario al efecto que tiene lugar respecto a la función de reducción de costes.

La segunda dirección se aproxima en parte a la primera. La reducción del coste total por pares podría considerarse una función general que es creciente en los esfuerzos e_{ij} y e_{ji}, es decir, $R_{ij}(e_{ij}, e_{ji})$. En nuestro modelo, esta función es separable aditivamente, es decir, $R_{ij}(e_{ij}, e_{ji}) \ = r_{ij}(e_{ji}) + r_{ji}(e_{ij})$. Sin embargo, como ya se ha mencionado, pueden darse situaciones de complementariedad o sustituibilidad estratégica en las que los esfuerzos de los agentes se refuercen o compensen mutuamente. En ese caso, la función $R_{ij}(e_{ij}, e_{ji})$ no sería separable. Esta es también una cuestión interesante para el análisis.

Otra dirección está relacionada con el supuesto de interacción bilateral entre agentes. Esto tiene la ventaja de ser analíticamente más manejable y se aplica ampliamente en la práctica (por ejemplo, Fang y Wang 2019; Amin et al. 2020, Park et al. 2010), pero la interacción global entre agentes, dependiente de los grupos, es un factor importante que creemos que no afecta al éxito de la cooperación. Una posible extensión futura sería investigar el modelo cooperativo con reducción de costes múltiples y el impacto de los esfuerzos realizados en esas reducciones de costes.

Por último, identificamos una gran familia de asignaciones centrales con reducción ponderada por pares que contiene el valor de Shapley y el Nucleolo y que siempre proporciona un nivel de esfuerzo eficiente en equilibrio. Esta familia es muy rica en sí misma, como concepto de solución de conjunto para nuestro modelo cooperativo. La investigación sobre esta familia de asignación de núcleos puede profundizarse mediante un análisis en profundidad de su estructura y su relación geométrica con el núcleo.

BIBLIOGRAPHY

Algaba, E., Fragnelli, V., & Sánchez-Soriano, J. (Eds.). (2019). Handbook of the Shapley value. CRC Press.

Amin, W., Huang, Q., Afzal, M., Khan, A. A., Umer, K., and Ahmed, S. A. (2020). A converging non-cooperative & cooperative game theory approach for stabilizing peer-to-peer electricity trading. Electric Power Systems Research, 183, 106278.

Arrow, K. J. (1951). Alternative approaches to the theory of choice in risk-taking situations. Econometrica: Journal of the Econometric Society, 404-437.

Aumann's, R., & Schelling's, T. (2005). Contributions to game theory: analyses of conflict and cooperation. Information on the Bank of Sweden Prize Economic Sciences in Memory of Alfred Nobel, Stockholm.

Bajada, C., Schneider, F. (2018). Size, Causes and Consequences of the Underground Economy: an international perspective. Routledge.

Bashir, Z., Mahnaz, S., & Abbas Malik, M. G. (2021). Conflict resolution using game theory and rough sets. International Journal of Intelligent Systems, 36(1), 237-259.

Bernstein F., Kok A.G., and Meca A. (2015). Cooperation in assembly systems: The role of knowledge sharing networks. European Journal of Operational Research, 240, 160-171.

Bertrand, J. (1883). Review of "Theorie mathematique de la richesse sociale" and of "Recherches sur les principles mathematiques de la theorie des richesses". Journal de savants, 67.

Bilbao, J. M., Fernández, J. R., Jiménez, N., and López, J. J. (2008). The Shapley value for bicooperative games. Annals of Operations Research, 158(1), 99-115.

Bonome, M. G. (2010). La Racionalidad en la toma de Decisiones: Analisis la Teoría de la decisión de Herbert A. Simon. Netbiblo.

Borel, É. (1920). La statistique et l'organisation de la présidence du conseil des ministres. Journal de la Société de Statistique de Paris, 61, 9-13.

Borzaga, C., & Tortia, E. C. (2017). Co-operation as Co-ordination Mechanism: a new approach to the economics of co-operative enterprises.

Bracken, J., and McGill, J.T. (1973). Mathematical Programs with Optimization Problems in the Constraints. Operations Research, 21(1), 37-44.

Bradley, R. (2017). Decision theory with a human face. Cambridge University Press.

Brams, S. J. (1994). Game theory and literature. Games and Econ. Behavior, 6(1), 32-54.

Brandenburger, A., and Stuart, H. (2007). Biform games. Management science, 53(4), 537-549.

Braund, M. (2005). The possibility of progress (p. 108). Shepheard-Walwyn.

Cachon, G. P., and Netessine, S. (2006). Game theory in supply chain analysis. In Models, methods, and applications for innovative decision making (pp. 200-233). INFORMS.

Caritat, M. J. A. N. (1785). Essai sur l'application de l'analyse à la probabilité des décisions, rendues à la pluralité des voix.

Chen, J. (2024). Information Asymmetric Cooperative Game with Agreements Implemented by a Third Party. Contemporary Mathematics, 2328-2364.

Cherington, P. T. (1913). Advertising as a business force. Page for the associated advertising clubs of America. Doubleday

Cournot, A. A. (1838). Recherches sur les principes mathématiques de la théorie des richesses (Vol. 48). L. Hachette.

Curiel I., Pederzoli G., and Tijs S. (1989). Sequencing games, European Journal of Operational Research, 40, 344-351.

Dell'Anno, R. (2009). Tax evasion, tax morale and policy maker's effec-tiveness. The Journal of Socio-Economics, 38 (6), 988-997.

Dempe, S., (2002). Foundations of Bilevel Programming. Kluwer Academic Publishers.

Dewatripont, M., Jewitt, I. and and Tirole, J. (1999). The economics of career concerns, Part I: Comparing information structures. Review of Economic Studies 66(1), 183-198.

Dowling, M. (2020). Coopetition. In Oxford Research Encyclopedia of Business and Management.

Edgeworth, F. Y. (1881). Mathematical psychics: An essay on the application of mathematics to the moral sciences (No. 10). CK Paul.

Edhan, O. (2014). Value Solutions in Cooperative Games, Journal of Economic Literature 52(1), 211-249.

Fang, D., and Wang, J., (2019). Horizontal capacity sharing between asymmetric competitors. Omega, in press, Article 102109.

Feige, E. (2016). Reflections on the Meaning and Measurement of Unobserved Economies: What Do We Really Know About the 'Shadow Economy'. Journal of Tax Administration, 2, 5-41.

Fiestras-Janeiro, M. G., García-Jurado, I., and Mosquera, M. A. (2011). Cooperative games and cost allocation problems. Top, 19(1), 1-22.

Freeman, E. (2013). Managing for stakeholders. In D. Arnold, T. Beauchamp & N. Bowie, Ethical theory and business (pp. 57-68). New Jersey: Pearson.

González-Díaz, J., García-Jurado, I., & Fiestras-Janeiro, M. G. (2023). An Introductory Course on Mathematical Game Theory and Applications (Vol. 238). American Mathematical Society.

Guajardo, M., and Rönnqvist, M. (2016). A review on cost allocation methods in collaborative transportation. International transactions in operational research, 23(3), 371-392.

Holmstrom, B. (1982). Moral hazard in teams. Bell Journal of Economics 13(2), 324-40.

Holmstrom, B. (1999). Managerial incentive problems: A dynamic perspective. Review of Economic Studies 66(1), 169-182.

Kar, A., Mitra, M., and Mutuswami, S. (2009). On the coincidence of the prenucleolus with the Shapley value. Mathematical Social Sciences, 57(1):16–25.

Keen, M., Slemrod, J. (2017). Optimal tax administration. Journal of Public Economics, 152, 133-142.

Kimms, A.,and Kozeletskyi, I. (2016). Shapley value-based cost allocation in the cooperative traveling salesman problem under rolling horizon planning. EURO Journal on Transportation and Logistics, 5(4), 371-392.

Le, P. H., Nguyen, T. D., and Bektaş, T. (2018). Efficient computation of the Shapley value for large-scale linear production games. Annals of Operations Research, 1-21.

Leibniz, G. W., & Brunschwig, J. (1966). Nouveaux essais sur l'entendement humain (p. 293). Paris: Garnier-Flammarion.

Li, M., Sun, H., Huang, Y., & Chen, H. (2024). Shapley value: from cooperative game to explainable artificial intelligence. Autonomous Intelligent Systems, 4(1), 2.

Li, S., and Zhang, Q. (2009). A simplified expression of the Shapley function for fuzzy game. European Journal of Operational Research, 196(1), 234-245.

Littlechild S.C. and Owen G., 1973, A simple expression for the Shapley value in a special case, Management Science, 20, 370-372.

Loomis, L. H. (1946). On a theorem of von Neumann. Proceedings of the National Academy of Sciences, 32(8), 213-215.

Lozano, S., Moreno, P., Adenso-Díaz, B., and Algaba, E. (2013). Cooperative game theory approach to allocating benefits of horizontal cooperation. European Journal of Operational Research, 229(2), 444-452.

Luo, C., Zhou, X., & Lev, B. (2022). Core, shapley value, nucleolus and nash bargaining solution: A Survey of recent developments and applications in operations management. Omega, 110, 102638.

Manuel, C. M., & Martín, D. (2020). A monotonic weighted Shapley value. Group Decision and Negotiation, 29, 627-654.

Marlon Monticelli, J., Leite, E., & F. Chim-Miki, A. (2024). Coopetition in the business landscape: Shaping strategies, paradoxes, and future prospects. Brazilian Administration Review, 21(1).

Meca, A.,and Sošić, G. (2014). Benefactors and Beneficiaries: The Effects of Giving and Receiving on Cost-Coalitional Problems. Production and Operations Management, 23(9), 1549-1560.

Meca, A., García-Martínez, J. A., and Mayor-Serra, A. J. (2019).The Shapley value of corporation tax games with dual benefactors. In E. Algaba, F.V. Fragnelli and J. Sánchez-Soriano (Eds.), Handbook of the Shapley value. Londres, Reino Unido: CRC Press. Taylor and Francis Group.

Meca, A., Varela-Peña, J. C. (2018). Corporation Tax Games: An Appli-cation of Linear Cost Games to Managerial Cost Accounting. In: Game Theory in Management Accounting pp. 361-378. Springer.

Mehlmann, A. (2000). The game's afoot!: game theory in myth and paradox. Providence: American Mathematical Society.

Monsalve, S. (2003). John Nash y la teoría de juegos. Lecturas matemáticas, 24(2), 137-149.

Moretti, S., and Patrone, F. (2008). Transversality of the Shapley value. Top, 16(1), 1.

Moulin, H. (1987). Equal or proportional division of a surplus, and other methods. International Journal of Game Theory, 16(3), 161-186.

Moulin, H. (2004). Fair Division and Collective Welfare. Cambridge, Massachusetts: MIT Press.

Mozaffari, F., & Eghbal, N. (2020). To do or not to do: Game theory and literature. International Journal of Nonlinear Analysis and Applications, 11(2), 63-69.

Myerson, R. B. (1992). Fictitious-transfer solutions in cooperative game theory. In Rational Interaction: Essays in Honor of John C. Harsanyi (pp. 13-33). Berlin, Heidelberg: Springer Berlin Heidelberg.

Nalebuff, B. J., & Brandenburger, A. M. (1997). Co-opetition: Competitive and cooperative business strategies for the digital economy. Strategy & leadership, 25(6), 28-33.

Nash, J. F. (1950). Equilibrium points in n-person games. Proceedings of the national academy of sciences, 36(1), 48-49.

Nash, J. (1951). Non-cooperative games. Annals of mathematics, 286-295.

Neumann J., & Morgenstern, O. (1944). Theory of games and economic behavior.

Park, J., and Van der Schaar, M. (2010). A game theoretic analysis of incentives in content production and sharing over peer-to-peer networks. IEEE Journal of Selected Topics in Signal Processing, 4(4), 704-717.

Peters, H. (2015). Cooperative games with transferable utility. Game Theory: A Multi-Leveled Approach, 151-169.

Poveda, C. E. P., & Bualó, J. F. (2024). La ética subyacente en la teoría económica. Perspectivas de las Ciencias Económicas y Jurídicas, 14(1), 119-130.

Rapoport, A. (Ed.). (2012). Game theory as a theory of conflict resolution (Vol. 2). Springer Science & Business Media.

Rémond de Montmort, (1713) Pierre Essay d'analyse sur les jeux de hazard. Paris, Universität Bielefeld

Riker, W. H. (1992). Teoría de juegos y de las coaliciones políticas. VV. AA., Diez textos básicos de ciencia política, Barcelona: Ariel Ciencia Política, 151-169.

Sandmo, A. (2012). An evasive topic: theorizing about the hidden econ-omy. International tax and public finance, 19, 5-24.

Schelling, T. C. (1958). The strategy of conflict. Prospectus for a reorientation of game theory. Journal of Conflict Resolution, 2(3), 203-264.

Schmeidler, D. (1969). The nucleolus of a characteristic function game. SIAM Journal on applied mathematics, 17(6), 1163-1170.

Schneider, F., Enste, D. H. (2000). Shadow economies: size, causes, and consequences. Journal of economic literature, 38(1), 77-114.

Serrano, R. (2009), Cooperative Games: Core and Shapley Value, in R. Meyers (ed.), Encyclopedia of Complexity and Systems Science, Springer, New York

Sexton, R.J. (1986), The Formation of Cooperatives: A Game-Theoretic Approach with Implications for Cooperative Finance, Decision Making, and Stability. American Journal of Agricultural Economics, 68: 214-225.

Shapley, L. S. (1953). A value for n-person games. Contributions to the Theory of Games, 2(28), 307-317.

Shapley, L. S. (1971). Cores of convex games. International journal of game theory, 1(1), 11-26.

Shapley L.S. (1967): On balanced sets and cores. Naval Res. Logist. Quart. 14:453-460.

da Silva, A., & Cardoso, A. M. (2024). Advancing BIM and Sustainability with Coopetition: Evidence from the Portuguese Stone Industry.

Slemrod, J., & Yitzhaki, S. (2002). Tax avoidance, evasion, and adminis-tration. In Handbook of public economics (Vol. 3, pp. 1423-1470). Else-vier.

Slikker, M., and Van den Nouweland, A. (2012). Social and economic networks in cooperative game theory (Vol. 27). Springer Science & Business Media.

Sohrabi, M. K., & Azgomi, H. (2020). A survey on the combined use of optimization methods and game theory. Archives of Computational Methods in Engineering, 27(1), 59-80.

Steinhaus, H. (1925). Definicje potrzebne do teorji gry i poscigu. Mysl. Akad. Lwów, 1(1), 13-14.

Thomson, W. (2010). Fair Allocation Rules. Chapter 21 in the Handbook of Social Choice and Welfare, vol 2, 393-492.

Todeva, E., and Knoke, D. (2005). Strategic Alliances & Models of Collaboration. Management Decision, 43(1).

Torgler, B. (2011). Tax morale and compliance : review of evidence and case studies for Europe. Policy Research working paper; no. WPS 5922. Washington, DC: World Bank.

Villalón, Á. F. T., & Caraballo, A. M. M. (2015). Un paseo por la historia de la Teoría de Juegos. Boletín de Matemáticas, 22(1), 77-95.

Williams, C. C. (2017). Tackling employment in the informal economy: A critical evaluation of the neoliberal policy approach. Economic and Industrial Democracy, 38(1), 145-169.

Zermelo, E., & Borel, E. (1913). On an application of set theory to the theory of the game of chess. In Congress of Mathematicians (pp. 501-504). CUP.

Zhang, M., Chen, S., Du, W., Cao, X., Li, D., Zhang, J., & Havlin, S. (2023). Risky cascading transitions in international relationships. PNAS nexus, 2(1), pgac289.

Zhou, L. (1994). A new bargaining set of an n-person game and endogenous coalition formation. Games and Economic Behavior, 6(3), 512-526.

APPENDICES

APPENDIX A

Proof of Lemma 1.1

Consider agent $k\prime \in N$ and any set $T \subset N\backslash\{k\prime\}$. To prove Lemma 1.1, we first consider that agent $k\prime$ is a country and compare the cost of agents in T and in $T \cup \{\bar{k}\}$, and second we consider that agent $k\prime$ is a firm, and we do the same analysis. Note that agents in T could be either countries or firms:

1. Consider that agent k' is a country i', then

(a) For all $i \in T \cap P$, $c_i^T = g_i(w_i^T)$ and $c_i^{T\cup\{i'\}} = g_i(w_i^{T\cup\{i'\}})$, where $w_i^T = w_i^{T\cup\{i'\}}$ because $T \subset T \cup \{i'\}$ and $i' \in P$. Consequently, as g_i is increasing, $c_i^T > c_i^{T\cup\{i'\}}$

(b) For all $j \in T \cap E$,

$$c_j^T = \sum_{i\in P\cap T} \bar{S}_j^i + \sum_{i\in P\backslash(P\cap T)} S_j^i = \sum_{i\in P\cap T} \bar{S}_j^i + S_j^{i'} + \sum_{i\in P\backslash(P\cap(T\cup\{i'\}))} S_j^i,$$

and

$$c_j^{T\cup\{i'\}} = \sum_{i\in P\cap(T\cup\{i'\})} \bar{S}_j^i + \sum_{i\in P\backslash(P\cap(T\cup\{i'\}))} S_j^i = \sum_{i\in P\cap T} \bar{S}_j^i + \bar{S}_j^{i'} + \sum_{i\in P\backslash(P\cap(T\cup\{i'\}))} S_j^i.$$

Consequently,

$c_j^T > c_j^{T\cup\{i'\}}$ because $S_j^{i'} > \bar{S}_j^{i'}$.

2. Consider that agent k' is a firm j', then

(a) For all $i \in T \cap P$, $g_i(w_i^T)$ and $c_i^T = g_i(w_i^T)$ and $c_i^{T\cup\{j'\}} = g_i(w_i^{T\cup\{j'\}})$ where $w_i^T = w_i^{T\cup\{j'\}}$ because $T \subseteq T \cup \{j'\}$ and $j' \in E$. Consequently, $c_i^T = c_i^{T\cup\{j'\}}$

(b) For all $j \in T \cap E$

$c_j^T = \sum_{i \in P \cap T} \bar{S}_j^i + \sum_{i \in P \setminus (P \cap T)} S_j^i$, and

$c_j^{T \cup \{j'\}} = \sum_{i \in P \cap (T \cup \{j'\})} \bar{S}_j^i + \sum_{i \in P \setminus (P \cap (T \cup \{j'\}))} S_j^i = \sum_{i \in P \cap T} \bar{S}_j^i + \sum_{i \in P \setminus (P \cap T)} S_j^i$. Therefore, $c_j^T = c_j^{T \cup \{i'\}}$

Point 1 implies that countries are benefactors, and point 2 implies that firms are not benefactor. Point 1 and 2 imply that countries and firms can be ben-eficiaries and an agent $k \in N$ is a benefactor if and only if it is a country. ■

There are agents that are dual in the sense that they are benefactors and beneficiaries, these are the countries. However, the firms are exclusively beneficiaries.

Proof of Lemma 1.2

Note that by Lemma 1.1 only countries can be benefactors, then consider any $T \subset N$ such that $T \cap P \neq \emptyset$ where $i\prime \in T \cap P$. To prove Lemma 1.2, we compare the costs in set T and in set $T \setminus \{i\prime\}$. Agents in $T \setminus \{i\prime\}$ can be either countries or firms.

First, if the agent is a country, $i \in (T \setminus \{i\prime\}) \cap P$,

then $c_i^T = g_i(w_i^T) < c_i^{T \setminus \{i'\}} = g_i(w_i^{T \setminus \{i'\}})$because g_i is is increasing,

and $w_i^T < w_i^{T \setminus \{i'\}}$because $\bar{S}_j^{i'} < S_j^{i'}$.

Second, if the agent in $T \setminus \{i'\}$ is a firm, $j \in (T \setminus \{i'\}) \cap E$,

then $c_j^T = \sum_{i \in P \cap T} \bar{S}_j^i + \sum_{i \in P \setminus (P \cap T)} S_j^i = \sum_{i \in P \cap T \setminus \{i'\}} \bar{S}_j^i + \bar{S}_j^{i'} + \sum_{i \in P \setminus (P \cap T)} S_j^i$, and

$c_j^{T \setminus \{i'\}} = \sum_{i \in P \cap (T \setminus \{i'\})} \bar{S}_j^i + \sum_{i \in P \setminus P \cap (T \setminus \{i'\})} S_j^i = \sum_{i \in P \cap (T \setminus \{i'\})} \bar{S}_j^i + S_j^{i'} + \sum_{i \in P \setminus (P \cap T)} S_j^i$.

Consequently, $c_j^T < c_j^{T \setminus \{i'\}}$ because $\bar{S}_j^{i'} < S_j^{i'}$.■

Proof of Lemma 1.3

Consider two sets such that $S \subset T \subseteq N$. Any agent in S has to be either a country or a firm.

First, if the agent is a country $i \in S \cap P$, then always $c_i^S = g_i(w_i^S)$ and $c_i^T = g_i(w_i^T)$,which implies that $c_i^S \geq c_i^T$. Note that, g_i is an increasing

function, and $w_i^S \geq w_i^T$ because $S \subset T$.

Second, if the agent in S is a firm $j \in S \cap E$, then

$c_j^S = \sum_{i \in P \cap S} \bar{S}_j^i + \sum_{i \in P \setminus (P \cap S)} S_j^i = \sum_{i \in P \cap S} \bar{S}_j^i + \sum_{i \in P \cap (T \setminus S)} S_j^i + \sum_{i \in P \setminus (P \cap T)} S_j^i,$

and

$c_j^T = \sum_{i \in P \cap T} \bar{S}_j^i + \sum_{i \in P \setminus (P \cap T)} S_j^i = \sum_{i \in P \cap S} \bar{S}_j^i + \sum_{i \in P \cap (T \setminus S)} \bar{S}_j^i + \sum_{i \in P \setminus (P \cap T)} S_j^i$

Note that, if in $T \setminus S$ there is at least a country, then $c_j^S > c_j^T$ because $S_j^i > \bar{S}_j^i$, otherwise $c_j^S = c_j^T$

■

Proof of Lemma 1.4

First, we prove (1). Take a coalition $T \subseteq N$, and a firm $j \in E \cap T$.

Then,

$$c(T) - c(T\backslash\{j\}) = \sum_{k\in T} c_k^T - \sum_{k\in T\backslash\{j\}} c_k^{T\backslash\{j\}} = c_j^T + \sum_{k\in T\backslash\{j\}} (c_k^T - c_k^{T\backslash\{j\}}).$$

Now we prove that

$\sum_{k\in T\backslash\{j\}} (c_k^T - c_k^{T\backslash\{j\}}) = 0$, and so $c(T) - c(T\backslash\{j\}) = c_j^T$.

Indeed,

$$\sum_{k\in T\backslash\{j\}} (c_k^T - c_k^{T\backslash\{j\}}) = \sum_{i\in P\cap(T\backslash\{j\})} (c_i^T - c_i^{T\backslash\{j\}}) + \sum_{j'\in E\cap(T\backslash\{j\})} (c_{j'}^T - c_{j'}^{T\backslash\{j\}}).$$

We know that

$c_i^T - c_i^{T\backslash\{j\}} = g_i(w_i^T) - g_i(w_i^{T\backslash\{j\}}) = 0$, since $w_i^{T\backslash\{j\}} = w_i^T$.

Moreover,

$$c_{j'}^T - c_{j'}^{T\backslash\{j\}} = \sum_{i\in P\cap T} \bar{S}_{j'}^i + \sum_{i\in P\backslash(P\cap T)} S_{j'}^i - \sum_{i\in P\cap(T\backslash\{j\})} \bar{S}_{j'}^i - \sum_{i\in P\backslash(P\cap T\backslash\{j\})} \bar{S}_{j'}^i = 0.$$

Then,

$\sum_{i\in P\cap(T\backslash\{j\})} (c_i^T - c_i^{T\backslash\{j\}}) = 0$, and $\sum_{j'\in E\cap(T\backslash\{j\})} (c_{j'}^T - c_{j'}^{T\backslash\{j\}}) = 0.$

Hence, we conclude that $\sum_{k\in T\backslash\{j\}} (c_k^T - c_k^{T\backslash\{j\}}) = 0.$

Second,we prove (2). Take a coalition $T \subseteq N$, and a country $i \in P \cap T$

Then,

$$c(T) - c(T\backslash\{i\}) = \sum_{k\in T} c_k^T - \sum_{k\in T\backslash\{i\}} c_k^{T\backslash\{i\}} = c_i^T - \sum_{k\in T\backslash\{i\}} (c_k^{T\backslash\{j\}} - c_k^T).$$

We know that

$$\sum_{k\in T\backslash\{i\}} (c_k^{T\backslash\{i\}} - c_k^T) = \sum_{i'\in P\cap(T\backslash\{i\})} (c_{i'}^{T\backslash\{i\}} - c_{i'}^T) + \sum_{j\in E\cap(T\backslash\{i\})} (c_j^{T\backslash\{i\}} - c_j^T).$$

We prove now that

$$c_j^{T\backslash\{i\}} - c_j^T = \left(S_j^i + \sum_{i'\in P\cap(T\backslash\{i\})} \bar{S}_j^{i'} + \sum_{i'\in P\backslash P\cap(T\backslash\{i\})} S_j^{i'} \right) - \bar{S}_j^i + \sum_{i'\in P\cap(T\backslash\{i\})} \bar{S}_j^{i'} + \sum_{i'\in P\backslash P\cap(T\backslash\{i\})} S_j^{i'} \Bigg) = S_j^i - \bar{S}_j^i.$$

We know, by definition, that

$c_{i'}^{T\backslash\{i\}} - c_{i'}^T = g_{i'}(w_{i'}^{T\backslash\{i\}}) - g_{i'}(w_{i'}^T)$

Hence we can conclude that

$$c(T) - c(T\backslash\{i\}) = c_i^T - \sum_{j\in E\cap T}(S_j^i - \bar{S}_j^i) - \sum_{i'\in P\cap(T\backslash\{i\})}(g_{i'}(w_{i'}^{T\backslash\{i\}}) - g_{i'}(w_{i'}^T)).$$

■

Proof of Theorem 1.1

Here we have to prove that the marginal contribution of an agent k diminishes as a coalition grows. Any agent k can only be either a firm or a country, and Lemma 1.4 provided its marginal contribution.

If the agent is a firm j, then for all $T \subseteq T'$, $j \in T$, $c_j^T \geq c_j^{T'}$, and so $c_j^T = c(T) - c(T\backslash\{j\}) \geq c(T') - c(T'\backslash\{j\}) = c_j^{T'}$

On the other hand, if the agent is a country i, again for all $T \subset T'$, $c_i^T \geq c_i^{T'}$

In addition, $\sum_{j\in E\cap T}(S_j^i - \bar{S}_j^i) \leq \sum_{j\in E\cap T'}(S_j^i - \bar{S}_j^i)$ because all the countries in$T$ are also in $T\prime$, and if $T\prime$ there is at least one more than in T, then the

inequality is strict.

Finally, for the same reason $\sum_{i'\in P\cap T\backslash\{i\}} zi'i \leq \sum_{i'\in P\cap T'\backslash\{i\}} zi'i$.

Hence, we can conclude that for all $T \subset T\prime$ and for all $i \in P \cap T$,

$c(T) - c(T\backslash\{i\}) = c_i^T - \sum_{j\in E\cap T}(S_j^i - \bar{S}_j^i) - \sum_{i'\in P\cap T\backslash\{i\}} zi'i \geq c_i^{T'} - \sum_{j\in E\cap T'}(S_j^i - \bar{S}_j^i) - \sum_{i'\in P\cap T'\backslash\{i\}} zi'i = c(T) - c(T'\backslash\{i\})$.■

Proof of Theorem 1.2

(1) First, we prove that for all $j \in E$, $\phi_j(N,c) = c_j^N + \frac{1}{2}\sum\limits_{i\in P}(S_j^i - \bar{S}_j^i)$.

Take $j \in E$. As detailed before, we know that

$\phi_j(N,c) = \sum\limits_{T\subseteq N; j\in T} \gamma(t)\ c_j^T$.

We can separate coalitions $j \in T \subseteq N$ into mixed coalitions ($j \in T \subseteq N, T\cap P \neq \emptyset$, $T \cap E \neq \emptyset$)

and coalitions with only firms ($j \in T \subseteq N$,$T \cap P = \emptyset$, $T \cap E \neq \emptyset$)

Then,

$\phi_j(N,c) =$

$$\sum_{j\in T\subseteq N, T\cap P=\emptyset, T\cap E\neq\emptyset} \gamma(t)(\sum_{i\in P} S_j^i) + \sum_{j\in T\subseteq N, T\cap P\neq\emptyset, T\cap E\neq\emptyset} \gamma(t)(\sum_{i\in P\setminus P\cap T'} \bar{S}_j^i).$$

Taking into account that $\sum\limits_{T\subseteq N; j\in T} \gamma(t) = 1$, we have that

$$\sum_{j\in T\subseteq N, T\cap P=\emptyset, T\cap E\neq\emptyset} \gamma(t) = 1 - \sum_{j\in T\subseteq N, T\cap P\neq\emptyset, T\cap E\neq\emptyset} \gamma(t),$$

and then,

$$\begin{aligned}
\phi_j(N,c) &= (1 - \sum_{j\in T\subseteq N, T\cap P\neq\emptyset, T\cap E\neq\emptyset} \gamma(t)(\sum_{i\in P} S_j^i) + \\
&+ \sum_{j\in T\subseteq N, T\cap P\neq\emptyset, T\cap E\neq\emptyset} \gamma(t)(\sum_{i\in P\setminus P\cap T} S_j^i + \sum_{i\in P\cap T} \bar{S}_j^i) \\
&= \sum_{i\in P} S_j^i + \sum_{j\in T\subseteq N, T\cap P\neq\emptyset, T\cap E\neq\emptyset} \gamma(t)(\sum_{i\in P\setminus P\cap T} S_j^i + \sum_{i\in P\cap T} \bar{S}_j^i - \sum_{i\in P} S_j^i) \\
&= \sum_{i\in P} S_j^i - \sum_{j\in T\subseteq N, T\cap P\neq\emptyset, T\cap E\neq\emptyset} \gamma(t) \sum_{i\in P\cap T} (S_j^i - \bar{S}_j^i).
\end{aligned}$$

Now, we prove for all coalitions that contain $j \in T\cap E$ and a particular country $i \in T\cap P$,

$$\sum_{j\in T\subseteq N, T\cap P\neq\emptyset, T\cap E\neq\emptyset} \gamma(t) = \tfrac{1}{2},$$

and then,

$$\phi_j(N,c) = \sum_{i\in P} S_j^i - \tfrac{1}{2} \sum_{i\in P\cap T} (S_j^i - \bar{S}_j^i) = \tfrac{1}{2} \sum_{i\in P} (S_j^i - \bar{S}_j^i).$$

Indeed,

$$\sum_{j\in T\subseteq N, T\cap P\neq\emptyset, T\cap E\neq\emptyset} \gamma(t) = \sum_{t=2}^{n} \begin{pmatrix} n-2 \\ t-2 \end{pmatrix} \gamma(t) =$$

$$\sum_{t=2}^{n} \frac{(t-1)}{n(n-1)} = \frac{\sum_{k=1}^{n} k - n}{n(n-1)} = \frac{1}{2}$$

where $\begin{pmatrix} n-2 \\ t-2 \end{pmatrix}$

is the number of coalitions in which there is j and a particular country i'.

Finally, doing some algebra, we have that

$\phi_j(N,c) = \sum\limits_{i\in P}(S_j^i - \bar{S}_j^i) = c_j^N + \frac{1}{2}\sum\limits_{i\in P}(S_j^i - \bar{S}_j^i),$

and so, we conclude that

$\phi_j(N,c) = c_j^N + \frac{1}{2}\sum\limits_{i\in P}(S_j^i - \bar{S}_j^i)$

(2) Second, we demonstrate that for all $i \in P$,

$\phi_i(N,c) = c_i^N - \frac{1}{2}\sum\limits_{j\in E}(S_j^i - \bar{S}_j^i) + \frac{1}{2}\sum\limits_{i'\in P\backslash\{i\}}(zii' - zi'i).$

Take $i \in P$. As demonstrated, we know that

$\phi_i(N,c) =$

$\sum\limits_{i\in T\subseteq N}\gamma(t)\left(c_i^T - \sum\limits_{j\in E\cap T}(S_j^i - \bar{S}_j^i) - \sum\limits_{i'\in P\cap T\backslash\{i\}}(g_{i'}(w_{i'}^{T\backslash\{i\}}) - g_{i'}(w_{i'}^T))\right).$

Let's calculate each of the addens separately.

(2.1) First, taking into account that

$c_i^T = c_i - \sum\limits_{i'\in P\cap T\backslash\{i\}} zii'$, for all $T \in N$, and

$\sum_{t=2}^{n}\begin{pmatrix} n-2 \\ t-2 \end{pmatrix}\gamma(t) = \frac{1}{2},$

we obtain that

$\sum\limits_{i\in T\subseteq N}\gamma(t)c_i^T = c_i - \sum_{t=2}^{n}\begin{pmatrix} n-2 \\ t-2 \end{pmatrix}\gamma(t)\sum\limits_{i'\in P\cap T\backslash\{i\}} zii' = c_i^N + \frac{1}{2}\sum\limits_{i'\in P\cap T\backslash\{i\}} zii',$

where $\begin{pmatrix} n-2 \\ t-2 \end{pmatrix}$ is now the number of coalitions that contain i and a particular country $i\prime$.

(2.2) Second, by a similar argument,

$\sum\limits_{i\in T\subseteq N}\gamma(t)\sum\limits_{j\in E\cap T}(S_j^i - \bar{S}_j^i) = \sum_{t=2}^{n}\begin{pmatrix} n-2 \\ t-2 \end{pmatrix}\gamma(t)\sum\limits_{j\in E}(S_j^i - \bar{S}_j^i) =$

$\frac{1}{2}\sum\limits_{j\in E}(S_j^i - \bar{S}_j^i) = 0.$

(2.3) Third, by the same argument,

$\sum\limits_{i\in T\subseteq N}\gamma(t)\sum\limits_{i'\in P\cap T\backslash\{i\}}(g_{i'}(w_{i'}^{T\backslash\{i\}}) - g_{i'}(w_{i'}^T)) =$

$\sum_{t=2}^{n} \left(\begin{array}{c} n-2 \\ \\ t-2 \end{array} \right) \gamma(t) \sum_{i' \in P \setminus \{i\}} zi'i = -\frac{1}{2} \sum_{i' \in P \setminus \{i\}} zi'i.$

Finally, adding the above three expressions, we obtain that

$\phi_i(N, c) = c_i^N + \frac{1}{2} \sum_{i' \in P \cap T \setminus \{i\}} zii' - \frac{1}{2} \sum_{j \in E} (S_j^i - \bar{S}_j^i) - \frac{1}{2} \sum_{i' \in P \setminus \{i\}} zi'i =$
$c_i^N - \frac{1}{2} \sum_{j \in E} (S_j^i - \bar{S}_j^i) + \frac{1}{2} \sum_{i' \in P \setminus \{i\}} (zii' - zi'i).$ ■

APPENDIX B

Proposition 2.1 in section 2.3, shows that PE-games are always concave. To prove this, the class of unanimity games must be described. Shapley (1953) proves that the family of unanimity games $\{(N, u_T), T \subseteq N\}$ forms a basis of the vector space of all games with set of players N, where (N, u_T) is defined for each $S \subseteq N$ as follows:

$$u_T(S) = \begin{cases} 1, & T \subseteq S \\ 0, & otherwise \end{cases}$$

Hence, for each cost game (N, c) there are unique real coefficients $(\alpha_T)_{T \subseteq N}$ such that $c = \sum_{T \subseteq N} \alpha_T u_T$. Many different classes of games, including airport games (Littlechild and Owen, 1973) and sequencing games (Curiel et al., 1989), can be characterized through constraints on these coefficients.

Proof of Proposition 2.1

Let $(N, e, \{c_i(e_i), \{r_{ji}(e_{ij})\}_{j \in N \setminus \{i\}}\}_{i \in N})$ be a PE-situation and (N, e, c) the associated PE-game. First, we prove that this game can be rewritten as a weighted sum of unanimity games $u_{\{i\}}$ and $u_{\{i,j\}}$ for all $i, j \in N$ as follows:

$$c = \sum_{i \in N} c_i(e_i) u_{\{i\}} - \sum_{i,j \in N; i \neq j} r_{ij}(e_{ji}) u_{\{i,j\}}. \tag{7.1}$$

Indeed, for all $S \subseteq N$,

$$
\begin{aligned}
c(S) &= \sum_{i \in N} c_i(e_i) u_{\{i\}}(S) - \sum_{i,j \in N; i \neq j} r_{ij}(e_{ji}) u_{\{i,j\}}(S) = \\
&= \sum_{i \in S} c_i(e_i) - \sum_{i,j \in S; i \neq j} r_{ij}(e_{ji}) = \\
&= \sum_{i \in S} c_i(e_i) - \sum_{i \in S} \sum_{j \in S \setminus \{i\}} r_{ij}(e_{ji}).
\end{aligned}
$$

It is easily shown that the additive game $\sum_{i \in N} c_i(e_i) u_{\{i\}}$ is concave and that $u_{\{i,j\}}$ is convex. Thus, the game $-\sum_{i,j \in N; i \neq j} r_{ij}(e_{ji}) u_{\{i,j\}}$ is concave because of $r_{ij}(e_{ji}) > 0$ for all $i, j \in N$. Finally, the concavity of (N, e, c) follows from the fact that game c is the sum of two concave games. ■

Theorem 2.1, in section 2.3, shows that the Shapley value reduces the individual cost of an agent by half the total reduction that it obtains from the others ($R_i(N)$) plus a half of the total reduction that it provides to the rest of the agents, which is $G_i(N) = \sum_{j \in N \setminus \{i\}} r_{ji}(e_{ij})$.

The Shapley value is the only allocation rule that satisfies the four properties of Efficiency, Equal treatment of equals, Linearity and Null player. Next, we describe all of these properties of the Shapley value, which are useful in demonstrating the Theorem 2.1.

(EFF) *Efficiency.* The sum of the Shapley values of all agents equals the value of the grand coalition, so all the gain is allocated to the agents:

$$\sum_{i \in N} \phi_i(c) = c(N). \tag{7.2}$$

(ETE) *Equal treatment of equals.* If i and j are two agents who are equivalent in the sense that $c(S \cup \{i\}) = c(S \cup \{j\})$ for every coalition S of N which contains neither i nor j, then $\phi_i(c) = \phi_j(c)$.

(LIN) *Linearity.* If two cost games c and c^* are combined, then the cost allocation should correspond to the costs derived from c and the costs derived from c^*:

$$\phi_i(c + c^*) = \phi_i(c) + \phi_i(c^*), \forall i \in N. \tag{7.3}$$

Also, for any real number a,

$$\phi_i(ac) = a\phi_i(c), \forall i \in N. \tag{7.4}$$

(NUP) *Null Player*. The Shapley value $\phi_i(c)$ of a null player i in a game c is zero. A player i is null in c if $c(S \cup \{i\}) = c(S)$ for all coalitions S that do not contain i.

Proof of Theorem 2.1

Consider the PE-game (N, e, c) rewritten as a weighted sum of unanimity games given by the expression 2.5, i.e.

$$c = \sum_{i \in N} c_i(e_i) u_{\{i\}} - \sum_{i,j \in N; i \neq j} r_{ij}(e_{ji}) u_{\{i,j\}}.$$

Take an agent $k \in N$. By the (LIN) property of the Shapley value, $\phi_k(e)$, it follows that

$$\begin{aligned} \phi_k(e) &= \phi_k \left. \sum_{i \in N} c_i(e_i) u_{\{i\}} \right) - \phi_k \left(\sum_{i,j \in N; i \neq j} r_{ij}(e_{ji}) \left(u_{\{i,j\}}\right) \right) \\ &= \sum_{i \in N} c_i(e_i) \phi_k \left(u_{\{i\}}\right) - \sum_{i \in N} \sum_{j \in N \setminus \{i\}} r_{ij}(e_{ji}) \phi_k \left(u_{\{i,j\}}\right). \end{aligned} \tag{7.5}$$

In addition, it is known from the (NUP) property that

$$\phi_k \left(u_{\{i\}}\right) = \begin{cases} 1, & i = k \\ 0, & otherwise \end{cases} \tag{7.6}$$

and from (ETE) and (NUP), that

$$\phi_k \left(u_{\{i,j\}}\right) = \begin{cases} 1/2, & i = k, j = k, i \neq j \\ 0, & otherwise \end{cases} \tag{7.7}$$

Consequently, by substituting the values (7.6) and (7.7) in equation (7.5), the following is obtained:

$$\begin{aligned}\phi_k(e) &= c_k(e_k) - \sum_{j\in N\setminus\{k\}} r_{kj}(e_{jk})\phi_k\left(u_{\{k,j\}}\right) - \sum_{j\in N\setminus\{k\}} r_{jk}(e_{kj})\phi_k\left(u_{\{j,k\}}\right)\\ &= c_k(e_k) - \frac{1}{2}\sum_{j\in N\setminus\{k\}}[r_{kj}(e_{jk}) + r_{jk}(e_{kj})].\end{aligned}$$

Finally, it can be concluded that, for each agent $k \in N$,

$$\phi_k(e) = c_k(e_k) - \frac{1}{2}[R_k(N) + G_k(N)].$$

■

Proof of Proposition 2.2

To prove that the Shapley value coincides with the Nucleolus for PE-games, it is first necessary to describe the class of PS-games introduced by Kar et al (2009).

Denote by $M_ic(T)$ the marginal contribution of player $i \in T$, that is $M_ic(T) = c(T) - c(T \setminus \{i\})$, for all $i \in T \subseteq N$. A cost game (N, c) satisfies the PS property if for all $i \in N$ there exists $k_i \in \mathbb{R}$ such that $M_ic(T \cup \{i\}) + M_ic(N \setminus T) = k_i$, for all $i \in N$ and all $T \subseteq N \setminus \{i\}$. Kar et al (2009) show that for PS games, the Shapley value coincides with the Nucleolus, i.e. $\phi_i(c) = \nu_i(c) = \frac{k_i}{2}$, for all $i \in N$.

Therefore, it only remains to show that (N, e, c) is a PS-game with $k_i = [c_i(e_i) - R_i(N)] + [c_i(e_i) - G_i(N)]$, for all $i \in N$.

