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PhD Dissertation

Sobre la caracterización y robustez de los atractores de sistemas dinámicos multivaluados

On the characterization and robustness of the attractors of multivalued dynamical systems

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INFORMA:

Que D. Rubén Caballero Toro ha realizado bajo mi supervisión el trabajo titulado "Sobre la caracterización y robustez de los atractores de sistemas dinámicos multivaluados" conforme a los términos y condiciones definidos en su Plan de Investigación y de acuerdo al Código de Buenas Prácticas de la Universidad Miguel Hernández de Elche, cumpliendo los objetivos previstos de forma satisfactoria para su defensa pública como tesis doctoral.

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Que D. Rubén Caballero Toro ha realizado bajo la supervisión de nuestro Programa de Doctorado el trabajo titulado "Sobre la caracterización y robustez de los atractores de sistemas dinámicos multivaluados" conforme a los términos y condiciones definidos en su Plan de Investigación y de acuerdo al Código de Buenas Prácticas de la Universidad Miguel Hernández de Elche, cumpliendo los objetivos previstos de forma satisfactoria para su defensa pública como tesis doctoral.

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Contents

Abstract

The objective of the present thesis is studying multivalued dynamical systems. In particular, we pretend to obtain results related with the structure of the attractors in order to describe the behaviour of solutions for different equations. Therefore, our research may be situated in the field of Applied Mathematics.

Specifically, Chapter 1 deals with robustness of dynamically gradient multivalued semiflows. As an application, we describe the dynamical properties of a family of Chafee-Infante problems approximating a differential inclusion, proving that the weak solutions of these problems generate a dynamically gradient multivalued semiflow with respect to suitable Morse sets.

Chapter 2 focus on a more general equation called nonlocal reaction-diffusion equation in which the diffusion depends on the gradient of the solution. Firstly, we prove the existence and uniqueness of regular and strong solutions. Secondly, we obtain the existence of global attractors in both situations under rather weak assumptions by defining a multivalued semiflow. We finish this section characterizing the attractor either as the unstable manifold of the set of stationary points or as the stable one when we consider solutions only in the set of bounded complete trajectories.

In the last chapter we study the structure of the global attractor for the multivalued semiflow generated by a nonlocal reaction-diffusion equation in which we cannot guarantee uniqueness of the Cauchy problem. We start analysing the existence and properties of stationary points, showing that the problem undergoes the same cascade of bifurcations as in the Chafee-Infante equation. To conclude, we

study the stability of the fixed points and establish that the semiflow is dynamically gradient. Also, we prove that the attractor consists of the stationary points and their heteroclinic connections and analyse some of the possible connections.

Apart from these three chapters, the manuscript contains an unnumbered section, *Introduction* (and its Spanish version), as a preamble, where the work as well as the objetives that we pretend to cover are exposed. Subsequently, we have included the preliminary Chapter 0 in order to detail the framework and the previous results needed to achieve the proposed objectives. To end this work, we have created two unnumbered sections, *Appendix A* and *Conclusions and future work* (and its Spanish version). In the first one, details about generalization of the lap number property are given whilst in the other one main contributions of our research and some comments on future research lines are summarized.

Resumen

El objetivo de esta tesis es estudiar sistemas dinámicos multivaluados. En particular, pretendemos obtener resultados relacionados con la estructura de los atractores para describir el comportamiento de las soluciones de diferentes ecuaciones. Por tanto, nuestra investigación puede situarse en el área de Matemática Aplicada.

Más concretamente, el Capítulo 1 versa sobre la robustez de los semiflujos multivaluados dinámicamente gradientes. Para aplicar este resultado describimos las propiedades dinámicas de una familia de problemas Chafee-Infante aproximando una inclusión diferencial, demostrando que las soluciones débiles de estos problemas generan un semiflujo multivaluado dinámicamente gradiente con respecto a unos conjuntos de Morse.

El Capítulo 2 se centra en una ecuación más general llamada ecuación de reacción-difusión no local, donde el término de difusión depende del gradiente de la solución. En primer lugar, demostramos la existencia y unicidad de soluciones regulares y fuertes. En segundo lugar, obtenemos la existencia de atractores globales en ambas situaciones bajo supuestos bastante débiles al definir un semiflujo multivaluado. Terminamos esta sección caracterizando al atractor como la variedad inestable del conjunto de puntos estacionarios o como la estable cuando consideramos soluciones sólo en el conjunto de trayectorias completas acotadas.

En el último capítulo estudiamos la estructura del atractor global para el semiflujo multivaluado generado por una ecuación de reacción-difusión no local donde no podemos garantizar la unicidad del problema de Cauchy. Comenzamos analizando la existencia y propiedades de los puntos estacionarios, mostrando que el

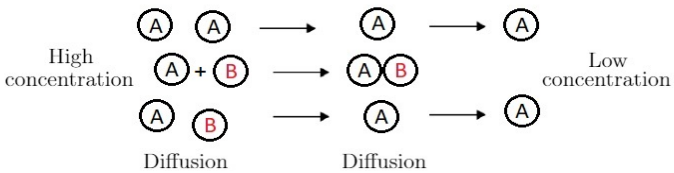
problema sufre la misma cascada de bifurcaciones que en la ecuación de Chafee-Infante. Para concluir, estudiamos la estabilidad de los puntos fijos y establecemos que el semiflujo es dinámicamente gradiente. Además, probamos que el atractor está formado por los puntos estacionarios y sus conexiones heteroclínicas y analizamos algunas de las posibles conexiones.

Además de estos tres capítulos, este trabajo contiene un apartado no numerado, *Introduction* (y su versión en español), a modo de preámbulo, donde se exponen tanto el trabajo como los objetivos que pretendemos alcanzar. Posteriormente, hemos incluido el Capítulo 0 preliminar para detallar el marco y los resultados previos necesarios para obtener los objetivos propuestos. Para terminar el trabajo, hemos creado dos secciones sin numerar, *Appendix A* y *Conclusions and future work* (y su versión en español de esta última). En el primero se dan detalles sobre la generalización de la propiedad *lap number* mientras que en el otro se aportan las principales contribuciones de nuestra investigación y algunos comentarios sobre futuras líneas de investigación.

Introduction

Differential equations play a more than very important role not only in Mathematics but also in other sciences. They have been used for centuries in fields such as Physics, Chemistry or Biology. There are two important equations which are used to describe various processes occurring around us in this world. Most common processes incorporate the variety of the concentration of at least one substance in time and in space under the impact of two responses, which are, as the name suggests, diffusion and reaction.

The diffusion equation stands for the procedure that makes things (molecules, atoms, heat) move from a high concentration part to a low concentration part to achieve balanced concentration. A simple example of diffusion in gases appears when we spray a perfume and after a few minutes its smell spreads throughout the room. Simple diffusion also occurs continuously in the human body while we breathe, since gas exchange occurs between our lungs and the air that we breathe. The term reaction refers to the procedure which changes the concentration of the concerned substance.



Diffusion process when two components interact. Here components A and B are diffusing and at the same time they are reacting to produce a complex AB which itself also diffuses.

A generalised formulation of the reaction-diffusion equation for a single substance in one spatial dimension is as follows:

$$\frac{\partial}{\partial t}C(x,t) = D\frac{\partial}{\partial x^2}C(x,t) + R(C),$$

where $C(x,t)$ is the concentration of the substance at a specifed x-coordinate and time t, $\frac{\partial}{\partial x^2}C(x,t)$ represents the diffusive transport of the substance, $R(C)$ is the reaction function which represents the production or destruction of the substance resulting from reactions among it, and D is the diffusion coefficient. This simple case of the reaction-diffusion equation is known as the Kolmogorov-Petrovsky-Piskunov equation [60].

Since in the classical works [60] and [48] the reaction-diffusion model was introduced to describe the propagation of an advantageous gene within a population, a great deal of work has been carried out to extend their model to take into account other biological, chemical or physical factors.

In fact, applications in Economics can be found. In particular, capital accumulation distribution in space and time following spatial extensions of the continuous Ramsey model [74] by Brito [14–16] and others later uses the semilinear parabolic equation

$$\partial_t u - \alpha\Delta u = f(u) - c.$$

This spatiality introduces important issues about the steady states distribution as well as the dynamic evolution, convergence, local interaction among local agents, and so on.

One of the most beatiful and visual application of this model was obtained by Turing in [82] where he described how patterns in nature, such as stripes and spots, can arise naturally and autonomously from a homogeneous, uniform state. In this work, Turing introduced the concept of pattern to study the behaviour of a system in which two diffusible substances interact with each other. He found that such a reaction-diffusion system is able to generate a spatially periodic pattern even from a random or almost uniform initial condition.

It is possible to find also applications in areas of medicine such as models about cancer mechanism for cancer invasion [50]. Recently, research has been developed and a reaction-diffusion model describing more accurately the spatial distribution and temporal development of tumor tissue is presented (see [47] and references therein).

In this line, there are models in epidemiology that make it possible to predict the characteristics of the spreading of an infectious disease. A general modelling technique are compartmental models where population is assigned to compartments with labels. The SIR model is one of the simplest with three compartments: S, the number of Susceptible individuals; I, the number of Infectious individuals; and R, the number of Removed (and immune) or deceased individuals.

This model is reasonably predictive for infectious diseases that are transmitted from human to human and try to predict things such as how a disease spreads, the total number infected, the duration of an epidemic [91]. To allow for spatial dynamics, disease-spreading theories such as the SIR model have been extended to reaction–diffusion equations (see [43], [71], [9], [87], [12], [69], [73]).

Therefore, the main aim is to show how different public health interventions may affect the outcome of the epidemic, e.g., which is the most efficient technique for issuing a limited number of vaccines in a given population.

As it is well known, in December 2019, in the Chinese city of Wuhan, an outbreak of a disease caused by a new coronavirus was reported. It rapidly spreaded to other regions of China and the whole world. Subsequently, the World Health Organization officially recognized the new coronavirus as SARS-CoV-2 and named the disease COVID-19. Since then, the disease has caused millions of deaths.

Consequently, studies using reaction-diffusion models about the spreading trend, long-term dynamic behavior, effects of social distancing, home quarantine or lockdown were carried out to understand how these factors affect the epidemic spreading of the COVID-19 (see, e.g., [76], [78], [94], [95] and references therein). This gives us an idea of the mathematical relevance of the reaction-diffusion model and the need to continue deepening a broader knowledge of the equation.

Once several applications of the differential equations, specifically the reaction-diffusion model, have been seen, it is necessary to focus on the technical part. Hence, approaching the study part of this thesis, it should be noted that very often it is important to know how the solutions of differential equations behave with respect to some parameter and many interesting phenomena can be hidden in such behavior.

As an example, we can mention various perturbations of differential equations generating plenty of interesting and intriguing scenarios for the behavior of the solution, studying of asymptotics of spectral characteristics for various differential operators, stability and bifurcations in dynamical systems, homogenization of boundary value problems and many others.

In this thesis, we restrict our attention on reaction-diffusion equations without uniqueness of solutions of the associated Cauchy problem. Afterward, we analyze with more precision the structure of the attractor for equations of Chafee–Infante type which has been extensively studied, starting with the article of the authors who give name to this equation [33]. Its most interesting feature is a bifurcation in the system parameter which considerably changes the dynamics. Existence and regularity of its solutions have been investigated, as well as the fine structure of the attractor. We refer to the classical books [80], [75], [54], [53] and the references therein.

We recall some properties of its longtime dynamics and in particular the structure of its attractor following the classical Chafee-Infante equation, although in [54] the reaction term is more general and all results are proved.
The equation is given by

$$\begin{aligned}
&\frac{\partial u}{\partial t} - \Delta u + \lambda(u^3 - u) = 0, && t > 0,\ x \in [0,1],\\
&u(t,0) = u(t,1) = 0, && t > 0,\\
&u(0,x) = u_0, && x \in [0,1].
\end{aligned}$$

Existence, uniqueness and regularity results are well known [80, pg. 84].

The solution flow $(t, x) \mapsto u(t; x)$ is continuous in t and x and defines a dynamical system in $H_0^1(0,1)$. Moreover, we have continuity with respect to the initial data in $H_0^1(0,1)$.

As regards the main features of the steady states, a detailed exposition of the bifurcation on the elliptic problem can be found in [75]. However, we summarize them in the following result.

Theorem. *Let be $\lambda \leq \pi^2$. Then there is a unique stable fixed point $v \equiv 0$. For $\lambda > \pi^2$ there are always two stable fixed points $\phi^\pm \in C^\infty([0,1])$. More precisely, if $(n\pi)^2 < \lambda \leq ((n+1)\pi)^2, n \in \mathbb{N}$ there are 2 stable and $(2n-1)$ unstable fixed points $\{0, \phi^\pm, \phi_1^\pm, \cdots, \phi_{n-1}^\pm\}$. Thus, the set of steady states Ξ_λ has the following shape*

$$\Xi_\lambda := \begin{cases} \{0\} & 0 < \lambda \leq \pi^2, \\ \\ \{0, \phi^\pm\} & \pi^2 < \lambda \leq (2\pi)^2, \\ \\ \{0, \phi^\pm, \phi_1^\pm, \ldots, \ldots, \phi_{n-1}^\pm\}, & (n\pi)^2 < \lambda \leq ((n+1)\pi)^2, \quad n \geq 2. \end{cases}$$

Moreover, for any initial value $u_0 \in H_0^1(0,1)$ the trajectory $t \mapsto u(t; u_0)$ converges to an element of Ξ_λ [54]. This fact relies on the existence of an energy functional called Lyapunov function for the equation. This will be crucial for our work as we well see.

We also have to focus on the properties of the global attractor of the system. We need to further specify the fine structure of the attractor of the Chafee–Infante equation. Note that it depends crucially on the bifurcation parameter λ.

The dynamical system induced by the solution flow of Chafee–Infante equation is well-known to have a global attractor $\mathcal{A} \in L^2(0,1), C([0,1])$ and $H_0^1(0,1)$ [80]. Let

$$M^u(v) := \{u_0 \in H_0^1(0,1) : \text{ there exists a global solution } u(t) \text{ in } H_0^1(0,1)$$
$$\text{such that } \exists t_0 \in \mathbb{R} : u_0 = u(t_0) \text{ and } \lim_{t \to -\infty} u(t) = v\}$$

be the unstable manifold of $v \in \Xi_\lambda$.

We define for $v, w \in \Xi_\lambda$ the set of complete connecting orbits

$$C(v,w) := \{u_0 \in H_0^1(0,1) : \text{ there exists a solution } u(t) \text{ in } H_0^1(0,1) \\ \text{such that } \exists t_0 \in \mathbb{R} : \lim_{t\to\infty} u(t) = w \text{ and } \lim_{t\to-\infty} u(t) = v\},$$

when it is non-empty. If such an orbit does not exist, $C(v,w) = \emptyset$.
The attractor $\mathcal{A}_\lambda$ consists of all fixed points and all global bounded trajectories $\{u(t), t \in \mathbb{R}\}$. For $v, w \in \Xi_\lambda, v \neq w$, using the notation

$$v \rightsquigarrow w \;\longleftrightarrow\; C(v,w) \neq \emptyset,$$

from [42] we have

$$\mathcal{A}_\lambda = \Xi_\lambda \cup \bigcup_{v\in\Xi_\lambda} M^u(v), \quad \text{where} \quad M^u(v) = \{v\} \cup \bigcup_{\substack{w\in\Xi_\lambda \\ v\rightsquigarrow w}} C(v,w),$$

for $\lambda > 0$.
In other words,

$$\mathcal{A}_\lambda = \{\phi^+, \phi^-\} \cup \bigcup_{v\in\Xi_\lambda\setminus\{\phi^+,\phi^-\}} \left\{ \{v\} \cup \bigcup_{\substack{w\in\Xi_\lambda \\ v\rightsquigarrow w}} C(v,w) \right\}.$$

As we will see later, a connection from a fixed point to another is allowed only if the number of zeros of the first one is greater. By this way, it is possible to have always a connection from the null equilibrium point to another equilibria.