First, it is straightforward to prove that $M_ic(T) = c_i(e_i) - \sum_{j\in T\setminus\{i\}}[r_{ji}(e_{ij}) + r_{ij}(e_{ji})]$ for all $i \in T \subseteq N$. Second, we show that $M_ic(T \cup \{i\}) + M_ic(N \setminus T) = [c_i(e_i) - R_i(N)] + [c_i(e_i) - G_i(N)]$ for all $i \in N$ and $T \subseteq N \setminus \{i\}$.

Indeed, take a coalition $T \subseteq N$ and an agent $i \in T$. It is shown that $M_ic(T \cup \{i\}) = c_i(e_i) - \sum_{j\in T}\left(r_{ji}(e_{ij}) + r_{ij}(e_{ji})\right)$, and $M_ic(N\setminus T) = c_i(e_i) - \sum_{j\in N\setminus(T\cup\{i\})}\left(r_{ji}(e_{ij}) + r_{ij}(e_{ji})\right)$.

Therefore,

$M_ic(T \cup \{i\}) + M_ic(N \setminus T) = 2c_i(e_i) - \sum_{j\in N\setminus\{i\}}\left(r_{ji}(e_{ij}) + r_{ij}(e_{ji})\right) =$
$\left[c_i(e_i) - \sum_{j\in N\setminus\{i\}} r_{ij}(e_{ji})\right] + \left[c_i(e_i) - \sum_{j\in N\setminus\{i\}} r_{ji}(e_{ij})\right].$

Hence, $M_ic(T \cup \{i\}) + M_ic(N \setminus T) = [c_i(e_i) - R_i(N)] + [c_i(e_i) - G_i(N)] = k_i)$, and so (N, e, c) is a PS game. ■

Proof of Theorem 2.2

Consider the PE-game (N, e, c) associated with the PE-situation

$(N, e, \{c_i(e_i), \{r_{ij}(e_{ij})\}_{j\in N\backslash\{i\}}\}_{i\in N})$.

Take a family of weights $\omega^i_{ij}, \omega^i_{ji} \in [0,1]$, for all $j \in N\backslash\{i\}$, such that $\omega^i_{ij} = 1 - \omega^j_{ij}$ and $\omega^i_{ji} = 1 - \omega^j_{ji}$, and $\Omega(e)$ the corresponding cost allocation with weighted pairwise reduction with $\Omega_i(e) = c_i(e_i) - \sum_{j\in N\backslash\{i\}}[\omega^i_{ij} r_{ij}(e_{ji}) + \omega^i_{ji} r_{ji}(e_{ij})]$, for all $i \in N$. To prove that $\Omega(e)$ belongs to the core of (N, e, c) it must be checked that (1) $\sum_{i\in N}\Omega_i(e) = c(N)$, (2) $\sum_{i\in S}\Omega_i(e) \leq c(S)$, for all $S \subset N$.

We start by checking (1). Notice that $\sum_{i\in N}\Omega_i(e) = c(N)$ is equivalent to

$\sum_{i\in N}\sum_{j\in N\backslash\{i\}}[\omega^i_{ij} r_{ij}(e_{ji}) + \omega^i_{ji} r_{ji}(e_{ij})] = \sum_{i\in N}\sum_{j\in N\backslash\{i\}} r_{ij}(e_{ji})$.

Indeed,

$\sum_{i\in N}\sum_{j\in N\backslash\{i\}}[\omega^i_{ij} r_{ij}(e_{ji}) + \omega^i_{ji} r_{ji}(e_{ij})] = \sum_{i\in N}\sum_{j\in N\backslash\{i\}}(\omega^i_{ij} + \omega^j_{ij}) r_{ij}(e_{ji}) =$

$= \sum_{i\in N}\sum_{j\in N\backslash\{i\}} r_{ij}(e_{ji})$,

where the last equality is due to $\omega^i_{ij} + \omega^j_{ij} = 1$ for all $i, j \in N$.

Next we check (2). Take $S \subset N$. Notice now that $\sum_{i\in S}\Omega_i(e) \leq c(S)$ is equivalent to

$\sum_{i\in S}\sum_{j\in N\backslash\{i\}}[\omega^i_{ij} r_{ij}(e_{ji}) + \omega^i_{ji} r_{ji}(e_{ij})] - \sum_{i\in S}\sum_{j\in S\backslash\{i\}} r_{ij}(e_{ji}) \geq 0$.

Indeed, an argument similar to that used in (1) leads to

$\sum_{i\in S}\sum_{j\in N\backslash\{i\}}[\omega^i_{ij} r_{ij}(e_{ji}) + \omega^i_{ji} r_{ji}(e_{ij})] - \sum_{i\in S}\sum_{j\in S\backslash\{i\}} r_{ij}(e_{ji}) =$

$\sum_{i\in S}\sum_{j\in S\backslash\{i\}}[\omega^i_{ij} r_{ij}(e_{ji}) + \omega^i_{ji} r_{ji}(e_{ij})] + \sum_{i\in S}\sum_{j\in N\backslash S\cup\{i\}}[\omega^i_{ij} r_{ij}(e_{ji}) + \omega^i_{ji} r_{ji}(e_{ij})] - \sum_{i\in S}\sum_{j\in S\backslash\{i\}} r_{ij}(e_{ji}) =$

$\sum_{i\in S}\sum_{j\in S\backslash\{i\}} r_{ij}(e_{ji}) + \sum_{i\in S}\sum_{j\in N\backslash S\cup\{i\}}[\omega^i_{ij} r_{ij}(e_{ji}) + \omega^i_{ji} r_{ji}(e_{ij})] - \sum_{i\in S}\sum_{j\in S\backslash\{i\}} r_{ij}(e_{ji}) =$

$\sum_{i\in S}\sum_{j\in N\backslash S\cup\{i\}}[\omega^i_{ij} r_{ij}(e_{ji}) + \omega^i_{ji} r_{ji}(e_{ij})] \geq 0$.

■

Proof of Proposition 2.3

To prove this result it is necessary to analyze $c(N)$ as a function of e. First, It is easy to prove that $c(N)$ is strictly convex in e_{ij} for all $i,\, j \in N, i \neq j$. Indeed, $\frac{\partial^2 c(N)}{\partial e_{ij}^2} = \frac{\partial^2 c_i(e_i)}{\partial e_{ij}^2} - \frac{\partial^2 r_{ji}(e_{ij})}{\partial e_{ij}^2} > 0$, because $\frac{\partial^2 c_i(e_i)}{\partial e_{ij}^2} > 0$ and $\frac{\partial^2 r_{ji}(e_{ij})}{\partial e_{ij}^2} < 0$. Thus, there is a unique effort profile $\tilde{e}$ that minimizes $c(N)$.

Second, we focus on finding this efficient effort profile $\tilde{e}$. Note that the derivative $\frac{\partial c(N)}{\partial e_{ij}} = \frac{\partial c_i(e_i)}{\partial e_{ij}} - \frac{\partial r_{ji}(e_{ij})}{\partial e_{ij}}$ only depends on e_{ij} because $\frac{\partial c_i^2(e_i)}{\partial e_{ij} \partial e_{ih}} = 0$ for all $h \neq i, j$. Therefore, if $\frac{\partial c_i(e_i)}{\partial e_{ij}} > \frac{\partial r_{ji}(e_{ij})}{\partial e_{ij}}$ for all $e_{ij} \in [0, 1]$, then the function $c(N)$ is increasing in e_{ij}, which implies that $\tilde{e}_{ij} = 0$. Analogously, if $\frac{\partial c_i(e_i)}{\partial e_{ij}} > \frac{\partial r_{ji}(e_{ij})}{\partial e_{ij}}$ for all $e_{ij} \in [0, 1]$, then $\tilde{e}_{ij} = 1$. Finally, if there is a solution of $\frac{\partial c_i(e_i)}{\partial e_{ij}} = \frac{\partial r_{ji}(e_{ij})}{\partial e_{ij}}$, that solution is $\tilde{e}_{ij}$. ■

Proof of Lemma 2.1

Consider the non-cooperative game $(N, \{E_i\}_{i \in N}, \{\Omega_i\}_{i \in N})$. To learn the optimal level of effort $\hat{e}_{ij}$ that agent i must exert to reduce the costs of agent j in this game, it is necessary to analyze the function $\Omega_i(e) = c_i(e_i) - \sum_{j \in N \setminus \{i\}} [\omega_{ij}^i r_{ij}(e_{ji}) + \omega_{ji}^i r_{ji}(e_{ij})]$ for all $i \in N$ with $\omega_{ij}^i, \omega_{ji}^i \in [0, 1]$, $i,\, j \in N, i \neq j$, such that $\omega_{ij}^i = 1 - \omega_{ij}^j$ and $\omega_{ji}^i = 1 - \omega_{ji}^j$.

As above, we also prove that the function $\Omega_i(e)$ is strictly convex in e_{ij}. Indeed, $\frac{\partial_i^2 \Omega(e)}{\partial e_{ij}^2} = \frac{\partial^2 c_i(e_i)}{\partial e_{ij}^2} - \omega_{ji}^i \frac{\partial^2 r_{ji}(e_{ij})}{\partial e_{ij}^2} > 0$ because $\frac{\partial^2 c_i(e_i)}{\partial e_{ij}^2} > 0$ and $\frac{\partial^2 r_{ji}(e_{ij})}{\partial e_{ij}^2} < 0$. Hence, there is a unique optimal level of effort $\hat{e}$.

Again, we focus on finding this optimal level of effort $\hat{e}$. We know that $\frac{\partial \Omega_i(e)}{\partial e_{ij}} = \frac{\partial c_i(e_i)}{\partial e_{ij}} - \omega_{ji}^i \frac{\partial r_{ji}(e_{ij})}{\partial e_{ij}}$, but $\frac{\partial c_i(e_i)}{\partial e_{ij}}$ only depends on e_{ij}, because $\frac{\partial c_i^2(e_i)}{\partial e_{ij} \partial e_{ih}} = 0$ for all $h \neq i, j$. Moreover, for all $e_{ij} \in [0, 1]$, $\frac{\partial \Omega_{ii}(e)}{\partial e_{ij}} \geq 0 \iff \frac{\partial c_i(e_i)}{\partial e_{ij}} \geq \omega_{ji}^i \frac{\partial r_{ji}(e_{ij})}{\partial e_{ij}}$.

Therefore, if $\frac{\partial c_i(e_i)}{\partial e_{ij}} > \omega_{ji}^i \frac{\partial r_{ji}(e_{ij})}{\partial e_{ij}}$ for all $e_{ij} \in [0, 1]$, then $\hat{e}_{ij} = 0$. If $\frac{\partial c_i(e_i)}{\partial e_{ij}} < \omega_{ji}^i \frac{\partial r_{ji}(e_{ij})}{\partial e_{ij}}$ for all $e_{ij} \in [0, 1]$, then $\hat{e}_{ij} = 1$. Finally, if there is a solution of $\frac{\partial c_i(e_i)}{\partial e_{ij}} = \omega_{ji}^i \frac{\partial r_{ji}(e_{ij})}{\partial e_{ij}}$, that solution is $\hat{e}_{ij}$ and is unique. Hence, there is a unique optimal level of effort.

■

Proof of Theorem 2.3

Now consider the non-cooperative game $(N, \{E_i\}_{i\in N}, \{H_i\}_{i\in N})$. Note that, both derivative functions $\frac{\partial c_i(e_i)}{\partial e_{ij}}$ and $\frac{\partial r_{ji}(e_{ij})}{\partial e_{ij}}$ only depend on e_{ij}. Thus, by Lemma, the optimal level of effort of a particular agent $i \in N$ with another particular agent $j \in N\backslash\{i\}$, i.e. $\hat{e}_{ij}$, is independent of any other effort made by i or by any other agent. Thus, the equilibrium is also characterized by Lemma with $\omega^i_{ji} = 1$ for $i, j \in N, i \neq j$. Comparing Lemma 2.1 with Proposition, it follows directly that the equilibrium must also be efficient. ■

Proof of Corollary 2.1

This is straightforward from the proof of **Theorem 2.3**. ■

Proof of Proposition 2.4

Take $A^*(e)$ the allocation rule in WPAR with α^*_{ij} for all $i, j \in N$ which induces the effort profile e^{*A^*} that minimizes the cost of the grand coalition. Since WPAR is a subfamily of WPR in which $\omega^i_{ij} = \omega^j_{ij} = \alpha_{ij} \in [0,1]$ for all $i, j \in N$, by Lemma 2.1 the optimal level of effort for $A^*(e)$ can be also characterized.

Thus, the efforts are optimal in equilibrium and so e^{*A^*} must hold that

$e^{*A^*}_{ij} = 0$ if and only if $\frac{\partial c_i(e_i)}{\partial e_{ij}} > \alpha^*_{ij}\frac{\partial r_{ji}(e_{ij})}{\partial e_{ij}}$, for all $e_{ij} \in [0,1]$,

$e^{*A^*}_{ij} = 1$ if and only if $\frac{\partial c_i(e_i)}{\partial e_{ij}} < \alpha^*_{ij}\frac{\partial r_{ji}(e_{ij})}{\partial e_{ij}}$, for all $e_{ij} \in [0,1]$,

Otherwise, $e^{*A^*}_{ij} \in (0,1)$ so $\left.\frac{\partial c_i(e_i)}{\partial e_{ij}}\right|_{e_{ij}=e^{*A^*}_{ij}} = \alpha^*_{ij} \left.\frac{\partial r_{ji}(e_{ij})}{\partial e_{ij}}\right|_{e_{ij}=e^{*A^*}_{ij}}$ holds.

Comparing the above expressions with Proposition 2.3 and taking into account that $\frac{\partial c_i(e_i)}{\partial e_{ij}}$ is a positive increasing function, $\frac{\partial r_{ji}(e_{ij})}{\partial e_{ij}}$ a positive decreasing function, and $\alpha^*_{ij} \in [0,1]$, it can be concluded that $\tilde{e}_{ij} \geq e^{*A^*}_{ij}$ for all $i, j \in N$. ■

APPENDIX C

Theorem 3.1, in section 3.2, characterizes all possible types of effort equilibrium according to the value of the parameter α_{ij}, for all $i, j \in N, i \neq j$. Before proving this theorem, we consider a previous Lemma that is very useful for latter results. It characterizes the optimal effort level for agent $i \in N$ in the first stage non-cooperative game.

Lemma 3.1

Let $(N, \{E_i\}_{i\in N}, \{A_i\}_{i\in N})$ *be the effort game, with* $\hat{e}_{ij}$ *being the optimal level of effort that agent* i *exerts to reduce the costs of agent* j*. Thus,*

1. $\hat{e}_{ij} = 0$ if and only if $\alpha_{ij} \leq \underline{\alpha}_{ij}$
2. There is a unique $\hat{e}_{ij} \in (0, 1)$ that holds $c_i'(\hat{e}_{ij}) - \alpha_{ij} r_{ji}'(\hat{e}_{ij}) = 0$ if and only if $\underline{\alpha}_{ij} < \alpha_{ij} < \bar{\alpha}_{ij}$.
3. $\hat{e}_{ij} = 1$ if and only if $\alpha_{ij} \geq \bar{\alpha}_{ij}$.

Proof

First, remember that the cost function $A_i(e)$ is convex for all $i \in N$. To obtain the optimal effort, the derivative of this function can be analyzed with respect to e_{ij} for any $j \in N \backslash \{i\}$. It must be noted that $\frac{\partial A_i(e)}{\partial e_{ij}} > 0 \iff c_i'(e_{ij}) > \alpha_{ij} r_{ji}'(e_{ij})$ for all $e_{ij} \in [0, 1]$, which is a necessary and sufficient condition for $\hat{e}_{ij} = 0$ to be the optimal effort. [1]

[1] This occurs because $A_i(e)$ is an increasing function in e_{ij} and the minimum value is obtained for $\hat{e}_{ij} = 0$, which is the optimal effort for agent i.

We begin by proving point 1. Note that $\underline{\alpha}_{ij} = \frac{c_i'(0)}{r_{ji}'(0)} < \frac{c_i'(e_{ij})}{r_{ji}'(e_{ij})}$ because $c_i' > 0$, $r_{ji}' > 0$, $c_i'' > 0$, and $r_{ji}'' < 0$. Thus, $c_i'(e_{ij})$ is a positive and increasing function, and $r_{ji}'(e_{ij})$ a positive and decreasing function, so for any $e_{ij} > 0$, $c_i'(0) < c_i'(e_{ij})$ and $r_{ji}'(0) > r_{ji}'(e_{ij})$. Therefore, $\alpha_{ij} \leq \underline{\alpha}_{ij} \iff c_i'(e_{ij}) > \alpha_{ij} r_{ji}'(e_{ij})$ for all $e_{ij} > 0 \iff \hat{e}_{ij} = 0$.

The demonstration in point 3 is similar to that of point 1. The above arguments are the same and only the signs of the inequalities change.

To end the proof, we prove point 2. First, we show that there is a unique $\hat{e}_{ij} \in (0, 1)$ such that $c_i'(\hat{e}_{ij}) = \alpha_{ij} r_{ji}'(\hat{e}_{ij})$, which is the unique optimal effort because $\left.\frac{\partial A_i(e)}{\partial e_{ij}}\right|_{e_i = \hat{e}_{ij}} = 0$ and $A_i(e)$ is a convex function. In addition, $c_i'(e_{ij})$ is a positive increasing function and $r_{ji}'(e_{ij})$ a positive decreasing function, in $e_{ij} \in [0, 1]$. This means that equation $\frac{\partial A_i(e)}{\partial e_{ij}} = c_i'(e_{ij}) - \alpha_{ij} r_{ji}'(e_{ij}) = 0$ has a unique root, which belongs to $(0, 1)$ if and only if $\alpha_{ij} \in (\underline{\alpha}_{ij}, \bar{\alpha}_{ij})$. Note that if $\alpha_{ij} \in (\underline{\alpha}_{ij}, \bar{\alpha}_{ij})$ then $c_i'(0) < \alpha_{ij} r_{ji}'(0)$ and $c_i'(1) > \alpha_{ij} r_{ji}'(1)$, and so there is a unique point $\hat{e}_{ij}$ where $c_i'(\hat{e}_{ij}) = \alpha_{ij} r_{ji}'(\hat{e}_{ij})$.$\blacksquare$

Proof of Theorem 3.1

As we already mention, the optimum $\hat{e}_{ij}$ is independent of other efforts. Therefore, the equilibrium effort is determined by Lemma 3.1. In addition, we want to characterize the effort equilibrium according to the value of the parameter α_{ij}. Thus, in the case of agent j, $\underline{\alpha}_{ji} < \alpha_{ji} < \bar{\alpha}_{ji} \Leftrightarrow \underline{\alpha}_{ji} < 1 - \alpha_{ij} < \bar{\alpha}_{ji} \Leftrightarrow 1 - \bar{\alpha}_{ji} < \alpha_{ij} < 1 - \underline{\alpha}_{ji}$. $\blacksquare$

The next corollary shows how the pairwise equilibrium efforts e_{ij}^* depend on α_{ij}, for all $i, j \in N$,$i \neq j$.

As expected, as the proportion of aggregate cost reduction obtained by an agent increases, the effort that agent exerts also increases (or at least stays the same).

Corollary 3.1

Let $(N, \{E_i\}_{i\in N}), \{A_i\}_{i\in N})$ *be the effort game and* (e^*_{ij}, e^*_{ji}) *the pairwise efforts equilibrium. Thus,*

- $\frac{\partial e^*_{ij}}{\partial \alpha_{ij}} > 0$, if $\alpha_{ij} \in (\underline{\alpha}_{ij}, \bar{\alpha}_{ij})$; $\quad \frac{\partial e^*_{ij}}{\partial \alpha_{ij}} = 0$ otherwise
- $\frac{\partial e^*_{ji}}{\partial \alpha_{ij}} < 0$, if $\alpha_{ij} \in (1-\bar{\alpha}_{ji}, 1-\underline{\alpha}_{ji})$; $\frac{\partial e^*_{ji}}{\partial \alpha_{ij}} = 0$ otherwise

Proof

By the implicit function theorem, $\frac{\partial e^*_{ij}}{\partial \alpha_{ij}} = -\frac{\frac{\partial(c'_i(e^*_{ij})-\alpha_{ij}r'_{ji}(e^*_{ij}))}{\partial \alpha_{ij}}}{\frac{\partial(c'_i(e^*_{ij})-\alpha_{ij}r'_{ji}(e^*_{ij}))}{\partial e^*_{ij}}} = \frac{r'_{ji}(e^*_{ij})}{c''_i(e^*_{ij})-\alpha_{ij}r''_{ji}(e^*_{ij})} > 0$, because $r'_{ji}(e^*_{ij}) > 0$, $c''_i(e^*_{ij}) > 0$, and $r''_{ji}(e^*_{ij}) < 0$. Thus, for any $\alpha_{ij} \leq \underline{\alpha}_{ij}$,

Lemma 3.1 implies that $e^*_{ij} = 0$, thus, $\frac{\partial e^*_{ij}}{\partial \alpha_{ij}} = 0$. However, if $\alpha_{ij} \in (\underline{\alpha}_{ij}, \bar{\alpha}_{ij})$, then $e^*_{ij} \in (0,1)$ and $\frac{\partial e^*_{ij}}{\partial \alpha_{ij}} > 0$. Finally, if $\alpha_{ij} \geq \bar{\alpha}_{ij}$, then $e^*_{ij} = 1$ and $\frac{\partial e^*_{ij}}{\partial \alpha_{ij}} = 0$. Analogously, if $\alpha_{ji} \leq \underline{\alpha}_{ji} \iff \alpha_{ij} \geq 1-\underline{\alpha}_{ji}$, then $e^*_{ji} = 0$ and $\frac{\partial e^*_{ji}}{\partial \alpha_{ij}} = 0$, if $\alpha_{ji} \in (\underline{\alpha}_{ji}, \bar{\alpha}_{ji}) \iff \alpha_{ij} \in (1-\bar{\alpha}_{ji}, 1-\underline{\alpha}_{ji})$, then $e^*_{ji} \in (0,1)$ and $\frac{\partial e^*_{ji}}{\partial \alpha_{ij}} < 0$. Finally, if $\alpha_{ji} \geq \bar{\alpha}_{ji} \iff \alpha_{ij} \leq 1-\bar{\alpha}_{ji}$, then $e^*_{ij} = 1$ and $\frac{\partial e^*_{ji}}{\partial \alpha_{ij}} = 0$. ■

Theorem 3.2, in Section 3.2, provides the weights α_{ij} that minimizes function $L^*_{ij}(\alpha_{ij})$, and the efficient effort equilibrium. To solve the above optimization problem it is necessary to know the function $L^*_{ij}(\alpha_{ij})$ very accurately.

To demonstrate Theorem 3.2, three technical lemmas are needed first. Lemmas 3.2, 3.3, and 3.4 characterize the derivatives $\frac{\partial(A^*_i(\alpha_{ij}))}{\partial \alpha_{ij}}$, $\frac{\partial(L^*_{ij}(\alpha_{ij}))}{\partial \alpha_{ij}}$, and $\frac{\partial^2(L^*_{ij}(\alpha_{ij}))}{\partial \alpha^2_{ij}}$ respectively.

The first lemma shows how the optimal cost function of agent $i \in N$ depends on α_{ij}. Henceforth, to simplify notation, we consider that for any $i, j \in N$, $\frac{\partial r_{ij}(e^*_{ji})}{\partial e^*_{ji}}$ and $\frac{\partial c_i(e^*_i)}{\partial e^*_{ij}}$ stand for derivatives $\frac{\partial r_{ij}(e_{ji})}{\partial e_{ji}}$ and $\frac{\partial c_i(e_i)}{\partial e_{ij}}$ evaluated in the unique effort equilibrium.

Lemma 3.2

Let $(N, \{E_i\}_{i\in N}, \{A_i\}_{i\in N})$ *be the effort game and* e^* *the effort equilibrium. Thus,*

1. $\frac{\partial(A_i(e^*))}{\partial\alpha_{ij}} = \frac{\partial(A_i^*(\alpha_{ij}))}{\partial\alpha_{ij}} =$
$$\begin{cases} -r_{ij}(e_{ji}^*) - \alpha_{ij}\frac{\partial r_{ij}(e_{ji}^*)}{\partial e_{ji}^*}\frac{\partial e_{ji}^*}{\partial\alpha_{ij}} - r_{ji}(e_{ij}^*), & if \quad \alpha_{ij} \in (\underline{\alpha}_{ij}, \bar{\alpha}_{ij}) \\ -r_{ij}(e_{ji}^*) - r_{ji}(e_{ij}^*) < 0, & \text{otherwise} \end{cases}$$

2. $\frac{\partial(A_j(e^*))}{\partial\alpha_{ij}} = \frac{\partial(A_j^*(1-\alpha_{ij}))}{\partial\alpha_{ij}} =$
$$\begin{cases} r_{ji}(e_{ij}^*) - (1-\alpha_{ij})\frac{\partial r_{ji}(e_{ij}^*)}{\partial e_{ij}^*}\frac{\partial e_{ij}^*}{\partial\alpha_{ij}} + r_{ij}(e_{ji}^*), & if \quad \alpha_{ij} \in (1-\bar{\alpha}_{ji}, 1-\underline{\alpha}_{ji}) \\ r_{ji}(e_{ij}^*) + r_{ij}(e_{ji}^*) > 0, & \text{otherwise.} \end{cases}$$

Proof

It is known that $A_i(e^*) = c_i(e_i^*) - \sum_{z\in N\setminus\{i\}} \alpha_{iz}(r_{iz}(e_{zi}^*) + r_{zi}(e_{iz}^*))$, and $A_i^*(\alpha_{ij}) =$ $c_i(e_i^*) - \alpha_{ij}(r_{ij}(e_{ji}^*) + r_{ji}(e_{ij}^*))$, thus

$$\begin{aligned} \frac{\partial(A_i(e^*))}{\partial\alpha_{ij}} &= \frac{\partial(A_i^*(\alpha_{ij}))}{\partial\alpha_{ij}} = \frac{\partial c_i(e_i^*)}{\partial e_{ij}^*}\frac{\partial e_{ij}^*}{\partial\alpha_{ij}} - r_{ij}(e_{ji}^*) - \alpha_{ij}\frac{\partial r_{ij}(e_{ji}^*)}{\partial e_{ji}^*}\frac{\partial e_{ji}^*}{\partial\alpha_{ij}} - r_{ji}(e_{ij}^*) - \alpha_{ij}\frac{\partial r_{ji}(e_{ij}^*)}{\partial e_{ij}^*}\frac{\partial e_{ij}^*}{\partial\alpha_{ij}}, \\ &= \left(\frac{\partial c_i(e_i^*)}{\partial e_{ij}^*} - \alpha_{ij}\frac{\partial r_{ji}(e_{ij}^*)}{\partial e_{ij}^*}\right)\frac{\partial e_{ij}^*}{\partial\alpha_{ij}} - r_{ij}(e_{ji}^*) - \alpha_{ij}\frac{\partial r_{ij}(e_{ji}^*)}{\partial e_{ji}^*}\frac{\partial e_{ji}^*}{\partial\alpha_{ij}} - r_{ji}(e_{ij}^*). \end{aligned}$$

The first term of the above expression is always zero, i.e. $\left(\frac{\partial c_i(e_i^*)}{\partial e_{ij}^*} - \alpha_{ij}\frac{\partial r_{ji}(e_{ij}^*)}{\partial e_{ij}^*}\right)\frac{\partial e_{ij}^*}{\partial\alpha_{ij}} = 0$. To see this, note that if $\alpha_{ij} \in (\underline{\alpha}_{ij}, \bar{\alpha}_{ij})$, then $e_{ij}^* \in (0,1)$ by Lemma 3.1, so $\left(\frac{\partial c_i(e_i^*)}{\partial e_{ij}^*} - \alpha_{ij}\frac{\partial r_{ji}(e_{ij}^*)}{\partial e_{ij}^*}\right) =$ 0 because it is evaluated in equilibrium. In the other case, where $\alpha_{ij} \leq \underline{\alpha}_{ij}$ or $\alpha_{ij} \geq \bar{\alpha}_{ij}$, $e_{ij}^* = 0$, so $\frac{\partial e_{ji}^*}{\partial\alpha_{ij}} = 0$. Therefore, $\frac{\partial(A_i(e^*))}{\partial\alpha_{ij}} = -r_{ij}(e_{ji}^*) - \alpha_{ij}\frac{\partial r_{ij}(e_{ji}^*)}{\partial e_{ji}^*}\frac{\partial e_{ji}^*}{\partial\alpha_{ij}} - r_{ji}(e_{ij}^*)$.

It is known by assumption that $r_{ij}(e_{ji}^*) \geq 0$, $\frac{\partial r_{ij}(e_{ji}^*)}{\partial e_{ji}^*} > 0$. If $\alpha_{ij} \in (1-\bar{\alpha}_{ji}, 1-\underline{\alpha}_{ji})$, then by Proposition in Lemma 3.2, $\frac{\partial e_{ji}^*}{\partial\alpha_{ij}} < 0$. However, if $\alpha_{ij} \notin (1-\bar{\alpha}_{ji}, 1-\underline{\alpha}_{ji})$ then, $\frac{\partial e_{ji}^*}{\partial\alpha_{ij}} = 0$, so $\frac{\partial(A_i(e^*))}{\partial\alpha_{ij}} = -r_{ij}(e_{ji}^*) - r_{ji}(e_{ij}^*)$.

The proof is analogous for $\frac{\partial(A_j(e^*))}{\partial\alpha_{ij}}$. ■

Notice that the effect of α_{ij} on the cost function of agent i could be positive or negative because of two simultaneous effects. First effect: As expected, if α_{ij} increases so does the proportion of cost reduction that agent i can obtain, and thus the cost function, $A_i(e^*)$, decreases. This decrease is measured by the term $-r_{ij}(e_{ji}^*) - r_{ji}(e_{ij}^*) < 0$ in the derivative.

Second effect: When α_{ij} increases, the effort of agent j decreases in equilibrium, so the cost function of agent i increases. The term $-\alpha_{ij}\frac{\partial r_{ij}(e^*_{ji})}{\partial e^*_{ji}}\frac{e^*_{ji}}{\partial \alpha_{ij}} > 0$ measures this second effect. The sum of these two effects determines the sign of the derivative. Therefore, an increase in the proportion of the aggregate cost reduction that an agent obtains could increase the cost of that agent if the second effect dominates the first. This is an interesting result: Giving too much to a particular agent could be not only worse for the aggregate cost but also for that particular agent.

The second lemma calculates the derivative of the aggregate cost function $L^*_{ij}(\alpha_{ij})$ in the effort equilibrium for any $i, j \in N$.

Lemma 3.3

Let $(N, \{E_i\}_{i\in N}, \{A_i\}_{i\in N})$ *be the effort game, and* e^* *the effort equilibrium.*

Thus,

$$\frac{\partial(L^*_{ij}(\alpha_{ij}))}{\partial \alpha_{ij}} = \left(\frac{\partial c_i(e^*_i)}{\partial e^*_{ji}} - \frac{\partial r_{ij}(e^*_{ji})}{\partial e^*_{ji}}\right)\frac{\partial e^*_{ji}}{\partial \alpha_{ij}} I_j + \left(\frac{c_j(e^*_j)}{\partial e^*_{ij}} - \frac{\partial r_{ji}(e^*_{ij})}{\partial e^*_{ij}}\right)\frac{\partial e^*_{ij}}{\partial \alpha_{ij}} I_i,$$

where $I_i = \begin{cases} 1 \quad if \quad \alpha_{ij} \in (\underline{\alpha}_{ij}, \bar{\alpha}_{ij}) \\ 0 \qquad \text{otherwise} \end{cases}$ and $I_j = \begin{cases} 1 \quad if \quad \alpha_{ij} \in (1-\bar{\alpha}_{ji}, 1-\underline{\alpha}_{ji}) \\ 0 \qquad \text{otherwise} \end{cases}$.

Therefore, there are four possible cases:

- $\frac{\partial(L^*_{ij}(\alpha_{ij}))}{\partial \alpha_{ij}}$ can be positive and/or negative if $\alpha_{ij} \in (\underline{\alpha}_{ij}, \bar{\alpha}_{ij}) \cap (1-\bar{\alpha}_{ji}, 1-\underline{\alpha}_{ji})$
- $\frac{\partial(L^*_{ij}(\alpha_{ij}))}{\partial \alpha_{ij}} = 0$, if $\alpha_{ij} \notin (\underline{\alpha}_{ij}, \bar{\alpha}_{ij}) \cup (1-\bar{\alpha}_{ji}, 1-\underline{\alpha}_{ji})$
- $\frac{\partial(L^*_{ij}(\alpha_{ij}))}{\partial \alpha_{ij}} > 0$ if $\alpha_{ij} \in (1-\bar{\alpha}_{ji}, 1-\underline{\alpha}_{ji}) \cap \big((0, \underline{\alpha}_{ij}) \cup (\bar{\alpha}_{ij}, 1)\big)$
- $\frac{\partial(L^*_{ij}(\alpha_{ij}))}{\partial \alpha_{ij}} < 0$ if $\alpha_{ij} \in \big((0, 1-\bar{\alpha}_{ji}) \cup \big(1-\underline{\alpha}_{ji}, 1\big)\big) \cap \big(\underline{\alpha}_{ij}, \bar{\alpha}_{ij}\big)\big)$

Proof

From (3.2), we calculate that $\frac{\partial(L^*_{ij}(\alpha_{ij}))}{\partial\alpha_{ij}} = \left(\frac{\partial c_i(e^*_i)}{\partial e^*_{ji}} - \frac{\partial r_{ij}(e^*_{ji})}{\partial e^*_{ji}}\right)\frac{\partial e^*_{ji}}{\partial\alpha_{ij}} + \left(\frac{c_j(e^*_j)}{\partial e^*_{ij}} - \frac{\partial r_{ji}(e^*_{ij})}{\partial e^*_{ij}}\right)\frac{\partial e^*_{ij}}{\partial\alpha_{ij}}$. Simplifying for the different subsets of α_{ij}, the following emerges:

1. if $\alpha_{ij} \in (\underline{\alpha}_{ij}, \bar{\alpha}_{ij}) \cap (1-\bar{\alpha}_{ji}, 1-\underline{\alpha}_{ji})$ then, by Theorem 3.1, $e^*_{ji} \in (0,1)$ and $e^*_{ij} \in (0,1)$, thus, by Corollary 3.1, $\frac{\partial e^*_{ji}}{\partial\alpha_{ij}} < 0$ and $\frac{\partial e^*_{ij}}{\partial\alpha_{ij}} > 0$. In addition, since $\frac{\partial c_i(e^*_i)}{\partial e^*_{ji}} - \alpha_{ij}\frac{\partial r_{ij}(e^*_{ji})}{\partial e^*_{ji}} = 0$ and $\frac{c_j(e^*_j)}{\partial e^*_{ij}} - (1-\alpha_{ij})\frac{\partial r_{ji}(e^*_{ij})}{\partial e^*_{ij}} = 0$, it follows that $\frac{\partial c_i(e^*_i)}{\partial e^*_{ji}} - \frac{\partial r_{ij}(e^*_{ji})}{\partial e^*_{ji}} < 0$ and $\frac{c_j(e^*_j)}{\partial e^*_{ij}} - \frac{\partial r_{ji}(e^*_{ij})}{\partial e^*_{ij}} < 0$. Therefore, $\frac{\partial(L^*_{ij}(\alpha_{ij}))}{\partial\alpha_{ij}} = \left(\frac{\partial c_i(e^*_i)}{\partial e^*_{ji}} - \frac{\partial r_{ij}(e^*_{ji})}{\partial e^*_{ji}}\right)\frac{\partial e^*_{ji}}{\partial\alpha_{ij}} + \left(\frac{c_j(e^*_j)}{\partial e^*_{ij}} - \frac{\partial r_{ji}(e^*_{ij})}{\partial e^*_{ij}}\right)\frac{\partial e^*_{ij}}{\partial\alpha_{ij}}$, which can be positive or negative in this case.

2. if $\alpha_{ij} \notin (\underline{\alpha}_{ij}, \bar{\alpha}_{ij}) \cup (1-\bar{\alpha}_{ji}, 1-\underline{\alpha}_{ji})$ then, by Theorem 3.1, $e^*_{ji} \in \{0,1\}$ and $e^*_{ij} \in \{0,1\}$, and by Corollary 3.1, $\frac{\partial e^*_{ji}}{\partial\alpha_{ij}} = \frac{\partial e^*_{ij}}{\partial\alpha_{ij}} = 0$. Therefore, $\frac{\partial(L^*_{ij}(\alpha_{ij}))}{\partial\alpha_{ij}} = 0$.

3. if $\alpha_{ij} \in (1-\bar{\alpha}_{ji}, 1-\underline{\alpha}_{ji}) \cap \left((0,\underline{\alpha}_{ij}) \cup (\bar{\alpha}_{ij}, 1)\right)$, then, as above, $\frac{\partial(L^*_{ij}(\alpha_{ij}))}{\partial\alpha_{ij}} = \left(\frac{\partial c_i(e^*_i)}{\partial e^*_{ji}} - \frac{\partial r_{ij}(e^*_{ji})}{\partial e^*_{ji}}\right)\frac{\partial e^*_{ji}}{\partial\alpha_{ij}} > 0$.

4. if $\alpha_{ij} \in \left((0, 1-\bar{\alpha}_{ji}) \cup \left(1-\underline{\alpha}_{ji}, 1\right)\right) \cap \left(\underline{\alpha}_{ij}, \bar{\alpha}_{ij}\right)$

   then $\frac{\partial(L^*_{ij}(\alpha_{ij}))}{\partial\alpha_{ij}} = \left(\frac{c_j(e^*_j)}{\partial e^*_{ij}} - \frac{\partial r_{ji}(e^*_{ij})}{\partial e^*_{ij}}\right)\frac{\partial e^*_{ij}}{\partial\alpha_{ij}} < 0$.

■

The derivative is a piecewise function and there are intervals where its sign is independent of the particular form of the functions of the game. For those cases, it is straightforward to find the optimal α_{ij} that minimizes the function $L^*_{ij}(\alpha_{ij})$. In those intervals, the derivative is either positive, negative or zero throughout the interval. These cases are respectively $\frac{\partial(L^*_{ij}(\alpha_{ij}))}{\partial\alpha_{ij}} = \left(\frac{\partial c_i(e^*_i)}{\partial e^*_{ji}} - \frac{\partial r_{ij}(e^*_{ji})}{\partial e^*_{ji}}\right)\frac{\partial e^*_{ji}}{\partial\alpha_{ij}} > 0$, $\frac{\partial(L^*_{ij}(\alpha_{ij}))}{\partial\alpha_{ij}} = \left(\frac{c_j(e^*_j)}{\partial e^*_{ij}} - \frac{\partial r_{ji}(e^*_{ij})}{\partial e^*_{ij}}\right)\frac{\partial e^*_{ij}}{\partial\alpha_{ij}} < 0$, and $\frac{\partial(L^*_{ij}(\alpha_{ij}))}{\partial\alpha_{ij}} = 0$. However, there is an interval where the sign of the derivative depends on the particular form of functions of the game. In this particular case $\frac{\partial(L^*_{ij}(\alpha_{ij}))}{\partial\alpha_{ij}} = \left(\frac{\partial c_i(e^*_i)}{\partial e^*_{ji}} - \frac{\partial r_{ij}(e^*_{ji})}{\partial e^*_{ji}}\right)\frac{\partial e^*_{ji}}{\partial\alpha_{ij}} + \left(\frac{c_j(e^*_j)}{\partial e^*_{ij}} - \frac{\partial r_{ji}(e^*_{ij})}{\partial e^*_{ij}}\right)\frac{\partial e^*_{ij}}{\partial\alpha_{ij}}$. This occurs when $\alpha_{ij} \in (\underline{\alpha}_{ij}, \bar{\alpha}_{ij}) \cap (1-\bar{\alpha}_{ji}, 1-\underline{\alpha}_{ji})$, which implies that in equilibrium simultaneously $0 < e^*_{ij} < 1$ and $0 < e^*_{ji} < 1$.

Therefore, in this case only, the derivative may be zero for some α_{ij} within this interval. In that case, the second derivative is needed to solve the optimization problem.

The third Lemma shows that the aggregate cost function $L^*_{ij}(\alpha_{ij})$ is convex in α_{ij}. Two additional assumptions about third derivatives need to be introduced.

Lemma 3.4

Let $(N, \{E_i\}_{i\in N}, \{A_i\}_{i\in N})$ *be the effort game,* e^* *the effort equilibrium, and* $\frac{\partial^3 c_i(e^*_i)}{\partial e^{*3}_{ij}} > 0$ *and* $\frac{\partial^3 r_{ji}(e^*_{ij})}{\partial e^{*3}_{ij}} < 0$, *for any* $i, j \in N$.
Thus $\frac{\partial^2 L^*_{ij}(\alpha_{ij}))}{\partial \alpha^{*2}_{ij}} > 0$ *for all* $\alpha_{ij} \in (\underline{\alpha}_{ij}, \bar{\alpha}_{ij}) \cap (1-\bar{\alpha}_{ji}, 1-\underline{\alpha}_{ji})$.

Proof

Take $\alpha_{ij} \in (\underline{\alpha}_{ij}, \bar{\alpha}_{ij}) \cap (1-\bar{\alpha}_{ji}, 1-\underline{\alpha}_{ji})$. Thus,

$$\frac{\partial^2(L^*_{ij}(\alpha_{ij}))}{\partial\alpha^2_{ij}} = \frac{\partial^2\left[\left(\frac{\partial c_i(e^*_i)}{\partial e^*_{ji}} - \frac{\partial r_{ij}(e^*_{ji})}{\partial e^*_{ji}}\right)\frac{\partial e^*_{ji}}{\partial\alpha_{ij}} + \left(\frac{c_j(e^*_j)}{\partial e^*_{ij}} - \frac{\partial r_{ji}(e^*_{ij})}{\partial e^*_{ij}}\right)\frac{\partial e^*_{ij}}{\partial\alpha_{ij}}\right]}{\partial\alpha^2_{ij}}$$

$$\left(\frac{\partial^2 c_i(e^*_i)}{\partial e^*_{ji}\partial\alpha_{ij}} - \frac{\partial^2 r_{ij}(e^*_{ji})}{\partial e^*_{ji}\partial\alpha_{ij}}\right)\frac{\partial e^*_{ji}}{\partial\alpha_{ij}} + \left(\frac{\partial c_i(e^*_i)}{\partial e^*_{ji}} - \frac{\partial r_{ij}(e^*_{ji})}{\partial e^*_{ji}}\right)\frac{\partial^2 e^*_{ji}}{\partial\alpha^2_{ij}}$$
$$+\left(\frac{\partial^2 c_j(e^*_j)}{\partial e^*_{ij}\partial\alpha_{ij}} - \frac{\partial^2 r_{ji}(e^*_{ij})}{\partial e^*_{ij}\partial\alpha_{ij}}\right)\frac{\partial e^*_{ij}}{\partial\alpha_{ij}} + \left(\frac{\partial c_j(e^*_j)}{\partial e^*_{ij}} - \frac{\partial r_{ji}(e^*_{ij})}{\partial e^*_{ij}}\right)\frac{\partial^2 e^*_{ij}}{\partial\alpha^2_{ij}}$$
$$= \left(\frac{\partial^2 c_i(e^*_i)}{\partial^2 e^*_{ji}}\frac{\partial e^*_{ji}}{\partial\alpha_{ij}} - \frac{\partial^2 r_{ij}(e^*_{ji})}{\partial^2 e^*_{ji}}\frac{\partial e^*_{ji}}{\partial\alpha_{ij}}\right)\frac{\partial e^*_{ji}}{\partial\alpha_{ij}} + \left(\frac{\partial c_i(e^*_i)}{\partial e^*_{ji}} - \frac{\partial r_{ij}(e^*_{ji})}{\partial e^*_{ji}}\right)\frac{\partial^2 e^*_{ji}}{\partial\alpha^2_{ij}}$$
$$+\left(\frac{\partial^2 c_j(e^*_j)}{\partial^2 e^*_{ij}}\frac{\partial e^*_{ij}}{\partial\alpha_{ij}} - \frac{\partial^2 r_{ji}(e^*_{ij})}{\partial^2 e^*_{ij}}\frac{\partial e^*_{ij}}{\partial\alpha_{ij}}\right)\frac{\partial e^*_{ij}}{\partial\alpha_{ij}} + \left(\frac{\partial c_j(e^*_j)}{\partial e^*_{ij}} - \frac{\partial r_{ji}(e^*_{ij})}{\partial e^*_{ij}}\right)\frac{\partial^2 e^*_{ij}}{\partial\alpha^2_{ij}}$$
$$= \left(\frac{\partial^2 c_i(e^*_i)}{\partial^2 e^*_{ji}} - \frac{\partial^2 r_{ij}(e^*_{ji})}{\partial^2 e^*_{ji}}\right)\left(\frac{\partial e^*_{ji}}{\partial\alpha_{ij}}\right)^2 + \left(\frac{\partial c_i(e^*_i)}{\partial e^*_{ji}} - \frac{\partial r_{ij}(e^*_{ji})}{\partial e^*_{ji}}\right)\frac{\partial^2 e^*_{ji}}{\partial\alpha^2_{ij}}$$
$$+\left(\frac{\partial^2 c_j(e^*_j)}{\partial^2 e^*_{ij}} - \frac{\partial^2 r_{ji}(e^*_{ij})}{\partial^2 e^*_{ij}}\right)\left(\frac{\partial e^*_{ij}}{\partial\alpha_{ij}}\right)^2 + \left(\frac{\partial c_j(e^*_j)}{\partial e^*_{ij}} - \frac{\partial r_{ji}(e^*_{ij})}{\partial e^*_{ij}}\right)\frac{\partial^2 e^*_{ij}}{\partial\alpha^2_{ij}} > 0$$

Now we prove that $\frac{\partial^2 e^*_{ji}}{\partial\alpha^2_{ij}} < 0$ and $\frac{\partial^2 e^*_{ij}}{\partial\alpha^2_{ij}} < 0$, so $\frac{\partial^2(L^*_{ij}(\alpha_{ij}))}{\partial\alpha^2_{ij}} > 0$.

We first prove that $\frac{\partial^2 e^*_{ji}}{\partial\alpha^2_{ij}} < 0$. It is known that

$$\frac{\partial A_j(e^*)}{\partial e_{ji}} = \frac{\partial c_j(e^*_j)}{\partial e^*_{ji}} - (1-\alpha_{ij})\frac{\partial r_{ij}(e^*_{ji})}{\partial e^*_{ji}} = 0$$

We now derive the second term regarding α_{ij}.

$$\frac{\partial^2 c_j(e^*_j)}{\partial e^{*2}_{ji}}\frac{\partial e^*_{ji}}{\partial\alpha_{ij}} + \frac{\partial r_{ij}(e^*_{ji})}{\partial e^*_{ji}} - (1-\alpha_{ij})\frac{\partial^2 r_{ij}(e^*_{ji})}{\partial e^{*2}_{ji}}\frac{\partial e^*_{ji}}{\partial\alpha_{ij}} = 0$$

We now do the same for α_{ij}.

$$\left(\frac{\partial^3 c_j(e^*_j)}{\partial e^{*3}_{ji}}\left(\frac{\partial e^*_{ji}}{\partial\alpha_{ij}}\right)^2 + \frac{\partial^2 c_j(e^*_j)}{\partial e^{*2}_{ji}}\frac{\partial^2 e^*_{ji}}{\partial\alpha^2_{ji}}\right) + \frac{\partial^2 r_{ij}(e^*_{ji})}{\partial e^{*2}_{ji}}\frac{\partial e^*_{ji}}{\partial\alpha_{ij}}$$
$$-(1-\alpha_{ij})\left(\frac{\partial^3 r_{ij}(e^*_{ji})}{\partial e^{*3}_{ji}}\left(\frac{\partial e^*_{ji}}{\partial\alpha_{ij}}\right)^2 + \frac{\partial^2 r_{ij}(e^*_{ji})}{\partial e^{*2}_{ji}}\frac{\partial^2 e^*_{ji}}{\partial\alpha^2_{ji}}\right) = 0$$

$$\left(\frac{\partial^2 c_j(e_j^*)}{\partial e_{ji}^{*2}} - (1-\alpha_{ij})\frac{\partial^2 r_{ij}(e_{ji}^*)}{\partial e_{ji}^{*2}}\right)\frac{\partial^2 e_{ji}^*}{\partial \alpha_{ji}^2} + \frac{\partial^2 r_{ij}(e_{ji}^*)}{\partial e_{ji}^{*2}}\frac{\partial e_{ji}^*}{\partial \alpha_{ij}}$$

$$+\left(\frac{\partial^3 c_j(e_j^*)}{\partial e_{ji}^{*3}} - (1-\alpha_{ij})\frac{\partial^3 r_{ij}(e_{ji}^*)}{\partial e_{ji}^{*3}}\right)\left(\frac{\partial e_{ji}^*}{\partial \alpha_{ij}}\right)^2 = 0$$

$$\frac{\partial^2 e_{ji}^*}{\partial \alpha_{ij}^2} = \frac{-\frac{\partial^2 r_{ij}(e_{ji}^*)}{\partial e_{ji}^{*2}}\frac{\partial e_{ji}^*}{\partial \alpha_{ij}} - \left(\frac{\partial^3 c_j(e_j^*)}{\partial e_{ji}^{*3}} - (1-\alpha_{ij})\frac{\partial^3 r_{ij}(e_{ji}^*)}{\partial e_{ji}^{*3}}\right)\left(\frac{\partial e_{ji}^*}{\partial \alpha_{ij}}\right)^2}{\frac{\partial^2 c_j(e_j^*)}{\partial e_{ji}^{*2}} - (1-\alpha_{ij})\frac{\partial^2 r_{ij}(e_{ji}^*)}{\partial e_{ji}^{*2}}}$$

Clearly, this expression is lower than zero if $\frac{\partial^3 c_j(e_j^*)}{\partial e_{ji}^{*3}} > 0$ and $\frac{\partial^3 r_{ij}(e_{ji}^*)}{\partial e_{ji}^{*3}} < 0$; note that $\frac{\partial e_{ji}^*}{\partial \alpha_{ij}} < 0$ by Proposition.

Analogously, we obtain

$$\frac{\partial^2 e_{ij}^*}{\partial \alpha_{ij}^2} = \frac{\frac{\partial^2 r_{ji}(e_{ij}^*)}{\partial e_{ij}^{*2}}\frac{\partial e_{ij}^*}{\partial \alpha_{ij}} - \left(\frac{\partial^3 c_i(e_i^*)}{\partial e_{ij}^{*3}} - \alpha_{ij}\frac{\partial^3 r_{ji}(e_{ij}^*)}{\partial e_{ij}^{*3}}\right)\left(\frac{\partial e_{ij}^*}{\partial \alpha_{ij}}\right)^2}{\frac{\partial^2 c_i(e_i^*)}{\partial e_{ij}^{*2}} - \alpha_{ij}\frac{\partial^2 r_{ji}(e_{ij}^*)}{\partial e_{ij}^{*2}}} < 0.$$

■

Lemma 3.4 enables us to state that in any interval where the piecewise derivative function takes the value $\frac{\partial(L_{ij}^*(\alpha_{ij}))}{\partial \alpha_{ij}} = -\alpha_{ij}\frac{\partial r_{ij}(e_{ji}^*)}{\partial e_{ji}^*}\frac{\partial e_{ji}^*}{\partial \alpha_{ij}} - (1-\alpha_{ij})\frac{\partial r_{ji}(e_{ij}^*)}{\partial e_{ij}^*}\frac{\partial e_{ij}^*}{\partial \alpha_{ij}}$, the function $L_{ij}^*(\alpha_{ij})$ is convex (see also Lemma 3.3).

The following proposition shows that, according to the value of the effort equilibrium, the cost function $L_{ij}^*(\alpha_{ij})$ is a continuous piecewise function with four types of piece. This result characterizes all of those pieces, showing the shape of $L_{ij}^*(\alpha_{ij})$ and the optimal α_{ij} in each type of piece.

Proposition 3.1

Consider the effort game $(N, \{E_i\}_{i\in N}, \{A_i\}_{i\in N})$ and e^* as the effort equilibrium. Let $\alpha_{ij} \in [a,b]$ be a piece of $L_{ij}^*(\alpha_{ij})$ with $0 \le a < b \le 1$, $L_{ij}^*(\alpha_{ij})$ can have only four types of piece:

1. **Constant** (e_{ij}^*, e_{ji}^*) is either $(0,0)$, $(1,0)$, $(0,1)$ or $(1,1)$. Thus $\frac{\partial(L_{ij}^*(\alpha_{ij}))}{\partial \alpha_{ij}} = 0$ and $L_{ij}^*(\alpha_{ij})$ is always constant. Therefore, any $\alpha_{ij} \in [a,b]$ minimizes $L_{ij}^*(\alpha_{ij})$.

2. **Increasing:** e_{ij}^* is either 0 or 1, and $0 < e_{ji}^* < 1$. Thus $\frac{\partial(L_{ij}^*(\alpha_{ij}))}{\partial \alpha_{ij}} = \left(\frac{\partial c_i(e_i^*)}{\partial e_{ji}^*} - \frac{\partial r_{ij}(e_{ji}^*)}{\partial e_{ji}^*}\right)\frac{\partial e_{ji}^*}{\partial \alpha_{ij}} >$ 0 and $L_{ij}^*(\alpha_{ij})$ is always increasing. Therefore, $\alpha_{ij} = a$ minimizes $L_{ij}^*(\alpha_{ij})$.

3. **Decreasing:** : $0 < e_{ij}^* < 1$, and e_{ji}^* is either 0 or 1. Thus $\frac{\partial(L_{ij}^*(\alpha_{ij}))}{\partial \alpha_{ij}} = \left(\frac{c_j(e_j^*)}{\partial e_{ij}^*} - \frac{\partial r_{ji}(e_{ij}^*)}{\partial e_{ij}^*}\right)\frac{\partial e_{ij}^*}{\partial \alpha_{ij}} <$ 0 and $L_{ij}^*(\alpha_{ij})$ is always decreasing. Therefore, $\alpha_{ij} = b$ minimizes $L_{ij}^*(\alpha_{ij})$.

4. Depending on cost function shape: $0 < e^*_{ij} < 1$ and $0 < e^*_{ji} < 1$. Thus,

$$\frac{\partial(L^*_{ij}(\alpha_{ij}))}{\partial\alpha_{ij}} = \left(\frac{\partial c_i(e^*_i)}{\partial e^*_{ji}} - \frac{\partial r_{ij}(e^*_{ji})}{\partial e^*_{ji}}\right)\frac{\partial e^*_{ji}}{\partial\alpha_{ij}} + \left(\frac{c_j(e^*_j)}{\partial e^*_{ij}} - \frac{\partial r_{ji}(e^*_{ij})}{\partial e^*_{ij}}\right)\frac{\partial e^*_{ij}}{\partial\alpha_{ij}}.$$

In this case, there is always a unique $\check{\alpha}^{[a,b]}_{ij} \in [a,b]$ that minimizes $L^*_{ij}(\alpha_{ij})$, which is:

$$\check{\alpha}^{[a,b]}_{ij} = \begin{cases} a & \text{if } \frac{\partial(L^*_{ij}(\alpha_{ij}))}{\partial\alpha_{ij}} > 0 \text{ for all } \alpha_{ij} \in [a,b] \\ b & \text{if } \frac{\partial(L^*_{ij}(\alpha_{ij}))}{\partial\alpha_{ij}} < 0 \text{ for all } \alpha_{ij} \in [a,b] \\ \text{Solution of } \frac{\partial(L^*_{ij}(\alpha_{ij}))}{\partial\alpha_{ij}} = 0 & \text{otherwise} \end{cases}$$

Proof

The proof of Lemma 3.3 shows four possible cases for $L^*_{ij}(\alpha_{ij})$. The point 2. of the proof of Lemma 3.3 proves the point 1. (Constant). The point 3. proves the point 2. (Increasing), and point 4. proves point 3 (decreasing). Finally, to prove the point 4. (Depending on cost function shape) we need the point 1 of Lemma 3.3 and Lemma 3.4 which proves that $L^*_{ij}(\alpha_{ij})$ is convex in this case. Therefore, in this last case, it is also straightforward to show that $\frac{\partial(L^*_{ij}(\alpha_{ij}))}{\partial\alpha_{ij}}$ is continuous, so there is always a unique α_{ij} that minimizes $L^*_{ij}(\alpha_{ij})$ in such pieces. The procedure for calculating $\check{\alpha}^{[a,b]}_{ij}$ is the following: First, by Theorem, we calculate e^*_{ij} and e^*_{ji} as a function of α_{ij} from $c'_i(e_{ij}) - \alpha_{ij}r'_{ji}(e_{ij}) = 0$ and $c'_j(e_{ji}) - \alpha_{ji}r'_{ij}(e_{ji}) = 0$. Second, we build the function $L^*_{ij}(\alpha_{ij})$ with the $e^*_{ij}(\alpha_{ij})$ and $e^*_{ji}(\alpha_{ij})$ previously calculated. Finally, we calculate $\frac{\partial(L^*_{ij}(\alpha_{ij}))}{\partial\alpha_{ij}}$ and obtain $\check{\alpha}^{[a,b]}_{ij}$.

Finally, Theorem 3.2 characterizes the optimal α^*_{ij}, for all $i,j \in N$ with $i \neq j$, which incentivizes an efficient effort equilibrium, which is also provided.

Proof of Theorem 3.2

As $L^*_{ij}(\alpha_{ij})$ is a continuous piecewise function, we analyze the five pieces that define it in each case. Lemma 3.3, 3.4 and Proposition 3.1 enable the type of piece to be determined, thus giving the value of α_{ij} that minimizes $L^*_{ij}(\alpha_{ij})$ in each piece. Comparing the pieces gives the α^*_{ij} that minimizes the aggregate cost for each of the six cases. This value need not be unique. Note, in addition, that $\underline{\alpha}_{ij}$, $\bar{\alpha}_{ij}$, $\bar{\alpha}_{ji}$ and $\underline{\alpha}_{ji}$ are always greater than zero, but any of them may be greater than one, which implies that some pieces of certain cases may not exist. We prove the theorem case by case:

Case A $(\underline{\alpha}_{ij} < \bar{\alpha}_{ij} < 1 - \bar{\alpha}_{ji} < 1 - \underline{\alpha}_{ji})$

Note that those thresholds are always greater than zero, so $0 < \underline{\alpha}_{ij} < \bar{\alpha}_{ij} < 1 - \bar{\alpha}_{ji} < 1 - \underline{\alpha}_{ji} < 1$. By Lemma 3.3,

- if $\alpha_{ij} \in \left(0, \underline{\alpha}_{ij}\right)$, then $L^*_{ij}(\alpha_{ij})$ is constant in this interval.
- If $\alpha_{ij} \in \left(\underline{\alpha}_{ij}, \bar{\alpha}_{ij}\right)$, then $L^*_{ij}(\alpha_{ij})$ is decreasing, which implies that $\alpha_{ij} = 1 - \bar{\alpha}_{ji}$ minimizes $L^*_{ij}(\alpha_{ij})$.
- If $\alpha_{ij} \in (\bar{\alpha}_{ij}, 1 - \bar{\alpha}_{ji})$, then $L^*_{ij}(\alpha_{ij})$ is constant in this interval.
- If $\alpha_{ij} \in \left(1 - \bar{\alpha}_{ji}, 1 - \underline{\alpha}_{ji}\right)$, then $L^*_{ij}(\alpha_{ij})$ is increasing, which implies that $1 - \bar{\alpha}_{ji}$ minimizes $L^*_{ij}(\alpha_{ij})$.
- If $\alpha_{ij} \in \left(1 - \underline{\alpha}_{ji}, 1\right)$, then $L^*_{ij}(\alpha_{ij})$ is constant in this interval.

Therefore, α^*_{ij} is equal to any $\alpha_{ij} \in [\bar{\alpha}_{ij}, 1 - \bar{\alpha}_{ji}]$.

Case B $(\underline{\alpha}_{ij} < 1 - \bar{\alpha}_{ji} < \bar{\alpha}_{ij} < 1 - \underline{\alpha}_{ji})$

Analogously, $0 < \underline{\alpha}_{ij} < 1 - \bar{\alpha}_{ji} < \bar{\alpha}_{ij} < 1 - \underline{\alpha}_{ji} < 1$, and by Lemma Lemma 3.3, 3.4 and Proposition 3.1,

- if $\alpha_{ij} \in \left(0, \underline{\alpha}_{ij}\right)$, then $L^*_{ij}(\alpha_{ij})$ is constant in this interval.
- If $\alpha_{ij} \in \left(\underline{\alpha}_{ij}, 1 - \bar{\alpha}_{ji}\right)$, then $L^*_{ij}(\alpha_{ij})$ is decreasing, which implies that $\alpha_{ij} = 1 - \bar{\alpha}_{ji}$ minimizes $L^*_{ij}(\alpha_{ij})$.
- If $\alpha_{ij} \in (1 - \bar{\alpha}_{ji}, \bar{\alpha}_{ij})$, then $\check{\alpha}_{ij}$ minimizes $L^*_{ij}(\alpha_{ij})$, where $\check{\alpha}_{ij}$ is defined-64 in Proposition 3.1.
- If $\alpha_{ij} \in \left(\bar{\alpha}_{ij}, 1 - \underline{\alpha}_{ji}\right)$, then $L^*_{ij}(\alpha_{ij})$ is increasing, which implies that $\bar{\alpha}_{ij}$ minimizes $L^*_{ij}(\alpha_{ij})$.
- If $\alpha_{ij} \in \left(1 - \underline{\alpha}_{ji}, 1\right)$, then $e^*_{ij} = 1$, $e^*_{ji} = 0$, and $L^*_{ij}(\alpha_{ij})$ is constant in this interval.

Therefore, $\alpha_{ij}^{*} = \check{\alpha}_{ij}^{[1-\bar{\alpha}_{ji},\bar{\alpha}_{ij}]}$.

Case C $(\underline{\alpha}_{ij} < 1 - \bar{\alpha}_{ji} < 1 - \underline{\alpha}_{ji} < \bar{\alpha}_{ij})$

It may happen here that either $\bar{\alpha}_{ij} < 1$ or $\bar{\alpha}_{ij} \geq 1$. Thus there are two subcases:

$0 < \underline{\alpha}_{ij} < 1 - \bar{\alpha}_{ji} < 1 - \underline{\alpha}_{ji} < \bar{\alpha}_{ij} < 1$

$0 < \underline{\alpha}_{ij} < 1 - \bar{\alpha}_{ji} < 1 - \underline{\alpha}_{ji} < 1 < \bar{\alpha}_{ij}$

Starting with the first subcase, by Lemma 3.3, 3.4 and Proposition 3.1

- if $\alpha_{ij} \in \big(0, \underline{\alpha}_{ij}\big)$, then $L_{ij}^{*}(\alpha_{ij})$ is constant in this interval.
- If $\alpha_{ij} \in \big(\underline{\alpha}_{ij}, 1 - \bar{\alpha}_{ji}\big)$, then $L_{ij}^{*}(\alpha_{ij})$ is decreasing, which implies that $\alpha_{ij} = 1 - \bar{\alpha}_{ji}$ minimizes $L_{ij}^{*}(\alpha_{ij})$.
- If $\alpha_{ij} \in \big(1 - \bar{\alpha}_{ji}, 1 - \underline{\alpha}_{ji}\big)$, then $\check{\alpha}_{ij}$ minimizes $L_{ij}^{*}(\alpha_{ij})$.
- If $\alpha_{ij} \in \big(1 - \underline{\alpha}_{ji}, \bar{\alpha}_{ij}\big)$, then $L_{ij}^{*}(\alpha_{ij})$ is decreasing, which implies that $\bar{\alpha}_{ij}$ minimizes $L_{ij}^{*}(\alpha_{ij})$.
- If $\alpha_{ij} \in (\bar{\alpha}_{ij}, 1)$, then $L_{ij}^{*}(\alpha_{ij})$ is constant, in this interval.

However, in the second subcase $\bar{\alpha}_{ij} > 1$, which implies that the last interval described above does not exist. The rest of the analysis is similar to the first subcase.

Therefore, $\alpha_{ij}^{*} = \arg min\{L_{ij}^{*}(\check{\alpha}_{ij}^{[1-\bar{\alpha}_{ji},1-\underline{\alpha}_{ji}]}), L_{ij}^{*}(\Lambda(\bar{\alpha}_{ij}))\}$. Note that, if $\alpha_{ij}^{*} = \Lambda(\bar{\alpha}_{ij})$ and $\bar{\alpha}_{ij} < 1$, then α_{ij}^{*} is equal to any $\alpha_{ij} \in (\bar{\alpha}_{ij}, 1)$.

Case D $(1-\bar{\alpha}_{ji} < \underline{\alpha}_{ij} < \bar{\alpha}_{ij} < 1-\underline{\alpha}_{ji})$

It may happen here that either $1-\bar{\alpha}_{ji} > 0$ or $1-\bar{\alpha}_{ji} \leq 0$. Thus there are two subcases:

$0 < 1-\bar{\alpha}_{ji} < \underline{\alpha}_{ij} < \bar{\alpha}_{ij} < 1-\underline{\alpha}_{ji} < 1$

$1-\bar{\alpha}_{ji} < 0 < \underline{\alpha}_{ij} < \bar{\alpha}_{ij} < 1-\underline{\alpha}_{ji} < 1$

Starting with the first subcase, by Lemma 3.3, 3.4 and Proposition 3.1

- if $\alpha_{ij} \in (0, 1-\bar{\alpha}_{ji})$, then $e^*_{ij} = 0$, $e^*_{ji} = 1$, and $L^*_{ij}(\alpha_{ij})$ is constant in this interval.
- If $\alpha_{ij} \in \left(1-\bar{\alpha}_{ji}, \underline{\alpha}_{ij}\right)$, then $L^*_{ij}(\alpha_{ij})$ is increasing, which implies that $\alpha_{ij} = 1-\bar{\alpha}_{ji}$ minimizes $L^*_{ij}(\alpha_{ij})$.
- If $\alpha_{ij} \in \left(\underline{\alpha}_{ij}, \bar{\alpha}_{ij}\right)$, then $\check{\alpha}_{ij}$ minimizes $L^*_{ij}(\alpha_{ij})$.
- If $\alpha_{ij} \in \left(\bar{\alpha}_{ij}, 1-\underline{\alpha}_{ji}\right)$, then $e^*_{ij} = 1$, $0 < e^*_{ji} < 1$, and $L^*_{ij}(\alpha_{ij})$ is increasing, which implies that $\bar{\alpha}_{ij}$ minimizes $L^*_{ij}(\alpha_{ij})$.
- If $\alpha_{ij} \in (\bar{\alpha}_{ij}, 1)$, then $e^*_{ij} = 1$, $e^*_{ji} = 0$, and $L^*_{ij}(\alpha_{ij})$ is constant in this interval.

However, if $1-\bar{\alpha}_{ji} < 0$ the first interval above does not exist. Again, the rest of the analysis is similar to the first subcase.

Therefore, $\alpha^*_{ij} = \arg min\{L^*_{ij}(\Lambda(1-\bar{\alpha}_{ji})), L^*_{ij}(\check{\alpha}_{ij}^{[\underline{\alpha}_{ij}, \bar{\alpha}_{ij}]})\}$. Note that if $\alpha^*_{ij} = \Lambda(1-\bar{\alpha}_{ji})$ and $1-\bar{\alpha}_{ji} > 0$, then α^*_{ij} is equal to any $\alpha_{ij} \in [0, 1-\bar{\alpha}_{ji}]$.

Case E $(1 - \bar{\alpha}_{ji} < \underline{\alpha}_{ij} < 1 - \underline{\alpha}_{ji} < \bar{\alpha}_{ij})$

In this case, it may happen that either $1 - \bar{\alpha}_{ji} > 0$ or $1 - \bar{\alpha}_{ji} \leq 0$, and either $\bar{\alpha}_{ij} < 1$ or $\bar{\alpha}_{ij} \geq 1$. Thus there are four subcases:

$0 < 1 - \bar{\alpha}_{ji} < \underline{\alpha}_{ij} < 1 - \underline{\alpha}_{ji} < \bar{\alpha}_{ij} < 1$

$1 - \bar{\alpha}_{ji} < 0 < \underline{\alpha}_{ij} < 1 - \underline{\alpha}_{ji} < \bar{\alpha}_{ij} < 1$

$0 < 1 - \bar{\alpha}_{ji} < \underline{\alpha}_{ij} < 1 - \underline{\alpha}_{ji} < 1 < \bar{\alpha}_{ij}$

$1 - \bar{\alpha}_{ji} < 0 < \underline{\alpha}_{ij} < 1 - \underline{\alpha}_{ji} < 1 < \bar{\alpha}_{ij}$

Focusing on the first subcase, by Lemma 3.3, 3.4 and Proposition 3.1

- if $\alpha_{ij} \in (0, 1 - \bar{\alpha}_{ji})$, then $L^*_{ij}(\alpha_{ij})$ is constant in this interval.
- If $\alpha_{ij} \in \big(1 - \bar{\alpha}_{ji}, \underline{\alpha}_{ij}\big)$, then $L^*_{ij}(\alpha_{ij})$ is increasing, which implies that $\alpha_{ij} = 1 - \bar{\alpha}_{ji}$ minimizes $L^*_{ij}(\alpha_{ij})$.
- If $\alpha_{ij} \in \big(\underline{\alpha}_{ij}, 1 - \underline{\alpha}_{ji}\big)$, then $\check{\alpha}_{ij}$ minimizes $L^*_{ij}(\alpha_{ij})$.
- If $\alpha_{ij} \in \big(1 - \underline{\alpha}_{ji}, \bar{\alpha}_{ij}\big)$, then $L^*_{ij}(\alpha_{ij})$ is decreasing, which implies that $\bar{\alpha}_{ij}$ minimizes $L^*_{ij}(\alpha_{ij})$.
- If $\alpha_{ij} \in (\bar{\alpha}_{ij}, 1)$, then $e^*_{ij} = 1$, $e^*_{ji} = 0$, and $L^*_{ij}(\alpha_{ij})$ is constant in this interval

In the other three subcases, the first and/or last interval may not exist. Once again, the rest of the analysis for those subcases is similar to the first one.

Therefore, $\alpha^*_{ij} = \arg min\{L^*_{ij}(\Lambda(1 - \bar{\alpha}_{ji})), \check{\alpha}_{ij}^{[\underline{\alpha}_{ij}, 1 - \underline{\alpha}_{ji}]}, L^*_{ij}(\Lambda(\bar{\alpha}_{ij}))\}$. Note that if $\alpha^*_{ij} = \Lambda(1 - \bar{\alpha}_{ji})$ and $1 - \bar{\alpha}_{ji} > 0$ then α^*_{ij} is equal to any $\alpha_{ij} \in [0, 1 - \bar{\alpha}_{ji}]$, and if $\alpha^E_{ij} = \Lambda(\bar{\alpha}_{ij})$ and $\bar{\alpha}_{ij} < 1$, then α^*_{ij} is equal to any $\alpha_{ij} \in [\bar{\alpha}_{ij}, 1]$

Case F $(1-\bar{\alpha}_{ji} < 1-\underline{\alpha}_{ji} < \underline{\alpha}_{ij} < \bar{\alpha}_{ij})$

This is the most general case and anything could happen with thresholds greater than one. Thus there are nine subcases.

First consider the case $0 < 1-\bar{\alpha}_{ji} < 1-\underline{\alpha}_{ji} < \underline{\alpha}_{ij} < \bar{\alpha}_{ij} < 1$:

If $\alpha_{ij} \in (0, 1-\bar{\alpha}_{ji})$, then $L^*_{ij}(\alpha_{ij})$ is constant in this interval.

If $\alpha_{ij} \in \left(1-\bar{\alpha}_{ji}, 1-\underline{\alpha}_{ji}\right)$, then $L^*_{ij}(\alpha_{ij})$ is increasing, which implies that $\alpha_{ij} = 1-\bar{\alpha}_{ji}$ minimizes $L^*_{ij}(\alpha_{ij})$.

If $\alpha_{ij} \in \left(1-\underline{\alpha}_{ji}, \underline{\alpha}_{ij}\right)$, then $L^*_{ij}(\alpha_{ij})$ is constant in this interval.

If $\alpha_{ij} \in \left(\underline{\alpha}_{ij}, \bar{\alpha}_{ij}\right)$, then $L^*_{ij}(\alpha_{ij})$ is decreasing, which implies that $\alpha_{ij} = \bar{\alpha}_{ij}$ minimizes $L^*_{ij}(\alpha_{ij})$.

If $\alpha_{ij} \in (\bar{\alpha}_{ij}, 1)$, then $L^*_{ij}(\alpha_{ij})$ is constant in this interval.

In any other subcase, the first, second, to last, and last intervals considered above, may not exist. The rest of the analysis for those subcases is similar to the first one.

Therefore, $\alpha^*_{ij} = \arg Min\{L^*_{ij}(\Lambda(1-\bar{\alpha}_{ji})), L^*_{ij}(\Lambda(\bar{\alpha}_{ij}))\}$. Note that, if $\alpha^*_{ij} = \Lambda(1-\bar{\alpha}_{ji})$ and $1-\bar{\alpha}_{ji} > 0$, then α^*_{ij} is equal to any $\alpha_{ij} \in [0, 1-\bar{\alpha}_{ji}]$, but if $\alpha^*_{ij} = \Lambda(\bar{\alpha}_{ij})$ and $\bar{\alpha}_{ij} < 1$, then α^*_{ij} is equal to any $\alpha_{ij} \in [\bar{\alpha}_{ij}, 1]$. Additionally, if $1-\underline{\alpha}_{ji} < 0$ and $\bar{\alpha}_{ij} > 1$, then $L^*_{ij}(\Lambda(1-\bar{\alpha}_{ji})) = L^*_{ij}(\Lambda(\bar{\alpha}_{ij})$, so α^*_{ij} is equal to any $\alpha_{ij} \in [0, 1]$.

APPENDIX D

Table 1: Notation summary

$N = \{1, 2, ..n\}$	Agents
$E_i = [0,1]^{n-1}$	Strategy space of agent i of the non-cooperative game
$E = \prod_{i \in N} E_i = [0,1]^{n(n-1)}$	Strategy profile space of the non-cooperative game
$e_{ij} \in [0,1]$	Effort exerted by agent i to reduce the cost of agent j
$e_i = (e_{ij})_{j \neq i} \in E_i$	Efforts exerted by agent i
$e \in E$	Effort profile
c_i: $E_i \rightarrow R_+$	Cost function for agent i with $c_i(e_i)$ the cost of effort e_i
r_{ij}: $[0,1] \rightarrow R_+$	Cost reduction function of agent i given by agent j
$r_{ij}(e_{ji})$	Cost reduction for agent i due to effort e_{ji}
$c : 2^N \rightarrow R$	Characteristic function of the cooperative cost game
$S \subseteq N$	Coalition of agents
$c^S(\{i\}) = c_i(e_i) - \sum_{j \in S \setminus \{i\}} r_{ij}(e_{ji})$	The reduced cost of agent i in coalition S
$c(S) = \sum_{i \in S} c^S(\{i\})$	The reduced cost for coalition S
ψ_i: $E \rightarrow R$	Allocation to agent i
$\psi(e) = (\psi_i(e))_{i \in N}$	Allocation rule, with $\sum_{i \in N} \psi_i(e) = c(N)$
$\Omega_i(e) = c_i(e_i) - \sum_{j \in N \setminus \{i\}} [\omega^i_{ij} r_{ij}(e_{ji}) + \omega^i_{ji} r_{ji}(e_{ij})]$	WPR allocation for agent i, where $\omega^i_{ij} \in [0,1]$, and $\omega^i_{ji} = 1 - \omega^j_{ji}$ with $i, j \in N, i \neq j$
$A_i(e) = c_i(e_i) - \sum_{j \in N \setminus \{i\}} \alpha_{ij} [r_{ij}(e_{ji}) + r_{ji}(e_{ij})]$	WPAR allocation for agent i, where $\alpha_{ij} \in [0,1]$ and $\alpha_{ji} = 1 - \alpha_{ij}$.
$\alpha = (\alpha_i)_{i \in N}$ with $\alpha_i = (\alpha_{ij})_{j \in N \setminus \{i\}}$	Weights of WPAR allocation
$\phi(c)$	Shapley value
$\nu(e)$	Nucleolus

Table 2: Summary of optimization problems

$\tilde{e}$	Efficient effort profile	$\tilde{e} = \arg \min\limits_{e \in [0,1]^{n(n-1)}} c(N)$
$\hat{e}_i$	Optimal efforts of agent i given efforts of other agents	$\hat{e}_i = \arg \min\limits_{e_i \in [0,1]^{(n-1)}} A_i(e)$
e_i^*	Equilibrium strategy of agent i	$e_i^* = \hat{e}_i$
α^*	Optimal weights of WPAR allocation	$\alpha^* = \arg \min\limits_{\alpha \in [0,1]^{n(n-1)}} \sum_{i \in N} A_i(e^*)$ $\Updownarrow$ $\alpha_{ij}^* = \arg \min\limits_{\alpha_{ij} \in [0,1]} L_{ij}^*(\alpha_{ij})$ for $i \neq j \in N$ with $L_{ij}^*(\alpha_{ij}) = c_i(e_i^*) + c_j(e_j^*)$

APPENDIX E

The Shapley value of corporation tax games with dual benefactors

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1.1 Introduction

In recent years, as a result of an eminently globalized environment, the debate on the necessary cooperation among states and firms has been intensified. The absence of this cooperation among countries can cause both a race to the bottom tax competition in fiscal policies and opacity or financial secrecy. On the part of firms or individuals, it can cause underground economy, tax evasion or fiscal fraud. All of them are inefficient behaviors.

In particular, the underground economy is a significant problem and difficult to deal with. The causes and negative effects of the underground economy have been debated by authors as [9], [3], [10], and [2], among other authors. The solutions to be adopted to detect and reduce the underground economy have been studied, for example, by [14], [15], [5], [16], and [1]. Three solutions of particular relevance are the design of optimal tax systems, the increase in transparency and information, and a greater severity of the punishments. These elements allow to increase the capability to detect and discourage the infringing behaviors. These efforts not only benefit the states themselves by allowing an increase in tax collection, but also benefit all the firms that act in accordance with the law, since it eliminates the competitors that acted in a submerged manner.

However, carry out effective policies focused at combating the underground economy, requires a high economic cost in human and material resources that must be faced by the countries governments. Cooperation among countries and firms could reduce these costs. For example, cooperation among countries could be based on the desire for transparency and the transfer of information in order to facilitate the detection of fraudulent behavior, allowing a reduction of costs. In addition, beyond the mandatory legal requirement, a firm can make an effort to improve the transparency of its financial practice. The firm can also just share any kind of relevant information with the tax authorities. This cooperation could be rewarded by a tax reduction.

Inspired by the Spanish tax system, [7] introduce a cooperative model, where the Government is considered the only benefactor, as it keeps costs at the same level, zero cost, while reduce the costs of those investors who act legally (beneficiaries). Investors may decide to cooperate or not cooperate with the Government. If they decide to cooperate, the Government will provide a framework of legal certainty, which is in their benefit. On the contrary, if investors decide not cooperate with the Government and try to defraud the system by tax evasion, they can be detected and charged with unlawful behavior. Once this irregular behavior is demonstrated, they will be punished and required to return all amount defrauded plus a penalty. This means that the costs of not cooperating with the Government would be higher than cooperate, and so all investors are willing to pay the lowest taxes under legal protection of the Government. The authors present the class of corporation tax games as an application of linear cost games to the corporate tax reduction system.

Linear cost games were introduced by [6] as a particular case of k-norm cost games with benefactor and beneficiaries, when $k = 1$. The authors introduce a class of cost-coalitional problems, which are based on a priori information about the cost faced by each agent in each set that it could belong to. Then, they focus on problems with decreasingly monotonic coalitional costs. Their paper study the effects of giving and receiving, on cost-coalitional problems, when there exist players whose participation in an alliance always contributes to the savings of all alliance members (benefactors), and there also exist players whose cost decreases in such an alliance (beneficiaries).

[6] show that when there are multiple benefactors, an agent sees the same individual costs in any coalition that contains at least one benefactor and is not all-inclusive. Thus, with a single benefactor all the members of a coalition may see their cost increase if he leaves the group; they say that he is irreplaceable.

On the other hand, when there are several benefactors, the cost of a member of the coalition remains the same as long as there is another benefactor in the coalition; they say then that each benefactor in this case is replaceable. They study separately the two cases, and use linear and quadratic norm cost games to analyze the role played by benefactors and beneficiaries in achieving stability of different cooperating alliances. Different notions of stability, the core and the bargaining set, are considered there and provided conditions for stability of the grand coalition which leads to minimum value of total cost incurred by all agents.

In this paper, we present a new model of corporate tax system with several firms and countries (multiple dual benefactors). Countries are dual in the sense they are benefactors (they reduce the cost of both firms and other countries) and beneficiaries (the information provided by others countries reduce its cost). They are also irreplaceable benefactors because all the members of a coalition may see their cost increase if one of them leaves the group. It differs from the corporate tax system given by [7] in the following three points. First, there is a single benefactor there. Moreover, the definition of benefactor given by [7] is a particular case of the definition of dual and irreplaceable benefactor given here. We can say that dual benefactors here generalize benefactors there. Second, the concept of beneficiary in [7] is less restrictive than the one considered here. We can say that a beneficiary here is a beneficiary in the corporate tax system given there (see Section 2 for more details). And third, we propose here the Shapley value [11] a as stable allocation rule for sharing the reduced total costs. [6] and [7], proved that the grand coalition is stable in the sense of the core, but they didn't study the Shapley Value. Here we present a simple expression for the Shapley value of multiple corporation tax games that benefits all agents and, in particular, compensates the benefactors for their dual role and irreplaceable character. A recent survey on this allocation rule is [8].

The outline of the paper is as follow. First, in Section 2, the cost-coalitional problems with multiple dual and irreplaceable benefactors and some of their properties are described. After that, in Section 3, we introduce the class of cooperative cost games associated to cost-coalitional problems with multiple dual and irreplaceable benefactors, the so called multiple corporation tax games. Section 4 presents a simple and easily computable expression for the Shapley value of multiple corporation tax games. An example illustrating the model and the role played by dual and irreplaceable benefactors is given in Section 5. Finally some concluding remarks and highlights for further research are collected in Section 6.

1.2 Cost-coalitional problems with multiple dual and irreplaceable benefactors

Let $E = \{1, 2, .., e\}$ be a set of firms, and $P = \{1, 2, .., p\}$ be a set of countries, with $S_j^i \geq 0$ and $\bar{S}_j^i \geq 0$ be respectively a tax and a reduced tax that firm j pays in country i, with $S_j^i > \bar{S}_j^i$. Let $N = E \cup P$ denote the set of all agents (firms and countries), with $|N| = n = e+p$, where $e \geq 1$ and $p \geq 2$. We define $T \subseteq N$ as an arbitrary set of agents in N. If two given countries are in a coalition T, then they cooperate and share information, which implies that they can reduce their levels of tax evasion and underground economy. The size of the reduction depends on how much information a country has and how relevant it is for the other country. Note that, for a country i, the more countries are in a coalition with it, the more relevant information this country gathers, and consequently, the smaller the degree of tax evasion and underground economy it has. Formally, let w_i^T be a measure of the underground economy and tax evasion of country i when it is in a coalition T, thus, given two sets $T \subseteq T' \subseteq N$, we assume that always $w_i^T > w_i^{T'}$ if $(T' \backslash T) \cap P \neq \emptyset$, and $w_i^T = w_i^{T'}$ otherwise. Therefore, always $w_i^T \geq w_i^{T'}$. We denote by w_i the countrys' stand alone measure of tax evasion, i.e., $w_i = w_i^{\{i\}}$.

Any agent $k \in T$ incurs certain non-negative cost, which depends on the subset T. We denote this cost by c_k^T, and by c_k an agents' stand alone cost, i.e., $c_k = c_k^{\{k\}}$. For any coalition $T \subseteq N$, the cost of agents are:

1. $c_j^T = \sum\limits_{i \in P \cap T} \bar{S}_j^i + \sum\limits_{i \in P \backslash (P \cap T)} S_j^i$ for all $j \in T \cap E$.

2. $c_i^T = g_i\left(w_i^T\right)$ for all $i \in T \cap P$.