If λ passes $(n\pi)^2$ from the left, the connection structure of the elements of Ξ_λ for $((n-1)\pi)^2 < \lambda < (n\pi)^2$ is retained in $\mathcal{A}_\lambda$ for $\lambda > (n\pi)^2$ as a substructure, but two new unstable fixed points $\phi_{n-1}^\pm$ appear in Ξ_λ. In addition, new connecting orbits emerge in the attractor: $2(2n-3)$ ones linking the $2n-3$ previously unstable fixed points $\{0, \phi_1^\pm, \ldots, \phi_{n-2}^\pm\}$ with each of the new ones $\{\phi_{n-1}^+, \phi_{n-1}^-\}$, and 4 trajectories directed from each the latter ones to each of the stable points $\{\phi^+, \phi^-\}$ and hence $4n-2$ newly connected orbits.

In particular the number of connecting orbits for $\lambda \in ((\pi(n-1))^2, (\pi n)^2)$ is exactly

$$\sum_{k=1}^{n-1} (4k-2) = 2(n-1)^2.$$

We show in Figure 1 the qualitative shape of the attractor. For $((n\pi)^2 < \lambda < ((n+1)\pi)^2$ the elements of Ξ_λ as well as the entire set $\mathcal{A}_\lambda$ depend continuosly on λ.

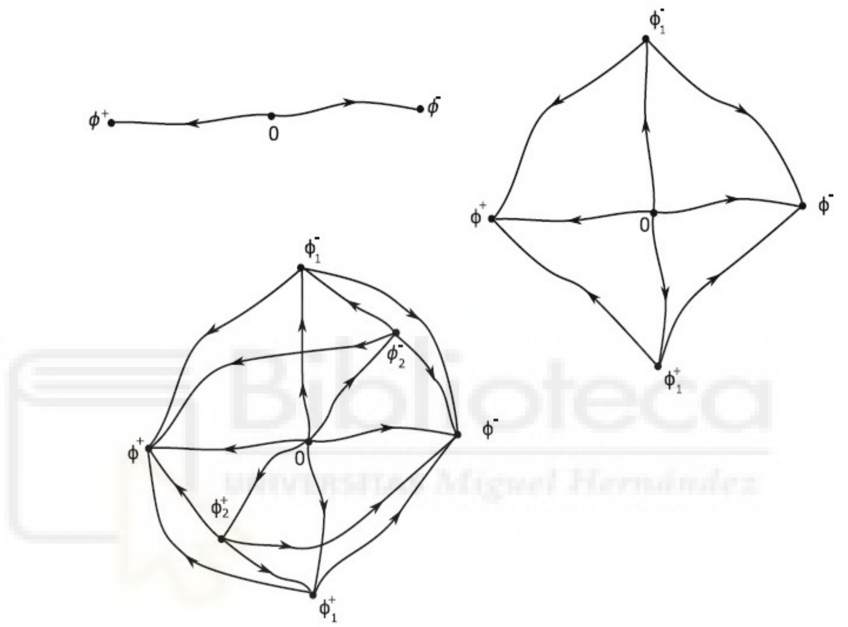


Figura 1: Sketch of $\mathcal{A}_\lambda$ for $\pi^2 < \lambda < (2\pi)^2,\ (2\pi)^2 < \lambda < (3\pi)^2,\ (3\pi)^2 < \lambda < (4\pi)^2$.

As we have just seen, one of the main goals of the theory of dynamical systems is to characterize the structure of global attractors. It is possible to find a wide literature about this problem for semigroups; however, it has been recently when new results in this direction for multivalued dynamical systems have been proved [7], [57], [58]. As one of the novelties, this thesis works with multivalued dynamical systems where the uniqueness of the associated Cauchy problem cannot be guaranteed.

By this way and focusing now on what will be done throughout this work, the first chapter is devoted to present definitions and basic results in the framework of multivalued dynamical systems. We also describe some elements of the theory of Morse decomposition which play an important role in this area of dynamical systems. In fact, the existence of a Lyapunov function, the property of being a dynamically gradient semiflow and the existence of a Morse decomposition are shown to be equivalent for multivalued dynamical systems in [44].

The second chapter in this thesis focuses on showing under suitable assumptions that a dynamically gradient multivalued semiflow is stable under perturbations.

For a fixed dynamically gradient multivalued semiflow with a global attractor we also analyze the rearrangement of a pairwise disjoint finite family of isolated weakly invariant sets, included in the attractor, in such a way that the dynamically gradient property is satisfied in the stronger sense of [64].

These results extend previous ones in the single-valued framework in [5, 6, 30] to the case where uniqueness of solution does not hold. Additionally, it is worth saying that the m-semiflows here are not supposed to be general dynamical systems as in [64], where a robustness theorem for Morse decompositions of multivalued dynamical systems is also proved under a suitable continuity assumption.

We also apply this general robustness theorem in order to show that a family of Chafee-Infante problems approximating a differential inclusion is dynamically gradient if it is close enough to the original problem.

Moving onto next question tackled on this thesis, the reaction-diffusion models studied before are generalized and we introduce a nonlocal term in the diffusion coeficient.

The study of this model is motivated because in real applications there might exist several nonlocal effects that influence the evolution of a system. For instance, usually we do not have enough information about the systems under study and its features at every point. In reality, the measurements are not made pointwise but through some local average. Actually, during the last decades many mathematicians have been studying nonlocal problems motivated by its various applications in Physics, Biology or population dynamics [35–39, 66].

Firstly we might comment about extensions by using some nonlocal operators acting in the right-hand side of the PDE and/or the boundary conditions as integral operators, leading to integro-differential equations. Among others we can cite [4] for a system coupling capital and pollution stock model (pollutants towards which the environment has low absorptive capacity), a population dynamic model in [46]

$$\partial_t u - \alpha \Delta u = u \left(f(u) - \alpha \int_{\mathbb{R}^N} g(x-y) u(y,t) dy \right),$$

the elliptic (stationary) counterpart in population/physics models as the Fischer-KPP [1], or a logistic model [45]. Secondly, we wish to point out that the nonlocal extensions have also been performed on the diffusion operators as well. The literature about fractional laplacian is vast nowadays. However, let us concentrate in an intermediate step. Coming originally from modeling of bacteria population in Biology, the introduction of a nonlocal viscosity in front of the laplacian has become an interesting problem for different applications and for its mathematical study, as for example occurs in the equation

$$u_t - a(\int_\Omega g(y) u(t,y) dy) \Delta u = f(t).$$

In this way, the spreading (or aggregating/concentrating) effects are given by the increasing (resp. non-increasing) function a as a viscosity nonlocal coefficient.

In this sense, let consider the problem of finding a function $u(t,x)$ such that

$$\begin{cases} u_t - a(\int_\Omega u(t,x)dx)\Delta u = g(t,u), \text{ in } \Omega \times (0,\infty), \\ u = 0 \quad \text{on } \partial\Omega \times (0,\infty), \\ u(0) = u_0 \quad \text{in } \Omega. \end{cases} \tag{1}$$

Here Ω is a bounded open subset in $\mathbb{R}^n$, $n \geq 1$, with smooth boundary and a is some function from $\mathbb{R}$ to $(0, +\infty)$. In such equation u could describe the density of a population subject to spreading. The diffusion coefficient a is then supposed to depend on the entire population in the domain rather than on the local density.

A wide literature with significant results about (1) has been developed during the last few decades (see for example [36, 39, 66]). However, it is possible to distinguish two basic cases of the following more general equation

$$\begin{cases} u_t - a(u)\Delta u = g(t,u), & t > 0, \ x \in \Omega, \\ u = 0, \quad \text{on } \partial\Omega \times (0,\infty), \\ u(0,x) = u_0(x) \quad x \in \Omega. \end{cases}$$

Some authors consider a depending on a linear functional $l(u)$, i.e.,

$$a(u) = a(l(u))$$

with

$$l(u) = \int_\Omega \Phi(x)u(x,t)dx,$$

where $\Phi(x)$ is a given function in $L^2(\Omega)$.
For $g(t,u) = f(t)$ the existence and uniqueness of solutions and their asymptotic behavior are studied for example in [37, 38, 40, 93].
For $g(t,u) = f(u) + h(t)$ the existence, uniqueness and asymptotic behaviour of solutions are studied in [3, 23, 25, 26]. Moreover, the authors prove the existence of pullback attractors in $L^2(\Omega)$ and $H_0^1(\Omega)$. Extensions in this direction for equations governed by the p-laplacian operator instead of the laplacian operator Δ are given in [24, 27], whereas nonclassical diffusion equations are considered in [72].

On the other hand, it is possible to consider a function a such that $a(u) = a(\|u\|^2_{H_0^1})$. The existence and uniqueness of solutions of the following problem

$$\begin{cases} u_t - a(\|u\|^2_{H_0^1})\Delta u = f, & t > 0, \ x \in \Omega, \\ u = 0, \ \text{on } \partial\Omega \times (0,\infty), \\ u(0,x) = u_0(x) \quad x \in \Omega. \end{cases}$$

is proved in [41, 93], where $f \in L^2(\Omega)$, $u_0 \in H_0^1(\Omega)$ and $a = a(s)$ is a continuous function such that $0 < m \le a(s) \le M$.

By this way, the following problem will be considered throughout chapters two and three

$$\begin{cases} u_t - a(\|u\|^2_{H^1_0})\Delta u = f(u) + h(t), \text{ in } \Omega \times (0,\infty), \\ u = 0 \quad \text{on } \partial\Omega \times (0,\infty), \\ u(0,x) = u_0\,(x) \quad \text{in } \Omega, \end{cases} \tag{2}$$

where $h \in L^2(0,T;L^2(\Omega))$, for all $T > 0$, $a : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function such that $a\,(s) \geq m > 0$ and f is a continuous function satisfying standard dissipative and growth conditions (see (2.1.5)).

More precisely, the aim of the third chapter is three-fold. First, we will prove the existence of solutions for problem (2) under different assumptions on the nonlinear function f. Second, we will obtain the existence of attractors for the semiflows generated by either regular or strong solutions in the autonomous case, that is, when h does not depend on time. Third, we establish that the global attractor can be characterized by the unstable manifold of the set of stationary points. It is important to notice that the proof of this last fact requires the existence of a Lyapunov function on the attractor, and for this aim the term $a(\|u\|^2_{H^1_0})$ is crucial. In the case when $a(u) = a(l(u))$ it is not known whether such a function exists or not.

We prove the existence of strong solutions by assuming that either the function f is continuously differentiable and $f'\,(s) \leq \eta$ or that it satisfies a more strict growth condition. Supposing additionaly that the function a has sublinear growth we prove the existence of regular solutions as well. Moreover, when $f'\,(s) \leq \eta$ and the function $s \mapsto a\,(s^2)\,s$ is non-decreasing, uniqueness is proved.

When studying the asymptotic behaviour of solutions, new challenging difficulties arise for problem (2). For this problem we consider the autonomous situation, that is, $h \in L^2\,(\Omega)$ does not depend on t.

If uniqueness holds, then we define classical semigroups (one for regular solutions and one for strong solutions) and prove the existence of the global attractor. Under some extra assumptions on the functions a, h we are able to obtain that the global attractor is bounded in $H^2\,(\Omega)$ and $L^\infty\,(\Omega)$.

If uniqueness is not known to be true, then we have to define a (possibly) multivalued semiflow. Then the existence of the global attractor is proved for regular solutions in the topology of the space $L^2(\Omega)$ and for strong solutions in the topology of the space $H_0^1(\Omega)$ (or $H_0^1(\Omega) \cap L^p(\Omega)$), extending in this way the known results for the local problem [57].

The structure of the global attractor is an important feature as it gives us an insight into the long-term dynamics of the solutions. In the multivalued situation it is a challenging problem that has not been completely understood yet. So far in the local case several results in this direction have been obtained for reaction-diffusion equations without uniqueness [7, 18, 57, 58].

In our nonlocal problem for both situations (for regular and strong solutions) we are able under some conditions to define a Lyapunov function on the attractor and to prove that it is characterized as the unstable set of the stationary points. Also, the attractor is equal to the stable set of the stationary points when we consider solutions only in the set of bounded complete trajectories.

If we consider the general equation

$$u_t - a(\Phi_\Omega(u(t))\Delta u = f(u), \tag{3}$$

equilibria are difficult to analyse. Here the functional Φ_Ω may represent a general nonlocal functional acting over the whole domain Ω, for instance

$$\|u(t)\|_{H_0^1}^2 \quad \text{or} \quad \int_\Omega g(y)u(t,y)dy.$$

Opposite to ordinary differential equations, the analysis of existence of stationary states for the above problem is much more involved. Also, comparing with reaction-diffusion equations with local diffusion, another difficulty is that in general a Lyapunov functional is not known to exist in most cases. One should cite Prof. Chipot and his collaborators [36–41, 93] among others for a detailed analysis including existence, uniqueness, steady states and convergence of evolutionary solutions to equilibria, in the particular case where f is constant.

If we consider the non-local equation

$$\frac{\partial u}{\partial t} - a(\|u\|^2_{H^1_0})\frac{\partial^2 u}{\partial x^2} = \lambda f(u) \tag{4}$$

with Dirichlet boundary conditions, then it is possible to define a suitable Lyapunov functional. As we shall see in the third chapter, regular and strong solutions generate (possibly) multivalued semiflows having a global attractor which is described by the unstable set of the stationary points.

Although this is already a good piece of information, our goal in the last chapter of this thesis is to describe the structure of the attractor as accurately as possible. For this aim, in [20] we focus on studying the particular situation where the domain is one-dimensional and the function f is of the type of the standard Chafee-Infante problem, for which the dynamics inside the attractor has been completely understood [55].

The first point in the last chapter studies the existence of strong solutions of the Cauchy problem in the space H^1_0. As well, we prove that strong solutions generate a multivalued semiflow in H^1_0 having a global attractor which is equal to the unstable set of the stationary points.

When we study the structure of the attractor, we need to analyse the stationary points. In the case where the function f is odd and equation (4) generates a continuous semigroup the existence of fixed points of the type given in the Chafee-Infante problem was established in [31]. Moreover, if a is non-decreasing, then they coincide with the ones in the Chafee-Infante problem and, also, in [32] the stability and hyperbolicity of the fixed points are studied.

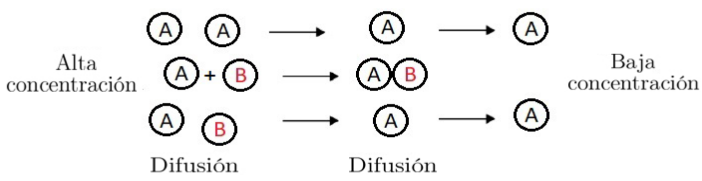
In this fourth chapter we extend these results for a more general function f (not necessarily odd and for which we do not known whether the Cauchy problem has a unique solution or not), showing that equation (4) undergoes the same cascade of bifurcations as the Chafee-Infante equation. Moreover, when we allow the function a to decrease, though the problem possesses at least the same fixed points as in the Chafee-Infante problem, we show that more equilibria can appear. For a non-decreasing function a and an odd function f we prove also that even when uniqueness fails the stability of the fixed points is the same as for the corresponding ones in the Chafee-Infante problem.

Finally, we are able to prove that in this last case the semiflow is dynamically gradient with respect to the disjoint family of isolated weakly invariant sets generated by the equilibria, which is ordered by the number of zeros of the fixed points. More precisely, the attractor consists of the set of equilibria and their heteroclinic connections and a connection from a fixed point to another is allowed only if the number of zeros of the first one is greater.

Introducción

Las ecuaciones diferenciales juegan un papel muy importante no solo en Matemáticas sino también en otras ciencias. Se han utilizado durante siglos en campos como Física, Química o Biología. Hay dos ecuaciones importantes que se utilizan para describir varios procesos que ocurren a nuestro alrededor. Los procesos más comunes incorporan la variedad de concentración de al menos una sustancia en el tiempo y en el espacio bajo el impacto de dos respuestas, que son, como su nombre indica, difusión y reacción.