Where firm $j \in T$ must pay a tax $\bar{S}_j^i$ to country i if $i \in T$, and S_j^i if $i \notin T$. In addition, g_i is a strictly increasing function such that for all $i, i' \in P$ and for all $T \subseteq N$, where $i, i' \in P \cap T$, always it holds that $g_i\left(w_i^{T \backslash \{i'\}}\right) - g_i\left(w_i^T\right) = z_{ii'}$, with $z_{ii'} > 0$ being how much the country i' reduces the cost of i with the information i' shares with i. [1]

Next, we identify two special roles that all the agents can play in the model, being benefactors and beneficiaries.

Definition 1.1 *A **benefactor** is an agent $\bar{k} \in N$ such that for any set $T \subset N \backslash \bar{k}$ and for all $k \in T$, $c_k^T \geq c_k^{T \cup \{\bar{k}\}}$, in addition, for at least one agent $k \in T$, $c_k^T > c_k^{T \cup \{\bar{k}\}}$. The agents whose cost decreases in an alliance with a benefactor are denoted by **beneficiaries**.*

[1] We assume $z_{ii'} > 0$, thus, countries are always benefactors. However, $z_{ii'}$ could be as close to zero as we want, i.e., the information that a country shares with other country can be negligible. Therefore, in the limit case in which $z_{ii'} = 0$, the results should hold. In any case, a wider generalization of this model will be consider in future research.

The following lemma characterizes the agents of the game as benefactors and beneficiaries.

Lemma 1.1 *An agent k is a benefactor if and only if it is a country. However, both firms and countries can be beneficiaries.*

Proof. Consider agent $k' \in N$ and any set $T \subset N \backslash \{k'\}$. To prove Lemma 1.1, we first consider that agent k' is a country and compare the cost of agents in T and in $T \cup \{\bar{k}\}$, and second we consider that agent k' is a firm, and we do the same analysis. Note that agents in T could be either countries or firms:

1. Consider that agent k' is a country i', then
   (a) For all $i \in T \cap P$, $c_i^T = g_i\left(w_i^T\right)$ and $c_i^{T\cup\{i'\}} = g_i\left(w_i^{T\cup\{i'\}}\right)$, where $w_i^T > w_i^{T\cup\{i'\}}$ because $T \subseteq T \cup \{i'\}$ and $i' \in P$. Consequently, as g_i is increasing, $c_i^T > c_i^{T\cup\{i'\}}$.
   (b) For all $j \in T \cap E$,
   $c_j^T = \sum_{i\in P\cap T} \bar{S}_j^i + \sum_{i\in P\backslash(P\cap T)} S_j^i = \sum_{i\in P\cap T} \bar{S}_j^i + S_j^{i'} + \sum_{i\in P\backslash(P\cap(T\cup\{i'\}))} S_j^i$,
   and
   $c_j^{T\cup\{i'\}} = \sum_{i\in P\cap(T\cup\{i'\})} \bar{S}_j^i + \sum_{i\in P\backslash(P\cap(T\cup\{i'\}))} S_j^i = \sum_{i\in P\cap T} \bar{S}_j^i + \bar{S}_j^{i'} + \sum_{i\in P\backslash(P\cap(T\cup\{i'\}))} S_j^i$. Consequently, $c_j^T > c_j^{T\cup\{i'\}}$ because $S_j^{i'} > \bar{S}_j^{i'}$.

2. Consider that agent k' is a firm j', then,
   (a) For all $i \in T \cap P$, $c_i^T = g_i\left(w_i^T\right)$ and $c_i^{T\cup\{j'\}} = g_i\left(w_i^{T\cup\{j'\}}\right)$, where, $w_i^T = w_i^{T\cup\{j'\}}$ because $T \subseteq T \cup \{j'\}$ and $j' \in E$. Consequently, $c_i^T = c_i^{T\cup\{j'\}}$.
   (b) For all $j \in T \cap E$,
   $c_j^T = \sum_{i\in P\cap T} \bar{S}_j^i + \sum_{i\in P\backslash(P\cap T)} S_j^i$, and
   $c_j^{T\cup\{j'\}} = \sum_{i\in P\cap(T\cup\{j'\})} \bar{S}_j^i + \sum_{i\in P\backslash(P\cap(T\cup\{j'\}))} S_j^i = \sum_{i\in P\cap T} \bar{S}_j^i + \sum_{i\in P\backslash(P\cap T)} S_j^i$. Therefore, $c_j^T = c_j^{T\cup\{i'\}}$.

Point 1 implies that countries are benefactors, and point 2 implies that firms are not benefactor. Point 1 and 2 imply that countries and firms can be beneficiaries and an agent $k \in N$ is a benefactor if and only if it is a country. ■

There are agents that are dual in the sense that they are benefactors and beneficiaries, these are the countries. However, the firms are exclusively beneficiaries.

The following definition is a relevant property of a benefactor.

Definition 1.2 *A benefactor $\bar{k} \in T \subseteq N$ is irreplaceable if $c_k^T \neq c_k^{T \backslash \bar{k}}$ for at least an agent $k \in T \backslash \bar{k}$.*

The following lemma states that our benefactors are irreplaceable.

Lemma 1.2 *Countries are irreplaceable benefactors.*

Proof. Note that by Lemma 1.1 only countries can be benefactors, then consider any $T \subset N$ such that $T \cap P \neq \emptyset$ where $i' \in T \cap P$. To prove Lemma 1.2, we compare the costs in set T and in set $T \backslash \{i'\}$. Agents in $T \backslash \{i'\}$ can be either countries or firms. First, if the agent is a country, $i \in (T \backslash \{i'\}) \cap P$, then $c_i^T = g_i\left(w_i^T\right) < c_i^{T \backslash \{i'\}} = g_i\left(w_i^{T \backslash \{i'\}}\right)$ because g_i is increasing, and $w_i^T < w_i^{T \backslash \{i'\}}$ because $T \backslash \{i'\} \subset T$.

Second, if the agent in $T \backslash \{i'\}$ is a firm, $j \in (T \backslash \{i'\}) \cap E$, then $c_j^T = \sum\limits_{i \in P \cap T} \bar{S}_j^i + \sum\limits_{i \in P \backslash (P \cap T)} S_j^i = \sum\limits_{i \in P \cap T \backslash \{i'\}} \bar{S}_j^i + \bar{S}_j^{i'} + \sum\limits_{i \in P \backslash (P \cap T)} S_j^i$, and $c_j^{T \backslash \{i'\}} = \sum\limits_{i \in P \cap (T \backslash \{i'\})} \bar{S}_j^i + \sum\limits_{i \in P \backslash P \cap (T \backslash \{i'\})} S_j^i = \sum\limits_{i \in P \cap (T \backslash \{i'\})} \bar{S}_j^i + S_j^{i'} + \sum\limits_{i \in P \backslash (P \cap T)} S_j^i$. Consequently, $c_j^T < c_j^{T \backslash \{i'\}}$ because $\bar{S}_j^{i'} < S_j^{i'}$. ■

We denote the vector of individual agents' costs in all possible subsets by $c^N = \left(c_k^T\right)_{k \in T, \emptyset \neq T \subseteq N}$. Thus, the set of agents N and the cost coalitional vector c^N define a cost-coalitional problem with multiple dual and irreplaceable benefactors $\left(N, c^N\right)$.

A desirable property is that cooperation is beneficial. This can be guaranteed if the cost in large subsets do not exceed their cost in smaller ones. The following definition formalize this idea.

Definition 1.3 *A cost-coalitional vector c^N satisfies **cost monotonicity** if $c_k^T \geq c_k^{T'}$ for all $k \in T$, with $T \subset T' \subseteq N$.*

The following lemma shows that the cost-coalitional problem with multiple dual benefactors has this property.

Lemma 1.3 *The cost coalitional problem $\left(N, c^N\right)$ has the property of cost monotonicity.*

Proof. Consider two sets such that $S \subset T \subseteq N$. Any agent in S has to be either a country or a firm.

First, if the agent is a country $i \in S \cap P$, then always $c_i^S = g_i\left(w_i^S\right)$ and $c_i^T = g_i\left(w_i^T\right)$, which implies that $c_i^S \geq c_i^T$. Note that, g_i is an increasing function, and $w_i^S \geq w_i^T$ because $S \subset T$.

Second, if the agent in S is a firm $j \in S \cap E$, then

$c_j^S = \sum\limits_{i \in P \cap S} \bar{S}_j^i + \sum\limits_{i \in P \backslash (P \cap S)} S_j^i = \sum\limits_{i \in P \cap S} \bar{S}_j^i + \sum\limits_{i \in P \cap (T \backslash S)} S_j^i + \sum\limits_{i \in P \backslash (P \cap T)} S_j^i$,

and

$$c_j^T = \sum_{i \in P \cap T} \bar{S}_j^i + \sum_{i \in P \setminus (P \cap T)} S_j^i = \sum_{i \in P \cap S} \bar{S}_j^i + \sum_{i \in P \cap (T \setminus S)} \bar{S}_j^i + \sum_{i \in P \setminus (P \cap T)} S_j^i.$$

Note that, if in $T \backslash S$ there is at least a country, then $c_j^S > c_j^T$ because $S_j^i > \bar{S}_j^i$, otherwise $c_j^S = c_j^T$. ■

We now define cost games related to our cost-coalitional problem with multiple dual benefactors and prove the cooperation in beneficial for all the agents in the model, benefactors and beneficiaries.

1.3 Multiple corporation tax games

For a given cost-coalitional problem with multiple dual and irreplaceable benefactors $\left(N, c^N\right)$ we define the multiple corporation tax game (N, c), where $c(T) = \sum_{k \in T} c_k^T$ for all $T \subseteq N$, and $c(\emptyset) = 0$.

We consider now the following issue. Is it profitable for the agents in N to form the grand coalition to pay lower taxes and so reduce the degree of tax evasion? Here, we prove that the answer to this question is positive because (N, c) is a subadditive game, in the sense that $c\left(T \cup T'\right) \leq c\left(T\right) + c\left(T'\right)$, for any T, $T' \subset N$, and $T \cap T' = \emptyset$. Notice that the superadditivity condition implies that if N is partitioned into disjoint coalitions (whose integrants reduce the degree of tax evasion) the corresponding cost will not decrease.

In fact we prove that (N, c) is not only subadditive but also concave, in the sense that for all $k \in N$ and all $T, T' \subset N$ such that $T \subset T' \subset N$ with $k \in T$, then $c(T) - c(T \setminus \{k\}) \geq c(T') - c(T' \setminus \{k\})$. It is a well-known result in cooperative game theory that every concave game is subadditive. Moreover, the concavity property provides us with additional information about the game: the marginal contribution of an agent diminishes as a coalition grows. It is well-known as the snowball effect. For more details on cooperative game theory see, for example, [4].

First, in Lemma 1.4, we found out which are the cost marginal contributions of the agents (firms and countries).

Lemma 1.4 *Let $\left(N, c^N\right)$ be a cost-coalitional problem with multiple dual and irreplaceable benefactors and (N, c) the associated multiple corporation tax game. Then,*

1. *For all $T \subseteq N$, for all $j \in E \cap T$,*

   $c(T) - c(T \setminus \{j\}) = c_j^T.$

2. For all $T \subseteq N$, for all $i \in P \cap T$,

$$c(T) - c(T \setminus \{i\}) = c_i^T - \sum_{j \in E \cap T} (S_j^i - \bar{S}_j^i) - \sum_{i' \in P \cap (T \setminus \{i\})} \left(g_{i'}(w_{i'}^{T \setminus \{i\}}) - g_{i'}(w_{i'}^T) \right).$$

Proof. First, we prove (1). Take a coalition $T \subseteq N$, and a firm $j \in E \cap T$. Then,

$$c(T) - c(T \setminus \{j\}) = \sum_{k \in T} c_k^T - \sum_{k \in T \setminus \{j\}} c_k^{T \setminus \{j\}} = c_j^T + \sum_{k \in T \setminus \{j\}} (c_k^T - c_k^{T \setminus \{j\}}).$$

Now, we prove that $\sum_{k \in T \setminus \{j\}} (c_k^T - c_k^{T \setminus \{j\}}) = 0$, and so $c(T) - c(T \setminus \{j\}) = c_j^T$.

Indeed,
$$\sum_{k \in T \setminus \{j\}} (c_k^T - c_k^{T \setminus \{j\}}) = \sum_{i \in P \cap (T \setminus \{j\})} (c_i^T - c_i^{T \setminus \{j\}}) + \sum_{j' \in E \cap (T \setminus \{j\})} (c_{j'}^T - c_{j'}^{T \setminus \{j\}}).$$

We know that $c_i^T - c_i^{T \setminus \{j\}} = g_i(w_i^T) - g_i(w_i^{T \setminus \{j\}}) = 0$, since $w_i^{T \setminus \{j\}} = w_i^T$.

Moreover, $c_{j'}^T - c_{j'}^{T \setminus \{j\}} = \sum_{i \in P \cap T} \bar{S}_{j'}^i + \sum_{i \in P \setminus (P \cap T)} S_{j'}^i - \sum_{i \in P \cap (T \setminus \{j\})} \bar{S}_{j'}^i - \sum_{i \in P \setminus (P \cap T \setminus \{j\})} S_{j'}^i = 0$.

Then,
$$\sum_{i \in P \cap (T \setminus \{j\})} (c_i^T - c_i^{T \setminus \{j\}}) = 0, \text{ and } \sum_{j' \in E \cap (T \setminus \{j\})} (c_{j'}^T - c_{j'}^{T \setminus \{j\}}) = 0.$$

Hence, we conclude that $\sum_{k \in T \setminus \{j\}} (c_k^T - c_k^{T \setminus \{j\}}) = 0$.

Second, we prove (2). Take a coalition $T \subseteq N$, and a country $i \in P \cap T$. Then,

$$c(T) - c(T \setminus \{i\}) = \sum_{k \in T} c_k^T - \sum_{k \in T \setminus \{i\}} c_k^{T \setminus \{i\}} = c_i^T - \sum_{k \in T \setminus \{i\}} (c_k^{T \setminus \{j\}} - c_k^T).$$

We know that,

$$\sum_{k \in T \setminus \{i\}} (c_k^{T \setminus \{i\}} - c_k^T) = \sum_{i' \in P \cap (T \setminus \{i\})} (c_{i'}^{T \setminus \{i\}} - c_{i'}^T) + \sum_{j \in E \cap (T \setminus \{i\})} (c_j^{T \setminus \{i\}} - c_j^T).$$

We prove now that

$$c_j^{T \setminus \{i\}} - c_j^T = \left(S_j^i + \sum_{i' \in P \cap (T \setminus \{i\})} \bar{S}_j^{i'} + \sum_{i' \in P \setminus P \cap (T \setminus \{i\})} S_j^{i'} \right) - \left(\bar{S}_j^i + \sum_{i' \in P \cap (T \setminus \{i\})} \bar{S}_j^{i'} + \sum_{i' \in P \setminus P \cap (T \setminus \{i\})} S_j^{i'} \right) = S_j^i - \bar{S}_j^i.$$

We know, by definition, that

$c_{i'}^{T\backslash\{i\}} - c_{i'}^{T} = g_{i'}(w_{i'}^{T\backslash\{i\}}) - g_{i'}(w_{i'}^{T})$

Hence, we can conclude that

$$c(T)-c(T\backslash\{i\}) = c_i^T - \sum_{j\in E\cap T}(S_j^i - \bar{S}_j^i) - \sum_{i'\in P\cap(T\backslash\{i\})}\left(g_{i'}(w_{i'}^{T\backslash\{i\}}) - g_{i'}(w_{i'}^{T})\right).$$

■

In point 1, this proposition states that a firm j always contributes to a coalition $T\setminus\{j\}$ exactly with its cost in coalition T, which is c_j^T. As a firm is always and exclusively a beneficiary in this model, it has not effect in the cost of others agents: either countries or firms. However, a country is a benefactor to both firms and others countries, therefore, its marginal contribution is smaller than its cost in coalition T. If country i is withdrawn from a coalition T, the individual cost of firms and others countries in coalition T increases.

The following Theorem states that our class of games are concave.

Theorem 1.1 *The multiple corporation tax games (N,c) are concave.*

Proof. Here we have to prove that the marginal contribution of an agent k diminishes as a coalition grows. Any agent k can only be either a firm or a country, and Lemma 1.4 provided its marginal contribution.

If the agent is a firm j, then for all $T\subseteq T', j\in T$, by Lemma 1.3, $c_j^T \geq c_j^{T'}$, and so $c_j^T = c(T)-c(T\setminus\{j\}) \geq c(T') - c(T'\setminus\{j\}) = c_j^{T'}$.

On the other hand, if the agent is a country i, again for all $T\subset T'$, by Lemma 1.3, $c_i^T \geq c_i^{T'}$.

In addition, $\sum_{j\in E\cap T}(S_j^i-\bar{S}_j^i) \leq \sum_{j\in E\cap T'}(S_j^i-\bar{S}_j^i)$ because all the countries in T are also in T', and if T' there is at least one more than in T, then the inequality is strict.

Finally, for the same reason $\sum_{i'\in P\cap T\backslash\{i\}} z_{i'i} \leq \sum_{i'\in P\cap T'\backslash\{i\}} z_{i'i}$.

Hence, we can conclude that for all $T\subset T'$ and for all $i\in P\cap T$,

$c(T)-c(T\backslash\{i\}) = c_i^T - \sum_{j\in E\cap T}(S_j^i-\bar{S}_j^i) - \sum_{i'\in P\cap T\backslash\{i\}} z_{i'i} \geq c_i^{T'} - \sum_{j\in E\cap T'}(S_j^i - \bar{S}_j^i) - \sum_{i'\in P\cap T'\backslash\{i\}} z_{i'i} = c(T)-c(T'\setminus\{i\})$. ■

So we proved that in a cost-coalitional problem with multiple dual and irreplaceable benefactors $\left(N,c^N\right)$ it is efficient that all firms pay lower taxes and all countries manage to jointly reduce their degrees of tax evasion. In that case, the reduced total cost is given by $c(N) = \sum_{i\in P} c_i^N + \sum_{j\in E} c_j^N$.

An allocation rule for multiple corporation tax games is a map ψ which assigns a vector $\psi(N,c)\in R^N$ to every (N,c), satisfying that $\sum_{k\in N}\psi_k(N,c) =$

$c(N)$. Each component $\psi_k(N,c)$ indicates the cost allocated to $k \in N$, so an allocation rule for multiple corporation tax games is a procedure to allocate the reduced total cost among the agents in N when they cooperate. An allocation rule should have good properties from the following points of view.

1. Computability. For a particular game the rule should be computable in a reasonable CPU time, even when the number of agents is large.
2. Coalitional Stability. It is very convenient that the rule proposes an allocation which belongs to the core of the cost game. This means that, for every multiple corporation tax game (N,c), ψ should satisfy the following:

   $\sum_{k \in T} \psi_k(N,c) \leq c(T)$, for every $T \subseteq N$.

   This condition assures that no group of agents T is disappointed with the proposal of the rule, because the cost allocated to it is less than or equal to the cost it would support if its members formed a coalition to pay lower taxes, and reduce the levels of tax evasion, independently of the agents in $N \setminus T$.
3. Acceptability. The rule must be understandable and acceptable by the agents.

A very natural allocation rule for multiple corporation tax games is $\psi_k(N,c) = c_k^N$, for all $k \in N$. It has good properties at least with respect to computability and coalitional stability. Notice that, for every $T \subseteq N$, $\sum_{k \in T} \psi_k(N,c) = \sum_{k \in T} c_k^N \leq \sum_{k \in T} c_k^T = c(T)$.

Nevertheless, the benefactors will have serious difficulties accepting the above allocation rule that rewards the beneficiaries excessively while they do not receive enough compensation for their dual role of giving and receiving.

Since the multiple corporation tax games are concave, cooperative game theory provides allocation rules for them with good properties at least with respect to items coalitional stability and acceptability. We highlight the Shapley value and the nucleolus, which always provide core allocations in this context (see [4] for details on them). Both allocations are, in general, hard to compute when the number of agents increases.

Next, we present a simple and easily calculated expression for the Shapley value of multiple corporation tax games that compensates the benefactors for their dual role and irreplaceable character.

1.4 The Shapley value

One of the most important allocation rules for cost games is the Shapley value (see [11]). As we already mentioned, the Shapley value is specially convenient for concave games: it is the barycenter of its core (see [13]).

We denote by $\phi(N,c)$ the shapley value of multiple corporation tax game (N,c), where for each agent $k \in N$, $\phi_k(N,c) = \sum_{T \subseteq N; k \in T} \gamma(T)\,[c(T) - c(T \setminus \{k\})]$, with $\gamma(t) = \frac{(n-t)!(t-1)!}{n!}$, $|T| = t$.

The following Theorem states that the Shapley value can be easily computed in the class of multiple corporation tax games. Moreover, it shows that the Shapley value provides an acceptable allocation for multiple corporation tax games: it increases the cost of a beneficiary in a half of the benefits it obtains from benefactors, and it decreases the cost of a benefactor in a half of the benefits it provided to the beneficiaries.

Theorem 1.2 *For any multiple corporation tax game (N,c), the Shapley value is*

1. *For all $j \in E$, $\phi_j(N,c) = c_j^N + \frac{1}{2}\sum_{i \in P}(S_j^i - \bar{S}_j^i)$*
2. *For all $i \in P$, $\phi_i(N,c) = c_i^N - \frac{1}{2}\sum_{j \in E}(S_j^i - \bar{S}_j^i) + \frac{1}{2}\sum_{i' \in P \setminus \{i\}}(z_{ii'} - z_{i'i})$*

Proof. (1) First, we prove that for all $j \in E$, $\phi_j(N,c) = c_j^N + \frac{1}{2}\sum_{i \in P}(S_j^i - \bar{S}_j^i)$.

Take $j \in E$. By Lemma 1.4, we know that

$\phi_j(N,c) = \sum_{T \subseteq N; j \in T} \gamma(t) c_j^T$.

We can separate coalitions $j \in T \subseteq N$ into mixed coalitions ($j \in T \subseteq N, T \cap P \neq \emptyset, T \cap E \neq \emptyset$) and coalitions with only firms ($j \in T \subseteq N, T \cap P = \emptyset, T \cap E \neq \emptyset$).

Then,

$$\begin{aligned}\phi_j(N,c) = & \sum_{j \in T \subseteq N, T \cap P = \emptyset, T \cap E \neq \emptyset} \gamma(t)(\sum_{i \in P} S_j^i) \\ & + \sum_{j \in T \subseteq N, T \cap P \neq \emptyset, T \cap E \neq \emptyset} \gamma(t)(\sum_{i \in P \setminus P \cap T'} S_j^i + \sum_{i \in P \cap T'} \bar{S}_j^i).\end{aligned}$$

Taking into account that $\sum_{T \subseteq N; j \in T} \gamma(t) = 1$, we have that

$$\sum_{j \in T \subseteq N, T \cap P = \emptyset, T \cap E \neq \emptyset} \gamma(t) = 1 - \sum_{j \in T \subseteq N, T \cap P \neq \emptyset, T \cap E \neq \emptyset} \gamma(t),$$

and then,

$$\begin{aligned}
\phi_j\left(N,c\right) =&(1-\sum_{j\in T\subseteq N, T\cap P\neq\emptyset, T\cap E\neq\emptyset}\gamma(t))(\sum_{i\in P}S_j^i)\\
&+\sum_{j\in T\subseteq N, T\cap P\neq\emptyset, T\cap E\neq\emptyset}\gamma(t)(\sum_{i\in P\setminus P\cap T}S_j^i+\sum_{i\in P\cap T}\bar{S}_j^i)\\
&=\sum_{i\in P}S_j^i+\sum_{j\in T\subseteq N, T\cap P\neq\emptyset, T\cap E\neq\emptyset}\gamma(t)(\sum_{i\in P\setminus P\cap T}S_j^i+\sum_{i\in P\cap T}\bar{S}_j^i-\sum_{i\in P}S_j^i)\\
&=\sum_{i\in P}S_j^i-\sum_{j\in T\subseteq N, T\cap P\neq\emptyset, T\cap E\neq\emptyset}\gamma(t)\sum_{i\in P\cap T}(S_j^i-\bar{S}_j^i).
\end{aligned}$$

Now, we prove that for all coalitions that contain $j\in T\cap E$ and a particular country $i\in T\cap P$,

$\sum\limits_{j\in T\subseteq N, T\cap P\neq\emptyset, T\cap E\neq\emptyset}\gamma(t)=1/2,$

and then,

$\phi_j\left(N,c\right)=\sum\limits_{i\in P}S_j^i-\frac{1}{2}\sum\limits_{i\in P\cap T}(S_j^i-\bar{S}_j^i)=\frac{1}{2}\sum\limits_{i\in P}(S_j^i+\bar{S}_j^i).$

Indeed,

$\sum\limits_{j\in T\subseteq N, T\cap P\neq\emptyset, T\cap E\neq\emptyset}\gamma(t)=\sum\limits_{t=2}^{n}\begin{pmatrix} n-2\\ t-2\end{pmatrix}\gamma(t)=$

$\sum\limits_{t=2}^{n}\frac{(t-1)}{n(n-1)}=\frac{\sum\limits_{k=1}^{n}k-n}{n(n-1)}=1/2,$

where $\begin{pmatrix} n-2\\ t-2\end{pmatrix}$ is the number of coalitions in which there is j and a particular country i'.

Finally, doing some algebra, we have that

$\frac{1}{2}\sum\limits_{i\in P}(S_j^i+\bar{S}_j^i)=c_j^N+\frac{1}{2}\sum\limits_{i\in P}(S_j^i-\bar{S}_j^i),$

and so, we conclude that

$\phi_j\left(N,c\right)=c_j^N+\frac{1}{2}\sum\limits_{i\in P}(S_j^i-\bar{S}_j^i).$

(2) Second, we demostrate that for all $i\in P$,

$\phi_i\left(N,c\right)=c_i^N-\frac{1}{2}\sum\limits_{j\in E}(S_j^i-\bar{S}_j^i)+\frac{1}{2}\sum\limits_{i'\in P\setminus\{i\}}\left(z_{ii'}-z_{i'i}\right).$

Take $i\in P$. By Lemma 1.4, we know that

$$\begin{aligned}
\phi_i\left(N,c\right)=&\sum_{i\in T\subseteq N}\gamma(t)\\
&\times\left(c_i^T-\sum_{j\in E\cap T}(S_j^i-\bar{S}_j^i)-\sum_{i'\in P\cap T\setminus\{i\}}\left(g_{i'}(w_{i'}^{T\setminus\{i\}})-g_{i'}(w_{i'}^T)\right)\right).
\end{aligned}$$

Let's calculate each of the addends separately.

(2.1) First, taking into account that $c_i^T = c_i - \sum\limits_{i' \in P \cap T \setminus \{i\}} z_{ii'}$, for all $T \in N$, and $\sum\limits_{t=2}^{n} \left(\begin{array}{c} n-2 \\ t-2 \end{array} \right) \gamma(t) = \frac{1}{2}$,

we obtain that

$$\sum_{i \in T \subseteq N} \gamma(t) c_i^T = c_i - \sum_{t=2}^{n} \left(\begin{array}{c} n-2 \\ t-2 \end{array} \right) \gamma(t) \sum_{i' \in P \cap T \setminus \{i\}} z_{ii'} = c_i^N + \frac{1}{2} \sum_{i' \in P \cap T \setminus \{i\}} z_{ii'},$$

where $\left(\begin{array}{c} n-2 \\ t-2 \end{array} \right)$ is now the number of coalitions that contain i and a particular country i'.

(2.2) Second, by a similar argument,

$$\sum_{i \in T \subseteq N} \gamma(t) \sum_{j \in E \cap T} (S_j^i - \bar{S}_j^i) = \sum_{t=2}^{n} \left(\begin{array}{c} n-2 \\ t-2 \end{array} \right) \gamma(t) \sum_{j \in E} (S_j^i - \bar{S}_j^i) = \frac{1}{2} \sum_{j \in E} (S_j^i - \bar{S}_j^i).$$

(2.3) Third, by the same argument,

$$\sum_{i \in T \subseteq N} \gamma(t) \sum_{i' \in P \cap T \setminus \{i\}} \left(g_{i'}(w_{i'}^{T \setminus \{i\}}) - g_{i'}(w_{i'}^{T}) \right) = \sum_{t=2}^{n} \left(\begin{array}{c} n-2 \\ t-2 \end{array} \right) \gamma(t) \sum_{i' \in P \setminus \{i\}} z_{i'i} = -\frac{1}{2} \sum_{i' \in P \setminus \{i\}} z_{i'i} .$$

Finally, adding the above three expressions, we obtain that

$$\phi_i(N, c) = c_i^N + \frac{1}{2} \sum_{i' \in P \cap T \setminus \{i\}} z_{ii'} - \frac{1}{2} \sum_{j \in E} (S_j^i - \bar{S}_j^i) - \frac{1}{2} \sum_{i' \in P \setminus \{i\}} z_{i'i} = c_i^N - \frac{1}{2} \sum_{j \in E} (S_j^i - \bar{S}_j^i) + \frac{1}{2} \sum_{i' \in P \setminus \{i\}} (z_{ii'} - z_{i'i}) . \quad \blacksquare$$

From Theorem 1.2 can be derived that Shapley value compensates benefactors. Note first that, the cost of a firm j in the grand coalition is c_j^N. This firm j is benefited from a country i in an amount which is $S_j^i - \overline{S}_j^i$. The Shapley value reduces this benefit exactly in a half, and consequently this is the amount in which the cost of firm j is increased, see point 1 of Theorem 1.2. In addition, the country i is compensated exactly in this amount, and consequently its cost is reduced, see point 2 of Theorem 1.2. However, a country in its relation with others countries is simultaneously benefactor and beneficiary. Let's first look at the role as beneficiary of i, in any coalition, the country i is benefited from country i' in a cost reduction of $z_{ii'}$, in this case, country i plays the role of beneficiary and i' of benefactor. Thus, the Shapley value reduces the benefit $z_{ii'}$ of country i in a half, in others words, it increases its cost in this amount. Nevertheless, at the same time, the country i benefits country i' in an amount equal to $z_{i'i}$. Now, country i is the benefactor and i'

the beneficiary. In this case, the Shapley value works in the same way, it compensates the benefactor and increasing the cost of the beneficiary in a half of $z_{i'i}$. Therefore, in the relation between two countries both are simultaneously benefactors and beneficiaries, however, if $z_{ii'} - z_{i'i} > 0$, then country i could be seen as a "net" beneficiary and i' as a "net" benefactor, on the contrary if $z_{ii'} - z_{i'i} < 0$. Thus, country i can be a "net" benefactor with some countries and a "net" beneficiary with others.

In conclusion, regarding to the individual cost in the grand coalition, the Shapley values increases the cost of a beneficiary in a half of the benefits it obtains from benefactors, and it decreases the cost of a benefactor in a half of the benefits it provided to the beneficiaries. As in our model there are dual agents (benefactors and beneficiaries), the final effect on these agents depends on which role is stronger.

1.5 An Example

In this example, we propose a simple situation with two countries A and B, and two firms 1 and 2 with activity in both countries. These countries are very concern about their own levels of underground economy, tax evasion, and fraud. To fight against this illegal behavior, these countries must to face a high economic cost in human and material resources. However, this cost can be reduced if both countries decide to cooperate and, for example, they share resources and/or information in its fight.

On the other hand, firms have to pay in each country a certain amount of taxes. Nevertheless, these firms can choose to cooperate with a particular country. For example, beyond the mandatory legal requirement, a firm can make an effort to improve the transparency of its financial practice. The firm can also just share any kind of relevant information with the tax authorities. This cooperation is rewarded by a tax reduction. In particular, country A will fix a reduction of 10%, and B will do it of 15%. Thus, each firm must pay either a tax (S_j^i) or a reduce tax ($\bar{S}_j^i$) as it is given in Table 1.1.

TABLE 1.1: Tax and reduced tax of each firm (in millions of euros)

$S_1^A = 2$	$S_1^B = 4$	$S_2^A = 5$	$S_2^B = 8$
$\overline{S}_1^A = 1.80$	$\overline{S}_1^B = 3.40$	$\overline{S}_2^A = 4.50$	$\overline{S}_2^B = 6.80$

We consider that the cost function of any country $c_i^T = g_i\left(w_i^T\right)$ has two terms. The first term does not depend on the type of coalition the country belong to. In other words, it does not depend on the information other countries could provide. This is a kind of fixed cost. The second term does depend on which coalition the country is. In particular, $g_A(w_A) = 4 + w_A^T$ and $g_B(w_B) = 8 + 2w_B^T$. In addition, the level of underground economy or tax evasion are normalized to 1 in any coalition with only one country, i.e., without the help of others countries. Thus, $w_i^T = 1$ for any $i \in P$, $T \subset N$ such that $P \cap T \backslash \{i\} = \emptyset$. However, in any coalition $T' \subset N$ such that $A, B \in T'$, $w_A^{T'} = 0.50$ and $w_B^{T'} = 0.60$.

Table 1.2 shows the cost-coalitional vector (columns 2-5) and corresponding cost game (last column); i.e. for any coalition $T \subseteq N$, the cost of each agent c_k^T, and the cost of this coalition $c(T)$

TABLE 1.2: Cost-coalitional vector and cost game

Coalition \ Agent	A	B	1	2	$c(T)$
$\{A\}$	5				5
$\{B\}$		10			10
$\{1\}$			6		6
$\{2\}$				13	13
$\{A, B\}$	4.5	9.2			13.70
$\{A, 1\}$	5		5.80		10.80
$\{A, 2\}$	5			12.50	17.50
$\{B, 1\}$		10	5.40		15.40
$\{B, 2\}$		10		11.80	21.80
$\{1, 2\}$			6	13	19
$\{A, B, 1\}$	4.50	9.20	5.20		18.90
$\{A, B, 2\}$	4.5	9.20		11.30	25
$\{A, 1, 2\}$	5		5.80	12.50	23.30
$\{B, 1, 2\}$		10	5.40	11.80	27.20
$\{A, B, 1, 2\}$	4.50	9.20	5.20	11.30	30.20

From the previous table, it is straightforward to obtain $z_{ii'}$, where $z_{ii'} = c_i^{T \backslash \{i'\}} - c_i^T$ for all $T \subseteq N$ such that $i, i' \in P \cap T$. Therefore, $z_{AB} = 0.50$ and $z_{BA} = 0.80$, i.e., country B reduces the cost of country A in 0.50 and country A reduces the cost of country B in 0.80. Consequently, country A is a net-benefactor with country B, and country B a net-beneficiary with country A.

We can calculate now the Shapley value by using the expressions from Theorem **1.2.** Note that, in this case, we only need the values of Table 1.1, the last row of Table 1.2 (c_A^N, c_B^N, c_1^N and c_2^N), and both values z_{AB} and z_{BA}. Therefore, Theorem **1.2** allows to reduces significantly the amount of information and time to compute Shapley value.

In Table 1.3, it is shown for any agent its individual cost, the cost in the grand coalition, the Shapley value, and the difference between the last two values.

Notice that costs in the grand coalition reduce the costs of each player. Regarding to the cost in the grand coalition, Shapley value decreases the cost

TABLE 1.3: Comparsison among individual costs, cost in the grand coalition and the Shapley value

Agent \ Value	$c(\{k\})$	$\psi_k(N,c)$	$\phi_k(N,c)$	$\psi_k(N,c) - \phi_k(N,c)$
A	5	4.50	4	0.50
B	10	9.20	8.45	0.75
1	6	5.20	5.60	-0.40
2	13	11.30	12.15	-0.85

of benefactors in a half of the benefits that it provided to the beneficiaries. Additionally, it increases the cost of beneficiaries in a half of the benefits that they obtain from benefactors. For example, for country A, $\phi_A(N,c) = c_A^N - \frac{1}{2}\left((S_1^A - \bar{S}_1^A) + (S_2^A - \bar{S}_2^A)\right) + \frac{1}{2}(z_{AB} - z_{BA})$. As $z_{AB} - z_{BA} = -0.30$, country A is a net-benefactor. Thus, Shapley value decreases its cost in a half of this difference. However, for country B, the cost is increased in the same amount because it is a net-beneficiary. In this example, there are only two countries, however, if there were more countries, a given country could be a net benefactor with some countries and a net beneficiary with others, this depends on the sign of $z_{ii'} - z_{i'i}$.

1.6 Conclusions

Corporation tax games were introduced by [7] as an application of linear cost games (see [6]) to a corporate tax reduction system. Motivated by the Spanish tax system, the authors considered that the Government, as benefactor, provided different group investment options which reduced the costs of those investors who acted legally (beneficiaries).

In this chapter, we have presented a new model of cooperation in corporate tax systems with several firms and countries (multiple dual and irreplaceable benefactors). Countries are dual in the sense they are benefactors (they reduce the costs of both firms and others countries), and beneficiaries (its cost is reduced by the information provided by others countries). They are also irreplaceable benefactors because all the members of a coalition may see their cost increase if one of them leaves the group.

The class of TU cooperative games corresponding to this model is called multiple corporation tax games. We have proved that these games are concave, i.e., the marginal contribution of a firm and a country diminishes as a coalition grows (snowball effect). Hence, the grand coalition is stable in the sense of the core. This means that firms have strong incentives to cooperate with the countries instead of being fraudsters. Then, we propose the Shapley value as an easily computable core-allocation that benefits all agents and, in particular, compensates the benefactors for their dual and irreplaceable role.

Our model here, distinguishes two groups of agents: dual benefactors (countries) and beneficiaries (firms), while the original model presented by [6], considered two disjoint groups of agents, benefactors and beneficiaries. A natural extension would be to consider that all agents can be dual (benefactors and beneficiaries). We believe that similar results to those obtained here could be achieved.

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Bibliography

[1] Dell'Anno, R. (2009). Tax evasion, tax morale and policy maker's effectiveness. The Journal of Socio-Economics, 38 (6), 988-997.

[2] Bajada, C., Schneider, F. (2018). Size, Causes and Consequences of the Underground Economy: an international perspective. Routledge.

[3] Feige, E. (2016). Reflections on the Meaning and Measurement of Unobserved Economies: What Do We Really Know About the 'Shadow Economy'. Journal of Tax Administration, 2, 5-41.

[4] González-Díaz, J., Garcí-Jurado, I., Fiestras-Janeiro, M.G. (2010). An introductory course on mathematical game theory. Graduate Studies in Mathematics, vol 115. American Mathematical Society.

[5] Keen, M., Slemrod, J. (2017). Optimal tax administration. Journal of Public Economics, 152, 133-142.

[6] Meca, A., Sosic, G. (2014). Benefactors and Beneficiaries: The Effects of Giving and Receiving on Cost-Coalitional Problems. Production and Operations Management, 23(9), 1549-1560.

[7] Meca, A., Varela-Pena, J. C. (2018). Corporation Tax Games: An Application of Linear Cost Games to Managerial Cost Accounting. In: Game Theory in Management Accounting pp. 361-378. Springer.

[8] Moretti S., Patrone F. (2008). Transversality of the Shapley value. Top 16, 1-41.

[9] Sandmo, A. (2012). An evasive topic: theorizing about the hidden economy. International tax and public finance, 19, 5-24.

[10] Schneider, F., Enste, D. H. (2000). Shadow economies: size, causes, and consequences. Journal of economic literature, 38(1), 77-114.

[11] Shapley, L. S. (1953): A value for n-person games. Contributions to the Theory of Games, 2(28), 307-317.

[12] Shapley L.S. (1967): On balanced sets and cores. Naval Res. Logist. Quart. 14:453-460.

[13] Shapley, L. S. (1971): Cores of convex games. International journal of game theory, 1(1), 11-26.

[14] Slemrod, J., & Yitzhaki, S. (2002). Tax avoidance, evasion, and administration. In Handbook of public economics (Vol. 3, pp. 1423-1470). Elsevier.

[15] Torgler, B. (2011). Tax morale and compliance : review of evidence and case studies for Europe. Policy Research working paper; no. WPS 5922. Washington, DC: World Bank.

[16] Williams, C. C. (2017). Tackling employment in the informal economy: A critical evaluation of the neoliberal policy approach. Economic and Industrial Democracy, 38(1), 145-169.

APPENDIX F

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Efficient effort equilibrium in cooperation with pairwise cost reduction[☆]

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ABSTRACT

There are multiple situations in which bilateral interaction between agents results in considerable cost reductions. The cost reduction that an agent obtains depends on the effort made by other agents. We model this situation as a bi-form game with two states. In the first stage, agents decide how much effort to exert. We model this first stage as a non-cooperative game, in which these efforts will reduce the cost of their partners in the second stage. This second stage is modeled as a cooperative game in which agents reduce each other's costs as a result of cooperation, so that the total reduction in the cost of each agent in a coalition is the sum of the reductions generated by the rest of the members of that coalition. The proposed cost allocation for the cooperative game in the second stage determines the payoff function of the non-cooperative game in the first stage. Based on this model, we explore the costs, benefits, and challenges associated with setting up a pairwise effort situation. We identify a family of cost allocations with weighted pairwise reductions which are always feasible in the cooperative game and contain the Shapley value. We also identify the cost allocation with weighted pairwise reductions that generate an efficient equilibrium effort level.

1. Introduction

The search for greater efficiency, access to new markets and greater competitiveness are some of the main factors that result in inter-organization or inter-corporate cooperation structures. There are different forms of cooperation depending on the degree of integration or interdependence of partners and on the intended goals of agreements. These forms have been widely studied in economic literature (see e.g. Todeva and Knoke [1] for a survey). There is one specific type of cooperation whose properties and characteristics differentiate it from the rest. It can occur between agents that share, for example, resources, knowledge or infrastructure. The common purpose is to obtain individual advantages such as reducing their respective individual costs. The particularity of this form of cooperation lies in the fact that the cost reduction is based on bilateral interactions.

We consider that form of cooperation here in which, given any pair of cooperating agents, one agent reduces the cost of the other by a certain amount which is independent of cooperation with other agents. This means that if there are more agents in the coalition the amount of the cost reduction does not change. This pairwise cost reduction is independent of the coalition to which the pair of agents may belong. Therefore, for any agent, the total cost reduction in any coalition can easily be calculated as the sum of the reductions obtained from each bilateral interaction with the other members of the coalition.