La ecuación de difusión representa el procedimiento que hace que las sustancias (moléculas, átomos, calor) se muevan desde altas a bajas concentraciones para lograr una concentración equilibrada. Un ejemplo simple de difusión en gases aparece cuando rociamos un perfume y después de unos minutos su olor se esparce por toda la estancia. La difusión simple también ocurre continuamente en el cuerpo humano mientras respiramos, ya que el intercambio de gases se produce entre nuestros pulmones y el aire que respiramos. El termino reacción se refiere al procedimiento que cambia la concentración de la sustancia en cuestión.



Proceso de difusión cuando interactúan dos componentes. Aquí los componentes A y B se difunden y al mismo tiempo reaccionan para producir un complejo AB que también se difunde.

Una formulación generalizada de la ecuación de reacción-difusión para una sola sustancia en una dimensión es la siguiente:

$$\frac{\partial}{\partial t}C(x,t) = D\frac{\partial}{\partial x^2}C(x,t) + R(C),$$

donde $C(x,t)$ es la concentración de la sustancia en un punto x y un tiempo t específicos, $\frac{\partial}{\partial x^2}C(x,t)$ representa el transporte difusivo de la sustancia, $R(C)$ es la función de reacción que representa la producción o destrucción de la sustancia resultante de las reacciones, and D es el coeficiente de difusión. Este caso simple de la ecuación de reacción-difusión se le conoce como la ecuación de Kolmogorov-Petrovsky-Piskunov [60].

Desde que en los trabajos clásicos [60] y [48] el modelo de reacción-difusión fuera introducido para describir la propagación de un gen predominante dentro de una población, se ha trabajado mucho en el modelo para tener en cuenta otros factores biológicos, químicos o físicos. De hecho, se pueden encontrar aplicaciones en Economía. En particular, la distribución de la acumulación de capital en el espacio y tiempo, siguiendo extensiones espaciales del modelo continuo de Ramsey [74] por Brito [14–16] y otros autores posteriores, usa la ecuación semilineal parabólica siguiente:

$$\partial_t u - \alpha\Delta u = f(u) - c.$$

Esta espacialidad introduce cuestiones importantes sobre la distribución de los estados estacionarios, así como sobre la evolución dinámica, convergencia o la interacción local entre agentes locales.

Una de las aplicaciones más hermosas y visuales de este modelo fue obtenida por Turing en [82] donde describió cómo patrones en la naturaleza, rayas y manchas, pueden surgir de forma natural y autónoma a partir de un estado homogéneo y uniforme. En este trabajo, Turing introdujo el concepto de patrón para estudiar el comportamiento de un sistema en el que dos sustancias, que presentan difusión, interactúan entre sí. Descubrió que tal sistema de reacción-difusión es capaz de generar un patrón espacialmente periódico incluso a partir de una condición inicial aleatoria o casi uniforme.

También es posible encontrar aplicaciones en áreas de la medicina, como modelos sobre el mecanismo del cáncer [50]. Recientemente, se ha desarrollado un modelo de reacción-difusión que describe con mayor precisión la distribución espacial y el desarrollo temporal del tejido tumoral (ver [47] y las referencias en el mismo).

En esta línea, existen modelos en epidemiología que permiten predecir las características de la propagación de una enfermedad infecciosa. Una técnica de modelado general son los modelos compartimentales en los que la población se asigna a compartimentos con etiquetas. El modelo SIR es uno de los más simples con tres compartimentos: S, el número de individuos susceptibles a la infección; I, el número de individuos infecciosos que infectan; y R, el número de personas recuperadas (curados o fallecidos).

Este modelo predictivo para enfermedades infecciosas que se transmiten de persona a persona trata de vaticinar cómo se propaga una enfermedad, el número total de infectados o la duración de una epidemia [91]. Para permitir un análisis de la dinámica espacial, las teorías de propagación de enfermedades como el modelo SIR se han extendido a ecuaciones de reacción-difusión (ver [43], [71], [9], [88], [12], [69], [73]).

Por lo tanto, el objetivo principal es mostrar cómo las diferentes intervenciones en la salud pública pueden afectar al resultado de la epidemia, por ejemplo, implementar la técnica más eficiente para emitir un número limitado de vacunas en una población determinada.

Como bien es sabido, en diciembre de 2019, en la ciudad china de Wuhan, se reportó un brote de una enfermedad provocada por un nuevo coronavirus. Extendida rápidamente a otras regiones de China y del mundo entero, la Organización Mundial de la Salud reconoció oficialmente el nuevo coronavirus como SARS-CoV-2 y nombró a la enfermedad COVID-19. Desde entonces, la enfermedad ha causado millones de muertes.

En consecuencia, estudios que utilizan modelos de reacción-difusión sobre la tendencia de propagación, comportamiento dinámico a largo plazo, efectos del distanciamiento social, cuarentena domiciliaria o confinamientos se llevaron a cabo para comprender cómo estos factores afectan a la propagación de la epidemia del COVID-19 (ver, p. ej., [76], [78], [94], [95] y las referencias en los mismos). Esto nos da una idea de la relevancia matemática del modelo de reacción-difusión y de la necesidad de seguir profundizando en un conocimiento más amplio de la ecuación.

Una vez vistas varias aplicaciones de las ecuaciones diferenciales, en particular el modelo de reacción difusión, es necesario centrarse en la parte técnica. En este sentido, y acercándonos al contenido de esta tesis, conviene señalar la importancia de saber cómo se comportan las soluciones de ecuaciones diferenciales con respecto a algún parámetro y los muchos fenómenos interesantes que pueden esconderse en dicho comportamiento.

Por ejemplo, podemos mencionar las perturbaciones de varias ecuaciones diferenciales que generan situaciones interesantes en el comportamiento de la solución, el estudio asintótico de las características espectrales para varios operadores diferenciales, estabilidad y bifurcaciones en sistemas dinámicos, homogeneización de problemas de valores límite y muchos otros.

En esta tesis, fijamos nuestra atención en las ecuaciones de reacción-difusión sin unicidad de soluciones del problema de Cauchy asociado. Posteriormente, analizamos con más precisión la estructura del atractor para ecuaciones de tipo Chafee-Infante que ha sido ampliamente estudiado, comenzando por el artículo de los autores que dan nombre a esta ecuación [33]. Su característica más interesante es una bifurcación en el parámetro del sistema que cambia considerablemente la dinámica. Existencia y regularidad de soluciones han sido estudiadas, así como la estructura fina del atractor. Para más detalle, se pueden consultar las fuentes clásicas [80], [75], [54], [53] y las referencias que contienen.

Vamos a recordar algunas propiedades de su dinámica a largo plazo y, en particular, la estructura de su atractor planteando la ecuación clásica de Chafee-Infante, aunque en [54] el término de reacción es más general y se prueban todos los resultados.

La ecuación está dada por

$$\begin{aligned} &\frac{\partial u}{\partial t} - \Delta u + \lambda(u^3 - u) = 0, && t > 0,\ x \in [0,1], \\ &u(t,0) = u(t,1) = 0, && t > 0, \\ &u(0,x) = u_0, && x \in [0,1]. \end{aligned}$$

Resultados sobre existencia, unicidad y regularidad son bien conocidos [80, pg. 84]. El flujo que genera la solución $(t, x) \mapsto u(t; x)$ es continuo en t y en x y define un sistema dinámico en el espacio $H_0^1(0,1)$. Además, tenemos la propiedad de la continuidad con respecto a la condición inicial en $H_0^1(0,1)$.

En cuanto a las características principales de los estados estacionarios, se puede encontrar una exposición detallada de la bifurcación en el problema elíptico en [75]. No obstante, en el siguiente resultado quedan recogidas.

Teorema. *Sea $\lambda \leq \pi^2$. Entonces existe un único punto fijo estable $v \equiv 0$. Para $\lambda > \pi^2$ hay siempre dos puntos fijos estables $\phi^{\pm} \in C^{\infty}([0,1])$. En concreto, si $(n\pi)^2 < \lambda \leq ((n+1)\pi)^2, n \in \mathbb{N}$ hay 2 puntos fijos estables y $(2n-1)$ inestables $\{0, \phi^{\pm}, \phi_1^{\pm}, \cdots, \phi_{n-1}^{\pm}\}$. De esta forma, el conjunto de los estados estacionarios Ξ_λ viene determinado por*

$$\Xi_\lambda := \begin{cases} \{0\} & 0 < \lambda \leq \pi^2, \\ \\ \{0, \phi^{\pm}\} & \pi^2 < \lambda \leq (2\pi)^2, \\ \\ \{0, \phi^{\pm}, \phi_1^{\pm}, \ldots, \ldots, \phi_{n-1}^{\pm}\}, & (n\pi)^2 < \lambda \leq ((n+1)\pi)^2, \quad n \geq 2. \end{cases}$$

Más aún, para cada valor inicial $u_0 \in H_0^1(0,1)$ la trayectoria $t \mapsto u(t; u_0)$ converge a un elemento de Ξ_λ [54]. Este hecho reside en la existencia de un funcional de energía asociado a la ecuación llamado función de Lyapunov. Como veremos más adelante, esta propiedad será crucial a lo largo de nuestro trabajo.

También tenemos que centrarnos en las propiedades del atractor global del sistema. Necesitamos especificar más la estructura fina del atractor de la ecuación de Chafee-Infante, ya que depende fundamentalmente del parámetro de bifurcación λ.

Se sabe que el sistema dinámico inducido por el flujo de la solución de la ecuación de Chafee–Infante tiene un atractor global $\mathcal{A} \in L^2(0,1), C([0,1])$ y $H_0^1(0,1)$ [80]. Sea

$$M^u(v) := \{u_0 \in H_0^1(0,1) : \text{ existe una solución global } u(t) \text{ en } H_0^1(0,1) \\ \text{tal que } \exists t_0 \in \mathbb{R} : u_0 = u(t_0) \text{ y } \lim_{t\to-\infty} u(t) = v\}$$

la variedad inestable de $v \in \Xi_\lambda$. Definimos para $v, w \in \Xi_\lambda$ el conjunto de las órbitas completas y conectadas como

$$C(v,w) := \{u_0 \in H_0^1(0,1) : \text{ there exists a solution } u(t) \text{ in } H_0^1(0,1) \\ \text{such that } \exists t_0 \in \mathbb{R} : \lim_{t\to\infty} u(t) = w \text{ and } \lim_{t\to-\infty} u(t) = v\},$$

cuando el conjunto es no vacío. Si tal órbita no existiera, $C(v,w) = \emptyset$.
El atractor $\mathcal{A}_\lambda$ está formado por todos los puntos fijos y todas las trayectorias globales y acotadas $\{u(t), t \in \mathbb{R}\}$. Para $v, w \in \Xi_\lambda, v \neq w$, usando la notación

$$v \rightsquigarrow w \quad \longleftrightarrow \quad C(v,w) \neq \emptyset,$$

en virtud de [42], tenemos que

$$\mathcal{A}_\lambda = \Xi_\lambda \cup \bigcup_{v\in\Xi_\lambda} M^u(v), \qquad \text{donde} \qquad M^u(v) = \{v\} \cup \bigcup_{\substack{w\in\Xi_\lambda \\ v\rightsquigarrow w}} C(v,w),$$

para $\lambda > 0$. En otras palabras,

$$\mathcal{A}_\lambda = \{\phi^+, \phi^-\} \cup \bigcup_{v\in\Xi_\lambda\setminus\{\phi^+,\phi^-\}} \left\{ \{v\} \cup \bigcup_{\substack{w\in\Xi_\lambda \\ v\rightsquigarrow w}} C(v,w) \right\}.$$

Como veremos posteriormente, las conexiones sólo están permitidas desde puntos fijos con más ceros a otros con menos. De esta forma, siempre es posible tener una conexión desde el punto de equilibrio nulo a otro cualquiera.

Si λ excede $(n\pi)^2$, la estructura de las conexiones de los elementos de Ξ_λ para $((n-1)\pi)^2 < \lambda < (n\pi)^2$ se mantiene en $\mathcal{A}_\lambda$ para $\lambda > (n\pi)^2$ como una subestructura, aunque dos nuevos puntos fijos inestables $\phi_{n-1}^{\pm}$ aparecen en Ξ_λ. Además, nuevas órbitas conectadas aparecen en el atractor: $2(2n-3)$ uniendo los $2n-3$ puntos fijos inestables previos $\{0, \phi_1^{\pm}, \ldots, \phi_{n-2}^{\pm}\}$ con cada uno de los nuevos $\{\phi_{n-1}^{+}, \phi_{n-1}^{-}\}$, y 4 trayectorias directas desde cada uno de los últimos hacia los puntos estables $\{\phi^+, \phi^-\}$ y, por tanto, $4n-2$ nuevas órbitas conectadas. En particular, el número de órbitas conectadas para $\lambda \in ((\pi(n-1))^2, (\pi n)^2)$ es exactamente

$$\sum_{k=1}^{n-1}(4k-2) = 2(n-1)^2.$$

En la Figura 2 se muestra la forma cualitativa del atractor. Para $((n\pi)^2 < \lambda < ((n+1)\pi)^2$, los elementos de Ξ_λ así como el conjunto completo $\mathcal{A}_\lambda$ depende continuamente de λ.

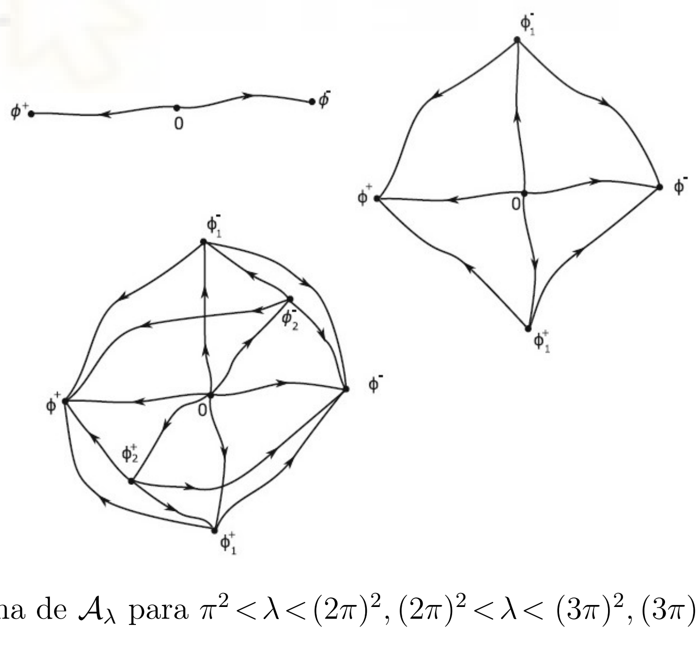


Figura 2: Esquema de $\mathcal{A}_\lambda$ para $\pi^2 < \lambda < (2\pi)^2$, $(2\pi)^2 < \lambda < (3\pi)^2$, $(3\pi)^2 < \lambda < (4\pi)^2$.

Como acabamos de ver, uno de los principales objetivos de la teoría de los sistemas dinámicos es caracterizar la estructura de los atractores globales. Es posible encontrar una amplia literatura sobre este problema para semigrupos; sin embargo, ha sido recientemente cuando nuevos resultados en esta dirección para sistemas dinámicos multivaluados han sido obtenidos [7], [57], [58]. Una de las novedades que pretende presentar esta tesis reside en la obtención de resultados trabajando con sistemas dinámicos multivaluados, donde no se puede garantizar la unicidad del problema de Cauchy asociado.

De esta manera y enfocándonos ahora en lo que se abordará a lo largo de este trabajo, el primer capítulo está dedicado a presentar definiciones y resultados básicos en el marco de sistemas dinámicos multivaluados. También describimos algunos elementos de la teoría de la descomposición de Morse que juegan un papel importante en esta área de sistemas dinámicos. De hecho, se demuestra en [44] que la existencia de una función de Lyapunov, la propiedad de ser un semiflujo dinámicamente gradiente y la existencia de una descomposición de Morse son propiedades equivalentes para sistemas dinámicos multivaluados.

El segundo capítulo de esta tesis se centra en mostrar, bajo determinadas condiciones, que un semiflujo multivaluado dinámicamente gradiente es estable bajo perturbaciones; es decir, la familia de semiflujos multivaluados perturbados permanece dinámicamente gradiente.

Para un semiflujo multivaluado dinámicamente gradiente con un atractor global, también analizamos el reordenamiento de una familia finita disjunta a pares de conjuntos aislados débilmente invariantes, incluidos en el atractor, de tal manera que la propiedad de ser dinámicamente gradiente se satisface en un sentido más fuerte que en [64].

Estos resultados amplían los anteriores en el marco univaluado (ver [5, 6, 30]) al caso donde la unicidad de la solución no se cumple. Además, los semiflujos multivaluados aquí no se suponen sistemas dinámicos generales como en [64], donde un teorema de robustez para la descomposición de Morse de un sistema dinámico multivaluado es obtenido bajo una condición específica de continuidad.

Adicionalmente, aplicamos este teorema de robustez general para demostrar que una familia de problemas de tipo Chafee-Infante que aproximan una inclusión diferencial es dinámicamente gradiente si está lo suficientemente cerca del problema original.

Pasando a la siguiente cuestión abordada en esta tesis, se generalizan los modelos de reacción-difusión estudiados anteriormente y se introduce un término no local en el coeficiente de difusión.

El estudio de este modelo viene motivado porque en las aplicaciones reales pueden existir varios efectos no locales que influyen en la evolución de un sistema. Por ejemplo, generalmente no tenemos suficiente información sobre los sistemas en estudio y sus características en cada punto. En realidad, las mediciones no se realizan puntualmente sino a través de algún promedio local. De hecho, durante las últimas décadas muchos matemáticos han estado estudiando problemas no locales motivados por sus diversas aplicaciones en Física, Biología o dinámica de poblaciones. [35–39, 66].

En primer lugar, podríamos hablar acerca de las extensiones utilizando algunos operadores no locales que actúan en el lado derecho de la EDP y/o en las condiciones de frontera como un operador integral, lo que lleva a unas ecuaciones de tipo integro-diferencial. Entre otros, cabe mencionar a [4] para un sistema de capital vinculado y un modelo de contaminantes de stock (contaminantes hacia los cuales el ambiente tiene una baja capacidad de absorción), [46] para un modelo dinámico de poblaciones

$$\partial_t u - \alpha \Delta u = u \left(f(u) - \alpha \int_{\mathbb{R}^N} g(x-y) u(y,t) dy \right),$$

el equivalente elíptico (estacionario) en modelos de poblaciones o en física como el Fischer-KPP [1], o el modelo logístico [45]. En segundo lugar, señalar que las extensiones no locales también se han realizado en los operadores de difusión. La literatura sobre el laplaciano fraccionario es muy amplia. Sin embargo, vamos a centrarnos en un paso intermedio.

Originario del modelo poblacional de bacterias en el campo de la Biología, la introducción de una viscosidad no local frente al laplaciano se ha convertido en un interesante problema para diferentes aplicaciones y para su estudio matemático, como ocurre por ejemplo en la siguiente ecuación

$$u_t - a(\int_\Omega g(y)u(t,y)dy)\Delta u = f(t).$$

De esta manera, los efectos de propagación (o agregación/concentración) están dados por la función creciente (resp. no creciente) a como un coeficiente de viscosidad no local.

En este sentido, considérese el problema de encontrar una función $u(t,x)$ tal que

$$\begin{cases} u_t - a(\int_\Omega u(t,x)dx)\Delta u = g(t,u), & \text{en } \Omega \times (0,\infty), \\ u = 0 \quad \text{on } \partial\Omega \times (0,\infty), \\ u(0) = u_0 \quad \text{en } \Omega. \end{cases} \tag{5}$$

Aquí Ω es un subconjunto abierto y acotado de $\mathbb{R}^n$, $n \geq 1$, con frontera suave y a es una función de $\mathbb{R}$ en $(0,+\infty)$. En esta ecuación u puede describir la densidad de una población sujeta a propagación. Por tanto, el coeficiente de difusión a depende de toda la población del dominio, en lugar de depender de la densidad local.

En las últimas décadas, una amplia literatura con resultados relevantes sobre (5) ha sido desarrollada (ver por ejemplo [36, 39, 66]). Sin embargo, es posible distinguir dos casos básicos de la siguiente ecuación más general

$$\begin{cases} u_t - a(u)\Delta u = g(t,u), \quad t > 0, \ x \in \Omega, \\ u = 0, \quad \text{en } \partial\Omega \times (0,\infty), \\ u(0,x) = u_0(x) \quad x \in \Omega. \end{cases}$$

Algunos autores consideran a dependiendo de un funcional lineal $l(u)$; es decir,

$$a(u) = a(l(u))$$

con

$$l(u) = \int_\Omega \Phi(x)u(x,t)dx,$$

donde $\Phi(x)$ es una función dada en $L^2(\Omega)$.
Para $g(t,u) = f(t)$, la existencia y unicidad de soluciones y el comportamiento asintótico es estudiado, por ejemplo, en [37, 38, 40, 93].
Para $g(t,u) = f(u) + h(t)$, en [3, 23, 25, 26] podemos encontrar los resultados sobre existencia, unicidad y comportamiento asintótico de las soluciones. Además, los autores prueban la existencia de atractores *pullback* en $L^2(\Omega)$ y en $H_0^1(\Omega)$. Extensiones en esta dirección para ecuaciones donde interviene el operador p-laplaciano en lugar del operador laplaciano clásico Δ se pueden encontrar en [24, 27], mientras que en [72] se consideran ecuaciones de difusión no clásicas.

Por otra parte, es posible considerar la función a como $a\,(u) = a(\|u\|^2_{H_0^1})$. De este modo, resultados acerca de la existencia y unicidad de soluciones del siguiente problema

$$\begin{cases} u_t - a(\|u\|^2_{H_0^1})\Delta u = f, \quad t > 0, \ x \in \Omega, \\ u = 0, \ \text{en } \partial\Omega \times (0, \infty)\,, \\ u(0,x) = u_0(x) \quad x \in \Omega. \end{cases}$$

se pueden encontrar en [41, 93], donde $f \in L^2(\Omega)$, $u_0 \in H_0^1(\Omega)$ y $a = a(s)$ es una función continua tal que $0 < m \le a(s) \le M$.

Vistos ambos casos, el siguiente problema se considerará a lo largo de los capítulos dos y tres

$$\begin{cases} u_t - a(\|u\|^2_{H_0^1})\Delta u = f(u) + h(t), \quad \text{en } \Omega \times (0,\infty), \\ u = 0 \quad \text{en } \partial\Omega \times (0,\infty), \\ u(0,x) = u_0\,(x) \quad \text{en } \Omega, \end{cases} \tag{6}$$

donde $h \in L^2(0,T;L^2(\Omega))$, para todo $T > 0$, $a : \mathbb{R}^+ \to \mathbb{R}^+$ es una función continua tal que $a\,(s) \ge m > 0$ y f es una función continua que cumple condiciones de disipación y crecimiento estándar (ver (2.1.5)).

Más precisamente, el objetivo del tercer capítulo es triple. Primero, probaremos la existencia de soluciones del problema (6) bajo diferentes hipótesis en la función no lineal f. Seguidamente, obtendremos la existencia de atractores para los semiflujos generados por soluciones regulares o fuertes en el caso autónomo; es decir, cuando h no depende del tiempo. Por último, estableceremos una caracterización para el atractor global en términos de las variedades inestables del conjunto de puntos fijos. Es importante remarcar que la prueba de este último hecho requiere de la existencia de una función de Lyapunov definida en el atractor, y para este objetivo el término $a(\|u\|^2_{H^1_0})$ resulta clave. En el caso cuando $a(u) = a(l(u))$ no se sabe si existe o no una función de Lyapunov.

Probamos la existencia de soluciones fuertes asumiendo que o bien la función f es continuamente diferenciable y $f'(s) \leq \eta$ o que satisface una condición de crecimiento más estricta. Suponiendo adicionalmente que la función a tiene un crecimiento sublineal, probamos la existencia de soluciones regulares. Además, cuando $f'(s) \leq \eta$ y la función $s \mapsto a(s^2)\, s$ es no decreciente, se tiene garantizada la unicidad.

Cuando estudiamos el comportamiento asintótico de las soluciones, nuevas dificultades desafiantes surgen para el problema (6). Para este problema, consideramos la situación autónoma, esto es, $h \in L^2(\Omega)$ no depende de t.

Si la propiedad de unicidad se mantiene, entonces podemos definir un semigrupo clásico (uno para soluciones regulares y otro para las fuertes) y es posible probar la existencia del atractor global. Bajo condiciones adicionales en las funciones a, h podemos obtener que el atractor global está acotado en $H^2(\Omega)$ y en $L^\infty(\Omega)$.

Si no se mantiene la propiedad de unicidad, entonces tenemos que definir un (posible) semiflujo multivaluado. Por tanto, la existencia del atractor global se prueba para soluciones regulares en la topología del espacio $L^2(\Omega)$ y para soluciones fuertes, en la topología del espacio $H^1_0(\Omega)$ (o $H^1_0(\Omega) \cap L^p(\Omega)$), extendiendo de esta manera los resultados conocidos para el local problema [57].

La estructura del atractor global es una característica importante ya que nos da una idea de la dinámica a largo plazo de las soluciones. En el marco multivaluado es un problema desafiante que todavía no ha sido entendido completamente. Hasta ahora, en el caso local, se han obtenido varios resultados en esta dirección para ecuaciones de reacción-difusión sin unicidad [7, 18, 57, 58].

En nuestro problema no local, para ambas situaciones (para soluciones regulares y fuertes) somos capaces, bajo algunas condiciones, de definir una función de Lyapunov en el atractor y demostrar que se caracteriza como el conjunto inestable de puntos estacionarios. Además, probamos que el atractor es igual al conjunto estable de los puntos estacionarios cuando consideramos soluciones sólo en el conjunto de trayectorias completas acotadas.

Si consideramos la ecuación general

$$u_t - a(\Phi_\Omega(u(t))\Delta u = f(u) \tag{7}$$

los puntos de equilibrio son difíciles de analizar. Aquí Φ_Ω puede representar un funcional no local general actuando sobre todo el dominio Ω, por ejemplo

$$\|u(t)\|^2_{H^1_0} \quad \text{o} \quad \int_\Omega g(y)u(t,y)dy.$$

A diferencia de las ecuaciones diferenciales ordinarias, el análisis de la existencia de estados estacionarios para el problema anterior es mucho más complicado. Además, comparando con las ecuaciones de reacción-difusión con difusión local, otra dificultad es que, en general, no se sabe que exista una función de Lyapunov en la mayoría de los casos. En este sentido, podemos citar al Prof. Chipot y sus colaboradores [36–41, 93], entre otros, para tener un análisis detallado, incluyendo existencia, unicidad, puntos fijos y convergencia de las soluciones hacia los puntos de equilibrio, en el caso particular en el que f es constante.

Si consideramos la ecuación no local

$$\frac{\partial u}{\partial t} - a(\|u\|^2_{H^1_0})\frac{\partial^2 u}{\partial x^2} = \lambda f(u) \tag{8}$$

con condiciones de frontera tipo Dirichlet, entonces es posible definir un funcional de Lyapunov adecuado. Como veremos en el tercer capítulo, soluciones regulares y fuertes generan (posibles) semiflujos multivaluados teniendo un atractor global que puede ser descrito por el conjunto inestable de puntos fijos.

Aunque esto representa un buen avance en el estudio, nuestro objetivo en el último capítulo de esta tesis es describir la estructura del atractor con la mayor precisión posible. Para este objetivo, en [20] centramos el estudio en la situación particular en la que el dominio es unidimensional y la función f es del tipo del problema estándar de Chafee-Infante, para el cual la dinámica dentro del atractor se ha entendido completamente [55].

En el primer punto del último capítulo se estudia la existencia de soluciones fuertes del problema de Cauchy en el espacio H_0^1. Asimismo, se demuestra que las soluciones fuertes generan un semiflujo multivaluado en el espacio H_0^1, teniendo un atractor global que es igual al conjunto inestable de los puntos estacionarios.

Cuando estudiamos la estructura del atractor, necesitamos analizar los puntos estacionarios. En el caso en el que la función f es impar y la ecuación (8) genera un semigrupo continuo, en [31] se establece la existencia de puntos fijos del tipo Chafee-Infante. Además, si a es no decreciente, entonces los puntos fijos coinciden con los mismos del problema clásico de Chafee-Infante y, adicionalmente, en [32] se estudia la estabilidad e hiperbolicidad de los puntos fijos.

En este cuarto capítulo ampliamos estos resultados para una función más general f (no necesariamente impar y no sabemos si el problema de Cauchy asociado tiene solución única o no), mostrando que la ecuación (8) sufre la misma cascada de bifurcaciones que la ecuación de Chafee-Infante. Además, en el caso de que la función a decrezca, aunque el problema posee al menos los mismos puntos fijos como en el problema de Chafee-Infante, demostramos que pueden aparecer más puntos fijos. Para una función no decreciente a y una función impar f, demostramos también que incluso cuando la unicidad falla, la estabilidad de los puntos fijos es la misma que para los correspondientes en el problema de Chafee-Infante.

Finalmente, podemos probar que en este último caso el semiflujo es dinámicamente gradiente con respecto a la familia disjunta de conjuntos aislados débilmente invariantes generados por los puntos fijos, que se ordena por el número de ceros de los puntos fijos. Más precisamente, el atractor consiste en el conjunto de puntos de equilibrio y sus conexiones heteroclínicas, donde la conexión de un punto fijo a otro sólo se permite si el número de ceros del primero es mayor.

Chapter 0

Preliminaries

This chapter tries to present the definitions and basic results in the framework of the multivalued dynamical systems. Also, in order to understand properties related to attractors for multivalued semiflows other results are contemplated.

0.1. Abstract theory of multivalued dynamical systems

Firstly, we introduce basic concepts and properties related to fixed points, complete trajectories and global attractors.

Consider a metric space (X, d) and a family of functions $\mathcal{R} \subset \mathcal{C}(\mathbb{R}_+; X)$. Denote by $P(X)$ the class of nonempty subsets of X. Then, define the multivalued map

$$G : \mathbb{R}_+ \times X \to P(X)$$

associated with the family $\mathcal{R}$ as follows

$$G(t, u_0) = \{u(t) : u(\cdot) \in \mathcal{R}, u(0) = u_0\}. \tag{0.1.1}$$

In this abstract setting, the multivalued map G is expected to satisfy some properties that fit in the framework of multivalued dynamical systems.

The first concept is given now, although a more axiomatic construction will be provided below.

Definition 0.1. A multivalued map $G : \mathbb{R}_+ \times X \to P(X)$ is a multivalued semiflow (or m-semiflow) if $G(0, x) = x$ for all $x \in X$ and $G(t + s, x) \subset G(t, G(s, x))$ for all $t, s \geq 0$ and $x \in X$.
If the above inclusion is an equality, it is said that the m-semiflow is strict.

Once a multivalued semiflow is defined, we recall the concepts of invariance and global attractor, with evident differences with respect to the single-valued case.

Definition 0.2. A map $\gamma : \mathbb{R} \to X$ is called a complete trajectory of $\mathcal{R}$ (resp. of G) if $\gamma(\cdot + h) \mid_{[0,\infty)} \in \mathcal{R}$ for all $h \in \mathbb{R}$ (resp. if $\gamma(t + s) \in G(t, \gamma(s))$ for all $s \in \mathbb{R}$ and $t \geq 0$).

Definition 0.3. A point $z \in X$ is a fixed point of $\mathcal{R}$ (resp. of G) if $\varphi(\cdot) \equiv z \in \mathcal{R}$ (resp. $z \in G(t, z)$ for all $t \geq 0$).

Definition 0.4. Given an m-semiflow G on a metric space (X, d) a set $B \subset X$ is said to be negatively (positively) invariant if $B \subset G(t, B)$ $(G(t, B) \subset B)$ for all $t \geq 0$, and strictly invariant (or, simply, invariant) if the above relation is not only an inclusion but an equality.