There are several situations where this kind of cooperation with pairwise cost reduction occurs and is profitable, e.g. strategic collaboration agreements between firms to reduce logistical operational costs. The need to increase market share requires logistics firms to expand their radius of action as far as possible. This means major investments in expensive infrastructures at new sites, which increase operational costs. Agreements are established between companies to reduce those costs while maintaining control of their

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respective markets and hindering access by new competitors. They offer the resources held by each firm in its respective area of influence under advantageous conditions. This enables them to expand their operating ranges with significant cost savings. Interactions occur bilaterally, with each company using the resources of the other. These cost reductions are independent of any cost reductions that can also be obtained by interacting with other agents in larger coalitions.

The second situation is that of bilateral free trade agreements between countries. In a globalized economy, free trade agreements are quite common. They facilitate trade in goods and services between countries, reducing trade barriers and consequently the cost of trade. These cost reductions are specific to each pair of countries, and are independent of any other agreements that either may decide to establish with other countries.

A third situation is the sharing of market data. Currently, information on customers and their purchase patterns is vitally important for firms. It enables them to maximize returns on advertising costs and focus on their ideal target markets. Cooperation between firms (usually from complementary sectors) consists of sharing information about their respective customers. This reduces the costs of each of the firms involved. The information that a particular firm provides is specific to it, so the value of the information that it receives from another specific firm is independent of information from other firms. Even if two firms provide information about the same customer, the information itself is different because it describes the purchase of a different good or service. This can increase the value of that particular customer as a target, which again boosts the value of this particular kind of cooperation.

The last situation presented here is that of inter-firm cooperation agreements to reduce costs by increasing the range of firms' respective telecommunication networks. In eminently competitive sectors such as mobile telephony and online services, cooperation between operators has become quite common. For example, they may share the locations of their respective antennas, which enables them to expand the reach of their networks. This means greater benefits thanks to the offering of a broader service, while avoiding the costs that would be entailed by each company installing its own structures. Here again, cost reduction is bilateral when two agents decide to share and use each other's antennas. These cost savings are independent of any collaboration agreements that each firm may have with other agents to share antennas in larger coalitions.

In this kind of cooperation, the cost reduction between agents may be highly asymmetric when they cooperate in pairs. For example, if two agents A and B decide to cooperate, agent A could provide a major reduction for agent B, while the reduction provided in the opposite direction could be more modest. These asymmetries can induce imbalances or discriminations that could jeopardize cooperation. A fair distribution mechanism for the costs generated by cooperation is undoubtedly needed to ensure the stability of any strategic partnership, as Thomson [2] points out.

In addition, it is quite common for this kind of cooperation to require the agents involved to make a set level of effort. It is natural to think that the amount by which one agent can reduce the costs of the other (if they decide to cooperate) could depend on the effort that the agent exerts. For example, if one country can obtain information relevant to another (e.g. information on tax evasion and the flight of capital involving its citizens), the amount and quality of the specific information may depend on the effort exerted by the first country in gathering it. This extends the situation beyond a cooperative model. For this reason, we model the sequence of decisions as a bi-form game ([3]). In the first stage of the bi-form game, agents decide how much (costly) effort they are willing to exert. This has a direct impact on their pairwise cost reductions. This first stage is modeled as a non-cooperative game in which agents determine the level of pairwise effort to reduce the costs of their partners. In the second stage, agents engage in bilateral interactions with multiple independent partners where the cost reduction brought by each agent to another agent is independent of any possible coalition. We study this bilateral cooperation in the second stage as a cooperative game in which cooperation leads agents to reduce their respective costs, so that the total reduction in costs for each agent in a coalition is the sum of the reductions generated by the rest of the members of that coalition. In the non-cooperative game of the first stage, the agents anticipate the cost allocation that will result from the cooperative game in the second stage by incorporating the effect of the effort made into their cost functions. Based on this model, we explore costs, benefits, and challenges associated with setting up a pairwise effort situation.

We investigate the impact of pairwise efforts on cost reductions and the resulting cost structure for this framework. In particular, we explore the design of a cost-allocation mechanism that efficiently allocates the gains from pairwise effort to all parties. To that end, we first compute the optimal level of cost reduction, taking into account the pairwise cost reductions collectively accrued by all agents. An ideal allocation scheme should encourage agents to participate in it and, at the same time, establish proper incentives to make efforts prior to cooperation. Specifically, we first show that it is profitable for all agents to participate in a pairwise effort situation. Then we study how the total reduction in costs should be allocated to the participants in such a situation. We do this by modeling the pairwise cost reduction between agents that takes place in the second stage as a cooperative game, which we refer to as the pairwise effort game or "PE-game".

We prove that the marginal contribution of an agent diminishes as a coalition grows in PE-games (i.e. they are concave games) and thus all-included cooperation is feasible, in the sense that there are possible cost reductions that make all agents better off or, at least, not worse off (i.e. PE-games are balanced, which means that the core is not empty). This all-included cooperation is also consistent (i.e. PE-games are totally balanced, which means the core of every subgame is non-empty). We identify various allocation mechanisms that enable all-included cooperation to be feasible (i.e. allocation mechanisms that belong to the core of PE-games). In particular, we discuss a family of cost allocations with weighted pairwise reduction which is always a subset of the core of PE-games. This is a broad family of core-allocations which includes the Shapley value, which is obtained when all the weights work out to a half. We provide a highly intuitive, simple expression for the Shapley value, which matches the Nucleolus in our model. To select one of these core-allocations in the second stage, we take into account the incentives that it generates in the efforts made by agents, and consequently in the aggregate cost of a coalition. We show that the Shapley value can induce inefficient effort strategies in equilibrium in the non-cooperative model. However, it is always possible to find core-allocations with weighted pairwise reductions that create appropriate incentives for agents to make optimal efforts that minimize aggregate costs, i.e. core-allocations that generate an efficient level of effort in equilibrium.

This paper contributes to the literature by presenting a doubly robust cost sharing mechanism. That mechanism not only has good properties regarding the cooperative game in the second stage but also creates appropriate incentives in the non-cooperative game in the first stage that enable efficiency to be achieved.

Cooperative game theory has developed a substantial mathematical framework for identifying and providing suitable cost sharing allocations (see, e.g., [4–6] for a survey). Multiple solutions have been proposed from a wide range of approaches (see, e.g., [7–16]). The Shapley value ([17]) is considered one of the most outstanding of them, and a suitable solution concept (see, e.g., [18,19] for a survey). As an allocation rule it has very good properties, such as efficiency, proportionality, and individual and coalitional rationality. However, it has the disadvantage of posing computational difficulties, which increase as the number of players increases. Nonetheless, there is a large body of literature in which the Shapley value is proposed as a simple, easy-to-apply solution in different economic scenarios (see, e.g., [20–25]). These papers give simplified solutions for different classes of games. They take the cost structure as given and do not consider the system externalities that arise when agents make efforts to give and receive cost reductions. Our paper here incorporates both the non-cooperative aspects of making efficient efforts (by modeling decisions related to pairwise cost reductions) and the cooperative nature of giving and receiving cost reductions in pairwise effort situations.

As in principal-agent literature, we refer to action by agents as "effort". In this setting, the concept of "effort" is widely used in analyzing different kinds of problem. One of the first was the moral hazard problems: See for example [26]. Other examples are Holmstrom [27] and Dewatripont et al. [28], who identify conditions under which more information can induce an agent to make less effort. The approach in our model is quite different, in that we do not consider any kind of principal. As far as we know, our model is novel in that it analyzes the incentive for agents to make efforts in a bi-form game: A non-cooperative stage where agents choose how much effort to make and a cooperative second stage. As mentioned, we show that the solution of the cooperative game determines the incentives of agents to make an effort in the first stage, and consequently the efficiency of the final outcome.

In [29], it is also used a bi-form model to analyze the role of process improvement in a decentralized assembly system in which an assembler lays in components from several suppliers. The assembler faces a deterministic demand and suppliers incur variable inventory costs and fixed production setup costs. In the first stage of the game suppliers invest in process improvement activities to reduce their fixed production costs. Upon establishing a relationship with suppliers, the assembler sets up a knowledge sharing network which is modeled as a cooperative game between suppliers in which all suppliers obtain reductions in their fixed costs. They compare two classes of allocation mechanism: Altruistic allocation enables non-efficient suppliers to keep the full benefits of the cost reductions achieved due to learning from the efficient supplier. The Tute allocation mechanism benefits a supplier by transferring the incremental benefit generated by including an efficient supplier in the network. They find that the system-optimal level of cost reduction is achieved under the Tute allocation rule. Our bi-form game is novel in terms of incentive for efforts by agents and is also richer in results: We find the allocation rule that generates the unique efficient effort in equilibrium in cooperation with pairwise cost reduction.

The paper is organized as follows. Section 2 presents the bi-form game and describes in detail the two stages in which the model is developed. Section 3 is devoted to analyzing the second stage which is a cooperative game. In this cooperative game, agents reduce each other's costs as a result of cooperation, so that the total reduction in the cost of each agent in a coalition is the sum of the reductions generated by the rest of the members of that coalition. In Section 4 the first stage is studied, that is the non-cooperative game that precedes the cooperative game in the second stage. Here, the agents anticipate the cost allocation that results from the cooperative game in the second stage by incorporating the effect of the effort exerted into their cost functions. We consider a family of cost allocation rules (in the second state) with pairwise reductions weighted separately (WPR family) and obtain the corresponding effort equilibria in the first state. Then, we develop a general and complete analysis of the efficient effort equilibria. Finally, in this section, we found the core-allocation rule in this WPR family that generates the unique efficient effort equilibria. Section 5 focuses on a subfamily of the WPR family in which pairwise reductions are not weighted separately, but are weighted as aggregated reduction, this is the WPAR family. We find out that the level of efficiency is lower than that attained when the pairwise reductions are weighted separately for each agent. Then, we found the rule, within this WPAR family, that generates the equilibrium efforts closest to the efficient ones. Finally, Section 6 completes the study of our model by comparing the two families of core-allocation analyzed. We complete the paper with a section of conclusions and four appendices containing the proofs of the results and tables of summaries (notation and optimization problems).

2. Model

We consider a model with a finite set of agents $N = \{1, 2, .n\}$, where each agent has a good (for example resources, knowledge or infrastructure) and has to perform a certain activity. The total cost of an agent's activity can be reduced if it cooperates with another agent, which means that the two agents share their goods. These cost reductions obtained by sharing goods in pairs depend on the effort made previously by each agent. Our model consists of two different stages. In the first stage, agents choose their effort levels as in a non-cooperative game. In the second stage, agents cooperate to reduce their costs, and allocate the minimum cost they achieve by pairwise cost reductions as in a cooperative game. The proposed cost allocation for the cooperative game in the second stage determines the payoff function of the non-cooperative game in the first stage. Therefore, we model the sequence of decisions as a bi-form game ([3]). The two stages of the model are described in detail below.

First Stage (non-cooperative game): Each agent $i \in N$ chooses in this state an effort level $e_i = (e_{i1}, \ldots, e_{i(i-1)}, e_{i(i+1)}, \ldots e_{in}) \in [0,1]^{n-1}$, where $e_{ij} \in [0,1]$ stands for the level of effort by agent i to reduce the cost of agent j if they cooperate in the second stage. These efforts have a cost $c_i(e_i) \in \mathbb{R}_+$ for any $i \in N$. We assume that $c_i(.) : [0,1]^{n-1} \to \mathbb{R}_+$ is a scalar field of class $C^2([0,1]^{n-1})$.[1] Moreover, for all $e_{ij} \in [0,1]$ with $j \in N\backslash\{i\}$, it is assumed that $\frac{\partial c_i(e_i)}{\partial e_{ij}} > 0$, $\frac{\partial^2 c_i(e_i)}{\partial e_{ij}^2} > 0$, and $\frac{\partial^2 c_i(e_i)}{\partial e_{ij}\partial e_{ih}} = 0$ for all $h \neq i, j$, which implies that the marginal cost $\frac{\partial c_i(e_i)}{\partial e_{ij}}$ is independent of the effort that i exerts with agents other than j.[2]

Second Stage (cooperative game): Given the effort made in the first stage, agents cooperate, so for any pair of cooperating agents $i, j \in N$ and a given effort e_{ij}, agent i reduces the total cost of agent j by an amount $r_{ji}(e_{ij}) \in R_+$, and vice versa. These particular reductions between agents $i, j \in N$ are independent of cooperation with other agents. We also assume for all $j \in N\backslash\{i\}$ that function $r_{ij}(.) : [0,1] \to R_+$ is class C^2, increasing and concave[3] at [0,1]. Thus, these agents participate in bilateral interactions with multiple independent partners whose cost reductions are coalitionally independent, i.e. the cost reduction given by each agent to another agent is independent of any possible coalition. This means that the total reduction in cost for each agent in a coalition $S \subset N$ is the sum of the pairwise cost reductions given to that agent by the rest

[1] A scalar field is said to be class C^2 at $[0,1]^{n-1}$ if its 2-partial derivatives exist at all points of $[0,1]^{n-1}$ and are continuous.

[2] This last assumption implies that the Hessian matrix is a diagonal matrix. In addition, note that, given our assumptions about c_i, w.l.o.g. we could consider that $c_i(e_i) = \sum_{j \in N\backslash\{i\}} c_{ij}(e_{ij})$ where $c_{ij}(.) : [0,1] \to \mathbb{R}_+$. We omit it from the paper so as not to introduce more notation into the model.

[3] $\partial r_{ji}(e_{ij})/\delta e_{ij} > 0$ (increasing) and $\partial^2 r_{ji}(e_{ij})/\delta e_{ij}^2 < 0$ (concave).

of the members of the coalition, i.e. for agent i, it is $\sum_{j\in S\setminus\{i\}} r_{ij}(e_{ji})$. We assume perfect information regarding agents' costs and cost reductions depending on efforts.

Given an effort profile $e=(e_1,e_2,\ldots,e_n)\in[0,1]^{n(n-1)}$ in the first stage, the second stage can be seen as a cooperative game, more specifically a transferable utility cost game (N,e,c), where N is the finite set of players, and $c:2^N\to R$ is the so-called characteristic function of the game, which assigns to each subset $S\subseteq N$ the cost $c(S)$ that is incurred if agents in S cooperate. By convention, $c(\emptyset)=0$. The cost of agent i in coalition $S\subseteq N$ is given by $c^S(i):=c_i(e_i)-\sum_{j\in S\setminus\{i\}} r_{ij}(e_{ji})$. This cost can be interpreted as the reduced cost of agent i in coalition S. Note that the larger the coalition, the greater the cost reduction it achieves, i.e. for all $i\in S\subseteq T\subseteq N,\ c^T(\{i\})\leq c^S(\{i\})$. The total reduced cost for coalition S is given by

$$c(S):=\sum_{i\in S} c^S(\{i\})=\sum_{i\in S}[c_i(e_i)-\sum_{j\in S\setminus\{i\}} r_{ij}(e_{ji})]. \tag{1}$$

When all agents cooperate, they form what is called the grand coalition, which is denoted by N. Thus, $c(N)$ is the aggregate cost of the grand coalition. The allocation of the grand coalition cost achieved through cooperation, in the second stage, assigns a reduced final cost to each agent, that is, $\psi_i(e)$, for all $i\in N$, where $\psi_i:E\to R$ with $E:=\prod_{i\in N}E_i$ and $E_i:=[0,1]^{n-1}$. Then, we define the cost allocation rule $\psi:E\to R^n$ s.t. $\psi(e)=(\psi_i(e))_{i\in N}$ and $\sum_{i\in N}\psi_i(e)=c(N)$.

The non-cooperative cost game in the first stage is defined through that cost allocation rule ψ by $(N,\{E_i\}_{i\in N},\{\psi_i\}_{i\in N})$, where E_i is the strategy space of agent $i\in N$ (its effort level space), and ψ_i is the payoff function of agent i, but in this case is a cost function. Hence, for an effort profile $e\in E$, the corresponding cost function is $\psi(e)$. That effort is made in anticipation of the result of the cooperative cost game that follows in the second stage. Therefore, we first analyze the second stage (see Section 3), and focus on different ways of allocating the grand coalition cost. We determine cost allocation rules with good computability properties and coalitional stability for this cooperative cost game. Notice that a given cost allocation rule will generate precise incentives in the first state and consequently particular equilibrium effort strategies[4] In turn, these particular effort strategies will have an associate cost of the grand coalition. At this point, a question about efficiency arises. The lower the cost of the grand coalition generated in equilibrium is, the more efficient the equilibrium effort strategies and the allocation rule considered will be.

Therefore, there are two dimensions to be considered. First, the cost allocation rule for the cooperative game should have good properties (computability and coalitional stability). Second, the allocation rule must induce the right incentives in the non-cooperative game to obtain the lowest cost of the grand coalition. This interesting, relevant question is analyzed in Section 4 and 5.

Throughout the paper, we consider the following assumptions:

(CA) Cost assumptions: $c_i\in C^2$, and $\frac{\partial c_i(e_i)}{\partial e_{ij}}>0$ (increasing), $\frac{\partial^2 c_i(e_i)}{\partial e_{ij}^2}>0$ (convex), and $\frac{\partial^2 c_i(e_i)}{\partial e_{ij}\partial e_{iK}}=0$, if $k\neq j$ (additively separable).

(RA) Reduction assumptions: $r_{ji}\in C^2$, and $\partial r_{ji}(e_{ij})/\delta e_{ij}>0$ (increasing), $\partial^2 r_{ji}(e_{ij})/\delta e_{ij}^2<0$ (concave).

A summary of the notation and the main optimization problems (Tables 1 and 2) can be found in Appendix D.

[4] An effort strategy profile is said to be in equilibrium when each agent has nothing to gain by changing only their own effort strategy given the strategies of all the other agents (Nash equilibrium).

3. Cooperation with pairwise cost reduction

This section presents the analysis of cooperation with pairwise cost reduction in the second stage. Agents make their efforts in pairwise sharing in the first stage, and initiate cooperation with efforts $e=(e_1,\ldots,e_i,\ldots,e_n)$. We model the PE-game as a multiple-agent cooperative game where each agent i incurs an initial cost of $c_i(e_i)$. All agents in a pairwise effort group (coalition) give cost reductions to and receive such reductions from other agents. As a result, all agents in the coalition reduce their initial costs to levels that depend on the efforts made in the first stage by the others. No agent outside the pairwise effort situation benefits from this pairwise cost reduction opportunity. We introduce all the game-theoretic concepts used in this paper, but readers are referred to [30] for more details on cooperative and non-cooperative games.

We refer to the pairwise effort situation as a PE-situation and denote it by the tuple $(N,e,\{c_i(e_i),\{r_{ji}(e_{ij})\}_{j\in N\setminus\{i\}}\}_{i\in N})$. We associate a cost game (N,e,c) with each PE-situation as defined by (1).

The class of PE-games has some similarities with the class of linear cost games introduced in [31]. They define the concept of cost-coalitional vectors as a collection of certain a priori information, available in the cooperative model, represented by the costs of the agents in all possible coalitions to which they could belong. The cost of a coalition in their study is thus the sum of the costs of all agents in that coalition. However, the PE-games considered here are significantly different from their linear cost games. Linear cost games focus on the role played by benefactors (giving) and beneficiaries (receiving) as two groups of disjoint agents, but PE-games consider that all agents could be dual benefactors, in the sense that they be benefactors and beneficiaries at the same time. In addition, PE-games are based on a bilateral cooperation between agents that enables both to reduce their costs but is coalitionally independent.

We now consider a PE-situation $(N,e,\{c_i(e_i),\{r_{ij}(e_{ij})\}_{j\in N\setminus\{i\}}\}_{i\in N})$ and consider whether it is profitable for the agents in N to form the grand coalition to obtain a significant reduction in costs. We find that the answer is yes, and show that the associated PE-game (N,e,c) is concave, in the sense that for all $i\in N$ and all $S,T\subseteq N$ such that $S\subseteq T\subset N$ with $i\in S$, so $c(S)-c(S\setminus\{i\})\geq c(T)-c(T\setminus\{i\})$. This concavity property provides additional information about the game: the marginal contribution of an agent diminishes as a coalition grows. This is well-known and is called the "snowball effect".

The first result in this section shows that PE-games are always concave. This means that the grand coalition can obtain a significant reduction in costs. In that case, the reduced total cost is given by $c(N)=\sum_{i\in N}c_i(e_i)-R(N)$, where $R(N)=\sum_{i\in N}\sum_{j\in N\setminus\{i\}} r_{ij}(e_{ji})$ is the total reduction produced by bilateral reductions between all agents in the situation, which turns out to be the total cost savings for all agents. The proof of Proposition 1, together with all our other proofs for this section, is shown in Appendix A.

Proposition 1. *Every PE-game is concave.*

An allocation rule for PE-games is a map ψ which assigns a vector $\psi(e)\in R^n$ to every (N,e,c), satisfying efficiency, that is, $\sum_{i\in N}\psi_i(e)=c(N)$. Each component $\psi_i(e)$ indicates the cost allocated to $i\in N$, so an allocation rule for PE-games is a procedure for allocating the reduced total to all the agents in N when they cooperate. It is a well-known result in cooperative game theory that concave games are totally balanced: The core of a concave game is non-empty, and since any subgame of a concave game is concave, the core of any subgame is also non-empty. That means that coalitionally stable allocation rules can always be found for PE-games. We interpret a non-empty core for PE-games as indicating a setting where all included cooperation is feasible, in the sense that there are possible cost reductions that make all agents better off (or, at least, not worse off). The totally balanced property suggests that this all-included cooperation is consistent, i.e. for every group of agents whole-group cooperation is also feasible.

A highly natural allocation rule for PE-games is $\varphi_i(e) = c^N(\{i\}) = c_i(e_i) - R_i(N)$, for all $i \in N$, with $R_i(N) = \sum_{j\in N\setminus\{i\}} r_{ij}(e_{ji})$ being the total reduction received by agent $i \in N$ from the rest of the agents $j \in N\setminus\{i\}$. It has good properties at least with respect to computability and coalitional stability in the sense of the core. Formally, the core of a PE-game (N, c) is defined as follows

$$Core(N,c) = \{x \in \mathbb{R}^n / \sum_{i\in N} x_i = c(N), \sum_{i\in S} x_i \leq c(S) \forall S \subseteq N\}. \tag{2}$$

Notice that $\varphi(e) \in Core(N,c)$. Indeed, $\sum_{i\in N} \varphi_i(e) \leq c(N)$ and for every $S \subseteq N, \sum_{i\in S} \varphi_i(e) = \sum_{i\in S} c^N(i) \leq \sum_{i\in S} c^S(i) = c(S)$. Nevertheless, the agents could argue that this allocation does not provide sufficient compensation for their dual role of giving and receiving. Note that the allocation only considers their role as receivers.

PE-games are concave, so cooperative game theory provides allocation rules for them with good properties, at least with respect to coalitional stability and acceptability of items. We highlight the Shapley value (see [17]), which assigns a unique allocation (among the agents) of a total surplus generated by the grand coalition. It measures how important each agent is to the overall cooperation, and what cost can it reasonably expect. The Shapley value of a concave game is the center of gravity of its core (see [32]). In general, this allocation becomes harder to compute as the number of agents increases. We present a simple expression here for the Shapley value of PE-games that takes into account all bilateral relations between agents and compensates them for their dual role of giving and receiving.

Given a general cost game (N, c), we denote the Shapley value by $\phi(c)$, where the corresponding cost allocation for each agent $i \in N$, is

$$\phi_i(c) = \sum_{i\in T\subseteq N} \frac{(n-t)!(t-1)!}{n!}[(c(T) - c(T\setminus\{i\})], \text{ with } |T| = t. \tag{3}$$

The Shapley value has many desirable properties, and it is also the only allocation rule that satisfies a certain subset of those properties (see [33]). For example, it is the only allocation rule that satisfies the four properties of Efficiency, Equal treatment of equals, Linearity and Null player ([17]).

Given a PE-game (N, e, c), we denote by $\phi(e)$ the Shapley value of the cost game. The following Theorem shows that the Shapley value provides an acceptable allocation for PE-games. Indeed, it reduces the individual cost of an agent by the average of the total reduction that it obtains from the others $(R_i(N))$ plus half of the total reduction that it provides to the rest of the agents, i.e. $G_i(N) = \sum_{j\in N\setminus\{i\}} r_{ji}(e_{ij})$.

Theorem 1. *Let (N, e, c) be a PE-game. For each agent $k \in N$, $\phi_k(e) = c_k(e_k) - \frac{1}{2}[R_k(N) + G_k(N)]$.*

From Theorem 1 it can be derived that the Shapley value, $\phi(e)$, considers the dual role of giving and receiving of all agents, and the final effect on those agents depends on which role is stronger. As mentioned above, if an allocation does not compensate them for their dual role of giving and receiving, and it only considers their role as receivers, as the individual cost in the grand coalition, $\varphi(e)$, does, the cooperation would not be desirable for those dual agents. This non-acceptability can be avoided by using the Shapley value, which also coincides with the Nucleolus ([34]) for PE-games.

The nucleolus selects the allocation in which the coalition with the smallest excess (the worst treated) has the highest possible excess. The nucleolus maximizes the "welfare" of the worst treated coalitions. Denote by $\nu(e) \in R^n$ the *Nucleolus* of the PE-game (N, e, c), associated with a PE-situation $(N, e, \{c_i(e_i), \{r_{ij}(e_{ij})\}_{j\in N\setminus\{i\}}\}_{i\in N})$. First, we define the excess of coalition S in (N, e, c) with respect to allocation x as $d(S, x) = c(S) - \sum_{i\in S} x_i$. This excess can be considered as an index of the "welfare" of coalition S at x: The greater $d(S, x)$, the better coalition S is at x. Let $d^*(x)$ be the vector of the 2^n excesses arranged in (weakly) increasing order, i.e., $d^*_i(x) \leq d^*_j(x)$ for all $i < j$. Second, we define the lexicographical order $\succ_l$. For any $x, y \in R^n$, $x \succ_l y$ if and only if there is an index k such that for any $i < k$, $x_i = y_i$ and $x_k > y_k$. The nucleolus of the PE-game (N, e, c) is the set

$$\nu(e) = \{x \in X : d^*(x) \succ_l d^*(y) \text{ for all } y \in X\} \tag{4}$$

with $X = \{x \in R^n : \sum_{i\in N} x_i = c(N), x_i \geq c(\{i\}) \text{ for all } i \in N\}$.

It is well known that the Nucleolus is a singleton for balanced games and that it is always a core-allocation.

The Proposition 2 proves that for PE-games the Shapley value matches the Nucleolus. This is a very good property that few cost games satisfy.

Proposition 2. *Let (N, e, c) be a PE-game. For each agent $k \in N$, $\nu_k(e) = \phi_k(e)$.*

Therefore, given an effort profile, the Shapley value is a very suitable way of allocating the reduced cost due to cooperation. Note that, the cost reduction as a result of cooperation between any pair of agents $i, j \in N$ is $r_{ij}(e_{ji}) + r_{ji}(e_{ij})$, and the Shapley value assigns one half of this amount to i and the other half to j. This seems a reasonable way to split this aggregate cost reduction. However, if agents knew before choosing their levels of efforts that the cost reductions resulting from their efforts were going to be allocated according to the Shapley value, the incentives created could generate inefficiencies. Some agents could find it optimal to exert too little effort and in some situations this could be inefficient.

For example, consider a PE-situation in which one agent has the ability to produce a substantial reduction in costs for other agents with a low effort cost and the rest of the agents have almost no ability to reduce costs for others even with a high effort cost. If the Shapley value is used as the allocation rule for this game, agents may not have incentives to make any level of effort. Note that in the first step agents have to decide how much effort to make. However, if the Shapley value is modified to give a greater portion of the pairwise cost reduction to the especially productive agent, it might make more effort and thus produce a greater reduction in cost for other agents. This change in the Shapley value generates new allocation rules, which can reduce the cost of the grand coalition regarding the Shapley allocation. The following example with three agents illustrates these ideas.

Example 1. Consider a pairwise inter-organizational situation with three firms, i.e. $N = \{1, 2, 3\}$. For any effort profile $e \in [0, 1]^6$, the PE-situation is given by the following initial costs,

$c_1(e_{12}, e_{13}) = 100 + 100e_{12} + 4e_{12}^2 + 100e_{13} + 4e_{13}^2$
$c_2(e_{21}, e_{23}) = 100 + 100e_{21} + 4e_{21}^2 + 100e_{23} + 4e_{23}^2$
$c_3(e_{31}, e_{32}) = 100 + 100e_{31} + 4e_{31}^2 + 100e_{32} + 4e_{32}^2$

and the following pairwise reduced costs, all of them in thousands of Euros,

$r_{i1}(e_{1i}) = 2 + 200e_{1i} - 3e_{1i}^2$ with $i = 2, 3$
$r_{i2}(e_{2i}) = 2 + 3e_{2i} - e_{2i}^2$ with $i = 1, 3$
$r_{i3}(e_{3i}) = 2 + 3e_{3i} - e_{3i}^2$ with $i = 1, 2$

If the allocation rule in the second stage is the Shapley value, the firms choose their levels of effort according to this cost allocation function. It is straight forward to show that in this case the unique effort equilibrium e^*, is one in which the three firms make no effort, i.e. $e^*_{ij} = 0$ for $i, j \in N$.[5] Thus, the Shapley value distributes the cost of the grand coalition $c^*(N) = 288$ equally, i.e. for each firm $i = 1, 2, 3$, $\phi_i(e^*) = c_i(e^*_i) - \frac{1}{2}\sum_{j\in N\setminus\{i\}}[r_{ij}(e^*_{ji}) + r_{ji}(e^*_{ij})] = 100 - \frac{1}{2}((2+2) + (2+2)) = 96$.

Note that, for example, in the relationship between firm 1 and 2, the pairwise cost reduction is $r_{12}(e_{21}) + r_{21}(e_{12})$, and the Shapley value gives $\frac{1}{2}$ of this amount to firm 1 and the other $\frac{1}{2}$ to firm 2. However, if the proportion that firm 1 obtains is increased, e.g. from $\frac{1}{2}$ to $\frac{3}{4}$, and the part for firm 2 is thus reduced to $\frac{1}{4}$, the incentive of firm 1 to make an effort can be increased. The same goes for firms 1 and 3 so that the incentive of firm 1 to make an effort for firm 3 is also increased. These changes in the Shapley value lead to a new allocation rule which we denote by $\Omega(e) = (\Omega_1(e), \Omega_2(e), \Omega_3(e))$ for any effort profile $e \in [0, 1]^6$. With this new allocation rule, the equilibrium efforts are zero for firms 2 and 3, and one for firm 1. That is, $e^{**}_{1j} = 1$, for $j = 2, 3$, $e^{**}_{2j} = 0$, for $j = 1, 3$, and $e^{**}_{3j} = 0$, for $j = 1, 2$. In this case, the grand coalition cost $c^{**}(N) = 102$ is allocated equally between firms 2 and 3, and the rest to firm 1. That is, $\Omega_i(e^{**}) = 100 - \frac{1}{4}[(2+200-3)+2] - \frac{1}{2}(2+2) = 47,75$ for $i = 2, 3$, and $\Omega_1(e^{**}) = 100 + 100 + 4 + 100 + 4 - \frac{3}{4}[(2 + (2+200-3)) + (2 + (2+200-3))] = 6,5$.

Hence, the new allocation rule $\Omega(e^{**})$ greatly reduces the grand coalition cost (by 136.000 Euros) as well as the costs of each firm; i.e. a reduction of 89.500 Euros for firm 1 and 23.250 Euros for firms 2 and 3. In relative terms, with the Shapley value each company pays 33.33% of the total cost. However, with the modified Shapley value agent 1 only pays 4.4% of the total cost, while agents 2 and 3 pay 47.8% each. Therefore, the modified Shapley value generates a more efficient outcome in the sense that it creates more appropriate incentives for firms.

To reach efficient effort strategies in equilibrium (henceforth EEE) in the first stage, we consider a new family of allocation rules, for PE-games (second stage), based on the Shapley value. This family consists of the rules $\Omega(e) \in R^n$, where for all $i \in N$,

$$\Omega_i(e) = c_i(e_i) - \sum_{j\in N\setminus\{i\}}[\omega^i_{ij} r_{ij}(e_{ji}) + \omega^i_{ji} r_{ji}(e_{ij})],$$

with $\omega^i_{ij}, \omega^i_{ji} \in [0, 1]$, for all $j \in N\setminus\{i\}$, such that $\omega^i_{ij} = 1 - \omega^j_{ij}$ and $\omega^i_{ji} = 1 - \omega^j_{ji}$. The Shapley value is a particular case of this family of rules in which $\omega^i_{ij} = \omega^i_{ji} = \frac{1}{2}$, for all $i \in N$ and all $j \in N\setminus\{i\}$. This family of cost allocation for PE-games is referred to as *cost allocation with weighted pairwise reduction*.

The Theorem below shows that the family of cost allocations with weighted pairwise reduction is always a subset of the core of PE-games. This property identifies a wide subset of the large core of PE-games, including the Shapley value (and thus the Nucleolus).

Theorem 2. *Let (N, e, c) be a PE-game. For every family of weights $\omega^i_{ij}, \omega^i_{ji} \in [0, 1]$, $i, j \in N, i \neq j$, such that $\omega^i_{ij} = 1 - \omega^j_{ij}$ and $\omega^i_{ji} = 1 - \omega^j_{ji}$, $\Omega(e)$ belongs to the core of (N, e, c).*

Now a complete analysis of the EEE for cooperation in pairwise cost reduction can be conducted.

[5] Theorem 3, in Section 4, shows the efforts of equilibrium in the non-cooperative game in the general case.

4. Efficiency, equilibrium strategies, and optimal rule

We first define an *efficient effort profile* as the effort profile that minimizes the cost of the grand coalition, $c(N) = \sum_{i\in N}[c_i(e_i) - \sum_{j\in N\setminus\{i\}} r_{ij}(e_{ji})]$.

Definition 1. An effort profile $\tilde{e} = (\tilde{e}_1, \ldots, \tilde{e}_i, \ldots, \tilde{e}_n)$ with $\tilde{e}_i = (\tilde{e}_{i1}, \ldots, \tilde{e}_{i(i-1)}, \tilde{e}_{i(i+1)}, \ldots \tilde{e}_{in}) \in [0, 1]^{n-1}$ is efficient if $\tilde{e} = \arg\min_{e\in[0,1]^{n(n-1)}} \sum_{i\in N}[c_i(e_i) - \sum_{j\in N\setminus\{i\}} r_{ij}(e_{ji})]$

An efficient effort profile $\tilde{e}$ is well defined because $c(N)$ as a function of e is strictly convex in e_{ij} for all $i, j \in N, i \neq j$.[6]

The following proposition shows that the effort e_{ij} is efficient if the marginal cost of that effort equals the marginal reduction that this effort generates; otherwise, the effort is zero or one. The proof of Proposition 3 appears in Appendix B, together with those of all the other proofs in this section.

Proposition 3. *There exists a unique efficient effort profile $\tilde{e} = (\tilde{e}_1, \ldots, \tilde{e}_i, \ldots, \tilde{e}_n)$ with $\tilde{e}_i = (\tilde{e}_{i1}, \ldots, \tilde{e}_{i(i-1)}, \tilde{e}_{i(i+1)}, \ldots \tilde{e}_{in}) \in [0, 1]^{n-1}$, such that*

- *$\tilde{e}_{ij} = 0$ if $\frac{\partial c_i(e_i)}{\partial e_{ij}} > \frac{\partial r_{ji}(e_{ij})}{\partial e_{ij}}$ for all $e_{ij} \in [0, 1]$,*
- *$\tilde{e}_{ij} = 1$ if $\frac{\partial c_i(e_i)}{\partial e_{ij}} < \frac{\partial r_{ji}(e_{ij})}{\partial e_{ij}}$ for all $e_{ij} \in [0, 1]$,*
- *$\tilde{e}_{ij} \in (0, 1)$ is the unique solution of $\left.\frac{\partial c_i(e_i)}{\partial e_{ij}}\right|_{e_{ij}=\tilde{e}_{ij}} = \left.\frac{\partial r_{ji}(e_{ij})}{\partial e_{ij}}\right|_{e_{ij}=\tilde{e}_{ij}}$, otherwise.*

We now focus on the non-cooperative effort game that arises under the family of *cost allocation with weighted pairwise reduction* (henceforth, WPR family). Then we analyze efficiency in equilibrium.

Consider the WPR family, i.e., $\Omega_i(e) = c_i(e_i) - \sum_{j\in N\setminus\{i\}}[\omega^i_{ij} r_{ij}(e_{ji}) + \omega^i_{ji} r_{ji}(e_{ij})]$ for all $i \in N$ with $\omega^i_{ij}, \omega^i_{ji} \in [0, 1]$, $i, j \in N, i \neq j$, such that $\omega^i_{ij} = 1 - \omega^j_{ij}$ and $\omega^i_{ji} = 1 - \omega^j_{ji}$. For each specification of these weights, a particular allocation rule can be obtained that induces a certain equilibrium effort strategy in the first stage, which in turn generates the associated cost allocation in equilibrium. The aim of this section is twofold. First, we identify the efficient allocation rule within the WPR family, i.e., that which results in the lowest cost of the grand coalition. Second, we show that the effort profile induced in equilibrium by this allocation rule coincides with the efficient effort profile of Proposition 3.

The non-cooperative cost game associated with $\Omega = (\Omega_i)_{i\in N}$ in the first stage is defined by $(N, \{E_i\}_{i\in N}, \{\Omega_i\}_{i\in N})$, where for every agent $i \in N$, $E_i := [0, 1]^{n-1}$ is the players' i strategy set, and for all effort profiles $e \in E := \prod_{i\in N} E_i$, and Ω_i is the cost function for agent $i \in N$. We call this an effort game.

In this game, we use the following definition of equilibrium.

Definition 2. The effort profile $e^* = (e^*_1, \ldots, e^*_n) \in E$ is an equilibrium for the game $(N, \{E_i\}_{i\in N}, \{\Omega_i\}_{i\in N})$ if e^*_i is the optimal effort for agent $i \in N$ given the strategies of all the other agents $j \in N\setminus\{i\}$.

First, note that the optimal effort for agent $i \in N$ given the strategies of all the other agents $j \in N\setminus\{i\}$ is the effort e_i that minimizes $\Omega_i(e_i, e_{-i})$. Note that the function $\Omega_i(e_i, e_{-i})$ is strictly convex in the effort e_{ij} that agent i exerts for any $j \in N\setminus\{i\}$.[7] This means that for agent i there is a unique optimal level of effort $\hat{e}_{ij}$

[6] Note that the second derivative in e_{ij} is equal to $\frac{\partial^2 c_i(e_i)}{\partial e^2_{ij}} - \frac{\partial^2 r_{ji}(e_{ij})}{\partial e^2_{ij}}$, which is always positive because $\frac{\partial^2 c_i(e_i)}{\partial e^2_{ij}} > 0$ and $\frac{\partial^2 r_{ji}(e_{ij})}{\partial e^2_{ij}} < 0$.

[7] Note that $\frac{\partial \Omega_i(e)}{\partial e_{ij}} = \frac{\partial c_i(e_i)}{\partial e_{ij}} - \omega^i_{ji}\frac{\partial r_{ji}(e_{ij})}{\partial e_{ij}}$ and $\frac{\partial^2 \Omega(e)}{\partial e^2_{ij}} = \frac{\partial^2 c_i(e_i)}{\partial e^2_{ij}} - \omega^i_{ji}\frac{\partial^2 r_{ji}(e_{ij})}{\partial e^2_{ij}} > 0$ because, as assumed above, $\frac{\partial^2 c_i(e_i)}{\partial e^2_{ij}} > 0$ and $\frac{\partial^2 r_{ji}(e_{ij})}{\partial e^2_{ij}} < 0$

for each $j \in N\backslash\{i\}$. That optimal level $\hat{e}_{ij}$ depends on the parameter ω^i_{ji}, on the marginal cost of agent i in regard to effort $\hat{e}_{ij}$ (i.e. $\frac{\partial c_i(e_i)}{\partial e_{ij}}$), and on the marginal cost-reduction for agent j in regard to effort $\hat{e}_{ij}$, (i.e. $\frac{\partial r_{ji}(e_{ij})}{\partial e_{ij}}$). Consequently, although the cost function of agent i depends on other agents' efforts (e_{ji} for all $j \in N\backslash\{i\}$), the optimal effort does not.

To obtain the optimal effort, we analyze the derivative of the convex function $\Omega_i(e)$ with respect to e_{ij}, for any $j \in N\backslash\{i\}$. It must be noted that $\frac{\partial \Omega_{ii}(e)}{\partial e_{ij}} \geq 0 \iff \frac{\partial c_i(e_i)}{\partial e_{ij}} \geq \omega^i_{ji}\frac{\partial r_{ji}(e_{ij})}{\partial e_{ij}}$ for all $e_{ij} \in [0, 1]$. The following result characterizes the optimal effort level for agent $i \in N$ in the first stage of the game.

Lemma 1. *Let $(N, \{E_i\}_{i\in N}, \{\Omega_i\}_{i\in N})$ be an effort game and $\hat{e}_{ij}$ be the optimal level of effort that agent i exerts to reduce the costs of agent j. Thus,*

- $\hat{e}_{ij} = 0$ *if and only if* $\frac{\partial c_i(e_i)}{\partial e_{ij}} > \omega^i_{ji}\frac{\partial r_{ji}(e_{ij})}{\partial e_{ij}}$, *for all* $e_{ij} \in [0, 1]$,
- $\hat{e}_{ij} = 1$ *if and only if* $\frac{\partial c_i(e_i)}{\partial e_{ij}} < \omega^i_{ji}\frac{\partial r_{ji}(e_{ij})}{\partial e_{ij}}$, *for all* $e_{ij} \in [0, 1]$,
- $\hat{e}_{ij} \in (0, 1)$ *that holds* $\left.\frac{\partial c_i(e_i)}{\partial e_{ij}}\right|_{e_{ij}=\hat{e}_{ij}} = \omega^i_{ji}\left.\frac{\partial r_{ji}(e_{ij})}{\partial e_{ij}}\right|_{e_{ij}=\hat{e}_{ij}}$, *otherwise.*

The following theorem shows the unique allocation rule of the WPR family that induces an efficient effort profile in equilibrium. This allocation rule gives all the reductions to the agent that generates them. Formally, let $H(e) := (H_i(e))_{i\in N}$ be the allocation rule in the WPR family with $\omega^i_{ji} = 1$ for $i, j \in N, i \neq j$, that is $H_i(e) = c_i(e_i) - \sum_{j\in N\backslash\{i\}} r_{ji}(e_{ij})$ for $i \in N$. We consider an allocation rule as efficient if it induces an efficient effort profile in equilibrium.

Theorem 3. *Consider the effort game $(N, \{E_i\}_{i\in N}, \{H_i\}_{i\in N})$. Let e^*_{ij} be the level of effort that an agent i exerts to reduce the costs of agent j in the unique equilibrium with i, $j \in N, i \neq j$. Thus,*

- $e^*_{ij} = 0$ *if and only if* $\left.\frac{\partial c_i(e_i)}{\partial e_{ij}}\right|_{e_{ij}=0} > \left.\frac{\partial r_{ji}(e_{ij})}{\partial e_{ij}}\right|_{e_{ij}=0}$
- $e^*_{ij} = 1$ *if and only if* $\left.\frac{\partial c_i(e_i)}{\partial e_{ij}}\right|_{e_{ij}=1} < \left.\frac{\partial r_{ji}(e_{ij})}{\partial e_{ij}}\right|_{e_{ij}=1}$
- $e^*_{ij} \in (0, 1)$ *that holds* $\left.\frac{\partial c_i(e_i)}{\partial e_{ij}}\right|_{e_{ij}=e^*_{ij}} = \left.\frac{\partial r_{ji}(e_{ij})}{\partial e_{ij}}\right|_{e_{ij}=e^*_{ij}}$, *otherwise.*

In addition, $e^*_{ij} = \tilde{e}_{ij}$ for i, $j \in N, i \neq j$ and $H_i(e)$ is the only allocation rule of the WPR family that always induces an efficient effort profile in equilibrium.

The next Corollary shows that the allocation rule H is not only the only efficient one within the WPR family, but that it induces the lowest possible grand coalition cost for any possible allocation rule.

Corollary 1. *Let Θ be the set of all allocation rules for PE-games. There is no $\psi \in \Theta$ such that the effort equilibrium profile induced in the non cooperative game $(N, \{E_i\}_{i\in N}, \{\psi_i\}_{i\in N})$ generates a lower cost of the grand coalition than allocation rule H.*

As mentioned, the effort e_{ij} is efficient when its marginal cost matches the marginal reduction that it generates; otherwise, the effort is zero or one. Allocation rule $H(e)$ aligns the incentives of agents in the first stage game with this idea. It gives all the reduction to the agent that generates it. In that case, the best response of any agent is to make its marginal cost equal to the marginal reduction that its effort generates; otherwise, this agent exerts the minimal or maximal effort depending on which is higher: the marginal cost or the marginal reduction.

We illustrate this analysis with the 3-firm case given in Example 2 in Section 6.

In this section we work out the allocation rule (in the second stage) within the WPR family that generates the unique efficient effort equilibrium (in the first stage). However, there are situations in which pairwise reductions cannot be weighted separately, i.e. it is not possible to assign different weights to what an agent gives and what the same agent receives in a pairwise interaction. For example, there may be situations in which there is a unique cost reduction for any pair of agents that depends on the effort exerted by both agents, i.e. an aggregate reduction. In that case they have to decide how to split the whole cost reduction. Such cases require a weight to be assigned to the pairwise aggregate reduction.

The question that arises in this new scenario is whether the level of efficiency maintained is the same as that attained when the pairwise reductions are weighted separately for each agent. Unfortunately, the answer is no: the level of efficiency decreases in this new scenario. The next section focuses on measuring the level of efficiency of efforts in equilibrium for a particular family of weighted pairwise aggregate reductions.

5. Measuring efficiency for pairwise aggregate reduction

Consider the family of cost allocation with weighted pairwise aggregate reduction $A(e) \in R^n$ defined as follows:

$$A_i(e) = c_i(e_i) - \sum_{j\in N\backslash\{i\}} \alpha_{ij}[r_{ij}(e_{ji}) + r_{ji}(e_{ij})], \tag{5}$$

with $\alpha_{ij} \in [0, 1]$. The interaction between agents i and j generates an aggregate cost reduction which is $r_{ij}(e_{ji}) + r_{ji}(e_{ij})$. The parameter α_{ij} measures the proportions in which this reduction is shared between agents i and j, i.e. α_{ij} is the proportion for agent i and $\alpha_{ji} = 1 - \alpha_{ij}$ for agent j.

Note that $A(e)$ is a subfamily of the WPR family $\Omega(e)$, where now $\omega^i_{ij} = \omega^j_{ij} = \alpha_{ij}$, for all $i, j \in N$. From now on we refer to this subfamily as the WPAR family. It is important to note that the Shapley value and the Nucleolus belong to the WPAR family with $\alpha_{ij} = \frac{1}{2}$ for all $i, j \in N$, $i \neq j$. We consider whether the allocation rule $H(e)$, which generates the efficient effort in equilibrium, is applicable in this situation. Unfortunately, $H(e)$ does not fit the scheme of pairwise aggregate reduction.

This section analyzes the non-cooperative effort game that arises in the first stage when cost allocations in the WPAR family are considered.

Our goal is to find out, within the WPAR family, a core-allocation in the cooperative game of the second stage that induce the effort equilibrium level in the first stage closest to the efficient one. We consider that an effort profile $e' \in E$ is more efficient than a profile $e'' \in E$ if the aggregate cost generated in the second stage by e' is lower than that generated by e''.

We therefore first study the non-cooperative effort game that arises under this new cost allocation $A(e)$, that is $(N, \{E_i\}_{i\in N}, \{A_i\}_{i\in N})$.

To simplify notation and analysis, we consider that for all $i \in N$ and $j \in N\backslash\{i\}$, $c'_i(e_{ij}) := \frac{\partial c_i(e_i)}{\partial e_{ij}}$, $c''_i(e_{ij}) := \frac{\partial^2 c_i(e_i)}{\partial e^2_{ij}}$, $r'_{ji}(e_{ij}) := \frac{\partial r_{ji}(e_{ij})}{\partial e_{ij}}$ and $r''_{ji}(e_{ij}) := \frac{\partial^2 r_{ji}(e_{ij})}{\partial e^2_{ij}}$. Note that, as the WPAR family is a subfamily of WPR, the properties of the latter apply to the former.

Before analyzing the EEE of the above non-cooperative effort game, we define thresholds of alpha parameters that enable them to be reached.

Definition 3. Given an effort game $(N, \{E_i\}_{i\in N}, \{A_i\}_{i\in N})$, we define the following lower and upper thresholds for each pair of agents i and j,

$\underline{\alpha}_{ij} := \frac{c_i'(0)}{r_{ji}'(0)}$, $\bar{\alpha}_{ij} := \frac{c_i'(1)}{r_{ji}'(1)}$, $\underline{\alpha}_{ji} := \frac{c_j'(0)}{r_{ij}'(0)}$, and $\bar{\alpha}_{ji} := \frac{c_j'(1)}{r_{ij}'(1)}$.

It is clear that $0 < \underline{\alpha}_{ij} < \bar{\alpha}_{ij}$ because c_i' is an increasing function and r_{ji}' decreasing one. Analogously, $0 < \underline{\alpha}_{ji} < \bar{\alpha}_{ji}$.

The first Theorem in this section characterizes all possible types of effort equilibrium according to the value of the parameter α_{ij}, for all $i, j \in N, i \neq j$. The proof of Theorem 4 appears in Appendix C, together with all the other proofs in this section.

Theorem 4. *Let $(N, \{E_i\}_{i\in N}, \{A_i\}_{i\in N})$ be an effort game. The pairwise efforts in any unique equilibrium (e_{ij}^*, e_{ji}^*) are given by*

$$e_{ij}^* = \begin{cases} 0 \text{ if and only if } \alpha_{ij} \leq \underline{\alpha}_{ij} \\ e^I \text{ if and only if } \underline{\alpha}_{ij} < \alpha_{ij} < \overline{\alpha}_{ij} \\ 1 \text{ if and only if } \alpha_{ij} \geq \overline{\alpha}_{ij} \end{cases}$$

$$e_{ji}^* = \begin{cases} 0 \text{ if and only if } \alpha_{ij} \geq 1 - \underline{\alpha}_{ji} \\ e^J \text{ if and only if } 1 - \bar{\alpha}_{ji} < \alpha_{ij} < 1 - \underline{\alpha}_{ji} \\ 1 \text{ if and only if } \alpha_{ij} \leq 1 - \underline{\alpha}_{ji} \end{cases}$$

where $e^I \in (0,1)$ is the unique solution of $c_i'(e_i) - \alpha_{ij} r_{ji}'(e_{ij}) = 0$ and $e^J \in (0,1)$ is the unique solution of $c_j'(e_j) - (1-\alpha_{ij}) r_{ij}'(e_{ji}) = 0$.

It is demonstrated in Appendix C that e^I increases with α_{ij} while e^J decreases, see Corollary 2. The findings of Corollary 2 are valuable when the objective is to incentivize agents $i, j \in N$ to increase their pairwise effort e_{ij} by adjusting the parameter α_{ij}. However, our aim is to go beyond this and achieve optimal efficiency within the WPAR family. In other words, we seek to determine the optimal values of α_{ij}^*, for all $i, j \in N$, which minimizes the aggregate cost function $\sum_{i\in N} A_i(e^*)$ at equilibrium, where both A_i and the effort equilibrium e^* depend on α_{ij}.

The search for alpha parameters which will lead to the EEE can be simplified by taking into account the bilateral independent interactions of agents. Note first that any pair of agents have a particular α_{ij}, and second that the optimal effort made by any agent $i \in N$ in regard to any agent $j \in N\backslash\{i\}$ is independent of the optimal effort that agent i exerts in regard to any other agent $h \in N\backslash\{i,j\}$. Thus, minimizing $\sum_{i\in N} A_i(e^*)$ in terms of α_{ij} is equivalent to minimizing $A_i(e^*) + A_j(e^*)$, since each particular α_{ij} only appears in $A_i(e^*)$ and $A_j(e^*)$. Fortunately, the problem can be further simplified: Note that, $A_i(e^*)$ and $A_j(e^*)$ are the sums of different terms, but α_{ij} only appears in those terms related to the interaction between i and j (see (5)). These terms are $c_i(e_i^*) - \alpha_{ij}(r_{ij}(e_{ji}^*) + r_{ji}(e_{ij}^*))$ from $A_i(e^*)$, and $c_j(e_j^*) - (1-\alpha_{ij})(r_{ji}(e_{ij}^*) + r_{ij}(e_{ji}^*))$ from $A_j(e^*)$. Thus, a new function $A_i^*(\alpha_{ij}) := c_i(e_i^*) - \alpha_{ij}(r_{ij}(e_{ji}^*) + r_{ji}(e_{ij}^*))$ can be considered, and analogously $A_j^*(1-\alpha_{ij})$. Note that $\frac{\partial^x (A_i(e^*))}{\partial \alpha_{ij}^x} = \frac{\partial^x (A_i^*(\alpha_{ij}))}{\partial \alpha_{ij}^x}$ and $\frac{\partial^x (A_j(e^*))}{\partial \alpha_{ij}^x} = \frac{\partial^x (A_j^*(1-\alpha_{ij}))}{\partial \alpha_{ij}^x}$ for $x = 1, 2, \ldots$. Therefore, for each pair i and j, it is possible to define the function $L_{ij}^*(\alpha_{ij}) := A_i^*(\alpha_{ij}) + A_j^*(1-\alpha_{ij})$. Hence, minimizing $\sum_{i\in N} A_i(e^*)$ is equivalent to minimizing $L_{ij}^*(\alpha_{ij})$, with

$$\begin{aligned} L_{ij}^*(\alpha_{ij}) &= c_i(e_i^*) + c_j(e_j^*) \\ &\quad - \left[\alpha_{ij}(r_{ij}(e_{ji}^*) + r_{ji}(e_{ij}^*)) + (1-\alpha_{ij})(r_{ji}(e_{ij}^*) + r_{ij}(e_{ji}^*))\right] \\ &= c_i(e_i^*) + c_j(e_j^*) - (r_{ij}(e_{ji}^*) + r_{ji}(e_{ij}^*)) \end{aligned} \quad (6)$$

The function $L_{ij}^*(\alpha_{ij})$ depends on α_{ij} through the equilibrium efforts e_{ij}^* and e_{ji}^* because they depend on α_{ij}. We now focus on finding the α_{ij} that minimizes function $L_{ij}^*(\alpha_{ij})$, and provide a procedure for finding the EEE for pairwise aggregate reduction.

We can summarize this reasoning as follows.[8] Let $\alpha = (\alpha_i)_{i\in N}$ and $\alpha_i = (\alpha_{ij})_{j\in N\backslash\{i\}}$, *then* $\alpha^* = arg \min_{\alpha\in[0,1]^{n(n-1)}} \sum_{i\in N} A_i(e^*) \Leftrightarrow \alpha_{ij}^* = arg \min_{\alpha_{ij}\in[0,1]} A_i(e^*) + A_j(e^*)$ for all $i \in N \iff \alpha_{ij}^* = \arg \min_{\alpha_{ij}\in[0,1]} c_i(e_i^*) - \alpha_{ij}(r_{ij}(e_{ji}^*) + r_{ji}(e_{ij}^*)) + c_j(e_j^*) - (1-\alpha_{ij})(r_{ji}(e_{ij}^*) + r_{ij}(e_{ji}^*))$ for all $i, j \in N, i \neq j \Leftrightarrow \alpha_{ij}^* = arg \min_{\alpha_{ij}\in[0,1]} c_i(e_i^*) + c_j(e_j^*) - (r_{ji}(e_{ij}^*) + r_{ij}(e_{ji}^*))$ for all $i, j \in N, i \neq j$. As $L_{ij}^*(\alpha_{ij}) = c_i(e_i^*) + c_j(e_j^*) - (r_{ij}(e_{ji}^*) + r_{ji}(e_{ij}^*))$, then $\alpha_{ij}^* = arg \min_{\alpha_{ij}\in[0,1]} L_{ij}^*(\alpha_{ij})$ for all $i\ j \in N, i \neq j$.

For any effort game considered here, there are only six possible distributions of the lower and upper thresholds of the alpha parameter.[9] These cases are

$$\begin{array}{ll} \textbf{Case A} & \underline{\alpha}_{ij} < \overline{\alpha}_{ij} < 1 - \overline{\alpha}_{ji} < 1 - \underline{\alpha}_{ji} \\ \textbf{Case B} & \underline{\alpha}_{ij} < 1 - \overline{\alpha}_{ji} < \overline{\alpha}_{ij} < 1 - \underline{\alpha}_{ji} \\ \textbf{Case C} & \underline{\alpha}_{ij} < 1 - \overline{\alpha}_{ji} < 1 - \underline{\alpha}_{ji} < \overline{\alpha}_{ij} \\ \textbf{Case D} & 1 - \overline{\alpha}_{ji} < \underline{\alpha}_{ij} < \overline{\alpha}_{ij} < 1 - \underline{\alpha}_{ji} \\ \textbf{Case E} & 1 - \overline{\alpha}_{ji} < \underline{\alpha}_{ij} < 1 - \underline{\alpha}_{ji} < \overline{\alpha}_{ij} \\ \textbf{Case F} & 1 - \overline{\alpha}_{ji} < 1 - \underline{\alpha}_{ji} < \underline{\alpha}_{ij} < \overline{\alpha}_{ij} \end{array} \quad (7)$$

The last theorem characterizes the optimal α_{ij}^* in cases A-F. Thus, Theorem 5 provides the α_{ij}^* that incentivizes an efficient effort equilibrium for WPAR.[10] In Theorem 5 we use the following notation:

1. $\breve{\alpha}_{ij}^{[a,b]} \in [a,b]$ with $0 \leq a < b \leq 1$ is:

$$\breve{\alpha}_{ij}^{[a,b]} = \begin{cases} a & \text{if } \frac{\partial(L_{ij}^*(\alpha_{ij}))}{\partial\alpha_{ij}} > 0 \text{ for all } \alpha_{ij} \in [a,b] \\ b & \text{if } \frac{\partial(L_{ij}^*(\alpha_{ij}))}{\partial\alpha_{ij}} < 0 \text{ for all } \alpha_{ij} \in [a,b] \\ \text{Solution of } \frac{\partial(L_{ij}^*(\alpha_{ij}))}{\partial\alpha_{ij}} = 0 & \text{otherwise} \end{cases}$$

2. $\Lambda(\alpha) = \begin{cases} 0 \text{ if } \alpha < 0 \\ \alpha \text{ if } \alpha \in (0,1) \\ 1 \text{ if } \alpha > 1 \end{cases}$

Theorem 5. *Let $(N, \{E_i\}_{i\in N}, \{A_i\}_{i\in N})$ be an effort game, and $L_{ij}^*(\alpha_{ij}) = c_i(e_i^*) + c_j(e_j^*) - (r_{ij}(e_{ji}^*) + r_{ji}(e_{ij}^*))$. The optimal solution $\alpha_{ij}^* = \arg \min_{\alpha_{ij}\in[0,1]} L_{ij}^*(\alpha_{ij})$ is in each case,*

Case A *α_{ij}^* is any element of $[\bar{\alpha}_{ij}, 1 - \bar{\alpha}_{ji}]$.*

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[8] In principle, this problem could be considered a bilevel optimization problem ([40]). The main characteristic of a bilevel programing problem is a kind of hierarchy, because its constraints are defined, in part, by a second optimization problem. In our case, the second level (lower level or follower' level) will be the problem $\min_{e_i\in[0,1]^{(n-1)}} A_i(e)$ with solution $e^* = (e_i^*)_{i\in N}$ where e^* depends on α. The first level (upper level or leader's problem) will be $\min_{\alpha\in[0,1]^{n(n-1)}} \sum_{i\in N} A_i(e^*)$. Thus, we can rewrite the problem as follows:

$$\begin{array}{ll} \min_{\alpha,e} \sum_{i\in N} A_i(\alpha, e) & \\ s.t. & (\alpha, e) \in [0,1]^{n(n-1)} \times [0,1]^{n(n-1)} \\ & e_i \in G_i(\alpha) \text{ for all } i \in N \\ & \text{with } e = (e_i)_{i\in N} \\ \text{where } & G_i(\alpha) = \arg\min_{e_i} A_i(\alpha, e) \\ & s.t.\ \ e_i \in [0,1]^{(n-1)},\ \alpha \in [0,1]^{n(n-1)} \end{array}$$

However, it is difficult to see this problem as a Stakelberg game, as described for example in [41], because α is not a strategy profile but a parameter of the reduction cost functions. We believe that our setting better fits a bi-form game that was introduced by [3].

[9] Note that $\underline{\alpha}_{ji} < \bar{\alpha}_{ji}$ and $\underline{\alpha}_{ij} < \bar{\alpha}_{ij}$.

[10] The function L_{ij}^* is a piecewise function, and although it is continuous in $\alpha_{ij} \in [0,1]$ it is not differentiable at all points in its domain. Since it is defined over intervals, it is generally non-differentiable at the endpoints of these intervals. Therefore, to compute the minimum, it is also necessary to evaluate the function at the interval endpoints. In addition, due to its convexity, the minimum can also be an interior point within any of the intervals. However, each interval entails a distinct derivative function, thereby contributing to the complexity of the computation process. The introduction of Theorem 5 streamlines the evaluation procedure by reducing the number of points to be assessed, presenting them in a case-by-case framework.

Case B $\alpha^*_{ij} = \breve{\alpha}_{ij}^{[1-\bar{\alpha}_{ji},\bar{\alpha}_{ij}]}$

Case C

$$\alpha^*_{ij} = \begin{cases} \text{any element of } [\bar{\alpha}_{ij}, 1] \text{ if } \alpha^C = \Lambda(\bar{\alpha}_{ij}) \text{ and } \Lambda(\bar{\alpha}_{ij}) < 1 \\ \alpha^C \text{ otherwise} \end{cases}$$

where

$$\alpha^C = \arg\min\{L^*_{ij}(\breve{\alpha}_{ij}^{[1-\bar{\alpha}_{ji},1-\underline{\alpha}_{ji}]}), L^*_{ij}(\Lambda(\bar{\alpha}_{ij}))\}.$$

Case D

$$\alpha^*_{ij} = \begin{cases} \text{any element of } [0, 1-\bar{\alpha}_{ji}] \text{ if } \alpha^D = \Lambda(1-\bar{\alpha}_{ji}) \text{ and } \Lambda(1-\bar{\alpha}_{ji}) > 0 \\ \alpha^D \text{ otherwise} \end{cases}$$

where

$$\alpha^D = \arg\min\{L^*_{ij}(\Lambda(1-\bar{\alpha}_{ji})), L^*_{ij}(\breve{\alpha}_{ij}^{[\underline{\alpha}_{ij},\bar{\alpha}_{ij}]})\}.$$

Case E

$$\alpha^*_{ij} = \begin{cases} \text{any element of } [0, 1-\bar{\alpha}_{ji}] \text{ if } \alpha^E = \Lambda(1-\bar{\alpha}_{ji}) \text{ and } \Lambda(1-\bar{\alpha}_{ji}) > 0 \\ \text{any element of } [\bar{\alpha}_{ij}, 1] \text{ if } \alpha^E = \Lambda(\bar{\alpha}_{ij}) \text{ and } \Lambda(\bar{\alpha}_{ij}) < 1 \\ \alpha^E \text{ otherwise} \end{cases}$$

where

$$\alpha^E = \arg\min\{L^*_{ij}(\Lambda(1-\bar{\alpha}_{ji})), \breve{\alpha}_{ij}^{[\underline{\alpha}_{ij},1-\underline{\alpha}_{ji}]}, L^*_{ij}(\Lambda(\bar{\alpha}_{ij}))\}.$$

Case F

$$\alpha^*_{ij} = \begin{cases} \text{any element of } [0, 1-\bar{\alpha}_{ji}] \text{ if } \alpha^F = \Lambda(1-\bar{\alpha}_{ji}) \text{ and } \Lambda(1-\bar{\alpha}_{ji}) > 0 \\ \text{any element of } [\bar{\alpha}_{ij}, 1] \text{ if } \alpha^F = \Lambda(\bar{\alpha}_{ij}) \text{ and } \Lambda(\bar{\alpha}_{ij}) < 1 \\ \alpha^F \text{ otherwise} \end{cases}$$

where

$$\alpha^F = \arg\min\{L^*_{ij}(\Lambda(1-\bar{\alpha}_{ji})), L^*_{ij}(\Lambda(\bar{\alpha}_{ij}))\}.$$

To conclude the section, we describe a procedure for finding an efficient effort in equilibrium induced by the WPAR family.

EEE Procedure

Given an effort game $(N, \{E_i\}_{i\in N}, \{A_i\}_{i\in N})$

1. we first calculate the lower and upper thresholds of the bilateral interaction between any pair of agents by using Definition ;
2. we then focus on the list (7) and determine which case (A-F) applies;
3. Theorem 5 provides an optimal α^*_{ij} for all $i, j \in N$ to minimize the centralized (aggregate) cost allocation $\sum_{i\in N} A_i(e^*)$;
4. with this α^*_{ij} Theorem 4 gives the associated efficient effort equilibrium (e^*_{ij}, e^*_{ji}) for every pair of agents, and thus an efficient effort equilibrium e^* for the game;
5. at this point the optimal cost allocation that incentivizes agents $i, j \in N$ to make an efficient effort equilibrium e^*_{ij} and e^*_{ji} is known, i.e.

   $$A^*_i(e^*) = c_i\big(e^*_i\big) - \sum_{j\in N\setminus\{i\}} \alpha^*_{ij}\big[r_{ij}\big(e^*_{ji}\big) + r_{ji}\big(e^*_{ij}\big)\big].$$

   We illustrate this procedure with the 3-firm case given in Example 2 in Section 6.

6. Comparison of WPR and WPAR families

We complete the study of our model of cooperation with pairwise cost reduction by comparing the two families of core-allocations analyzed. We find that there is a loss of efficiency when cooperation is restricted to a pairwise aggregate cost reduction. That loss of efficiency can be measured. In addition, we show that those agents who receive less than the total reduction generated and bear the total cost of this effort always exert less effort than the efficient agent.

As mentioned above, the allocation rule $H(e)$ induces an equilibrium effort e^{*H} that matches the efficient effort of Proposition 3, i.e. $e^{*H} = \tilde{e}$. This means that there is no rule that generates a lower cost of the grand coalition, see Corollary 1. However, as also mentioned above, WPAR is a subfamily of WPR, but $H(e)$ is not in WPAR, so e^{*A} is not always equal to e^{*H}.

Let $A^*(e)$ be the allocation rule in WPAR that induces the effort profile e^{*A^*} that minimizes the cost of the grand coalition, i.e. the efficient allocation in this subfamily. The difference, in terms of efficiency, between the cost of the grand coalition with e^{*A^*} and $\tilde{e}$ can be measured. Note that for any particular functions $c_i(e_i)$ and $r_{ij}(e_{ji})$ for $i, j \in N$, $i \neq j$, the associated e^{*A^*} and $\tilde{e}$ can be obtained. Let Δ be this difference or loss of efficiency, where

$$\Delta = \sum_{i\in N}[c_i(e_i^{*A^*}) - \sum_{j\in N\setminus\{i\}} r_{ij}(e_{ji}^{*A^*})] - \sum_{i\in N}[c_i(\tilde{e}_i) - \sum_{j\in N\setminus\{i\}} r_{ij}(\tilde{e}_{ji})]. \tag{8}$$

The following proposition shows the relation between efforts e^{*A^*} and $\tilde{e}$. The proof of Proposition appears in Appendix B.

Proposition 4. *Let $e^{*A^*}_{ij}$ for $i, j \in N$, $i \neq j$ be the equilibrium efforts of $A^*(e)$, that minimize the cost of the grand coalition in the family WPAR. Thus, the efficient effort $\tilde{e}_{ij} \geq e^{*A^*}_{ij}$ for all $i, j \in N$, $i \neq j$.*

As mentioned above, when an agent receives less than the total reduction that it generates and bears the total cost of that effort, then that agent always exerts less effort than the efficient one

Finally, readers may think that the rationale behind the efficient rule, $H(e)$, in the WPR family, could also apply to the WPAR family. However, this is not the case. To reach an efficient effort equilibrium in the WPR family, for each pair of agents $i, j \in N, i \neq j$, the weight ω^i_{ji} must be 1, because $\frac{\partial \Omega_i(e)}{\partial e_{ij}} = \frac{\partial c_i(e_i)}{\partial e_{ij}} - \omega^i_{ji}\frac{\partial r_{ji}(e_{ij})}{\partial e_{ij}}$, and ω^j_{ij} must also be 1, because $\frac{\partial \Omega_j(e)}{\partial e_{ji}} = \frac{\partial c_j(e_j)}{\partial e_{ji}} - \omega^j_{ij}\frac{\partial r_{ij}(e_{ji})}{\partial e_{ji}}$. However, this is no longer true for the WPAR family.[11]

The following example with three agents illustrates the comparison of the two core allocation families and completes the paper.

Example 2. Consider a pairwise inter-organizational situation with three firms, i.e. $N = \{1, 2, 3\}$. For any effort profile $e \in [0, 1]^6$, the PE-situation is given by the following initial costs,

$c_1(e_{12}, e_{13}) = 100 + 100e_{12} + 4e^2_{12} + 100e_{13} + 4e^2_{13}$
$c_2(e_{21}, e_{23}) = 100 + 100e_{21} + 4e^2_{21} + 100e_{23} + 4e^2_{23}$
$c_3(e_{31}, e_{32}) = 100 + 100e_{31} + 4e^2_{31} + 100e_{32} + 4e^2_{32}$

and the following pairwise reduced costs, all of them in thousands of Euros,

$r_{i1}(e_{1i}) = 2 + 110e_{1i} - 2e^2_{1i}$ with $i = 2, 3$
$r_{i2}(e_{2i}) = 2 + 105e_{2i} - 3e^2_{2i}$ with $i = 1, 3$
$r_{i3}(e_{3i}) = 2 + 105e_{3i} - 3e^2_{3i}$ with $i = 1, 2$

By Definition 3, the pair of firms $\{1, 2\}$ has the thresholds $\underline{\alpha}_{12} = 0.91$, $\bar{\alpha}_{12} = 1.02$, $\underline{\alpha}_{21} = 0.95$, and $\bar{\alpha}_{21} = 1.09$, which correspond to Case F in the Table 7. By using Theorem 5, it can easily be checked that $\alpha^F = \Lambda(\bar{\alpha}_{12}) < 1$ and $\alpha^*_{12} = 1$. Thus, by Theorem 4, $e^*_{12} = 0.833$, $e^*_{21} = 0$. As firms 2 and 3 are identical, $\alpha^*_{13} = 1$, $e^*_{13} = 0.833$ and $e^*_{31} = 0$. Finally, for the pair $\{2, 3\}$, $\underline{\alpha}_{23} = 0.95$, $\bar{\alpha}_{23} = 1.09$, $\underline{\alpha}_{32} = 0.95$, and $\bar{\alpha}_{32} = 1.09$. This is again Case F. Note that in case F, $\alpha^F = \arg\min\{L^*_{23}(\Lambda(1-\bar{\alpha}_{32})), L^*_{23}(\Lambda(\bar{\alpha}_{23}))\}$, where in this particular case $L^*_{23}(\Lambda(1-\bar{\alpha}_{32})) = L^*_{23}(\Lambda(\bar{\alpha}_{23}))$ with $\Lambda(1-\bar{\alpha}_{32}) = 0$ and

[11] In WPAR, for each pair of agents $i, j \in N, i \neq j$, the weight α_{ij} is not always 1, because $\frac{\partial A_i(e)}{\partial e_{ij}} = \frac{\partial c_i(e_i)}{\partial e_{ij}} - \alpha_{ij}\frac{\partial r_{ji}(e_{ij})}{\partial e_{ij}}$ and $\frac{\partial A_j(e)}{\partial e_{ji}} = \frac{\partial c_j(e_j)}{\partial e_{ji}} - \alpha_{ji}\frac{\partial r_{ij}(e_{ji})}{\partial e_{ji}}$ but $\alpha_{ij} = 1 - \alpha_{ji}$. Note that if $\alpha_{ij} = 1$, then $\alpha_{ji} = 0$ and the derivative conditions for efficiency in Proposition 3 would be violated. Bear in mind that the weights ω^i_{ji} that appear in each derivative $\frac{\partial \Omega_i(e)}{\partial e_{ij}}$ for $i, j \in N, i \neq j$ are independent of one another. However, the weights α_{ij} that appear in the each derivative $\frac{\partial A_i(e)}{\partial e_{ij}}$ for $i, j \in N, i \neq j$ are not, because $\alpha_{ij} = 1 - \alpha_{ji}$. In addition, it is known that $\omega^i_{ij} = \omega^i_{ji} = \alpha_{ij}$ in WPAR for all $i, j \in N, i \neq j$, where $\omega^i_{ij} = 1 - \omega^j_{ij}$ and $\omega^i_{ji} = 1 - \omega^j_{ji}$. The fact that pairwise cost reduction is aggregated by α_{ij} in the subfamily WPAR means that it is not possible to apply the efficient argument used for the WPR family.

$\Lambda(\bar{\alpha}_{23}) = 1$ Thus, two solutions emerge: (i) $e^*_{23} = 0.357$, $e^*_{32} = 0$, and $\alpha^*_{23} = 1$, and (ii) $e^*_{23} = 0$, $e^*_{32} = 0.357$, and $\alpha^*_{23} = 0$. Therefore, there are two EEE in WPAR.

(i) $e^*_{12} = e^*_{13} = 0.833$, $e^*_{21} = 0$, $e^*_{23} = 0.357$, $e^*_{31} = e^*_{32} = 0$

(ii) $e^*_{12} = e^*_{13} = 0.833$, $e^*_{21} = e^*_{23} = 0$, $e^*_{31} = 0$, $e^*_{32} = 0.357$

We now calculate the efficient efforts in this example by Proposition 3. They are the solutions of $c'_i(e_{ij}) - r'_{ji}(e_{ij}) = 0$, thus, $\tilde{e}_{12} = \tilde{e}_{13} = 0.833$, and $\tilde{e}_{21} = \tilde{e}_{23} = \tilde{e}_{31} = \tilde{e}_{32} = 0.357$. Note that by Theorem 3 these efforts are also the effort equilibrium obtained by the allocation rule $H(e)$.

This example is a particular subcase of Case F. This implies that α^*_{ij} is zero or one, which in turn implies that one of the agents makes no effort and the other makes the efficient value. However, they are never able to make the efficient effort simultaneously under WPAR. The loss of efficiency in WPAR with regard to WPR can be calculated with the help of (8).

$\Delta = \sum_{i\in N}[c_i(e_i^{*A^*}) - \sum_{j\in N\setminus\{i\}} r_{ij}(e_{ji}^{*A^*})] - \sum_{i\in N}[c_i(\tilde{e}_i) - \sum_{j\in N\setminus\{i\}} r_{ij}(\tilde{e}_{ji})] = 278.776 - 276.104 = 2.67.$

7. Conclusions and future research

This paper presents a model of cooperation with pairwise cost reduction. The direct impact of pairwise effort on cost reductions is investigated by means of a bi-form game. First, the agents determine the level of pairwise effort to be made to reduce the costs of their partners. Second, they participate in a bilateral interaction with multiple independent partners where the cost reduction that each agent gives to another agent is independent of any possible coalition. As a result of cooperation, agents reduce each other's costs. In the non-cooperative game that precedes cooperation, the agents anticipate the cost allocation that will result from the cooperative game by incorporating the effect of the effort made into their cost functions. We show that all-included cooperation is feasible, in the sense that there are possible cost reductions that make all agents better off (or, at least, not worse off), and consistent. We then identify a family of feasible cost allocations with weighted pairwise reduction. One of these cost allocations is selected by taking into account the incentives generated in the efforts that agents make, and consequently in the total cost of coalitions. Surprisingly, we find that the Shapley value, which coincides with the Nucleolus in this model, can induce inefficient effort strategies in equilibrium in the non-cooperative model. However, it is always possible to select a core-allocation with appropriate pairwise weights that can generate an efficient effort.

Future research could take any of several directions. First, this paper assumes that the individual effort cost function $c_i(e_i)$ is independent of the effort of other agents, and that the marginal cost $\frac{\partial c_i(e_i)}{\partial e_{ij}}$ is independent of the effort that i makes in regard to agents other than j, i.e. $\frac{\partial c_i^2(e_i)}{\partial e_{ij}\partial e_{ih}} = 0$. We make a similar assumption with the cost reduction function $r_{ij}(e^*_{ji})$. There is some degree of independence between efforts. This is a reasonable assumption in many contexts, but in some settings different assumptions might be needed. For example, there are situations with strategic complementarity in which the efforts of agents reinforce each other. In such cases the cost function is supermodular. In other cases there is strategic substitutability, so that efforts offset each other and the function is submodular. Focusing on the effort cost function of one agent, if $\frac{\partial c_i^2(e_i)}{\partial e_{ij}\partial e_{ih}} > 0$ then there is complementarity between the efforts, and if $\frac{\partial c_i^2(e_i)}{\partial e_{ij}\partial e_{ih}} < 0$, then there is substitutability. This is a very interesting future extension. It could also be worth considering this complementarity/substitutability not only between the different efforts that one agent makes in regard to other agents but also between the efforts made by different agents. This assumption can be made on both the effort cost functions and the cost reduction function. Obviously, complementarity on the effort cost function has the opposite effect to that on the cost reduction function.

The second direction is close to the first. The pairwise total cost reduction could be considered as a general function which is increasing in the efforts e_{ij} and e_{ji}, that is $R_{ij}(e_{ij}, e_{ji})$. In our model, this function is additively separable, i.e. $R_{ij}(e_{ij}, e_{ji}) = r_{ij}(e_{ji}) + r_{ji}(e_{ij})$. However, as mentioned above, there could be situations with strategic complementarity or substitutability in which the efforts of agents reinforce or offset each other. In that case, the function $R_{ij}(e_{ij}, e_{ji})$ would not be separable. This is also an interesting question for analysis.

Another direction is related to the assumption of bilateral interaction between agents. This has the advantage of being analytically more tractable and is widely applied in practice (e.g., [35–37]), but overall interaction between agents, dependent on groups, is an important factor that we believe does not affect the success of cooperation. One possible future extension would be to investigate the cooperative model with multiple cost reduction and the impact of the efforts made on those cost reductions.

Finally, we identify a large family of core-allocations with weighted pairwise reduction which contains the Shapley value and the Nucleolus and always provides a level of efficient effort in equilibrium. This family is very rich in itself, as a set solution concept for our cooperative model. Research into this core-allocation family can be furthered through an in-depth analysis of its structure and its geometric relationship to the core.

Declaration of Competing Interest

In accordance with Elsevier policy and our ethical obligation as researchers, we are reporting that we do not have received funding from any company that may be affected by the research reported in the enclosed paper. We have disclosed those interests fully to Elsevier, and we have in place an approved plan for managing any potential conflicts arising from that involvement.

CRediT authorship contribution statement

Jose A. García-Martínez: Conceptualization, Formal analysis, Methodology, Writing – original draft. **Antonio J. Mayor-Serra:** Conceptualization, Formal analysis, Methodology, Writing – original draft. **Ana Meca:** Conceptualization, Formal analysis, Methodology, Writing – original draft.

Data availability

No data was used for the research described in the article.

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Appendix A

Proposition 1, in Section 3, shows that PE-games are always concave. To prove this, the class of unanimity games must be described. In [39], it is proved that the family of unanimity games

$\{(N,u_T), T\subseteq N\}$ forms a basis of the vector space of all games with set of players N, where (N,u_T) is defined for each $S\subseteq N$ as follows:

$$u_T(S)=\begin{cases}1, & T\subseteq S\\ 0, & otherwise\end{cases}$$

Hence, for each cost game (N,c) there are unique real coefficients $(\alpha_T)_{T\subseteq N}$ such that $c=\sum_{T\subseteq N}\alpha_T u_T$. Many different classes of games, including airport games ([24]) and sequencing games ([38]), can be characterized through constraints on these coefficients.

Proof of Proposition 1

Proof. Let $(N,e,\{c_i(e_i),\{r_{ji}(e_{ij})\}_{j\in N\setminus\{i\}}\}_{i\in N})$ be a PE-situation and (N,e,c) the associated PE-game. First, we prove that this game can be rewritten as a weighted sum of unanimity games $u_{\{i\}}$ and $u_{\{i,j\}}$ for all $i,j\in N$ as follows:

$$c=\sum_{i\in N}c_i(e_i)u_{\{i\}}-\sum_{i,j\in N;i\neq j}r_{ij}(e_{ji})u_{\{i,j\}}. \tag{9}$$

Indeed, for all $S\subseteq N$,

$$\begin{aligned}c(S)&=\sum_{i\in N}c_i(e_i)u_{\{i\}}(S)-\sum_{i,j\in N;i\neq j}r_{ij}(e_{ji})u_{\{i,j\}}(S)\\&=\sum_{i\in S}c_i(e_i)-\sum_{i,j\in S;i\neq j}r_{ij}(e_{ji})=\sum_{i\in S}c_i(e_i)-\sum_{i\in S}\sum_{j\in S\setminus\{i\}}r_{ij}(e_{ji}).\end{aligned}$$

It is easily shown that the additive game $\sum_{i\in N}c_i(e_i)u_{\{i\}}$ is concave and that $u_{\{i,j\}}$ is convex. Thus, the game $-\sum_{i,j\in N;i\neq j}r_{ij}(e_{ji})u_{\{i,j\}}$ is concave because of $r_{ij}(e_{ji})>0$ for all $i,j\in N$. Finally, the concavity of (N,e,c) follows from the fact that game c is the sum of two concave games. □

The Theorem 1, in Section 3, shows that the Shapley value reduces the individual cost of an agent by half the total reduction that it obtains from the others ($R_i(N)$) plus a half of the total reduction that it provides to the rest of the agents, which is $G_i(N)=\sum_{j\in N\setminus\{i\}}r_{ji}(e_{ij})$.

The Shapley value is the only allocation rule that satisfies the four properties of Efficiency, Equal treatment of equals, Linearity and Null player. Next, we describe all of these properties of the Shapley value, which are useful in demonstrating the Theorem 1.

(EFF) *Efficiency*. The sum of the Shapley values of all agents equals the value of the grand coalition, so all the gain is allocated to the agents:

$$\sum_{i\in N}\phi_i(c)=c(N). \tag{10}$$

(ETE) *Equal treatment of equals*. If i and j are two agents who are equivalent in the sense that $c(S\cup\{i\})=c(S\cup\{j\})$ for every coalition S of N which contains neither i nor j, then $\phi_i(c)=\phi_j(c)$.

(LIN) *Linearity*. If two cost games c and c^* are combined, then the cost allocation should correspond to the costs derived from c and the costs derived from c^*:

$$\phi_i(c+c^*)=\phi_i(c)+\phi_i(c^*),\forall i\in N. \tag{11}$$

Also, for any real number a,

$$\phi_i(ac)=a\phi_i(c),\forall i\in N. \tag{12}$$

(NUP) *Null Player*. The Shapley value $\phi_i(c)$ of a null player i in a game c is zero. A player i is null in c if $c(S\cup\{i\})=c(S)$ for all coalitions S that do not contain i.

Proof of the Theorem 1. Consider the PE-game (N,e,c) rewritten as a weighted sum of unanimity games given by (9), i.e.

$$c=\sum_{i\in N}c_i(e_i)u_{\{i\}}-\sum_{i,j\in N;i\neq j}r_{ij}(e_{ji})u_{\{i,j\}}.$$

Take an agent $k\in N$. By the (LIN) property of the Shapley value, $\phi_k(e)$, it follows that

$$\begin{aligned}\phi_k(e)&=\phi_k\Big(\sum_{i\in N}c_i(e_i)u_{\{i\}}\Big)-\phi_k\Big(\sum_{i,j\in N;i\neq j}r_{ij}(e_{ji})\big(u_{\{i,j\}}\big)\Big)\\&=\sum_{i\in N}c_i(e_i)\phi_k\big(u_{\{i\}}\big)-\sum_{i\in N}\sum_{j\in N\setminus\{i\}}r_{ij}(e_{ji})\phi_k\big(u_{\{i,j\}}\big).\end{aligned} \tag{13}$$

In addition, it is known from the (NUP) property that

$$\phi_k\big(u_{\{i\}}\big)=\begin{cases}1, & i=k\\ 0, & otherwise\end{cases} \tag{14}$$

and from (ETE) and (NUP), that

$$\phi_k\big(u_{\{i,j\}}\big)=\begin{cases}1/2, & i=k, j=k, i\neq j\\ 0, & otherwise\end{cases} \tag{15}$$

Consequently, by substituting the values (14) and (15) in Eq. (13), the following is obtained:

$$\begin{aligned}\phi_k(e)&=c_k(e_k)-\sum_{j\in N\setminus\{k\}}r_{kj}(e_{jk})\phi_k\big(u_{\{k,j\}}\big)-\sum_{j\in N\setminus\{k\}}r_{jk}(e_{kj})\phi_k\big(u_{\{j,k\}}\big)\\&=c_k(e_k)-\frac{1}{2}\sum_{j\in N\setminus\{k\}}[r_{kj}(e_{jk})+r_{jk}(e_{kj})].\end{aligned}$$

Finally, it can be concluded that, for each agent $k\in N$,

$$\phi_k(e)=c_k(e_k)-\frac{1}{2}[R_k(N)+G_k(N)].$$

□

Proof of Proposition 2. To prove that the Shapley value coincides with the Nucleolus for PE-games, it is first necessary to describe the class of PS-games introduced by [39].