Definition 0.5. The set B is said to be weakly invariant if for any $x \in B$ there exists a complete trajectory γ of $\mathcal{R}$ contained in B such that $\gamma(0) = x$. We observe that weak invariance implies negative invariance.

Definition 0.6. A closed weakly invariant set B of X is isolated if there is a neighborhood O of B such that B is the maximal weakly invariant subset on $\mathcal{O}$. If B belongs to the global attractor $\mathcal{A}$ and $\mathcal{A}$ is compact, then it is compact. In this case, it is equivalent to use a δ-neighborhood $\mathcal{O}_\delta(B) = \{y \in X : dist\,(y, B) < \delta\}$.

Remark 0.7. If in this definition we use the stronger conditions that $\mathcal{O}$ is a $\delta-$neighborhood, then it follows from the proof of Lemma 19 in [44] that B is closed, so this assumption is not necessary.

Definition 0.8. We say that G is asymptotically compact if every sequence $y_n \in G(t_n, B)$, where $t_n \to \infty$ and $B \subset X$ is bounded, is relatively compact.

Definition 0.9. A set $\mathcal{A} \subset X$ is called a global attractor for an m-semiflow if it is negatively semi-invariant and it attracts all attainable sets through the m-semiflow starting in bounded subsets, i.e.,

$$dist_X(G(t,B),\mathcal{A}) \to 0 \quad \text{as } t \to \infty,$$

where

$$dist_X(A,B) = \sup_{a\in A} \inf_{b\in B} d(a,b)$$

is the Hausdorff semidistance from the $G(t,B)$ to $\mathcal{A}$. When $\mathcal{A}$ is compact, it is the minimal closed attracting set [68, Remark 5].

Definition 0.10. A subset $A \subset \mathcal{A}$ is a local attractor in $\mathcal{A}$ if for some $\varepsilon > 0$ it follows that $\omega(\mathcal{O}_\varepsilon(A) \cap \mathcal{A}) = A$.

Definition 0.11. Let A be a local attractor in $\mathcal{A}$. Then its repeller A^* is defined by

$$A^* = \{x \in \mathcal{A} : \omega(x)\backslash A \neq \emptyset\}.$$

Remark 0.12. A global attractor for an m-semiflow does not have to be unique, nor a bounded set. However, if a global attractor is bounded and closed, it is minimal among all closed sets that attract bounded sets [68, Remark 5]. In particular, a bounded and closed global attractor is unique.

In order to obtain a detailed characterization of the internal structure of a global attractor, we introduce an axiomatic set of properties on the set $\mathcal{R}$ (see [10] and [57]).

The set of axiomatic properties that we will deal with is the following.

(K1) For any $x \in X$ there exists at least one element $\varphi \in \mathcal{R}$ such that $\varphi(0) = x$.

(K2) $\varphi_\tau(\cdot) := \varphi(\cdot + \tau) \in \mathcal{R}$ for any $\tau \geq 0$ and $\varphi \in \mathcal{R}$ (translation property).

(K3) Let $\varphi_1, \varphi_2 \in \mathcal{R}$ be such that $\varphi_2(0) = \varphi_1(s)$ for some $s > 0$. Then, the function φ defined by

$$\varphi(t) = \begin{cases} \varphi_1(t) & 0 \leq t \leq s, \\ \varphi_2(t-s) & s \leq t, \end{cases}$$

belongs to $\mathcal{R}$ (concatenation property).

(K4) For any sequence $\{\varphi^n\} \subset \mathcal{R}$ such that $\varphi^n(0) \to x_0$ in X, there exist a subsequence $\{\varphi^{n_k}\}$ and $\varphi \in \mathcal{R}$ such that $\varphi^{n_k}(t) \to \varphi(t)$ for all $t \geq 0$.

In Chapter 1 we will need a stronger condition than $(K4)$. Namely, we shall consider the following stronger property.

$(\overline{K}4)$ For any sequence $\{\varphi^n\} \subset \mathcal{R}$ such that $\varphi^n(0) \to x_0$ in X, there exists a subsequence $\{\varphi^n\}$ and $\varphi \in \mathcal{R}$ such that φ^n converges to φ uniformly in bounded subsets of $[0, \infty)$.

Remark 0.13. If in assumption $(K1)$, for every $x \in X$, there exists a unique $\varphi \in \mathcal{R}$ such that $\varphi(0) = x$, then the set $\{\varphi \in \mathcal{R} : \varphi(0) = x\}$ consists of a single trajectory φ, and the equality $G(t, x) = \varphi(t)$ defines a classical semigroup $G : \mathbb{R}^+ \times X \to X$.

It is immediate to observe [29, Proposition 2] or [59, Lemma 9] that $\mathcal{R}$ fulfilling (K1) and (K2) gives rise to a m-semiflow G through (3.3.1), and if besides (K3) holds, then this m-semiflow is strict. In such a case, a global bounded attractor, supposing that it exists, is strictly invariant [68, Remark 8].

Several properties concerning fixed points, complete trajectories and global attractors are summarized in the following results [57].

Lemma 0.14. *Let (K1)-(K2) be satisfied. Then every fixed point (resp. complete trajectory) of $\mathcal{R}$ is also a fixed point (resp. complete trajectory) of G.*

If $\mathcal{R}$ fulfills (K1)-(K4), then the fixed points of $\mathcal{R}$ and G coincide. Besides, a map $\gamma : \mathbb{R} \to X$ is a complete trajectory of $\mathcal{R}$ if and only if it is continuous and a complete trajectory of G.

The standard well-known result in the single-valued case for describing the attractor as the union of bounded complete trajectories (see [61]) reads in the multivalued case as follows.

Theorem 0.15. *Consider $\mathcal{R}$ satisfying (K1) and (K2), and either (K3) or (K4). Assume also that G possesses a compact global attractor $\mathcal{A}$. Then*

$$\mathcal{A} = \{\gamma(0) : \gamma \in \mathbb{K}\} = \cup_{t\in\mathbb{R}}\{\gamma(t) : \gamma \in \mathbb{K}\}, \tag{0.1.2}$$

where $\mathbb{K}$ denotes the set of all bounded complete trajectories in $\mathcal{R}$. Hence, $\mathcal{A}$ is weakly invariant.

Before stating a general result about the existence of attractors, we need the following definition.

Definition 0.16. The map $t \mapsto G(t, x)$ is upper semicontinuous if for any $x \in X$ and any neighborhood $O(G(t, x))$ in X there exists $\delta > 0$ such that if $d(y, x) < \delta$, then $G(t, y) \subset O$.

Theorem 0.17. *[68, Theorem 4 and Remark 8] Let the map $t \mapsto G(t, x)$ be upper semicontinuous with closed values. If there exists a compact attracting set K, that is,*

$$dist_X(G(t, B), K) \to 0, \text{ as } t \to +\infty,$$

for any bounded set B, then G possesses a global compact attractor $\mathcal{A}$, which is the minimal closed attracting set. If, moreover, G is strict, then $\mathcal{A}$ is invariant.

We observe that, although in the papers [68], [57] the space X is assumed to be complete, the results are true in a non-complete space.

Now we recall the definitions of some important sets in the literature of dynamical systems. Let $B \subset X$ and let $\varphi \in \mathcal{R}$. We define the $\omega-$limit sets $\omega(B)$ and $\omega(\varphi)$ as follows:

$$\begin{aligned}\omega(B) =& \{y \in X : \text{ there are sequences } t_n \to \infty, y_n \in G(t_n, B) \text{ such that } y_n \to y\},\\ \omega(\varphi) =& \{y \in X : \text{ there is a sequence } t_n \to \infty \text{ such that } \varphi(t_n) \to y\}.\end{aligned}$$

If γ is a complete trajectory of $\mathcal{R}$, then the $\alpha-$limit set is defined by

$$\alpha(\gamma) = \{y \in X : \text{ there is a sequence } t_n \to -\infty \text{ such that } \gamma(t_n) \to y\}.$$

Some useful properties of these sets [10, Lemma 3.4 and Proposition 4.1] are summarized in the following lemma.

Lemma 0.18. *Assume that* $(K1), (K2)$ *and* $(K4)$ *hold. Let* G *be asymptotically compact. Then:*

1. *For any non-empty bounded set* $B, \omega(B)$ *is non-empty, compact, weakly invariant and*
$$dist_X(G(t, B), \omega(B)) \to 0, \text{ as } t \to +\infty.$$

2. *For any* $\varphi \in \mathcal{R}$, $\omega(\varphi)$ *is non-empty, compact, weakly invariant and*
$$dist_X(\varphi(t), \omega(\varphi)) \to 0, \text{ as } t \to +\infty.$$

3. *For any* $\gamma \in \mathbb{K}$, $\alpha(\gamma)$ *is non-empty, compact, weakly invariant and*
$$dist_X(\gamma(t), \alpha(\gamma)) \to 0, \text{ as } t \to -\infty.$$

4. *For any* $\varphi \in \mathcal{R}$, $\omega(\varphi)$ *is connected. If* ψ *is a complete trajectory then* $\alpha(\psi)$ *is connected.*

The following definition summarizes additional concepts. They are required in order to give a more detailed description of the internal structure of the attractor under special cases.

Definition 0.19. Consider a m-semiflow G.

1. We say that $\mathcal{S} = \{\Xi_1, \dots, \Xi_n\}$ is a disjoint finite family of isolated weakly invariant sets if there exists $\delta > 0$ such that
$$\mathcal{O}_\delta(\Xi_i) \cap \mathcal{O}_\delta(\Xi_j) = \emptyset \quad \text{for } 1 \leq i < j \leq n,$$

and each Ξ_i is the maximal weakly invariant subset in

$$\mathcal{O}_\delta(\Xi_i) := \{x \in X : dist_X(x, \Xi_i) < \delta\}.$$

2. For an m-semiflow G on (X, d) with a global attractor $\mathcal{A}$ and a finite number of weakly invariant sets $\mathcal{S}$, a homoclinic orbit in $\mathcal{A}$ is a collection

$$\{\Xi_{p(1)}, \ldots, \Xi_{p(k)}\} \subset \mathcal{S}$$

and a collection of complete trajectories $\{\gamma_i\}_{1 \le i \le k}$ of $\mathcal{R}$ in $\mathcal{A}$ such that (putting $p(k+1) := p(1)$)

$$\lim_{t \to -\infty} dist_X(\gamma_i(t), \Xi_{p(i)}) = 0, \lim_{t \to \infty} dist_X(\gamma_i(t), \Xi_{p(i+1)}) = 0,\ 1 \le i \le k,$$

and for each i there exists $t_i \in \mathbb{R}$ such

$$\gamma_i(t_i) \notin \Xi_{p(i)} \cup \Xi_{p(i+1)}.$$

3. We say that an m-semiflow G on (X, d) with the global attractor $\mathcal{A}$ is dynamically gradient if the following two properties hold:

(G1) there exists a disjoint finite family $\mathcal{S} = \{\Xi_1, \ldots, \Xi_n\}$ of isolated weakly invariant sets in $\mathcal{A}$ with the property that any complete trajectory γ of $\mathcal{R}$ in $\mathcal{A}$ satisfies

$$\lim_{t \to -\infty} dist_X(\gamma(t), \Xi_i) = 0,\ \lim_{t \to \infty} dist_X(\gamma(t), \Xi_j) = 0$$

for some $1 \le i, j \le n$;

(G2) $\mathcal{S}$ does not contain homoclinic orbits.

Remark 0.20. It is possible to establish the definition of being dynamically gradient in terms of α, $\omega-$limit sets. After reordering the sets Ξ_i, the two definitions are equivalent, as will be shown in Chapter 1. We say the m-semiflow $G\colon \mathbb{R}^+ \times X \to P(X)$ is dynamically gradient with respect to the disjoint family of isolated weakly invariant sets $\mathcal{S} = \{\Xi_1, \dots, \Xi_n\}$ if for every complete and bounded trajectory ψ of $\mathcal{R}$ we have that either

$$\psi(\mathbb{R}) \subset \Xi_j, \qquad \text{for some } j \in \{1, \dots, m\},$$

or

$$\alpha(\psi) \subset \Xi_i \quad \text{and} \quad \omega(\psi) \subset \Xi_j$$

with $1 \leq j < i \leq m$.

Definition 0.21. A disjoint family of isolated weakly invariant sets

$$\mathcal{S} = \{\Xi_1, \dots, \Xi_n\} \subset \mathcal{A}$$

is a Morse decomposition of the global compact attractor $\mathcal{A}$ if there is a sequence of local attractors

$$\emptyset = A_0 \subset A_1 \subset \ldots \subset A_n = \mathcal{A}$$

such that for every $k \in \{1, \dots, n\}$ it holds

$$\Xi_k = A_k \cap A^*_{k-1}.$$

Remark 0.22. The property of being a dynamically gradient semiflow and the existence of a Morse decomposition are shown to be equivalent for multivalued dynamical systems in [44] under conditions (K1)-(K3), $(\overline{K}4)$.

Remark 0.23. In the single-valued case, dynamically gradient semigroups have been called also gradient-like semigroups [30]. Observe that the above definitions are concerned with weakly invariant families, which need not to be unitary sets. This is to deal with the more general concept of generalized gradient-like semigroups [30], in contrast with gradient-like semigroups (when the invariant sets are unitary).

Now, we introduce the concept of unstable manifold, that will allow us to describe more precisely the structure of a global attractor of a dynamically gradient m-semiflow.

Definition 0.24. Let G be an m-semiflow. The unstable manifold of a set Ξ is

$$W^u(\Xi) = \{u_0 \in X : \text{ there exists complete trajectory } \gamma \text{ of } \mathcal{R} \text{ such that } \gamma(0) = u_0 \text{ and } \lim_{t\to-\infty} dist_X(\gamma(t), \Xi) = 0\}.$$

Now the following result, relating the global attractor with unstable manifolds, is standard. The first statement is straightforward to see. The second one, supposing that the global attractor is compact, follows directly from the structure described in Theorem 0.15 and the definition of dynamically gradient semiflows.

Lemma 0.25. *Consider a family $\mathcal{R} \subset \mathcal{C}(\mathbb{R}_+; X)$ satisfying (K1) and (K2). Suppose that the associated m-semiflow has a global attractor $\mathcal{A}$. Then, for any bounded set $\Xi \subset X, W^u(\Xi) \subset \bar{\mathcal{A}}$.*

Moreover, assume that $\mathcal{R}$ satisfies either (K3) or (K4), and that the global attractor $\mathcal{A}$ is compact. Suppose also that the associated m-semiflow G defined in (3.3.1) is dynamically gradient. Then

$$\mathcal{A} = \bigcup_{i=1}^{n} W^u(\Xi_i). \tag{0.1.3}$$

0.2. Notation

We end this chapter fixing basic questions about notation. Throughout this work we will denote by $\|\cdot\|_X$ the norm in the Banach space X. Let $\Omega \subset \mathbb{R}^n$ be a subset and $1/p + 1/q = 1$. Denote by $(\cdot, \cdot)$ the scalar product in $L^2(\Omega)$ and $\|\cdot\|_{H_0^1}$ the norm in $H_0^1(\Omega)$ associated to the scalar product of gradients in $L^2(\Omega)$ thanks to Poincaré's inequality. We also denote by $(\cdot, \cdot)$ the duality product between $L^p(\Omega)$ and $L^q(\Omega)$, where p is the conjugate exponent of q. As usual, let $H^{-1}(\Omega)$ be the dual space to $H_0^1(\Omega)$. Denote by $\langle \cdot, \cdot \rangle$ pairing between the space $L^p(\Omega) \cap H_0^1(\Omega)$ and its dual $L^q(\Omega) + H^{-1}(\Omega)$. Note also that we will use $\langle \cdot,\cdot \rangle$ for the duality between $H_0^1(\Omega)$ and $H^{-1}(\Omega)$.

Chapter 1

Robustness of dynamically gradient multivalued dynamical systems

The basic theory about properties related to fixed points, complete trajectories and global attractors has been introduced in Chapter 0. In this Chapter we present the main result about robustness of dynamically gradient multivalued semiflows. Further, we prove a theorem which allows us to reorder the family of weakly invariants sets, thus establishing an equivalent definition of dynamically gradient families.