Denote by $M_ic(T)$ the marginal contribution of player $i\in T$, that is $M_ic(T)=c(T)-c(T\setminus\{i\})$, for all $i\in T\subseteq N$. A cost game (N,c) satisfies the PS property if for all $i\in N$ there exists $k_i\in\mathbb{R}$ such that $M_ic(T\cup\{i\})+M_ic(N\setminus T)=k_i$, for all $i\in N$ and all $T\subseteq N\setminus\{i\}$. Kar et al. [39] show that for PS games, the Shapley value coincides with the Nucleolus, i.e. $\phi_i(c)=\nu_i(c)=\frac{k_i}{2}$, for all $i\in N$.

Therefore, it only remains to show that (N,e,c) is a PS-game with $k_i=[c_i(e_i)-R_i(N)]+[c_i(e_i)-G_i(N)]$, for all $i\in N$.

First, it is straightforward to prove that $M_ic(T)=c_i(e_i)-\sum_{j\in T\setminus\{i\}}[r_{ji}(e_{ij})+r_{ij}(e_{ji})]$ for all $i\in T\subseteq N$. Second, we show that $M_ic(T\cup\{i\})+M_ic(N\setminus T)=[c_i(e_i)-R_i(N)]+[c_i(e_i)-G_i(N)]$ for all $i\in N$ and $T\subseteq N\setminus\{i\}$. Indeed, take a coalition $T\subseteq N$ and an agent $i\in T$. It is shown that $M_ic(T\cup\{i\})=c_i(e_i)-\sum_{j\in T}\big(r_{ji}(e_{ij})+r_{ij}(e_{ji})\big)$, and $M_ic(N\setminus T)=c_i(e_i)-\sum_{j\in N\setminus(T\cup\{i\})}\big(r_{ji}(e_{ij})+r_{ij}(e_{ji})\big)$. Therefore,

$M_ic(T\cup\{i\})+M_ic(N\setminus T)=2c_i(e_i)-\sum_{j\in N\setminus\{i\}}\big(r_{ji}(e_{ij})+r_{ij}(e_{ji})\big)=\big[c_i(e_i)-\sum_{j\in N\setminus\{i\}}r_{ij}(e_{ji})\big]+\big[c_i(e_i)-\sum_{j\in N\setminus\{i\}}r_{ji}(e_{ij})\big]$.

Hence, $M_ic(T\cup\{i\})+M_ic(N\setminus T)=[c_i(e_i)-R_i(N)]+[c_i(e_i)-G_i(N)]=k_i$), and so (N,e,c) is a PS game. □

Proof of Theorem 2. Consider the PE-game (N,e,c) associated with the PE-situation $(N,e,\{c_i(e_i),\{r_{ij}(e_{ij})\}_{j\in N\setminus\{i\}}\}_{i\in N})$. Take a family of weights $\omega^i_{ij},\omega^i_{ji}\in[0,1]$, for all $j\in N\setminus\{i\}$, such that $\omega^i_{ij}=1-\omega^j_{ij}$ and $\omega^i_{ji}=1-\omega^j_{ji}$, and $\Omega(e)$ the corresponding cost allocation with weighted pairwise reduction with $\Omega_i(e)=c_i(e_i)-\sum_{j\in N\setminus\{i\}}[\omega^i_{ij}r_{ij}(e_{ji})+\omega^i_{ji}r_{ji}(e_{ij})]$, for all $i\in N$. To prove that $\Omega(e)$ belongs to the core of (N,e,c) it must be checked that (1) $\sum_{i\in N}\Omega_i(e)=c(N)$, (2) $\sum_{i\in S}\Omega_i(e)\leq c(S)$, for all $S\subset N$.

We start by checking (1). Notice that $\sum_{i\in N}\Omega_i(e)=c(N)$ is equivalent to

$\sum_{i\in N}\sum_{j\in N\setminus\{i\}}[\omega^i_{ij}r_{ij}(e_{ji})+\omega^i_{ji}r_{ji}(e_{ij})]=\sum_{i\in N}\sum_{j\in N\setminus\{i\}}r_{ij}(e_{ji})$.

Indeed,

$\sum_{i\in N}\sum_{j\in N\setminus\{i\}}[\omega^i_{ij}r_{ij}(e_{ji})+\omega^i_{ji}r_{ji}(e_{ij})]=\sum_{i\in N}\sum_{j\in N\setminus\{i\}}(\omega^i_{ij}+\omega^j_{ij})r_{ij}(e_{ji})=\sum_{i\in N}\sum_{j\in N\setminus\{i\}}r_{ij}(e_{ji})$, where the last equality is due to $\omega^i_{ij}+\omega^j_{ij}=1$ for all $i,j\in N$.

Next we check (2). Take $S\subset N$. Notice now that $\sum_{i\in S}\Omega_i(e)\le c(S)$ is equivalent to

$\sum_{i\in S}\sum_{j\in N\setminus\{i\}}[\omega^i_{ij}r_{ij}(e_{ji})+\omega^i_{ji}r_{ji}(e_{ij})]-\sum_{i\in S}\sum_{j\in S\setminus\{i\}}r_{ij}(e_{ji})\ge 0.$

Indeed, an argument similar to that used in (1) leads to

$\sum_{i\in S}\sum_{j\in N\setminus\{i\}}[\omega^i_{ij}r_{ij}(e_{ji})+\omega^i_{ji}r_{ji}(e_{ij})]-\sum_{i\in S}\sum_{j\in S\setminus\{i\}}r_{ij}(e_{ji})=$
$\sum_{i\in S}\sum_{j\in S\setminus\{i\}}[\omega^i_{ij}r_{ij}(e_{ji})+\omega^i_{ji}r_{ji}(e_{ij})]+$
$\sum_{i\in S}\sum_{j\in N\setminus S\cup\{i\}}[\omega^i_{ij}r_{ij}(e_{ji})+\omega^i_{ji}r_{ji}(e_{ij})]-\sum_{i\in S}\sum_{j\in S\setminus\{i\}}r_{ij}(e_{ji})=$
$\sum_{i\in S}\sum_{j\in S\setminus\{i\}}r_{ij}(e_{ji})+\sum_{i\in S}\sum_{j\in N\setminus S\cup\{i\}}[\omega^i_{ij}r_{ij}(e_{ji})+\omega^i_{ji}r_{ji}(e_{ij})]-\sum_{i\in S}\sum_{j\in S\setminus\{i\}}r_{ij}(e_{ji})=$
$\sum_{i\in S}\sum_{j\in N\setminus S\cup\{i\}}[\omega^i_{ij}r_{ij}(e_{ji})+\omega^i_{ji}r_{ji}(e_{ij})]\ge 0.$ □

Appendix B

Proof of Proposition 3

To prove this result it is necessary to analyze $c(N)$ as a function of e. First, It is easy to prove that $c(N)$ is strictly convex in e_{ij} for all $i,j\in N, i\neq j$. Indeed, $\frac{\partial^2 c(N)}{\partial e_{ij}^2}=\frac{\partial^2 c_i(e_i)}{\partial e_{ij}^2}-\frac{\partial^2 r_{ji}(e_{ij})}{\partial e_{ij}^2}>0$, because $\frac{\partial^2 c_i(e_i)}{\partial e_{ij}^2}>0$ and $\frac{\partial^2 r_{ji}(e_{ij})}{\partial e_{ij}^2}<0$. Thus, there is a unique effort profile $\tilde{e}$ that minimizes $c(N)$.

Second, we focus on finding this efficient effort profile $\tilde{e}$. Note that the derivative $\frac{\partial c(N)}{\partial e_{ij}}=\frac{\partial c_i(e_i)}{\partial e_{ij}}-\frac{\partial r_{ji}(e_{ij})}{\partial e_{ij}}$ only depends on e_{ij} because $\frac{\partial c_i^2(e_i)}{\partial e_{ij}\partial e_{ih}}=0$ for all $h\neq i,j$. Therefore, if $\frac{\partial c_i(e_i)}{\partial e_{ij}}>\frac{\partial r_{ji}(e_{ij})}{\partial e_{ij}}$ for all $e_{ij}\in[0,1]$, then the function $c(N)$ is increasing in e_{ij}, which implies that $\tilde{e}_{ij}=0$. Analogously, if $\frac{\partial c_i(e_i)}{\partial e_{ij}}>\frac{\partial r_{ji}(e_{ij})}{\partial e_{ij}}$ for all $e_{ij}\in[0,1]$, then $\tilde{e}_{ij}=1$. Finally, if there is a solution of $\frac{\partial c_i(e_i)}{\partial e_{ij}}=\frac{\partial r_{ji}(e_{ij})}{\partial e_{ij}}$, that solution is $\tilde{e}_{ij}$. □

Proof of Lemma 1

Consider the non-cooperative game $(N,\{E_i\}_{i\in N},\{\Omega_i\}_{i\in N})$. To learn the optimal level of effort $\hat{e}_{ij}$ that agent i must exert to reduce the costs of agent j in this game, it is necessary to analyze the function $\Omega_i(e)=c_i(e_i)-\sum_{j\in N\setminus\{i\}}[\omega^i_{ij}r_{ij}(e_{ji})+\omega^i_{ji}r_{ji}(e_{ij})]$ for all $i\in N$ with $\omega^i_{ij},\omega^i_{ji}\in[0,1]$, $i,j\in N, i\neq j$, such that $\omega^i_{ij}=1-\omega^j_{ij}$ and $\omega^i_{ji}=1-\omega^j_{ji}$.

As above, we also prove that the function $\Omega_i(e)$ is strictly convex in e_{ij}. Indeed, $\frac{\partial^2\Omega(e)}{\partial e_{ij}^2}=\frac{\partial^2 c_i(e_i)}{\partial e_{ij}^2}-\omega^i_{ji}\frac{\partial^2 r_{ji}(e_{ij})}{\partial e_{ij}^2}>0$ because $\frac{\partial^2 c_i(e_i)}{\partial e_{ij}^2}>0$ and $\frac{\partial^2 r_{ji}(e_{ij})}{\partial e_{ij}^2}<0$. Hence, there is a unique optimal level of effort $\hat{e}$.

Again, we focus on finding this optimal level of effort $\hat{e}$. We know that $\frac{\partial\Omega_i(e)}{\partial e_{ij}}=\frac{\partial c_i(e_i)}{\partial e_{ij}}-\omega^i_{ji}\frac{\partial r_{ji}(e_{ij})}{\partial e_{ij}}$, but $\frac{\partial c_i(e_i)}{\partial e_{ij}}$ only depends on e_{ij}, because $\frac{\partial c_i^2(e_i)}{\partial e_{ij}\partial e_{ih}}=0$ for all $h\neq i,j$. Moreover, for all $e_{ij}\in[0,1]$, $\frac{\partial\Omega_{ii}(e)}{\partial e_{ij}}\ge 0\iff\frac{\partial c_i(e_i)}{\partial e_{ij}}\ge\omega^i_{ji}\frac{\partial r_{ji}(e_{ij})}{\partial e_{ij}}$.

Therefore, if $\frac{\partial c_i(e_i)}{\partial e_{ij}}>\omega^i_{ji}\frac{\partial r_{ji}(e_{ij})}{\partial e_{ij}}$ for all $e_{ij}\in[0,1]$, then $\hat{e}_{ij}=0$. If $\frac{\partial c_i(e_i)}{\partial e_{ij}}<\omega^i_{ji}\frac{\partial r_{ji}(e_{ij})}{\partial e_{ij}}$ for all $e_{ij}\in[0,1]$, then $\hat{e}_{ij}=1$. Finally, if there is a solution of $\frac{\partial c_i(e_i)}{\partial e_{ij}}=\omega^i_{ji}\frac{\partial r_{ji}(e_{ij})}{\partial e_{ij}}$, that solution is $\hat{e}_{ij}$ and is unique. Hence, there is a unique optimal level of effort. □

Proof of Theorem 3

Now consider the non-cooperative game $(N,\{E_i\}_{i\in N},\{H_i\}_{i\in N})$. Note that, both derivative functions $\frac{\partial c_i(e_i)}{\partial e_{ij}}$ and $\frac{\partial r_{ji}(e_{ij})}{\partial e_{ij}}$ only depend on e_{ij}. Thus, by Lemma, the optimal level of effort of a particular agent $i\in N$ with another particular agent $j\in N\setminus\{i\}$, i.e. $\hat{e}_{ij}$, is independent of any other effort made by i or by any other agent. Thus, the equilibrium is also characterized by Lemma with $\omega^i_{ji}=1$ for $i,j\in N, i\neq j$. Comparing Lemma 1 with Proposition, it follows directly that the equilibrium must also be efficient. □

Proof of Corollary 1

This is straightforward from the proof of **Theorem 3** □

Proof of Proposition 4

Take $A^*(e)$ the allocation rule in WPAR with α^*_{ij} for all $i,j\in N$ which induces the effort profile e^{*A^*} that minimizes the cost of the grand coalition. Since WPAR is a subfamily of WPR in which $\omega^i_{ij}=\omega^j_{ij}=\alpha_{ij}\in[0,1]$ for all $i,j\in N$, by Lemma 1 the optimal level of effort for $A^*(e)$ can be also characterized.

Thus, the efforts are optimal in equilibrium and so e^{*A^*} must hold that

$e^{*A^*}_{ij}=0$ if and only if $\frac{\partial c_i(e_i)}{\partial e_{ij}}>\alpha^*_{ij}\frac{\partial r_{ji}(e_{ij})}{\partial e_{ij}}$, for all $e_{ij}\in[0,1]$,

$e^{*A^*}_{ij}=1$ if and only if $\frac{\partial c_i(e_i)}{\partial e_{ij}}<\alpha^*_{ij}\frac{\partial r_{ji}(e_{ij})}{\partial e_{ij}}$, for all $e_{ij}\in[0,1]$,

Otherwise, $e^{*A^*}_{ij}\in(0,1)$ so $\left.\frac{\partial c_i(e_i)}{\partial e_{ij}}\right|_{e_{ij}=e^{*A^*}_{ij}}=\alpha^*_{ij}\left.\frac{\partial r_{ji}(e_{ij})}{\partial e_{ij}}\right|_{e_{ij}=e^{*A^*}_{ij}}$ holds.

Comparing the above expressions with Proposition 3 and taking into account that $\frac{\partial c_i(e_i)}{\partial e_{ij}}$ is a positive increasing function, $\frac{\partial r_{ji}(e_{ij})}{\partial e_{ij}}$ a positive decreasing function, and $\alpha^*_{ij}\in[0,1]$, it can be concluded that $\tilde{e}_{ij}\ge e^{*A^*}_{ij}$ for all $i,j\in N$. □

Appendix C

Theorem 4, in Section 5, characterizes all possible types of effort equilibrium according to the value of the parameter α_{ij}, for all $i,j\in N, i\neq j$. Before proving this theorem, we consider a previous Lemma that is very useful for latter results. It characterizes the optimal effort level for agent $i\in N$ in the first stage non-cooperative game.

Lemma 2. *Let $(N,\{E_i\}_{i\in N},\{A_i\}_{i\in N})$ be the effort game, with $\hat{e}_{ij}$ being the optimal level of effort that agent i exerts to reduce the costs of agent j. Thus,*

1. *$\hat{e}_{ij}=0$ if and only if $\alpha_{ij}\le\underline{\alpha}_{ij}$*
2. *There is a unique $\hat{e}_{ij}\in(0,1)$ that holds $c'_i(\hat{e}_{ij})-\alpha_{ij}r'_{ji}(\hat{e}_{ij})=0$ if and only if $\underline{\alpha}_{ij}<\alpha_{ij}<\bar{\alpha}_{ij}$.*
3. *$\hat{e}_{ij}=1$ if and only if $\alpha_{ij}\ge\bar{\alpha}_{ij}$.*

Proof. First, remember that the cost function $A_i(e)$ is convex for all $i\in N$. To obtain the optimal effort, the derivative of this function can be analyzed with respect to e_{ij} for any $j\in N\setminus\{i\}$. It must be noted that $\frac{\partial A_i(e)}{\partial e_{ij}}>0\iff c'_i(e_{ij})>\alpha_{ij}r'_{ji}(e_{ij})$ for all $e_{ij}\in[0,1]$, which is a necessary and sufficient condition for $\hat{e}_{ij}=0$ to be the optimal effort.[12]

We begin by proving point 1. Note that $\underline{\alpha}_{ij}=\frac{c'_i(0)}{r'_{ji}(0)}<\frac{c'_i(e_{ij})}{r'_{ji}(e_{ij})}$ because $c'_i>0$, $r'_{ji}>0$, $c''_i>0$, and $r''_{ji}<0$. Thus, $c'_i(e_{ij})$ is a positive and increasing function, and $r'_{ji}(e_{ij})$ a positive and decreasing function, so for any $e_{ij}>0$, $c'_i(0)<c'_i(e_{ij})$ and $r'_{ji}(0)>r'_{ji}(e_{ij})$. Therefore, $\alpha_{ij}\le\underline{\alpha}_{ij}\iff c'_i(e_{ij})>\alpha_{ij}r'_{ji}(e_{ij})$ for all $e_{ij}>0\iff\hat{e}_{ij}=0$.

The demonstration in point 3 is similar to that of point 1. The above arguments are the same and only the signs of the inequalities change.

To end the proof, we prove point 2. First, we show that there is a unique $\hat{e}_{ij}\in(0,1)$ such that $c'_i(\hat{e}_{ij})=\alpha_{ij}r'_{ji}(\hat{e}_{ij})$, which is the

[12] This occurs because $A_i(e)$ is an increasing function in e_{ij} and the minimum value is obtained for $\hat{e}_{ij}=0$, which is the optimal effort for agent i.

unique optimal effort because $\frac{\partial A_i(e)}{\partial e_{ij}}\Big|_{e_i=\hat{e}_{ij}} = 0$ and $A_i(e)$ is a convex function. In addition, $c_i'(e_{ij})$ is a positive increasing function and $r_{ji}'(e_{ij})$ a positive decreasing function, in $e_{ij} \in [0,1]$. This means that equation $\frac{\partial A_i(e)}{\partial e_{ij}} = c_i'(e_{ij}) - \alpha_{ij} r_{ji}'(e_{ij}) = 0$ has a unique root, which belongs to (0,1) if and only if $\alpha_{ij} \in (\underline{\alpha}_{ij}, \bar{\alpha}_{ij})$. Note that if $\alpha_{ij} \in (\underline{\alpha}_{ij}, \bar{\alpha}_{ij})$ then $c_i'(0) < \alpha_{ij} r_{ji}'(0)$ and $c_i'(1) > \alpha_{ij} r_{ji}'(1)$, and so there is a unique point $\hat{e}_{ij}$ where $c_i'(\hat{e}_{ij}) = \alpha_{ij} r_{ji}'(\hat{e}_{ij})$. $\square$

Proof of Theorem 4. As we already mention, the optimum $\hat{e}_{ij}$ is independent of other efforts. Therefore, the equilibrium effort is determined by Lemma 2. In addition, we want to characterize the effort equilibrium according to the value of the parameter α_{ij}. Thus, in the case of agent j, $\underline{\alpha}_{ji} < \alpha_{ji} < \bar{\alpha}_{ji} \Leftrightarrow \underline{\alpha}_{ji} < 1 - \alpha_{ij} < \bar{\alpha}_{ji} \Leftrightarrow 1 - \bar{\alpha}_{ji} < \alpha_{ij} < 1 - \underline{\alpha}_{ji}$. $\square$

The next corollary shows how the pairwise equilibrium efforts e_{ij}^* depend on α_{ij}, for all $i, j \in N, i \neq j$. As expected, as the proportion of aggregate cost reduction obtained by an agent increases, the effort that agent exerts also increases (or at least stays the same).

Corollary 2. *Let $(N, \{E_i\}_{i\in N}, \{A_i\}_{i\in N})$ be the effort game and (e_{ij}^*, e_{ji}^*) the pairwise efforts equilibrium. Thus,*

- $\frac{\partial e_{ij}^*}{\partial \alpha_{ij}} > 0$, *if* $\alpha_{ij} \in (\underline{\alpha}_{ij}, \bar{\alpha}_{ij})$; $\frac{\partial e_{ij}^*}{\partial \alpha_{ij}} = 0$, *otherwise.*
- $\frac{\partial e_{ji}^*}{\partial \alpha_{ij}} < 0$*if* $\alpha_{ij} \in (1 - \bar{\alpha}_{ji}, 1 - \underline{\alpha}_{ji})$; $\frac{\partial e_{ji}^*}{\partial \alpha_{ij}} = 0$, *otherwise.*

Proof. By the implicit function theorem, $\frac{\partial e_{ij}^*}{\partial \alpha_{ij}} = -\frac{\frac{\partial (c_i'(e_{ij}^*) - \alpha_{ij} r_{ji}'(e_{ij}^*))}{\partial \alpha_{ij}}}{\frac{\partial (c_i'(e_{ij}^*) - \alpha_{ij} r_{ji}'(e_{ij}^*))}{\partial e_{ij}^*}} = \frac{r_{ji}'(e_{ij}^*)}{c_i''(e_{ij}^*) - \alpha_{ij} r_{ji}''(e_{ij}^*)} > 0$, because $r_{ji}'(e_{ij}^*) > 0$, $c_i''(e_{ij}^*) > 0$, and $r_{ji}''(e_{ij}^*) < 0$. Thus, for any $\alpha_{ij} \le \underline{\alpha}_{ij}$, Lemma 2 implies that $e_{ij}^* = 0$, thus, $\frac{\partial e_{ij}^*}{\partial \alpha_{ij}} = 0$. However, if $\alpha_{ij} \in (\underline{\alpha}_{ij}, \bar{\alpha}_{ij})$, then $e_{ij}^* \in (0,1)$ and $\frac{\partial e_{ij}^*}{\partial \alpha_{ij}} > 0$. Finally, if $\alpha_{ij} \ge \bar{\alpha}_{ij}$, then $e_{ij}^* = 1$ and $\frac{\partial e_{ij}^*}{\partial \alpha_{ij}} = 0$. Analogously, if $\alpha_{ji} \le \underline{\alpha}_{ji} \iff \alpha_{ij} \ge 1 - \underline{\alpha}_{ji}$, then $e_{ji}^* = 0$ and $\frac{\partial e_{ji}^*}{\partial \alpha_{ij}} = 0$, if $\alpha_{ji} \in (\underline{\alpha}_{ji}, \bar{\alpha}_{ji}) \iff \alpha_{ij} \in (1 - \bar{\alpha}_{ji}, 1 - \underline{\alpha}_{ji})$, then $e_{ji}^* \in (0,1)$ and $\frac{\partial e_{ji}^*}{\partial \alpha_{ij}} < 0$. Finally, if $\alpha_{ji} \ge \bar{\alpha}_{ji} \iff \alpha_{ij} \le 1 - \bar{\alpha}_{ji}$, then $e_{ij}^* = 1$ and $\frac{\partial e_{ji}^*}{\partial \alpha_{ij}} = 0$. $\square$

Theorem 5, in Section 5, provides the weights α_{ij} that minimizes function $L_{ij}^*(\alpha_{ij})$, and the efficient effort equilibrium. To solve the above optimization problem it is necessary to know the function $L_{ij}^*(\alpha_{ij})$ very accurately.

To demonstrate Theorem 5, three technical lemmas are needed first. Lemmas 3, 4, and 5 characterize the derivatives $\frac{\partial (A_i^*(\alpha_{ij}))}{\partial \alpha_{ij}}$, $\frac{\partial (L_{ij}^*(\alpha_{ij}))}{\partial \alpha_{ij}}$, and $\frac{\partial^2 (L_{ij}^*(\alpha_{ij}))}{\partial \alpha_{ij}^2}$ respectively.

The first lemma shows how the optimal cost function of agent $i \in N$ depends on α_{ij}. Henceforth, to simplify notation, we consider that for any $i, j \in N$, $\frac{\partial r_{ij}(e_{ji}^*)}{\partial e_{ji}^*}$ and $\frac{\partial c_i(e_i^*)}{\partial e_{ij}^*}$ stand for derivatives $\frac{\partial r_{ij}(e_{ji})}{\partial e_{ji}}$ and $\frac{\partial c_i(e_i)}{\partial e_{ij}}$ evaluated in the unique effort equilibrium.

Lemma 3. *Let $(N, \{E_i\}_{i\in N}, \{A_i\}_{i\in N})$ be the effort game and e^* the effort equilibrium. Thus,*

1. $$\frac{\partial (A_i(e^*))}{\partial \alpha_{ij}} = \frac{\partial (A_i^*(\alpha_{ij}))}{\partial \alpha_{ij}} = \begin{cases} -r_{ij}(e_{ji}^*) - \alpha_{ij} \frac{\partial r_{ij}(e_{ji}^*)}{\partial e_{ji}^*} \frac{\partial e_{ji}^*}{\partial \alpha_{ij}} - r_{ji}(e_{ij}^*), & if \quad \alpha_{ij} \in (\underline{\alpha}_{ij}, \bar{\alpha}_{ij}) \\ -r_{ij}(e_{ji}^*) - r_{ji}(e_{ij}^*) < 0, & \text{otherwise} \end{cases}$$

2. $$\frac{\partial (A_j(e^*))}{\partial \alpha_{ij}} = \frac{\partial (A_j^*(1-\alpha_{ij}))}{\partial \alpha_{ij}} = \begin{cases} r_{ji}(e_{ij}^*) - (1-\alpha_{ij}) \frac{\partial r_{ji}(e_{ij}^*)}{\partial e_{ij}^*} \frac{\partial e_{ij}^*}{\partial \alpha_{ij}} + r_{ij}(e_{ji}^*), & if \quad \alpha_{ij} \in (1 - \bar{\alpha}_{ji}, 1 - \underline{\alpha}_{ji}) \\ r_{ji}(e_{ij}^*) + r_{ij}(e_{ji}^*) > 0, & \text{otherwise.} \end{cases}$$

Proof. It is known that $A_i(e^*) = c_i(e_i^*) - \sum_{z\in N\setminus\{i\}} \alpha_{iz}(r_{iz}(e_{zi}^*) + r_{zi}(e_{iz}^*))$, and $A_i^*(\alpha_{ij}) = c_i(e_i^*) - \alpha_{ij}(r_{ij}(e_{ji}^*) + r_{ji}(e_{ij}^*))$, thus
$\frac{\partial (A_i(e^*))}{\partial \alpha_{ij}} = \frac{\partial (A_i^*(\alpha_{ij}))}{\partial \alpha_{ij}} = \frac{\partial c_i(e_i^*)}{\partial e_{ij}^*} \frac{\partial e_{ij}^*}{\partial \alpha_{ij}} - r_{ij}(e_{ji}^*) - \alpha_{ij} \frac{\partial r_{ij}(e_{ji}^*)}{\partial e_{ji}^*} \frac{\partial e_{ji}^*}{\partial \alpha_{ij}} - r_{ji}(e_{ij}^*) - \alpha_{ij} \frac{\partial r_{ji}(e_{ij}^*)}{\partial e_{ij}^*} \frac{\partial e_{ij}^*}{\partial \alpha_{ij}}$
$= \left(\frac{\partial c_i(e_i^*)}{\partial e_{ij}^*} - \alpha_{ij} \frac{\partial r_{ji}(e_{ij}^*)}{\partial e_{ij}^*} \right) \frac{\partial e_{ij}^*}{\partial \alpha_{ij}} - r_{ij}(e_{ji}^*) - \alpha_{ij} \frac{\partial r_{ij}(e_{ji}^*)}{\partial e_{ji}^*} \frac{\partial e_{ji}^*}{\partial \alpha_{ij}} - r_{ji}(e_{ij}^*)$.

The first term of the above expression is always zero, i.e. $\left(\frac{\partial c_i(e_i^*)}{\partial e_{ij}^*} - \alpha_{ij} \frac{\partial r_{ji}(e_{ij}^*)}{\partial e_{ij}^*} \right) \frac{\partial e_{ij}^*}{\partial \alpha_{ij}} = 0$. To see this, note that if $\alpha_{ij} \in (\underline{\alpha}_{ij}, \bar{\alpha}_{ij})$, then $e_{ij}^* \in (0,1)$ by Lemma 2, so $\left(\frac{\partial c_i(e_i^*)}{\partial e_{ij}^*} - \alpha_{ij} \frac{\partial r_{ji}(e_{ij}^*)}{\partial e_{ij}^*} \right) = 0$ because it is evaluated in equilibrium. In the other case, where $\alpha_{ij} \le \underline{\alpha}_{ij}$ or $\alpha_{ij} \ge \bar{\alpha}_{ij}$, $e_{ij}^* = 0$ by Proposition 2, so $\frac{\partial e_{ji}^*}{\partial \alpha_{ij}} = 0$. Therefore, $\frac{\partial (A_i(e^*))}{\partial \alpha_{ij}} = -r_{ij}(e_{ji}^*) - \alpha_{ij} \frac{\partial r_{ij}(e_{ji}^*)}{\partial e_{ji}^*} \frac{\partial e_{ji}^*}{\partial \alpha_{ij}} - r_{ji}(e_{ij}^*)$.

It is known by assumption that $r_{ij}(e_{ji}^*) \ge 0$, $\frac{\partial r_{ij}(e_{ji}^*)}{\partial e_{ji}^*} > 0$. If $\alpha_{ij} \in (1 - \bar{\alpha}_{ji}, 1 - \underline{\alpha}_{ji})$, then by Proposition 2, $\frac{\partial e_{ji}^*}{\partial \alpha_{ij}} < 0$. However, if $\alpha_{ij} \notin (1 - \bar{\alpha}_{ji}, 1 - \underline{\alpha}_{ji})$ then, by Proposition 2, $\frac{\partial e_{ji}^*}{\partial \alpha_{ij}} = 0$, so $\frac{\partial (A_i(e^*))}{\partial \alpha_{ij}} = -r_{ij}(e_{ji}^*) - r_{ji}(e_{ij}^*)$.

The proof is analogous for $\frac{\partial (A_j(e^*))}{\partial \alpha_{ij}}$. $\square$

Notice that the effect of α_{ij} on the cost function of agent i could be positive or negative because of two simultaneous effects. First effect: As expected, if α_{ij} increases so does the proportion of cost reduction that agent i can obtain, and thus the cost function, $A_i(e^*)$, decreases. This decrease is measured by the term $-r_{ij}(e_{ji}^*) - r_{ji}(e_{ij}^*) < 0$ in the derivative. Second effect: When α_{ij} increases, the effort of agent j decreases in equilibrium, so the cost function of agent i increases. The term $-\alpha_{ij} \frac{\partial r_{ij}(e_{ji}^*)}{\partial e_{ji}^*} \frac{e_{ji}^*}{\partial \alpha_{ij}} > 0$ measures this second effect. The sum of these two effects determines the sign of the derivative. Therefore, an increase in the proportion of the aggregate cost reduction that an agent obtains could increase the cost of that agent if the second effect dominates the first. This is an interesting result: Giving too much to a particular agent could be not only worse for the aggregate cost but also for that particular agent.

The second lemma calculates the derivative of the aggregate cost function $L_{ij}^*(\alpha_{ij})$ in the effort equilibrium for any $i, j \in N$.

Lemma 4. *Let $(N, \{E_i\}_{i\in N}, \{A_i\}_{i\in N})$ be the effort game, and e^* the effort equilibrium. Thus,*

$$\frac{\partial (L_{ij}^*(\alpha_{ij}))}{\partial \alpha_{ij}} = \left(\frac{\partial c_j(e_j^*)}{\partial e_{ji}^*} - \frac{\partial r_{ij}(e_{ji}^*)}{\partial e_{ji}^*} \right) \frac{\partial e_{ji}^*}{\partial \alpha_{ij}} I_j + \left(\frac{\partial c_i(e_i^*)}{\partial e_{ij}^*} - \frac{\partial r_{ji}(e_{ij}^*)}{\partial e_{ij}^*} \right) \frac{\partial e_{ij}^*}{\partial \alpha_{ij}} I_i$$

where $I_i = \begin{cases} 1 & if \quad \alpha_{ij} \in (\underline{\alpha}_{ij}, \bar{\alpha}_{ij}) \\ 0 & \text{otherwise} \end{cases}$ *and* $I_j = \begin{cases} 1 & if \quad \alpha_{ij} \in (1 - \bar{\alpha}_{ji}, 1 - \underline{\alpha}_{ji}) \\ 0 & \text{otherwise} \end{cases}$.

Therefore, there are four possible cases:

- $\frac{\partial (L_{ij}^*(\alpha_{ij}))}{\partial \alpha_{ij}}$ *can be positive and/or negative if* $\alpha_{ij} \in (\underline{\alpha}_{ij}, \bar{\alpha}_{ij}) \cap (1 - \bar{\alpha}_{ji}, 1 - \underline{\alpha}_{ji})$
- $\frac{\partial (L_{ij}^*(\alpha_{ij}))}{\partial \alpha_{ij}} = 0$*if* $\alpha_{ij} \notin (\underline{\alpha}_{ij}, \bar{\alpha}_{ij}) \cup (1 - \bar{\alpha}_{ji}, 1 - \underline{\alpha}_{ji})$
- $\frac{\partial (L_{ij}^*(\alpha_{ij}))}{\partial \alpha_{ij}} > 0$ *if* $\alpha_{ij} \in (1 - \bar{\alpha}_{ji}, 1 - \underline{\alpha}_{ji}) \cap \big((0, \underline{\alpha}_{ij}) \cup (\bar{\alpha}_{ij}, 1)\big)$
- $\frac{\partial (L_{ij}^*(\alpha_{ij}))}{\partial \alpha_{ij}} < 0$ *if* $\alpha_{ij} \in \big((0, 1 - \bar{\alpha}_{ji}) \cup (1 - \underline{\alpha}_{ji}, 1)\big) \cap \big(\underline{\alpha}_{ij}, \bar{\alpha}_{ij}\big)$

Proof. From (6), we calculate that $\frac{\partial(L^*_{ij}(\alpha_{ij}))}{\partial\alpha_{ij}} = \left(\frac{\partial c_j(e^*_j)}{\partial e^*_{ji}} - \frac{\partial r_{ij}(e^*_{ji})}{\partial e^*_{ji}}\right)\frac{\partial e^*_{ji}}{\partial \alpha_{ij}} + \left(\frac{\partial c_i(e^*_i)}{\partial e^*_{ij}} - \frac{\partial r_{ji}(e^*_{ij})}{\partial e^*_{ij}}\right)\frac{\partial e^*_{ij}}{\partial\alpha_{ij}}$. Simplifying for the different subsets of α_{ij}, the following emerges:

1. if $\alpha_{ij} \in (\underline{\alpha}_{ij}, \bar\alpha_{ij}) \cap (1-\bar\alpha_{ji}, 1-\underline{\alpha}_{ji})$ then, by Theorem 4, $e^*_{ji}\in(0,1)$ and $e^*_{ij}\in(0,1)$, thus, by Corollary 2, $\frac{\partial e^*_{ji}}{\partial\alpha_{ij}}<0$ and $\frac{\partial e^*_{ij}}{\partial\alpha_{ij}}>0$. In addition, since $\frac{\partial c_j(e^*_j)}{\partial e^*_{ji}} - (1-\alpha_{ij})\frac{\partial r_{ij}(e^*_{ji})}{\partial e^*_{ji}} = 0$ and $\frac{\partial c_i(e^*_i)}{\partial e^*_{ij}} - \alpha_{ij}\frac{\partial r_{ji}(e^*_{ij})}{\partial e^*_{ij}}=0$, it follows that $\frac{\partial c_j(e^*_j)}{\partial e^*_{ji}} - \frac{\partial r_{ij}(e^*_{ji})}{\partial e^*_{ji}} < 0$ and $\frac{\partial c_i(e^*_i)}{\partial e^*_{ij}} - \frac{\partial r_{ji}(e^*_{ij})}{\partial e^*_{ij}}<0$. Therefore, $\frac{\partial(L^*_{ij}(\alpha_{ij}))}{\partial\alpha_{ij}} = \left(\frac{\partial c_j(e^*_j)}{\partial e^*_{ji}} - \frac{\partial r_{ij}(e^*_{ji})}{\partial e^*_{ji}}\right)\frac{\partial e^*_{ji}}{\partial \alpha_{ij}} + \left(\frac{\partial c_i(e^*_i)}{\partial e^*_{ij}} - \frac{\partial r_{ji}(e^*_{ij})}{\partial e^*_{ij}}\right)\frac{\partial e^*_{ij}}{\partial\alpha_{ij}}$, which can be positive or negative in this case.
2. if $\alpha_{ij}\notin (\underline\alpha_{ij},\bar\alpha_{ij})\cup(1-\bar\alpha_{ji},1-\underline\alpha_{ji})$ then, by Theorem 4, $e^*_{ji}\in\{0,1\}$ and $e^*_{ij}\in\{0,1\}$, and by Corollary, $\frac{\partial e^*_{ji}}{\partial\alpha_{ij}} = \frac{\partial e^*_{ij}}{\partial\alpha_{ij}}=0$. Therefore, $\frac{\partial(L^*_{ij}(\alpha_{ij}))}{\partial\alpha_{ij}}=0$.
3. if $\alpha_{ij}\in(1-\bar\alpha_{ji},1-\underline\alpha_{ji})\cap\big((0,\underline\alpha_{ij})\cup(\bar\alpha_{ij},1)\big)$, then, as above, $\frac{\partial(L^*_{ij}(\alpha_{ij}))}{\partial\alpha_{ij}} = \left(\frac{\partial c_j(e^*_j)}{\partial e^*_{ji}} - \frac{\partial r_{ij}(e^*_{ji})}{\partial e^*_{ji}}\right)\frac{\partial e^*_{ji}}{\partial \alpha_{ij}} > 0$.
4. if $\alpha_{ij}\in\big((0,1-\bar\alpha_{ji})\cup(1-\underline\alpha_{ji},1)\big)\cap\big(\underline\alpha_{ij},\bar\alpha_{ij}\big)$ then $\frac{\partial(L^*_{ij}(\alpha_{ij}))}{\partial\alpha_{ij}} = \left(\frac{\partial c_i(e^*_i)}{\partial e^*_{ij}} - \frac{\partial r_{ji}(e^*_{ij})}{\partial e^*_{ij}}\right)\frac{\partial e^*_{ij}}{\partial\alpha_{ij}} < 0$.

□

The derivative is a piecewise function and there are intervals where its sign is independent of the particular form of the functions of the game. For those cases, it is straightforward to find the optimal α_{ij} that minimizes the function $L^*_{ij}(\alpha_{ij})$. In those intervals, the derivative is either positive, negative or zero throughout the interval. These cases are respectively $\frac{\partial(L^*_{ij}(\alpha_{ij}))}{\partial\alpha_{ij}} = \left(\frac{\partial c_j(e^*_j)}{\partial e^*_{ji}} - \frac{\partial r_{ij}(e^*_{ji})}{\partial e^*_{ji}}\right)\frac{\partial e^*_{ji}}{\partial \alpha_{ij}} > 0$, $\frac{\partial(L^*_{ij}(\alpha_{ij}))}{\partial\alpha_{ij}} = \left(\frac{\partial c_i(e^*_i)}{\partial e^*_{ij}} - \frac{\partial r_{ji}(e^*_{ij})}{\partial e^*_{ij}}\right)\frac{\partial e^*_{ij}}{\partial\alpha_{ij}} < 0$, and $\frac{\partial(L^*_{ij}(\alpha_{ij}))}{\partial\alpha_{ij}} = 0$. However, there is an interval where the sign of the derivative depends on the particular form of functions of the game. In this particular case $\frac{\partial(L^*_{ij}(\alpha_{ij}))}{\partial\alpha_{ij}} = \left(\frac{\partial c_j(e^*_j)}{\partial e^*_{ji}} - \frac{\partial r_{ij}(e^*_{ji})}{\partial e^*_{ji}}\right)\frac{\partial e^*_{ji}}{\partial \alpha_{ij}} + \left(\frac{\partial c_i(e^*_i)}{\partial e^*_{ij}} - \frac{\partial r_{ji}(e^*_{ij})}{\partial e^*_{ij}}\right)\frac{\partial e^*_{ij}}{\partial\alpha_{ij}}$. This occurs when $\alpha_{ij}\in(\underline\alpha_{ij},\bar\alpha_{ij})\cap(1-\bar\alpha_{ji},1-\underline\alpha_{ji})$, which implies that in equilibrium simultaneously $0<e^*_{ij}<1$ and $0<e^*_{ji}<1$. Therefore, in this case only, the derivative may be zero for some α_{ij} within this interval. In that case, the second derivative is needed to solve the optimization problem.

The third Lemma shows that the aggregate cost function $L^*_{ij}(\alpha_{ij})$ is convex in α_{ij}. Two additional assumptions about third derivatives need to be introduced.