Afterwards, we consider a Chafee-Infante problem, where the equivalence of weak and strong solutions is established. Once the set of fixed points is analyzed, we consider a family of Chafee-Infante equations, approximating the differential inclusion tackled in [7]. We check that this family of Chafee-Infante equations verifies the hypotheses of the robustness theorem in order to obtain, therefore, that the multivalued semiflows generated by the solutions of the approximating problems are dynamically gradient if this family is close enough to the original one.

1.1. Robustness of dynamically gradient m-semiflows

Our first main goal is to prove that a dynamically gradient multivalued semiflow is stable under suitable perturbations, that is, a family of perturbed multivalued semiflows remains dynamically gradient if it is close enough to the original semiflow, generalizing the corresponding result in the single-valued case [30]. This is rigorously formulated in the following theorem.

Theorem 1.1. *Consider a complete metric space* (X, d)*. Let* η *be a parameter in [0,1],* $\mathcal{R}_\eta \subset \mathcal{C}(\mathbb{R}_+; X)$ *fulfill (K1), (K2), (K3) and (K4), and let* G_η *be the corresponding m-semiflow on* X *having the global compact attractor* $\mathcal{A}_\eta$*. Assume that*

(H1) $\overline{\bigcup_{\eta\in[0,1]} \mathcal{A}_\eta}$ *is compact.*

(H2) G_0 *is a dynamically gradient m-semiflow with a disjoint finite family of isolated weakly invariant sets* $\mathcal{S}^0 = \{\Xi_1^0, \ldots, \Xi_n^0\}$*.*

(H3) $\mathcal{A}_\eta$ *has a disjoint finite family of isolated weakly invariant sets* $\mathcal{S}_\eta = \{\Xi_1^\eta, \ldots, \Xi_n^\eta\}$*,* $\eta \in [0,1]$*, which satisfy*

$$\lim_{\eta\to 0} \sup_{1\le i\le n} dist_X(\Xi_i^\eta, \Xi_i^0) = 0.$$

(H4) Any sequence $\{\gamma_\eta\}$ *with* $\gamma_\eta \in \mathcal{R}_\eta$ *such that* $\{\gamma_\eta(0)\}$ *converges for* $\eta \to 0^+$*, possesses a subsequence* $\{\gamma_{\eta'}\}$ *that converges uniformly in bounded intervals of* $[0,\infty)$ *to* $\gamma \in \mathcal{R}_0$*.*

(H5) There exists $\overline{\eta} > 0$ *and neighborhoods* V_i *of* Ξ_i^0 *such that* Ξ_i^η *is the maximal weakly invariant set for* G_η *in* V_i *for any* $i = 1, \ldots, n$ *and for each* $0 < \eta \le \overline{\eta}$*.*

Then there exists $\eta_0 > 0$ *such that for all* $\eta \le \eta_0$*,* $\{G_\eta\}$ *is a dynamically gradient m-semiflow. In particular, the structure of* $\mathcal{A}_\eta$ *is analogous to that given in (0.1.3).*

Proof. Observe that assumption (H5) concerning certain neighborhood V_i of Ξ_i^0 involves a hyperbolicity condition of G_0 w.r.t. each Ξ_i^0, and as far as (H3) is also assumed, there exist $\{\eta(V_i)\}_{i=1,\ldots,n}$ such that $\Xi_i^\eta \subset V_i$ for all $\eta \le \eta(V_i)$. W.l.o.g. assume that $\delta > 0$ is such that $\{x \in X : dist_X(x, \Xi_i^0) \le \delta\} \subset V_i$ for all $i = 1, \ldots, n$.

By Theorem 0.15, we have that $\mathcal{A}_\eta$ is composed by all the orbits of bounded complete trajectories of $\mathcal{R}_\eta$, $\mathbb{K}_\eta$.

We are going to prove by contradiction arguments that there exists $\eta_0 \in (0,1]$ such that $\{G_\eta\}_{\eta \leq \eta_0}$ is dynamically gradient.

Step 1: There exists $\eta_0 > 0$ such that for all $\eta < \eta_0$, any bounded complete trajectory ξ_η of $\mathcal{R}_\eta$ satisfies that there exist $i \in \{1, \dots, n\}$ and t_0 such that for all $t \geq t_0$, $dist_X(\xi_\eta(t), \Xi_i^0) \leq \delta$.

After proving the above claim, we consider the sets

$$B_\eta := \{\xi_\eta(s) : s \geq t_0\} \subset A = \{y : dist_X(y, \Xi_i^0) \leq \delta\}$$

and $\omega(\xi_\eta)$.
It follows that $\omega(\xi_\eta) \subset A$, since

$$dist_X(\xi_\eta(t), \omega(\xi_\eta)) \to 0 \quad \text{as t} \to +\infty.$$

On the other hand, by Lemma 0.18 $\omega(\xi_\eta)$ is a weakly invariant set of G_η contained in V_i. By assumption (H5) we have that $\omega(\xi_\eta) \subset \Xi_i^\eta$, whence the 'forward part' of property (G1) of a dynamically gradient m-semiflow will follow immediately.

We prove this Step 1 by contradiction. Suppose it does not hold. Then, there exist a sequence $\eta_k \to 0$ (as $k \to \infty$) and bounded complete trajectories ξ_k of $\mathcal{R}_{\eta_k}$ (therefore, from $\mathcal{A}_{\eta_k}$) such that

$$\sup_{t \geq t_0} dist_X(\xi_k(t), \mathcal{S}^0) > \delta \;\forall t_0 \in \mathbb{R}. \tag{1.1.1}$$

The set $\{\xi_k(0)\} \subset \overline{\cup_{\eta \in [0,1]} \mathcal{A}_\eta}$ is relatively compact from assumption (H1). So, there exists a converging subsequence (relabeled the same) in X. From (H4), there exist a subsequence (relabeled the same, again) and $\xi_0 \in \mathcal{R}_0$, such that $\{\xi_k|_{[0,\infty)}\}$ converges to ξ_0 in bounded intervals of $[0, \infty)$. Actually, if we argue similarly not for time 0, but now for times $-1, -2, \dots$, and use a diagonal argument, we have that $\xi_0 = \gamma_0|_{[0,\infty)}$ where $\gamma_0 \in \mathbb{K}_0$, and the convergence of (a subsequence of) $\{\xi_k\}$ toward γ_0 holds uniformly in bounded intervals $[a, b]$ of $\mathbb{R}$.

Since G_0 is dynamically gradient, there exists $i \in \{1, \dots, n\}$ such that

$$dist_X(\gamma_0(t), \Xi_i^0) \to 0 \text{ as } t \to \infty.$$

Therefore, for all $r \in \mathbb{N}$, there exist t_r and k_r such that

$$dist_X(\xi_k(t_r), \Xi_i^0) < 1/r \quad \text{for all } k \geq k_r.$$

Indeed, this is done as follows: $dist_X(\gamma_0(s), \Xi_i^0) < 1/r$ for all $s \geq t_r$ (for some t_r, w.l.o.g. $t_r \geq r > 1/\delta$); now, combining this with the uniform convergence on $[0, t_r]$ of ξ_k toward γ_0, the existence of k_r follows.

However, from (1.1.1), there exists $t'_r > t_r$ such that $dist_X(\xi_{k_r}(t), \Xi_i^0) < \delta$ for all $t \in [t_r, t'_r)$ and $dist_X(\xi_{k_r}(t'_r), \Xi_i^0) = \delta$.

Now we distinguish two cases and we will arrive to the same conclusion in both of them.

Case (1a): Suppose that $t'_r - t_r \to \infty$ as $r \to \infty$ (at least for a certain subsequence).

Since $\{\xi_{k_r}(t'_r)\}$ is also relatively compact (by (H1), again), and $\xi^1_{k_r}(\cdot) = \xi_{k_r}(t'_r + \cdot)$ is a bounded complete trajectory of $\mathcal{R}_{k_r}$, from (H4) we deduce that a subsequence (relabeled the same) is converging on bounded time-intervals of $[0, \infty)$, i.e. $\gamma_1(t) := \lim_{r\to\infty} \xi_{k_r}(t + t'_r)$ holds for certain $\gamma_1 \in \mathcal{R}_0$. Moreover, as before, a diagonal argument, using not t'_r above, but $t'_r - 1, t'_r - 2, \ldots$ implies that γ_1 can be extended to the whole real line (the function will still be denoted the same; and the convergence holds in bounded time-intervals of $\mathbb{R}$), in particular, by (H1) and (H4), $\gamma_1 \in \mathbb{K}_0$.

Moreover, by its construction, we have that $dist_X(\gamma_1(t), \Xi_i^0) \leq \delta$ for all $t \leq 0$. By Lemma 0.18 we have that the α-limit set $\alpha(\gamma_1)$ is weakly invariant.

As long as Ξ_i^0 is the biggest weakly invariant set contained in V_i, we deduce that $dist_X(\gamma_1(\tau), \Xi_i^0) \to 0$ when $\tau \to -\infty$.

On the other hand, from (G1) and (G2) we have that $dist_X(\gamma_1(t), \Xi_j^0) \to 0$ as $t \to \infty$ for $j \neq i$.

Case (1b): Suppose that there exists $C > 0$ such that $|t'_r - t_r| \leq C$ as $r \to \infty$. (W.l.o.g. we assume that $t'_r - t_r \to t_*$.)

Recall that $dist_X(\xi_{k_r}(t_r), \Xi_i^0) < 1/r$. By [44, Lemma 19] Ξ_i^0 is closed, so, up to a subsequence

$$\xi_{k_r}(t_r) \to y \in \Xi_i^0.$$

Denote $\xi^1_{k_r}(\cdot) = \xi_{k_r}(\cdot + t_r)$. From (H4), there exist a subsequence $\{\xi^1_{k_r}\}$ and $\xi^1 \in \mathcal{R}_0$ with $\xi^1(0) = y$ such that $\xi^1_{k_r}$ converge towards ξ^1 uniformly in bounded intervals of $[0,\infty)$. In particular,

$$\xi^1_{k_r}(t'_r - t_r) \to \xi^1(t_*),$$

so that

$$dist_X(\xi^1(t_*), \Xi^0_i) \geq \delta.$$

Since Ξ^0_i is weakly invariant, there exists $\gamma \in \mathbb{K}_0$ with $\gamma(0) = \xi^1(0)$ and $\gamma(t) \in \Xi^0_i$ for all $t \in \mathbb{R}$. By (K3) consider the concatenation

$$\gamma_1(t) := \begin{cases} \gamma(t), & \text{if } t \leq 0, \\ \xi^1(t), & \text{if } t \geq 0. \end{cases}$$

Then by (G1)-(G2) it follows that

$$dist_X(\gamma_1(t), \Xi^0_j) \to 0 \quad \text{ as } t \to \infty$$

with $j \neq i$. This is exactly the same conclusion we arrived in Case (1a).

Reasoning now with the subsequence $\{\xi^1_{k_r}\}$, and proceeding as above, we obtain the existence of $\gamma_2 \in \mathbb{K}_0$ such that

$$dist_X(\gamma_2(t), \Xi^0_j) \to 0 \quad \text{ as } t \to -\infty$$

and

$$dist_X(\gamma_2(t), \Xi^0_p) \to 0 \quad \text{ as } t \to \infty,$$

with $p \notin \{i, j\}$.

Thus, in a finite number of steps we arrive to a contradiction, since G_0 satisfies (G2). Therefore, (1.1.1) is absurd, and Step 1 is proved.

Step 2: There exists $\eta_1 > 0$ such that for all $\eta < \eta_1$, any bounded complete trajectory ξ_η of $\mathcal{R}_\eta$ satisfies that there exist $j \in \{1, \ldots, n\}$ and t_1 such that $dist_X(\xi_\eta(t), \Xi^0_j) \leq \delta$ for all $t \leq t_1$.

The above claim can be proved analogously as before, and since for any bounded complete trajectory $\xi_\eta \in \mathbb{K}_\eta$, by Lemma 0.18, $\alpha(\xi_\eta)$ is weakly invariant for G_η, and contained in some V_j, the 'backward part' of property (G1) of a dynamically gradient m-semiflow will follow immediately. The same argument is valid for the 'forward part', and so, for all suitable small η, $\{G_\eta(t) : t \geq 0\}$ satisfies (G1).

Step 3: There exists $\eta_2 > 0$ such that $\{G_\eta\}_{\eta \leq \eta_2}$ satisfies (G2).

If not, there exist a sequence $\eta_k \to 0$, with G_{η_k} having an homoclinic structure. We may suppose that the number of elements of weakly invariant subsets connected on each homoclinic chain in $\mathcal{S}_{\eta_k}$ is the same. Moreover, by assumption (H3) each $\Xi_j^{\eta_k}$ is contained in V_j for η_k small enough and w.l.o.g. the order in the route of the homoclinics visiting the V_j sets is the same.

Therefore, for $k \geq k_0$ there exist a sequence of subsets $\Xi_{p(1)}^{\eta_k}, \ldots \Xi_{p(l)}^{\eta_k}$ in $\mathcal{S}_{\eta_k}$ (with $p(l+1) = p(1)$), and a sequence of complete trajectories $\{\{\xi_i^k\}_{i=1}^l\}_k$, each collection of l elements in the corresponding attractor $\mathcal{A}_{\eta_k}$, with

$$\lim_{t \to -\infty} dist_X(\xi_i^k(t), \Xi_{p(i)}^{\eta_k}) = 0, \ \lim_{t \to \infty} dist_X(\xi_i^k(t), \Xi_{p(i+1)}^{\eta_k}) = 0, \quad 1 \leq i \leq l.$$

If we argue now as in the proof of (G1), we may construct a homoclinic structure of G_0, getting a contradiction with the fact that the m-semiflow G_0 is dynamically gradient. □

Remark 1.2. The above result also applies to the particular case of a dynamically gradient m-semiflow when the weakly invariant families of the original and perturbed problems are reduced to unitary sets (Remark 0.23 and [30, Theorem 1.5]).

1.2. An equivalent definition of dynamically gradient families.

We will give an equivalent definition of dynamically gradient families. For proving the main result in this section, we will need a stronger condition than $(K4)$, that is, the $(\overline{K4})$ property defined in Chapter 0.

Remark 1.3. We have seen that the property of being dynamically gradient for a disjoint family of isolated weakly invariant sets $\mathcal{S} = \{\Xi_1, \ldots, \Xi_n\} \subset \mathcal{A}$ is stable under perturbations. We observe that in [64] a slightly different definition was used for dynamically gradients families. Namely, instead of conditions $(G1)$-$(G2)$ it is assumed that any bounded complete trajectory $\gamma(\cdot)$ satisfies one of the following properties:

1. $\{\gamma(t) : t \in \mathbb{R}\} \subset \Xi_i$ for some i.
2. There are $i < j$ for which

$$\gamma(t) \underset{t\to\infty}{\rightarrow} \Xi_i, \ \gamma(t) \underset{t\to-\infty}{\rightarrow} \Xi_j.$$

These assumptions are clearly stronger than $(G1)$-$(G2)$ and imply that the sets Ξ_j are ordered. Our aim is to show that when $\mathcal{S}$ is a disjoint family of isolated weakly invariant sets, these conditions are equivalent. For this, the concept of local attractor and its repeller will be crucial. Therefore, some properties about local attractors and its repeller as well as the proof of the following lemmas can be found in [44].

Lemma 1.4. *Assume that* $(K1)-(K4)$ *hold. Then a local attractor* A *is invariant.*

Remark 1.5. Although in [44] the stronger assumption $(\overline{K}4)$ is assumed, the proof is valid for just $(K4)$.

Lemma 1.6. *Assume that* $(K1)$-$(K3)$, $(\overline{K}4)$ *hold and that a global compact attractor* $\mathcal{A}$ *exists. Then the repeller* A^* *of a local attractor* $A \subset \mathcal{A}$ *is weakly invariant and compact.*

Lemma 1.7. *Assume that* $(K1)$-$(K3)$, $(\overline{K}4)$ *hold and that a global compact attractor* $\mathcal{A}$ *exists. Let us consider the sequences* $x_k \in \mathcal{A}$, $t_k \to +\infty$ *and* $\varphi_k(\cdot) \in \mathcal{R}$ *such that* $\varphi_k(0) = x_k$. *Then from the sequence of maps* $\xi_k(\cdot) : [-t_k, +\infty) \to \mathcal{A}$ *defined by*

$$\xi_k(t) = \varphi_k(t + t_k)$$

one can extract a subsequence converging to some $\psi(\cdot) \in \mathbb{K}$ *uniformly on bounded subsets of* $\mathbb{R}$.

In order to prove the equivalent definition of dynamically gradient families, we have to ensure the existence of one local attractor in a family of isolated weakly invariant sets.

Lemma 1.8. *Assume that* $(K1)$-$(K3)$, $(\overline{K}4)$ *hold and that a global compact attractor* $\mathcal{A}$ *exists. Let* $\mathcal{S} = \{\Xi_1, \ldots, \Xi_n\} \subset \mathcal{A}$ *be a disjoint family of isolated weakly invariant sets. If* G *is dynamically gradient with respect to* $\mathcal{S}$*, then one of the sets* Ξ_j *is a local attractor in* $\mathcal{A}$*.*

Proof. Let $\delta_0 > 0$ be such that $\mathcal{O}_{\delta_0}(\Xi_i) \cap \mathcal{O}_{\delta_0}(\Xi_j) = \varnothing$ if $i \neq j$ and Ξ_j is the maximal weakly invariant set in $\mathcal{O}_{\delta_0}(\Xi_j)$ for all j. First we will prove the existence of $j \in \{1, ..., n\}$ such that for all $\delta \in (0, \delta_0)$ there exists $\delta' \in (0, \delta)$ satisfying

$$\cup_{t \geq 0} G(t, \mathcal{O}_{\delta'}(\Xi_j) \cap \mathcal{A}) \subset \mathcal{O}_{\delta}(\Xi_j). \tag{1.2.1}$$

If not, there would exist $0 < \delta < \delta_0$ and for each j sequences $t_k^j \in \mathbb{R}^+$, $x_k^j \in \mathcal{A}$, $\varphi_k^j \in \mathcal{R}$ with $\varphi_k^j(0) = x_k^j$ such that

$$\begin{aligned} d(x_k^j, \Xi_j) &< \frac{1}{k}, \\ d(\varphi_k^j(t_k^j), \Xi_j) &= \delta, \\ d(\varphi_k^j(t), \Xi_j) &< \delta \quad \text{ for all } t \in [0, t_k^j). \end{aligned}$$

We have to consider two cases: $t_k^j \to +\infty$ or $t_k^j \leq C$.

Case 1: Let $t_k^j \to +\infty$. We define the sequence

$$\psi_k^j(t) = \varphi_k^j(t + t_k^j) \text{ for } t \in [-t_k^j, \infty).$$

By Lemma 1.7 we obtain the existence of a complete trajectory of $\mathcal{R}$, $\psi^j(\cdot)$, such that a subsequence of ψ_k^j satisfies

$$\psi_k^j(t) \to \psi^j(t) \quad \text{ for every } t \in \mathbb{R}.$$

Hence,

$$d(\psi^j(t), \Xi_j) \leq \delta < \delta_0 \quad \text{for all } t \leq 0.$$

Therefore, as $\psi^j \in \mathbb{K}$, condition $(G1)$ implies that

$$d(\psi^j(t), \Xi_j) \to 0 \quad \text{as } t \to -\infty.$$

On the other hand, since $d(\psi^j(0), \Xi_j) = \delta$, conditions $(G1) - (G2)$ imply that

$$d(\psi^j(t), \Xi_i) \to 0 \quad \text{as } t \to +\infty,$$

where $i \neq j$.

Case 2: Let now $t_k^j \leq C$. We can assume that

$$t_k^j \to t^j.$$

By $(\overline{K}4)$ we obtain the existence of $\varphi^j \in \mathcal{R}$ such that φ_k^j converges to φ^j uniformly on bounded sets of $[0, \infty)$. It is clear then that $d(\varphi^j(t^j), \Xi_j) = \delta$. As $\varphi^j(0) \in \Xi_j$ and Ξ_j is weakly invariant, there exists a complete trajectory of $\mathcal{R}$, $\psi_j^-(\cdot)$, such that $\psi_j^-(0) = \varphi^j(0)$ and $\psi_j^-(t) \in \Xi_j$ for all $t \leq 0$. Concatenating ψ_j^- and φ^j we define

$$\psi^j(t) = \begin{cases} \psi_j^-(t) \text{ if } t \leq 0, \\ \varphi^j(t) \text{ if } t \geq 0, \end{cases}$$

which is a complete trajectory by $(K3)$. Again, conditions $(G1) - (G2)$ imply that

$$d(\psi^j(t), \Xi_i) \to 0 \quad \text{as } t \to +\infty,$$

where $i \neq j$.

We have obtained then a connection from Ξ_j to a different Ξ_i. Since this is true for any Ξ_j, we would obtain a homoclinic structure, which contradicts $(G2)$. Therefore, (1.2.1) holds for some j. It follows that

$$\omega(\mathcal{O}_{\delta'}(\Xi_j) \cap \mathcal{A}) \subset \overline{\mathcal{O}_\delta(\Xi_j)} \subset \mathcal{O}_{\delta_0}(\Xi_j).$$

Since $\omega(\mathcal{O}_{\delta'}(\Xi_j) \cap \mathcal{A})$ is weakly invariant, we obtain that $\omega(\mathcal{O}_{\delta'}(\Xi_j) \cap \mathcal{A}) \subset \Xi_j$. But $\Xi_j \subset G(t, \Xi_j) \subset G(t, \mathcal{O}_{\delta'}(\Xi_j) \cap \mathcal{A})$ for any $t \geq 0$ implies the converse inclusion, so that $\Xi_j = \omega(\mathcal{O}_{\delta'}(\Xi_j) \cap \mathcal{A})$. Thus, Ξ_j is a local attractor in $\mathcal{A}$. $\square$

Now we prove the main result of this section which allows us to establish the equivalent definition of dynamically gradient families.

Theorem 1.9. *Assume that (K1)-(K3), $(\overline{K}4)$ hold and that a global compact attractor $\mathcal{A}$ exists. Let $\mathcal{S} = \{\Xi_1, \ldots, \Xi_n\} \subset \mathcal{A}$ be a disjoint family of isolated weakly invariant sets. Then G is dynamically gradient with respect to $\mathcal{S}$ in the sense of Definition 0.19 if and only if $\mathcal{S}$ can be reordered in such a way that any bounded complete trajectory $\gamma(\cdot)$ satisfies one of the following properties:*

1. *$\{\gamma(t) : t \in \mathbb{R}\} \subset \Xi_i$ for some i.*

2. *There are $i < j$ for which*

$$\gamma(t) \underset{t\to\infty}{\to} \Xi_i, \ \gamma(t) \underset{t\to-\infty}{\to} \Xi_j.$$

Proof. It is obvious that conditions 1-2 imply that G is dynamically gradient. We shall prove the converse.

By Lemma 1.8 one of the sets Ξ_i is a local attractor. After reordering the sets, we can say that Ξ_1 is the local attractor. Let

$$\Xi_1^* = \{x \in \mathcal{A} : \omega(x) \backslash \Xi_1 \neq \varnothing\}$$

be its repeller, which is weakly invariant by Lemma 1.6. Since Ξ_j are closed (see Definition 0.6), weakly invariant and disjoint, we obtain that $\Xi_j \subset \Xi_1^*$ for $j \geq 2$.

We will consider only the dynamics inside the repeller Ξ_1^*, that is, we define the following set:

$$\mathcal{R}_1 = \{\varphi \in \mathcal{R} : \varphi(t) \in \Xi_1^* \ \forall t \geq 0\}.$$

Since Ξ_1^* is weakly invariant, $\mathcal{R}_1$ satisfies $(K1)$. Further, let

$$\varphi_\tau(\cdot) = \varphi(\cdot + \tau),$$

where $\varphi \in \mathcal{R}_1$ and $\tau \geq 0$. Then it is clear that

$$\varphi_\tau(t) \in \mathcal{R}_1, \quad \text{for all } t \geq 0$$

and then $(K2)$ holds.
If $\varphi_1(\cdot),\ \varphi_2(\cdot) \in \mathcal{R}_1$, it follows by $(K3)$ that the concatenation belongs also to $\mathcal{R}_1$. Finally, if

$$\varphi_n(0) \to \varphi_0$$

with $\varphi_n(0) \in \Xi_1^*$ and $\varphi_n(\cdot) \in \mathcal{R}_1$, then $\varphi_0 \in \Xi_1^*$ (as Ξ_1^* is closed) and by $(\overline{K}4)$ passing to a subsequence

$$\varphi_n(t_n) \to \varphi(t), \quad \text{for } t_n \to t \geq 0,$$

where $\varphi \in \mathcal{R}$. Again, the closedness of Ξ_1^* implies that $\varphi \in \mathcal{R}_1$. Hence, $(\overline{K}4)$ also holds.

We can define then the multivalued semiflow $G_1 : \mathbb{R}^+ \times \Xi_1^* \to P(\Xi_1^*)$:

$$G_1(t,x) = \{y \in \Xi_1^* : y = \varphi(t) \text{ for some } \varphi \in \mathcal{R}_1,\, \varphi(0) = x\},$$

which is strict by $(K3)$. This definition is equivalent to the following one:

$$\overline{G}_1(t,x) = G(t,x) \cap \Xi_1^* \text{ for } x \in \Xi_1^*.$$

Indeed, $G_1(t,x) \subset \overline{G}_1(t,x)$ is obvious. Conversely, let $y \in \overline{G}_1(t,x)$. Then, $y = \varphi(t),\ \varphi(\cdot) \in \mathcal{R}$, and $y \in \Xi_1^*$. We state that

$$\varphi(s) \in \Xi_1^* \quad \text{for all } 0 \leq s \leq t.$$

Assume by contradiction that $\varphi(s) \notin \Xi_1^*$ for $0 < s < t$. Therefore, $\omega(\varphi(s)) \subset \Xi_1$. But then by $(K3)$,

$$G(T,y) \subset G(T, G(t-s,\varphi(s))) \subset G(T+t-s,\varphi(s)) \to \Xi_1 \text{ as } T \to \infty,$$

which is a contradiction with $y \in \Xi_1^*$. Using again $(K3)$ one can define a function $\psi(\cdot) \in \mathcal{R}_1$ such that $\psi(0) = y$, so that $y \in G_1(t,x)$.

It is clear that G_1 possesses a global compact attractor, which is the union of all bounded complete trajectories of $\mathcal{R}_1$, and that G_1 is dynamically gradient with respect to $\{\Xi_2, \ldots, \Xi_n\}$. Then, again by Lemma 1.8 we can reorder the sets in such a way that Ξ_2 is a local attractor in Ξ_1^*. Let $\Xi_{2,1}^*$ be the repeller of Ξ_2 in Ξ_1^*. Then we restrict as before the dynamics to the set $\Xi_{2,1}^*$ and so on. Hence, we have reordered the sets Ξ_j in such a way that Ξ_1 is a local attractor and Ξ_j is a local attractor for the dynamics restricted to the repeller of the previous local attractor

$$\Xi_{j-1,j-2}^* \quad \text{for } j \geq 2,$$

and

$$\Xi_i \subset \Xi_{j-1,j-2}^* \quad \text{if } i \geq j,$$

where $\Xi_{1,0}^* = \Xi_1^*$.

Now, if $\gamma(\cdot)$ is a bounded complete trajectory such that

$$\gamma(t) \underset{t\to\infty}{\to} \Xi_i, \ \gamma(t) \underset{t\to-\infty}{\to} \Xi_j,$$

then we shall prove that $i \leq j$. Moreover, if $\gamma(\cdot)$ is not completely contained in some Ξ_k, then $i < j$.
If $i = 1$, then it is clear that $j \geq 1$. Also, if there exists $\gamma(t_0) \notin \Xi_1$, then $j > 1$, as Ξ_1 is a local attractor.
Let $i = 2$. Then

$$\gamma(t) \in \Xi_1^* \quad \text{for all } t \in \mathbb{R},$$

and then

$$\gamma(t) \underset{t\to-\infty}{\to} \Xi_1$$

is forbidden. Hence, $j \geq 2$.
Again, if there exists $\gamma(t_0) \notin \Xi_2$, then the fact that Ξ_2 is a local attractor in Ξ_1^* implies that $j > 2$.

Further, note that if $i \geq 3$, then $\gamma(t) \in \Xi_1^*$ for all $t \in \mathbb{R}$. Also, by induction, it follows that $\gamma(t) \in \Xi_{k,k-1}^*$ for all $t \in \mathbb{R}$ and $2 \leq k \leq i-1$. Indeed, let

$$\gamma(t) \in \Xi_{k-1,k-2}^* \quad \text{for all } t \in \mathbb{R}$$

with $2 \leq k \leq i-1$.

Then

$$\gamma(t) \underset{t\to\infty}{\rightarrow} \Xi_i$$

implies clearly that

$$\gamma(t) \in \Xi^*_{k,k-1} \quad \text{for all } t \in \mathbb{R}.$$

In particular,

$$\gamma(t) \in \Xi^*_{i-1,i-2} \quad \text{for all } t \in \mathbb{R}.$$

Hence, $\Xi_j \in \Xi^*_{i-1,i-2}$, so that $j \geq i$. Finally, if there exists $\gamma(t_0) \notin \Xi_i$, then $j > i$ as Ξ_i is a local attractor in $\Xi^*_{i-1,i-2}$.

□

To finish this section, it is worth noting that under conditions $(K1)$-$(K3)$,$(\overline{K}4)$, Theorem 1.9 implies that the family $\mathcal{S}$ generates a Morse decomposition if and only if G is dynamically gradient in view of Remark 0.22.

1.3. Application to a reaction-diffusion equation

In this section we will consider a Chafee-Infante problem where the reaction term is continuous but only differentiable in the origin. In fact, we have replaced the condition of being twice derivable by conditions of convexity and concavity type. This had never been considered in the literature until now.

In order to see how the previous theory is applyied, a study of the fixed points of this problem is needed. Later on, a family of Chafee-Infante problems depending on a parameter will be considered. We will check that they fulfill the hypotheses of Theorem 1.1. By this way, we will have seen that a dynamically gradient multivalued semiflow generated by the solutions of this Chafee-Infante problem is stable under perturbations.

We will consider the following Chafee-Infante problem

$$\begin{cases} \dfrac{\partial u}{\partial t} - \dfrac{\partial^2 u}{\partial x^2} = f(u), \ t > 0, \ x \in (0,1)\,, \\ u(t,0) = 0, \ u(t,1) = 0, \\ u(0,x) = u_0(x), \end{cases} \tag{1.3.1}$$

where f satisfies

(A1) $f \in C(\mathbb{R})$;

(A2) $f(0) = 0$;

(A3) $f'(0) > 0$ exists and is finite;

(A4) f is strictly concave if $u > 0$ and strictly convex if $u < 0$;

(A5) Growth condition:

$$|f(u)| \leq C_1 + C_2 |u|^{p-1},$$

where $p \geq 2, C_1, C_2 > 0$;

(A6) Dissipation condition:

a) If $p > 2$:

$$f(u)u \leq C_3 - C_4 |u|^p, \quad C_3, C_4 > 0.$$

b) If $p = 2$:

$$\limsup_{u \to \pm\infty} \frac{f(u)}{u} \leq 0.$$

Remark 1.10. Note that as a consequence of condition (A6)(b), we have that $f(u)u \leq (\lambda_1 - C_5)u^2 + C_6$, where $C_5, C_6 > 0$ and $\lambda_1 = \pi^2$ is the first eigenvalue of the operator $-\frac{\partial^2 u}{\partial x^2}$ with Dirichlet boundary conditions.