Lemma 5. *Let $(N,\{E_i\}_{i\in N},\{A_i\}_{i\in N})$ be the effort game, e^* the effort equilibrium, and $\frac{\partial^3 c_i(e^*_i)}{\partial e^{*3}_{ij}}>0$ and $\frac{\partial^3 r_{ji}(e^*_{ij})}{\partial e^{*3}_{ij}}<0$, for any $i,j\in N$. Thus $\frac{\partial^2 L^*_{ij}(\alpha_{ij}))}{\partial\alpha^{*2}_{ij}}>0$ for all $\alpha_{ij}\in(\underline\alpha_{ij},\bar\alpha_{ij})\cap(1-\bar\alpha_{ji},1-\underline\alpha_{ji})$.*

Proof. Take $\alpha_{ij}\in(\underline\alpha_{ij},\bar\alpha_{ij})\cap(1-\bar\alpha_{ji},1-\underline\alpha_{ji})$. Thus,

$$\frac{\partial^2(L^*_{ij}(\alpha_{ij}))}{\partial\alpha^2_{ij}} = \frac{\partial^2\left[\left(\frac{\partial c_j(e^*_j)}{\partial e^*_{ji}} - \frac{\partial r_{ij}(e^*_{ji})}{\partial e^*_{ji}}\right)\frac{\partial e^*_{ji}}{\partial \alpha_{ij}} + \left(\frac{\partial c_i(e^*_i)}{\partial e^*_{ij}} - \frac{\partial r_{ji}(e^*_{ij})}{\partial e^*_{ij}}\right)\frac{\partial e^*_{ij}}{\partial\alpha_{ij}}\right]}{\partial\alpha^2_{ij}}$$

$$\left(\frac{\partial^2 c_j(e^*_j)}{\partial e^*_{ji}\partial\alpha_{ij}} - \frac{\partial^2 r_{ij}(e^*_{ji})}{\partial e^*_{ji}\partial\alpha_{ij}}\right)\frac{\partial e^*_{ji}}{\partial\alpha_{ij}} + \left(\frac{\partial c_j(e^*_j)}{\partial e^*_{ji}} - \frac{\partial r_{ij}(e^*_{ji})}{\partial e^*_{ji}}\right)\frac{\partial^2 e^*_{ji}}{\partial\alpha^2_{ij}}$$
$$+\left(\frac{\partial^2 c_i(e^*_i)}{\partial e^*_{ij}\partial\alpha_{ij}} - \frac{\partial^2 r_{ji}(e^*_{ij})}{\partial e^*_{ij}\partial\alpha_{ij}}\right)\frac{\partial e^*_{ij}}{\partial\alpha_{ij}} + \left(\frac{\partial c_i(e^*_i)}{\partial e^*_{ij}} - \frac{\partial r_{ji}(e^*_{ij})}{\partial e^*_{ij}}\right)\frac{\partial^2 e^*_{ij}}{\partial\alpha^2_{ij}}$$

$$= \left(\frac{\partial^2 c_j(e^*_j)}{\partial^2 e^*_{ji}}\frac{\partial e^*_{ji}}{\partial\alpha_{ij}} - \frac{\partial^2 r_{ij}(e^*_{ji})}{\partial^2 e^*_{ji}}\frac{\partial e^*_{ji}}{\partial\alpha_{ij}}\right)\frac{\partial e^*_{ji}}{\partial\alpha_{ij}} + \left(\frac{\partial c_j(e^*_j)}{\partial e^*_{ji}} - \frac{\partial r_{ij}(e^*_{ji})}{\partial e^*_{ji}}\right)\frac{\partial^2 e^*_{ji}}{\partial\alpha^2_{ij}}$$
$$+\left(\frac{\partial^2 c_i(e^*_i)}{\partial^2 e^*_{ij}}\frac{\partial e^*_{ij}}{\partial\alpha_{ij}} - \frac{\partial^2 r_{ji}(e^*_{ij})}{\partial^2 e^*_{ij}}\frac{\partial e^*_{ij}}{\partial\alpha_{ij}}\right)\frac{\partial e^*_{ij}}{\partial\alpha_{ij}} + \left(\frac{\partial c_i(e^*_i)}{\partial e^*_{ij}} - \frac{\partial r_{ji}(e^*_{ij})}{\partial e^*_{ij}}\right)\frac{\partial^2 e^*_{ij}}{\partial\alpha^2_{ij}}$$
$$= \left(\frac{\partial^2 c_j(e^*_j)}{\partial^2 e^*_{ji}} - \frac{\partial^2 r_{ij}(e^*_{ji})}{\partial^2 e^*_{ji}}\right)\left(\frac{\partial e^*_{ji}}{\partial\alpha_{ij}}\right)^2 + \left(\frac{\partial c_j(e^*_j)}{\partial e^*_{ji}} - \frac{\partial r_{ij}(e^*_{ji})}{\partial e^*_{ji}}\right)\frac{\partial^2 e^*_{ji}}{\partial\alpha^2_{ij}}$$
$$+\left(\frac{\partial^2 c_i(e^*_i)}{\partial^2 e^*_{ij}} - \frac{\partial^2 r_{ji}(e^*_{ij})}{\partial^2 e^*_{ij}}\right)\left(\frac{\partial e^*_{ij}}{\partial\alpha_{ij}}\right)^2 + \left(\frac{\partial c_i(e^*_i)}{\partial e^*_{ij}} - \frac{\partial r_{ji}(e^*_{ij})}{\partial e^*_{ij}}\right)\frac{\partial^2 e^*_{ij}}{\partial\alpha^2_{ij}} > 0$$

Now we prove that $\frac{\partial^2 e^*_{ji}}{\partial\alpha^2_{ij}}<0$ and $\frac{\partial^2 e^*_{ij}}{\partial\alpha^2_{ij}}<0$, so $\frac{\partial^2(L^*_{ij}(\alpha_{ij}))}{\partial\alpha^2_{ij}}>0$.

We first prove that $\frac{\partial^2 e^*_{ji}}{\partial\alpha^2_{ij}}<0$. It is known that

$$\frac{\partial A_j(e^*)}{\partial e_{ji}} = \frac{\partial c_j(e^*_j)}{\partial e^*_{ji}} - (1-\alpha_{ij})\frac{\partial r_{ij}(e^*_{ji})}{\partial e^*_{ji}} = 0$$

We now derive the second term regarding α_{ij}.

$$\frac{\partial^2 c_j(e^*_j)}{\partial e^{*2}_{ji}}\frac{\partial e^*_{ji}}{\partial\alpha_{ij}} + \frac{\partial r_{ij}(e^*_{ji})}{\partial e^*_{ji}} - (1-\alpha_{ij})\frac{\partial^2 r_{ij}(e^*_{ji})}{\partial e^{*2}_{ji}}\frac{\partial e^*_{ji}}{\partial\alpha_{ij}} = 0$$

We now do the same for α_{ij}.

$$\left(\frac{\partial^3 c_j(e^*_j)}{\partial e^{*3}_{ji}}\left(\frac{\partial e^*_{ji}}{\partial\alpha_{ij}}\right)^2 + \frac{\partial^2 c_j(e^*_j)}{\partial e^{*2}_{ji}}\frac{\partial^2 e^*_{ji}}{\partial\alpha^2_{ji}}\right) + \frac{\partial^2 r_{ij}(e^*_{ji})}{\partial e^{*2}_{ji}}\frac{\partial e^*_{ji}}{\partial\alpha_{ij}}$$
$$-(1-\alpha_{ij})\left(\frac{\partial^3 r_{ij}(e^*_{ji})}{\partial e^{*3}_{ji}}\left(\frac{\partial e^*_{ji}}{\partial\alpha_{ij}}\right)^2 + \frac{\partial^2 r_{ij}(e^*_{ji})}{\partial e^{*2}_{ji}}\frac{\partial^2 e^*_{ji}}{\partial\alpha^2_{ji}}\right) = 0$$
$$\left(\frac{\partial^2 c_j(e^*_j)}{\partial e^{*2}_{ji}} - (1-\alpha_{ij})\frac{\partial^2 r_{ij}(e^*_{ji})}{\partial e^{*2}_{ji}}\right)\frac{\partial^2 e^*_{ji}}{\partial\alpha^2_{ji}} + \frac{\partial^2 r_{ij}(e^*_{ji})}{\partial e^{*2}_{ji}}\frac{\partial e^*_{ji}}{\partial\alpha_{ij}}$$
$$+\left(\frac{\partial^3 c_j(e^*_j)}{\partial e^{*3}_{ji}} - (1-\alpha_{ij})\frac{\partial^3 r_{ij}(e^*_{ji})}{\partial e^{*3}_{ji}}\right)\left(\frac{\partial e^*_{ji}}{\partial\alpha_{ij}}\right)^2 = 0$$
$$\frac{\partial^2 e^*_{ji}}{\partial\alpha^2_{ij}} = \frac{-\frac{\partial^2 r_{ij}(e^*_{ji})}{\partial e^{*2}_{ji}}\frac{\partial e^*_{ji}}{\partial\alpha_{ij}} - \left(\frac{\partial^3 c_j(e^*_j)}{\partial e^{*3}_{ji}} - (1-\alpha_{ij})\frac{\partial^3 r_{ij}(e^*_{ji})}{\partial e^{*3}_{ji}}\right)\left(\frac{\partial e^*_{ji}}{\partial\alpha_{ij}}\right)^2}{\frac{\partial^2 c_j(e^*_j)}{\partial e^{*2}_{ji}} - (1-\alpha_{ij})\frac{\partial^2 r_{ij}(e^*_{ji})}{\partial e^{*2}_{ji}}}$$

Clearly, this expression is lower than zero if $\frac{\partial^3 c_j(e^*_j)}{\partial e^{*3}_{ji}}>0$ and $\frac{\partial^3 r_{ij}(e^*_{ji})}{\partial e^{*3}_{ji}}<0$; note that $\frac{\partial e^*_{ji}}{\partial\alpha_{ij}}<0$ by Proposition.

Analogously, we obtain

$$\frac{\partial^2 e^*_{ij}}{\partial\alpha^2_{ij}} = \frac{\frac{\partial^2 r_{ji}(e^*_{ij})}{\partial e^{*2}_{ij}}\frac{\partial e^*_{ij}}{\partial\alpha_{ij}} - \left(\frac{\partial^3 c_i(e^*_i)}{\partial e^{*3}_{ij}} - \alpha_{ij}\frac{\partial^3 r_{ji}(e^*_{ij})}{\partial e^{*3}_{ij}}\right)\left(\frac{\partial e^*_{ij}}{\partial\alpha_{ij}}\right)^2}{\frac{\partial^2 c_i(e^*_i)}{\partial e^{*2}_{ij}} - \alpha_{ij}\frac{\partial^2 r_{ji}(e^*_{ij})}{\partial e^{*2}_{ij}}} < 0. \quad \square$$

Lemma 5 enables us to state that in any interval where the piecewise derivative function takes the value $\frac{\partial(L^*_{ij}(\alpha_{ij}))}{\partial\alpha_{ij}} = -\alpha_{ij}\frac{\partial r_{ij}(e^*_{ji})}{\partial e^*_{ji}}\frac{\partial e^*_{ji}}{\partial\alpha_{ij}} - (1-\alpha_{ij})\frac{\partial r_{ji}(e^*_{ij})}{\partial e^*_{ij}}\frac{\partial e^*_{ij}}{\partial\alpha_{ij}}$, the function $L^*_{ij}(\alpha_{ij})$ is convex (see also Lemma 4).

The following proposition shows that, according to the value of the effort equilibrium, the cost function $L^*_{ij}(\alpha_{ij})$ is a continuous piecewise function with four types of piece. This result characterizes all of those pieces, showing the shape of $L^*_{ij}(\alpha_{ij})$ and the optimal α_{ij} in each type of piece.

Proposition 5. *Consider the effort game $(N,\{E_i\}_{i\in N},\{A_i\}_{i\in N})$ and e^* as the effort equilibrium. Let $\alpha_{ij}\in[a,b]$ be a piece of $L^*_{ij}(\alpha_{ij})$ with $0\le a<b\le 1$, $L^*_{ij}(\alpha_{ij})$ can have only four types of piece:*

1. ***Constant:*** *(e^*_{ij},e^*_{ji}) is either (0,0), $(1,0)$, (0,1) or (1,1). Thus $\frac{\partial(L^*_{ij}(\alpha_{ij}))}{\partial\alpha_{ij}}=0$ and $L^*_{ij}(\alpha_{ij})$ is always constant. Therefore, any $\alpha_{ij}\in[a,b]$ minimizes $L^*_{ij}(\alpha_{ij})$.*
2. *Increasing: e^*_{ij} is either 0 or 1, and $0<e^*_{ji}<1$. Thus $\frac{\partial(L^*_{ij}(\alpha_{ij}))}{\partial\alpha_{ij}} = \left(\frac{\partial c_j(e^*_j)}{\partial e^*_{ji}} - \frac{\partial r_{ij}(e^*_{ji})}{\partial e^*_{ji}}\right)\frac{\partial e^*_{ji}}{\partial \alpha_{ij}} > 0$ and $L^*_{ij}(\alpha_{ij})$ is always increasing. Therefore, $\alpha_{ij}=a$ minimizes $L^*_{ij}(\alpha_{ij})$.*

3. ***Decreasing:*** *$0 < e^*_{ij} < 1$, and e^*_{ji} is either 0 or 1. Thus $\frac{\partial (L^*_{ij}(\alpha_{ij}))}{\partial \alpha_{ij}} = \left(\frac{\partial c_i(e^*_i)}{\partial e^*_{ij}} - \frac{\partial r_{ji}(e^*_{ij})}{\partial e^*_{ij}}\right)\frac{\partial e^*_{ij}}{\partial \alpha_{ij}} < 0$ and $L^*_{ij}(\alpha_{ij})$ is always decreasing. Therefore, $\alpha_{ij} = b$ minimizes $L^*_{ij}(\alpha_{ij})$.*
4. ***Depending on cost function shape:*** *$0 < e^*_{ij} < 1$ and $0 < e^*_{ji} < 1$. Thus, $\frac{\partial (L^*_{ij}(\alpha_{ij}))}{\partial \alpha_{ij}} = \left(\frac{\partial c_j(e^*_j)}{\partial e^*_{ji}} - \frac{\partial r_{ij}(e^*_{ji})}{\partial e^*_{ji}}\right)\frac{\partial e^*_{ji}}{\partial \alpha_{ij}} + \left(\frac{\partial c_i(e^*_i)}{\partial e^*_{ij}} - \frac{\partial r_{ji}(e^*_{ij})}{\partial e^*_{ij}}\right)\frac{\partial e^*_{ij}}{\partial \alpha_{ij}}$.*
In this case, there is always a unique $\breve{\alpha}^{[a,b]}_{ij} \in [a, b]$ that minimizes $L^*_{ij}(\alpha_{ij})$, which is:

$$\breve{\alpha}^{[a,b]}_{ij} = \begin{cases} a & \text{if } \frac{\partial (L^*_{ij}(\alpha_{ij}))}{\partial \alpha_{ij}} > 0 \text{ for all } \alpha_{ij} \in [a, b] \\ b & \text{if } \frac{\partial (L^*_{ij}(\alpha_{ij}))}{\partial \alpha_{ij}} < 0 \text{ for all } \alpha_{ij} \in [a, b] \\ \text{Solution of } \frac{\partial (L^*_{ij}(\alpha_{ij}))}{\partial \alpha_{ij}} = 0 & \text{otherwise} \end{cases}$$

Proof. The proof of Lemma 4 shows four possible cases for $L^*_{ij}(\alpha_{ij})$. The point 2. of the proof of Lemma 4 proves the point 1. (Constant). The point 3. proves the point 2. (Increasing), and point 4. proves point 3 (decreasing). Finally, to prove the point 4. (Depending on cost function shape) we need the point 1. of Lemma 4 and Lemma 5 which proves that $L^*_{ij}(\alpha_{ij})$ is convex in this case. Therefore, in this last case, it is also straightforward to show that $\frac{\partial (L^*_{ij}(\alpha_{ij}))}{\partial \alpha_{ij}}$ is continuous, so there is always a unique α_{ij} that minimizes $L^*_{ij}(\alpha_{ij})$ in such pieces. The procedure for calculating $\breve{\alpha}^{[a,b]}_{ij}$ is the following: First, by Theorem, we calculate e^*_{ij} and e^*_{ji} as a function of α_{ij} from $c'_i(e_{ij}) - \alpha_{ij} r'_{ji}(e_{ij}) = 0$ and $c'_j(e_{ji}) - \alpha_{ji} r'_{ij}(e_{ji}) = 0$. Second, we build the function $L^*_{ij}(\alpha_{ij})$ with the $e^*_{ij}(\alpha_{ij})$ and $e^*_{ji}(\alpha_{ij})$ previously calculated. Finally, we calculate $\frac{\partial (L^*_{ij}(\alpha_{ij}))}{\partial \alpha_{ij}}$ and obtain $\breve{\alpha}^{[a,b]}_{ij}$. □

Finally, Theorem 5 characterizes the optimal α^*_{ij}, for all $i, j \in N$ with $i \neq j$, which incentivizes an efficient effort equilibrium, which is also provided.

Proof of Theorem 5

Proof. As $L^*_{ij}(\alpha_{ij})$ is a continuous piecewise function, we analyze the five pieces that define it in each case. Lemma, and Proposition 5 enable the type of piece to be determined, thus giving the value of α_{ij} that minimizes $L^*_{ij}(\alpha_{ij})$ in each piece. Comparing the pieces gives the α^*_{ij} that minimizes the aggregate cost for each of the six cases. This value need not be unique. Note, in addition, that $\underline{\alpha}_{ij}$, $\bar{\alpha}_{ij}$, $\bar{\alpha}_{ji}$ and $\underline{\alpha}_{ji}$ are always greater than zero, but any of them may be greater than one, which implies that some pieces of certain cases may not exist. We prove the theorem case by case:

Case A ($\underline{\alpha}_{ij} < \bar{\alpha}_{ij} < 1 - \bar{\alpha}_{ji} < 1 - \underline{\alpha}_{ji}$)
Note that those thresholds are always greater than zero, so $0 < \underline{\alpha}_{ij} < \bar{\alpha}_{ij} < 1 - \bar{\alpha}_{ji} < 1 - \underline{\alpha}_{ji} < 1$. By Lemma 4,
if $\alpha_{ij} \in \left(0, \underline{\alpha}_{ij}\right)$, then $L^*_{ij}(\alpha_{ij})$ is constant in this interval.
If $\alpha_{ij} \in \left(\underline{\alpha}_{ij}, \bar{\alpha}_{ij}\right)$, then $L^*_{ij}(\alpha_{ij})$ is decreasing, which implies that $\alpha_{ij} = 1 - \bar{\alpha}_{ji}$ minimizes $L^*_{ij}(\alpha_{ij})$.
If $\alpha_{ij} \in \left(\bar{\alpha}_{ij}, 1 - \bar{\alpha}_{ji}\right)$, then $L^*_{ij}(\alpha_{ij})$ is constant in this interval.
If $\alpha_{ij} \in \left(1 - \bar{\alpha}_{ji}, 1 - \underline{\alpha}_{ji}\right)$, then $L^*_{ij}(\alpha_{ij})$ is increasing, which implies that $1 - \bar{\alpha}_{ji}$ minimizes $L^*_{ij}(\alpha_{ij})$.
If $\alpha_{ij} \in \left(1 - \underline{\alpha}_{ji}, 1\right)$, then $L^*_{ij}(\alpha_{ij})$ is constant in this interval.
Therefore, α^*_{ij} is equal to any $\alpha_{ij} \in [\bar{\alpha}_{ij}, 1 - \bar{\alpha}_{ji}]$.

Case B ($\underline{\alpha}_{ij} < 1 - \bar{\alpha}_{ji} < \bar{\alpha}_{ij} < 1 - \underline{\alpha}_{ji}$)
Analogously, $0 < \underline{\alpha}_{ij} < 1 - \bar{\alpha}_{ji} < \bar{\alpha}_{ij} < 1 - \underline{\alpha}_{ji} < 1$, and by Lemma 4, 5 and Proposition 5,
if $\alpha_{ij} \in \left(0, \underline{\alpha}_{ij}\right)$, then $L^*_{ij}(\alpha_{ij})$ is constant in this interval.
If $\alpha_{ij} \in \left(\underline{\alpha}_{ij}, 1 - \bar{\alpha}_{ji}\right)$, then $L^*_{ij}(\alpha_{ij})$ is decreasing, which implies that $\alpha_{ij} = 1 - \bar{\alpha}_{ji}$ minimizes $L^*_{ij}(\alpha_{ij})$.
If $\alpha_{ij} \in \left(1 - \bar{\alpha}_{ji}, \bar{\alpha}_{ij}\right)$, then $\breve{\alpha}_{ij}$ minimizes $L^*_{ij}(\alpha_{ij})$, where $\breve{\alpha}_{ij}$ is define in Proposition 5.
If $\alpha_{ij} \in \left(\bar{\alpha}_{ij}, 1 - \underline{\alpha}_{ji}\right)$, then $L^*_{ij}(\alpha_{ij})$ is increasing, which implies that $\bar{\alpha}_{ij}$ minimizes $L^*_{ij}(\alpha_{ij})$.
If $\alpha_{ij} \in \left(1 - \underline{\alpha}_{ji}, 1\right)$, then $e^*_{ij} = 1$, $e^*_{ji} = 0$, and $L^*_{ij}(\alpha_{ij})$ is constant in this interval.
Therefore, $\alpha^*_{ij} = \breve{\alpha}^{[1-\bar{\alpha}_{ji}, \bar{\alpha}_{ij}]}_{ij}$.

Case C ($\underline{\alpha}_{ij} < 1 - \bar{\alpha}_{ji} < 1 - \underline{\alpha}_{ji} < \bar{\alpha}_{ij}$)
It may happen here that either $\bar{\alpha}_{ij} < 1$ or $\bar{\alpha}_{ij} \geq 1$. Thus there are two subcases:
$0 < \underline{\alpha}_{ij} < 1 - \overline{\alpha}_{ji} < 1 - \underline{\alpha}_{ji} < \overline{\alpha}_{ij} < 1$;
$0 < \underline{\alpha}_{ij} < 1 - \overline{\alpha}_{ji} < 1 - \underline{\alpha}_{ji} < 1 < \overline{\alpha}_{ij}$.
Starting with the first subcase, by Lemma 4, 5 and Proposition 5
if $\alpha_{ij} \in \left(0, \underline{\alpha}_{ij}\right)$, then $L^*_{ij}(\alpha_{ij})$ is constant in this interval.
If $\alpha_{ij} \in \left(\underline{\alpha}_{ij}, 1 - \bar{\alpha}_{ji}\right)$, then $L^*_{ij}(\alpha_{ij})$ is decreasing, which implies that $\alpha_{ij} = 1 - \bar{\alpha}_{ji}$ minimizes $L^*_{ij}(\alpha_{ij})$.
If $\alpha_{ij} \in \left(1 - \bar{\alpha}_{ji}, 1 - \underline{\alpha}_{ji}\right)$, then $\breve{\alpha}_{ij}$ minimizes $L^*_{ij}(\alpha_{ij})$.
If $\alpha_{ij} \in \left(1 - \underline{\alpha}_{ji}, \bar{\alpha}_{ij}\right)$, then $L^*_{ij}(\alpha_{ij})$ is decreasing, which implies that $\bar{\alpha}_{ij}$ minimizes $L^*_{ij}(\alpha_{ij})$.
If $\alpha_{ij} \in \left(\bar{\alpha}_{ij}, 1\right)$, then $L^*_{ij}(\alpha_{ij})$ is constant, in this interval.
However, in the second subcase $\bar{\alpha}_{ij} > 1$, which implies that the last interval described above does not exist. The rest of the analysis is similar to the first subcase.
Therefore, $\alpha^*_{ij} = \arg\min\{L^*_{ij}(\breve{\alpha}^{[1-\bar{\alpha}_{ji}, 1-\underline{\alpha}_{ji}]}_{ij}), L^*_{ij}(\Lambda(\bar{\alpha}_{ij}))\}$. Note that, if $\alpha^*_{ij} = \Lambda(\bar{\alpha}_{ij})$ and $\bar{\alpha}_{ij} < 1$, then α^*_{ij} is equal to any $\alpha_{ij} \in (\bar{\alpha}_{ij}, 1)$.

Case D ($1 - \bar{\alpha}_{ji} < \underline{\alpha}_{ij} < \bar{\alpha}_{ij} < 1 - \underline{\alpha}_{ji}$)
It may happen here that either $1 - \bar{\alpha}_{ji} > 0$ or $1 - \bar{\alpha}_{ji} \leq 0$. Thus there are two subcases:
$0 < 1 - \overline{\alpha}_{ji} < \underline{\alpha}_{ij} < \overline{\alpha}_{ij} < 1 - \underline{\alpha}_{ji} < 1$;
$1 - \overline{\alpha}_{ji} < 0 < \underline{\alpha}_{ij} < \overline{\alpha}_{ij} < 1 - \underline{\alpha}_{ji} < 1$.
Starting with the first subcase, by Lemma 4, 5 and Proposition 5
if $\alpha_{ij} \in \left(0, 1 - \bar{\alpha}_{ji}\right)$, then $e^*_{ij} = 0$, $e^*_{ji} = 1$, and $L^*_{ij}(\alpha_{ij})$ is constant in this interval.
If $\alpha_{ij} \in \left(1 - \bar{\alpha}_{ji}, \underline{\alpha}_{ij}\right)$, then $L^*_{ij}(\alpha_{ij})$ is increasing, which implies that $\alpha_{ij} = 1 - \bar{\alpha}_{ji}$ minimizes $L^*_{ij}(\alpha_{ij})$.
If $\alpha_{ij} \in \left(\underline{\alpha}_{ij}, \bar{\alpha}_{ij}\right)$, then $\breve{\alpha}_{ij}$ minimizes $L^*_{ij}(\alpha_{ij})$.
If $\alpha_{ij} \in \left(\bar{\alpha}_{ij}, 1 - \underline{\alpha}_{ji}\right)$, then $e^*_{ij} = 1$, $0 < e^*_{ji} < 1$, and $L^*_{ij}(\alpha_{ij})$ is increasing, which implies that $\bar{\alpha}_{ij}$ minimizes $L^*_{ij}(\alpha_{ij})$.
If $\alpha_{ij} \in \left(\bar{\alpha}_{ij}, 1\right)$, then $e^*_{ij} = 1$, $e^*_{ji} = 0$, and $L^*_{ij}(\alpha_{ij})$ is constant in this interval.
However, if $1 - \bar{\alpha}_{ji} < 0$ the first interval above does not exist. Again, the rest of the analysis is similar to the first subcase.
Therefore, $\alpha^*_{ij} = \arg\min\{L^*_{ij}(\Lambda(1 - \bar{\alpha}_{ji})), L^*_{ij}(\breve{\alpha}^{[\underline{\alpha}_{ij}, \bar{\alpha}_{ij}]}_{ij})\}$. Note that if $\alpha^*_{ij} = \Lambda(1 - \bar{\alpha}_{ji})$ and $1 - \bar{\alpha}_{ji} > 0$, then α^*_{ij} is equal to any $\alpha_{ij} \in [0, 1 - \bar{\alpha}_{ji}]$.

Case E ($1 - \bar{\alpha}_{ji} < \underline{\alpha}_{ij} < 1 - \underline{\alpha}_{ji} < \bar{\alpha}_{ij}$)
In this case, it may happen that either $1 - \bar{\alpha}_{ji} > 0$ or $1 - \bar{\alpha}_{ji} \leq 0$, and either $\bar{\alpha}_{ij} < 1$ or $\bar{\alpha}_{ij} \geq 1$. Thus there are four subcases:
$0 < 1 - \overline{\alpha}_{ji} < \underline{\alpha}_{ij} < 1 - \underline{\alpha}_{ji} < \overline{\alpha}_{ij} < 1$;
$1 - \overline{\alpha}_{ji} < 0 < \underline{\alpha}_{ij} < 1 - \underline{\alpha}_{ji} < \overline{\alpha}_{ij} < 1$;
$0 < 1 - \overline{\alpha}_{ji} < \underline{\alpha}_{ij} < 1 - \underline{\alpha}_{ji} < 1 < \overline{\alpha}_{ij}$;
$1 - \overline{\alpha}_{ji} < 0 < \underline{\alpha}_{ij} < 1 - \underline{\alpha}_{ji} < 1 < \overline{\alpha}_{ij}$.
Focusing on the first subcase, by Lemma 4, 5 and Proposition 5.

if $\alpha_{ij} \in (0, 1-\bar{\alpha}_{ji})$, then $L^*_{ij}(\alpha_{ij})$ is constant in this interval.
If $\alpha_{ij} \in (1-\bar{\alpha}_{ji}, \underline{\alpha}_{ij})$, then $L^*_{ij}(\alpha_{ij})$ is increasing, which implies that $\alpha_{ij} = 1-\bar{\alpha}_{ji}$ minimizes $L^*_{ij}(\alpha_{ij})$.
If $\alpha_{ij} \in (\underline{\alpha}_{ij}, 1-\underline{\alpha}_{ji})$, then $\breve{\alpha}_{ij}$ minimizes $L^*_{ij}(\alpha_{ij})$.
If $\alpha_{ij} \in (1-\underline{\alpha}_{ji}, \bar{\alpha}_{ij})$, then $L^*_{ij}(\alpha_{ij})$ is decreasing, which implies that $\bar{\alpha}_{ij}$ minimizes $L^*_{ij}(\alpha_{ij})$.
If $\alpha_{ij} \in (\bar{\alpha}_{ij}, 1)$, then $e^*_{ij} = 1$, $e^*_{ji} = 0$, and $L^*_{ij}(\alpha_{ij})$ is constant in this interval.
In the other three subcases, the first and/or last interval may not exist. Once again, the rest of the analysis for those subcases is similar to the first one.
Therefore, $\alpha^*_{ij} = \arg\min\{L^*_{ij}(\Lambda(1-\bar{\alpha}_{ji})), \breve{\alpha}_{ij}^{[\underline{\alpha}_{ij}, 1-\underline{\alpha}_{ji}]}, L^*_{ij}(\Lambda(\bar{\alpha}_{ij}))\}$. Note that if $\alpha^*_{ij} = \Lambda(1-\bar{\alpha}_{ji})$ and $1-\bar{\alpha}_{ji} > 0$ then α^*_{ij} is equal to any $\alpha_{ij} \in [0, 1-\bar{\alpha}_{ji}]$, and if $\alpha^E_{ij} = \Lambda(\bar{\alpha}_{ij})$ and $\bar{\alpha}_{ij} < 1$, then α^*_{ij} is equal to any $\alpha_{ij} \in [\bar{\alpha}_{ij}, 1]$.

Case F $(1-\bar{\alpha}_{ji} < 1-\underline{\alpha}_{ji} < \underline{\alpha}_{ij} < \bar{\alpha}_{ij})$
This is the most general case and anything could happen with thresholds greater than one. Thus there are nine subcases. First consider the case $0 < 1-\bar{\alpha}_{ji} < 1-\underline{\alpha}_{ji} < \underline{\alpha}_{ij} < \bar{\alpha}_{ij} < 1$:
If $\alpha_{ij} \in (0, 1-\bar{\alpha}_{ji})$, then $L^*_{ij}(\alpha_{ij})$ is constant in this interval.
If $\alpha_{ij} \in (1-\bar{\alpha}_{ji}, 1-\underline{\alpha}_{ji})$, then $L^*_{ij}(\alpha_{ij})$ is increasing, which implies that $\alpha_{ij} = 1-\bar{\alpha}_{ji}$ minimizes $L^*_{ij}(\alpha_{ij})$.
If $\alpha_{ij} \in (1-\underline{\alpha}_{ji}, \underline{\alpha}_{ij})$, then $L^*_{ij}(\alpha_{ij})$ is constant in this interval.
If $\alpha_{ij} \in (\underline{\alpha}_{ij}, \bar{\alpha}_{ij})$, then $L^*_{ij}(\alpha_{ij})$ is decreasing, which implies that $\alpha_{ij} = \bar{\alpha}_{ij}$ minimizes $L^*_{ij}(\alpha_{ij})$.
If $\alpha_{ij} \in (\bar{\alpha}_{ij}, 1)$, then $L^*_{ij}(\alpha_{ij})$ is constant in this interval.
In any other subcase, the first, second, to last, and last intervals considered above, may not exist. The rest of the analysis for those subcases is similar to the first one.
Therefore, $\alpha^*_{ij} = \arg Min\{L^*_{ij}(\Lambda(1-\bar{\alpha}_{ji})), L^*_{ij}(\Lambda(\bar{\alpha}_{ij}))\}$. Note that, if $\alpha^*_{ij} = \Lambda(1-\bar{\alpha}_{ji})$ and $1-\bar{\alpha}_{ji} > 0$, then α^*_{ij} is equal to any $\alpha_{ij} \in [0, 1-\bar{\alpha}_{ji}]$, but if $\alpha^*_{ij} = \Lambda(\bar{\alpha}_{ij})$ and $\bar{\alpha}_{ij} < 1$, then α^*_{ij} is equal to any $\alpha_{ij} \in [\bar{\alpha}_{ij}, 1]$. Additionally, if $1-\underline{\alpha}_{ji} < 0$ and $\bar{\alpha}_{ij} > 1$, then $L^*_{ij}(\Lambda(1-\bar{\alpha}_{ji})) = L^*_{ij}(\Lambda(\bar{\alpha}_{ij}))$, so α^*_{ij} is equal to any $\alpha_{ij} \in [0, 1]$.

□

Appendix D

Table 1 and 2.

Table 2
Summary of optimization problems.

$\tilde{e}$	Efficient effort profile	$\tilde{e} = \arg \min_{e \in [0,1]^{n(n-1)}} c(N)$
$\hat{e}_i$	Optimal efforts of agent i given efforts of other agents	$\hat{e}_i = \arg \min_{e_i \in [0,1]^{(n-1)}} A_i(e)$
e^*_i	Equilibrium strategy of agent i	$e^*_i = \hat{e}_i$
α^*	Optimal weights of WPAR allocation	$\alpha^* = \arg \min_{\alpha \in [0,1]^{n(n-1)}} \sum_{i \in N} A_i(e^*)$ $\Updownarrow$ $\alpha^*_{ij} = \arg \min_{\alpha_{ij} \in [0,1]} L^*_{ij}(\alpha_{ij})$ for $i \neq j \in N$ with $L^*_{ij}(\alpha_{ij}) = c_i(e^*_i) + c_j(e^*_j)$

References

[1] Todeva E, Knoke D. Strategic alliances & models of collaboration. Management Decision 2005;43(1).
[2] Thomson W. Fair allocation rules. Chapter 21 in the Handbook of Social Choice and Welfare 2010;2:393–492.
[3] Brandenburger A, Stuart H. Biform games. Manage Sci 2007;53(4):537–49.
[4] Luo C, Zhou X, Lev B. Core, shapley value, nucleolus and Nash bargaining solution: a survey of recent developments and applications in oerations management. Omega 2022;110:102638.
[5] Fiestras-Janeiro MG, García-Jurado I, Mosquera MA. Cooperative games and cost allocation problems. Top 2011;19(1):1–22.
[6] Guajardo M, Rönnqvist M. A review on cost allocation methods in collaborative transportation. International transactions in operational research 2016;23(3):371–92.
[7] Ciardello F, Genovese A, Simpson A. Pollution responsibility allocation in supply networks: a game-theoretic approach and a case study. Int J Prod Econ 2019;217:211–17.
[8] Fardi K, Jafarzaheh-Ghouschi S, Hafezalkotob A. An extended robust approach for a cooperative inventory routing problem. Expert Syst Appl 2019;116:310–27.
[9] Lozano S, Moreno P, Adenso-Díaz B, Algaba E. Cooperative game theory approach to allocating benefits of horizontal cooperation. Eur J Oper Res 2013;229(2):444–52.
[10] Meng F, Tang J, An Q. Cooperative game based two-stage consensus adjustment mechanism for large-scale group decision making. Omega 2023;117:102842.
[11] Mitridati L, Kazempour J, Pinson P. Design and game-theoretic analysis of community-based market mechanisms in heat and electricity systems. Omega 2021;99:102177.
[12] Moulin H. Equal or proportional division of a surplus, and other methods. International Journal of Game Theory 1987;16(3):161–86.
[13] Omrani H, Shafaat K, Emrouznejad A. An integrated fuzzy clustering cooperative game data envelopment analysis model with application in hospital efficiency. Expert Syst Appl 2018;114:615–28.
[14] Slikker M, Van den Nouweland A. Social and economic networks in cooperative game theory (Vol. 27). Springer Science & Business Media; 2012.
[15] Tajbakhsh A, Hassini E. A game-theoretic approach for pollution control initiatives. Int J Prod Econ 2022;254:108651.

Table 1
Notation summary.

$N = \{1, 2, .n\}$	Agents
$E_i = [0, 1]^{n-1}$	Strategy space of agent i of the non-cooperative game
$E = \prod_{i \in N} E_i = [0, 1]^{n(n-1)}$	Strategy profile space of the non-cooperative game
$e_{ij} \in [0, 1]$	Effort exerted by agent i to reduce the cost of agent j
$e_i = (e_{ij})_{j \neq i} \in E_i$	Efforts exerted by agent i
$e \in E$	Effort profile
$c_i : E_i \to \mathbb{R}_+$	Cost function for agent i with $c_i(e_i)$ the cost of effort e_i
$r_{ij} : [0, 1] \to \mathbb{R}_+$	Cost reduction function of agent i given by agent j
$r_{ij}(e_{ji})$	Cost reduction for agent i due to effort e_{ji}
$c : 2^N \to \mathbb{R}$	Characteristic function of the cooperative cost game
$S \subseteq N$	Coalition of agents
$c^S(\{i\}) = c_i(e_i) - \sum_{j \in S \setminus \{i\}} r_{ij}(e_{ji})$	The reduced cost of agent i in coalition S
$c(S) = \sum_{i \in S} c^S(\{i\})$	The reduced cost for coalition S
$\psi_i : E \to \mathbb{R}$	Allocation to agent i
$\psi(e) = (\psi_i(e))_{i \in N}$	Allocation rule, with $\sum_{i \in N} \psi_i(e) = c(N)$
$\Omega_i(e) = c_i(e_i) - \sum_{j \in N \setminus \{i\}} [\omega^i_{ij} r_{ij}(e_{ji}) + \omega^i_{ji} r_{ji}(e_{ij})]$	WPR allocation for agent i, where $\omega^i_{ij} \in [0, 1]$, and $\omega^i_{ji} = 1 - \omega^j_{ji}$ with $i, j \in N, i \neq j$
$A_i(e) = c_i(e_i) - \sum_{j \in N \setminus \{i\}} \alpha_{ij}[r_{ij}(e_{ji}) + r_{ji}(e_{ij})]$	WPAR allocation for agent i, where $\alpha_{ij} \in [0, 1]$ and $\alpha_{ji} = 1 - \alpha_{ij}$.
$\alpha = (\alpha_i)_{i \in N}$ with $\alpha_i = (\alpha_{ij})_{j \in N \setminus \{i\}}$	Weights of WPAR allocation
$\phi(c)$	Shapley value
$\nu(e)$	Nucleolus

[16] Yu Y, Lou Q, Tang J, Wang J, Yu X. An exact decomposition method to save trips in cooperative pickup and delivery based on scheduled trips and profit distribution. Computers and Operations Research 2017;87:245–57.
[17] Shapley LS. A value for n-person games. Contributions to the Theory of Games 1953;2(28):307–17.
[18] Moretti S, Patrone F. Transversality of the shapley value. Top 2008;16(1):1.
[19] Serrano R. Cooperative games: core and shapley value. Encyclopedia of complexity and systems science. Meyers R, editor. New York: Springer; 2009.
[20] Bilbao JM, Fernández JR, Jiménez N, López JJ. The shapley value for bicooperative games. Ann Oper Res 2008;158(1):99–115.
[21] Kimms A, Kozeletskyi I. Shapley value-based cost allocation in the cooperative traveling salesman problem under rolling horizon planning. EURO Journal on Transportation and Logistics 2016;5(4):371–92.
[22] Le PH, Nguyen TD, Bektaş T. Efficient computation of the shapley value for large-scale linear production games. Ann Oper Res 2018:1–21.
[23] Li S, Zhang Q. A simplified expression of the shapley function for fuzzy game. Eur J Oper Res 2009;196(1):234–45.
[24] Littlechild SC, Owen G. A simple expression for the shapley value in a special case. Manage Sci 1973;20:370–2.
[25] Meca A, García-Martínez JA, Mayor-Serra AJ, Sánchez-Soriano J. The shapley value of corporation tax games with dual benefactors. Handbook of the Shapley value. Algaba E, Fragnelli FV, editors. Londres, Reino Unido: CRC Press. Taylor and Francis Group; 2019.
[26] Holmstrom B. Moral hazard in teams. Bell Journal of Economics 1982;13(2):324–40.
[27] Holmstrom B. Managerial incentive problems: a dynamic perspective. Review of Economic Studies 1999;66(1):169–82.
[28] Dewatripont M, Jewitt I, Tirole J. The economics of career concerns, part i: comparing information structures. Review of Economic Studies 1999;66(1):183–98.
[29] Bernstein F, Kok AG, Meca A. Cooperation in assembly systems: the role of knowledge sharing networks. Eur J Oper Res 2015;240:160–71.
[30] González-Díaz J, Fiestras-Janeiro MG, García-Jurado. An introductory course on mathematical game theory. graduate studies in mathematics, vol 115. American Mathematical Society; 2010.
[31] Meca A, Sošić G. Benefactors and beneficiaries: the effects of giving and receiving on cost-coalitional problems. Production and Operations Management 2014;23(9):1549–60.
[32] Shapley LS. Cores of convex games. International journal of game theory 1971;1(1):11–26.
[33] Moulin H. Fair division and collective welfare. Cambridge, Massachusetts: MIT Press; 2004.
[34] Schmeidler D. The nucleolus of a characteristic function game. SIAM J Appl Math 1969;17(6):1163–70.
[35] Amin W, Huang Q, Afzal M, Khan AA, Umer K, Ahmed SA. A converging non–cooperative & cooperative game theory approach for stabilizing peer-to-peer electricity trading. Electr Power Syst Res 2020;183:106278.
[36] Fang D, Wang J. Horizontal capacity sharing between asymmetric competitors. Omega 2020;97:102109.
[37] Park J, Van der Schaar M. A game theoretic analysis of incentives in content production and sharing over peer-to-peer networks. IEEE J Sel Top Signal Process 2010;4(4):704–17.
[38] Curiel I, Pederzoli G, Tijs S. Sequencing games. Eur J Oper Res 1989;40:344–51.
[39] Kar A, Mitra M, Mutuswami S. On the coincidence of the prenucleolus with the shapley value. Math Soc Sci 2009;57(1):16–25.
[40] Bracken J, McGill JT. Mathematical programs with optimization problems in the constraints. Oper Res 1973;21(1):37–44.
[41] Dempe S. Foundations of bilevel programming. Kluwer Academic Publishers; 2002.