Depending on the regularity, we can have different types of solutions.

Definition 1.11. The function $u(\cdot) \in C([0,T], L^2(\Omega))$ is called a strong solution of (1.3.1) on $[0,T]$ if:

1. $u(0) = u_0$;
2. $u(\cdot)$ is absolutely continuous on compact subsets of $(0,T)$;
3. $u(t) \in H^2(\Omega) \cap H_0^1(\Omega)$, $f(u(t)) \in L^2(\Omega)$ for a.e. $t \in (0,T)$ and

$$\frac{du(t)}{dt} - \Delta u = f(u(t)), \text{ a.e. } t \in (0,T);$$

where the equality is understood in the sense of the space $L^2(\Omega)$.

Definition 1.12. The function $u(\cdot) \in C([0,T], L^2(\Omega))$ is called a weak solution of (1.3.1) on $[0,T]$ if:

1. $u \in L^\infty(0,T; L^2(\Omega))$;
2. $u \in L^2(0,T; H_0^1(\Omega)) \cap L^p(0,T; L^p(\Omega))$;
3. The equality in (1.3.1) is understood in the weak sense, i.e.

$$\frac{d}{dt}(u(t), v) - \langle \Delta u, v \rangle = (f(u(t)), v), \ \forall v \in H_0^1(\Omega) \cap L^p(\Omega),$$

where the equality is understood in the sense of distributions.

Let us make some comments on the natural relation among the above two definitions. Let $u(\cdot)$ be a strong solution such that $f(u(\cdot)) \in L^2(0,T; L^2(\Omega))$. In view of [7, Proposition 2.2] we have that $u \in L^2(0,T; H_0^1(\Omega))$, so $\Delta u \in L^2(0,T; H^{-1}(\Omega))$ and then $\frac{du}{dt} \in L^2(0,T; H^{-1}(\Omega))$. Hence, by [75, Lemma 7.4] we get

$$\langle \frac{du}{dt}, v \rangle - \langle \Delta u, v \rangle = (f(u(t)), v), \ \forall v \in H_0^1(\Omega).$$

Using [79, p.250] we obtain

$$\frac{d}{dt}(u, v) - \langle \Delta u, v \rangle = (f(u(t)), v), \ \forall v \in H_0^1(\Omega),$$

so point 3 of Definition 1.12 is satisfied.

Finally, if $p > 2$ by condition (A6)(a) we have

$$|u(t,x)|^p \leq \frac{C_3}{C_4} - \frac{f(u(t,x))u(t,x)}{C_4}$$

Thus, $f(u)u \in L^1((0,T) \times \Omega)$ implies that $u \in L^p((0,T) \times \Omega) = L^p(0,T; L^p(\Omega))$. Hence, $u(\cdot)$ is a weak solution as well.

In view of [34, p.283], for any $u_0 \in L^2(\Omega)$ there exists at least one weak solution. Moreover, if $f(u(\cdot)) \in L^2(0,T; L^2(\Omega))$, then putting $g(\cdot) = f(u(\cdot))$ we obtain by [11, p.189] that the problem

$$\begin{cases} \dfrac{dv}{dt} - \Delta v = g(t), \\ v(0) = u_0, \end{cases}$$

possesses a unique strong solution $v(\cdot)$. Since this problem has also a unique weak solution $\tilde{v}(\cdot)$ and the strong solution is a weak solution as well, then $v(\cdot) = \tilde{v}(\cdot) = u(\cdot)$. Hence $u(\cdot)$ is also a strong solution of problem (1.3.1).

Therefore, we have checked that the sets of weak and strong solutions satisfying $f(u(\cdot)) \in L^2(0,T; L^2(\Omega))$ coincide.

1.3.1. Stationary points

We now focus on the properties of the stationary points. To this end, we have followed the classic procedure from [33], [54] and [55]. Moreover, we have also taken some ideas from [67].

Let $\mathcal{R} \subset C([0,\infty), L^2(\Omega))$ be the set of all weak solutions of problem (1.3.1). Properties $(K1) - (K4)$ are satisfied (cf. [57]), so that a multivalued semiflow is defined (see Section 0). It is shown in [57, Lemma 12] that v is a fixed point of $\mathcal{R}$ (equivalently, of G) if and only if $v \in H_0^1(\Omega)$ and

$$\frac{\partial^2 v}{\partial x^2} + f(v) = 0, \text{ in } H^{-1}(\Omega). \tag{1.3.2}$$

The inclusion $H_0^1(\Omega) \subset L^\infty(\Omega)$ implies that $f(v) \in L^\infty(\Omega)$, so that $v \in H^2(\Omega) \cap H_0^1(\Omega)$. Therefore, $v(\cdot)$ is a strong solution as well.

Let consider the function $F : \mathbb{R} \to \mathbb{R}$ defined by

$$F(s) = \int_0^s f(r)\mathrm{d}r, \;\; s \in \mathbb{R}.$$

We define

$$a_- = inf\{s < 0 : \; \mathrm{sgn}\; f(x) = \; \mathrm{sgn}\; x, \; \forall x; s < x < 0\}$$

and

$$a_+ = sup\{s > 0 : \; \mathrm{sgn}\; f(x) = \; \mathrm{sgn}\; x, \; \forall x; 0 < x < s\}.$$

If follows from conditions (A2) and (A3) of f that $-\infty \leq a_- < 0 < a_+ \leq +\infty$. Since f is positive on $(0, a_+)$ and negative on $(a_-, 0)$, we have that F is strictly increasing on $[0, a_+)$, strictly decreasing on $(a_-, 0]$ and $F(0) = 0$.

We consider $E_+, E_- \in [0, \infty]$ defined by

$$E_+ = \lim_{s \to a_+} F(s),$$

$$E_- = \lim_{s \to a_-} F(s).$$

Then, F has the inverse functions $U_+ : [0, E_+) \to [0, a_+)$, $U_- : [0, E_-) \to (a_-, 0]$.

We also define the following functions with domains $(0, E_+)$ and $(0, E_-)$, respectively, with values on $[0, \infty)$:

$$\tau_+(E) = \int_0^{U_+(E)} (E - F(u))^{-1/2} \; \mathrm{d}u, \;\; 0 < E < E_+,$$

$$\tau_-(E) = \int_{U_-(E)}^0 (E - F(u))^{-1/2} \; \mathrm{d}u, \;\; 0 < E < E_-.$$

Let us consider $v_0 \in \mathbb{R}$ and a solution u of

$$\begin{cases} \frac{\partial^2 u}{\partial x^2} + f(u) = 0, \\ u(0) = 0, u'(0) = v_0. \end{cases} \tag{1.3.3}$$

Note that the solution of the problem (1.3.3) is unique, since f is convex for $u < 0$ and concave for $u > 0$, so it is Lipschitz on compact intervals (see [89, p.4] or [52, p.8]).

If we define $E = v_0^2/2$, then:

$$\frac{(u'(x))^2}{2} + F(u(x)) = E.$$

On the other hand, the functions τ_+, τ_- evaluated in $E = v_0^2/2$ give us $\sqrt{2}$ the x-time necessary to go from the initial condition $u(0) = 0$, with initial velocity $v_0, -v_0$ respectively, to the point where $u'(T_+(E)) = 0$. Indeed, $u(x)$ satisfies

$$\frac{(u'(x))^2}{2} + F(u(x)) = E,$$

so

$$\frac{dx}{du} = \frac{1}{\sqrt{2}} \frac{1}{\sqrt{E - F(u)}}.$$

Since $u'(T_+(E)) = 0$ for $u = U_+(E)$, then

$$\sqrt{2} \int_0^{T_+(E)} 1 \ \mathrm{d}x = \int_0^{U_+(E)} \frac{1}{\sqrt{E - F(u)}} \mathrm{d}u = \tau_+(E).$$

By symmetry with respect to the u axis, the $x-$time it takes for $u(x)$ to go from $(U^+(E), 0)$ to $(0, -v_0)$ is $T_+(E)$. Hence, if $2T_+(E) = 1$, that is, $\tau^+(E) = \frac{1}{\sqrt{2}}$, then $u(\cdot)$ is a solution satisfying the boundary conditions $u(0) = u(1) = 0$. Applying a similar reasoning for $\tau^-(E)$, we obtain that u satisfies the boundary conditions if, and only if, E satisfies for some $k \in \mathbb{N}$ only one of the following conditions
:

$$k\tau_+(E) + (k-1)\tau_-(E) = \frac{1}{\sqrt{2}}, \tag{1.3.4}$$

$$k\tau_-(E) + (k-1)\tau_+(E) = \frac{1}{\sqrt{2}}, \tag{1.3.5}$$

$$k\tau_+(E) + k\tau_-(E) = \frac{1}{\sqrt{2}}. \tag{1.3.6}$$

Remark 1.13. Note that if E satisfies (1.3.4) or (1.3.5) for a certain k, then u has $2k$ zeros and if E satisfies (1.3.6), then u has $2k+1$ zeros. Our goal is to solve these equations for E as a function of $f'(0)$. To this end, we study the properties of $\tau_{\pm}$.

In order to obtain solutions of the equations (1.3.4), (1.3.5) and (1.3.6) it is necessary to make a change of variable for the functions $\tau_{\pm}$. Given $E \in (0, E_{\pm})$, we put

$$Ey^2 = F(u),\ 0 \leq y \leq 1,\ 0 \leq u \leq U_+(E)$$

and

$$Ey^2 = F(u),\ -1 \leq y \leq 0,\ U_-(E) \leq u \leq 0.$$

Hence, $du = (2yE/f(u))dy$ and $E - F(u) = E(1-y^2)$. By this change, we obtain

$$\tau_+(E) = 2\sqrt{E}\int_0^1 (1-y^2)^{-1/2}\frac{y}{f(u)}dy, 0 < E < E_+;\ u = U_+(Ey^2), 0 \leq y \leq 1;$$

$$\tau_-(E) = 2\sqrt{E}\int_{-1}^0 (1-y^2)^{-1/2}\frac{y}{f(u)}dy, 0 < E < E_-; u = U_-(Ey^2), -1 \leq y \leq 0.$$

The next results show some properties of these functions.

Theorem 1.14. *The functions $\tau_{\pm}$ satisfy*

$$\lim_{E\to 0^+} \tau_{\pm}(E) = \frac{\pi}{(2f'(0))^{1/2}}.$$

Proof. Since $f'(0) > 0$ and $f(0) = 0$, given $\varepsilon \in (0,1)$, there exists $\delta > 0$ such that

$$\begin{gathered} f'(0)(1-\varepsilon)u \leq f(u) \leq f'(0)(1+\varepsilon)u, \quad 0 \leq u \leq \delta. \\ \frac{1}{f'(0)(1+\varepsilon)} \leq \frac{u}{f(u)} \leq \frac{1}{f'(0)(1-\varepsilon)}, \quad 0 \leq u \leq \delta. \end{gathered} \tag{1.3.7}$$

Moreover, as $U_+(E)$ is continuous at 0, given $\delta > 0$, there exists $\eta > 0$ such that for $0 < E \leq \eta$, $U_+(E) \leq \delta$. Now, if we integrate (3.5.5) between 0 and u we obtain the following inequality

$$\frac{f'(0)}{2}(1-\varepsilon)u^2 \leq F(u) \leq \frac{f'(0)}{2}(1+\varepsilon)u^2, \ 0 \leq u \leq \delta.$$

Using the change of variable $Ey^2 = F(u)$, we have

$$\left(\frac{f'(0)(1-\varepsilon)}{2E}\right)^{1/2} u \leq y \leq \left(\frac{f'(0)(1+\varepsilon)}{2E}\right)^{1/2} u, \quad \text{for} \ \ 0 < E \leq \eta, \ 0 \leq y \leq 1.$$

Dividing the previous expression by $f(u)$ and using (3.5.5) we obtain

$$\left(\frac{1-\varepsilon}{2Ef'(0)(1+\varepsilon)^2}\right)^{1/2} \leq \frac{y}{f(u)} \leq \left(\frac{1+\varepsilon}{2Ef'(0)(1-\varepsilon)^2}\right)^{1/2}, \text{for} \ \ 0 < E \leq \eta, \ 0 \leq y \leq 1.$$

Now if we multiply by $2\sqrt{E}(1-y^2)^{-\frac{1}{2}}$ and integrate from 0 to 1, we get

$$\pi\left(\frac{1-\varepsilon}{2f'(0)(1+\varepsilon)^2}\right)^{1/2} \leq \tau_+(E) \leq \pi\left(\frac{1+\varepsilon}{2f'(0)(1-\varepsilon)^2}\right)^{1/2}, \text{for} \ \ 0 < E \leq \eta.$$

Finally, taking $\varepsilon \to 0$, the theorem follows. The proof for τ_- is analogous. □

Theorem 1.15. *The functions $\tau_\pm$ are strictly increasing on their domains.*

Proof. Let consider the expression of τ_+ and $0 < E_1 < E_2 < E_+$. Then,

$$\tau_+(E_2) - \tau_+(E_1) = \int_0^1 \frac{2y}{\sqrt{1-y^2}}\left[\frac{\sqrt{E_2}}{f(U^+(E_2y^2))} - \frac{\sqrt{E_1}}{f(U^+(E_1y^2))}\right] dy.$$

From [52, p.8] we have that the function f is differentiable almost everywhere in $\mathbb{R}$, so

$$\alpha(E) = \frac{\sqrt{E}}{f(U^+(Ey^2))}$$

is differentiable as well. Hence,

$$\alpha'(E) = \frac{f^2(U^+(Ey^2)) - 2y^2Ef'(U^+(Ey^2))}{2\sqrt{E}f^3(U^+(Ey^2))}.$$

Recall the change of variable $F(u) = Ey^2$. Consider the numerator of α', that is,

$\beta(u) = f^2(u) - 2F(u)f'(u)$. Then we obtain

$$\beta(u) = 2\int_0^u f(s)(f'(s) - f'(u))ds,\ 0 < s < u.$$

Since f is strictly concave, if $s < u$, then $f'(s) > f'(u)$ (cf. [89, p.5]). As a result, $\beta(u) > 0$.

In order to finish the proof rigorously, we have to justify the previous calculations. Indeed, from [52, p.5], we have that the function f is absolutely continuous and from [11, p.16], $f' \in L^1_{loc}$. Therefore, $\alpha' \in L^1_{loc}$ and $\alpha' > 0$ a.e., which implies that $\alpha(E)$ is strictly increasing and the proof is finished.

The claim for $\tau_-(E)$ follows analogously. □

Theorem 1.16. *The functions $\tau_\pm$ satisfy*

$$\lim_{E\to E^\pm} \tau_\pm(E) = \infty$$

Then, $\tau_\pm : (0, E^\pm) \to \left(\frac{\pi}{(2f'(0))^{1/2}}, \infty\right)$.

Proof. **Case** $a_+ < \infty$**.** Then, we have $f(a_+) = 0$ and $\bar{u}(x) = a_+$ is a constant solution to the problem $\frac{\partial^2 u}{\partial x^2} + f(u) = 0$. Let us consider $E_+ = F(a_+)$ and the solution u to this problem satisfying the conditions $u(0) = 0, u'(0) = v_0, E = \frac{1}{2}v_0^2$. As a_+ is a constant solution, by uniqueness $\tau_+(E^+) = \infty$. Therefore, given $T > 0$, there exists $\delta > 0$ such that if $E > E_+ - \delta$, then $\tau_+(E) > T$, which follows from the continuity of u with respect to its initial conditions.

Case $a_+ = \infty$. Note that if $p > 2$, then $a_+ < \infty$. Therefore, $p = 2$. In this case, $f(u) > 0$ for all $u \in (0, \infty)$. From condition (A5), there exist $\alpha, \beta > 0$ such that $f(u) \leq \alpha + \beta u$. For $u > 0$ we have

$$\frac{f(u)}{u^2} \leq \frac{\alpha}{u^2} + \frac{\beta}{u}.$$

Hence, $f(u)/u^2 \to 0$, as $u \to \infty$.

On the other hand,

$$\int_0^u f(s)ds \le \int_0^u (\alpha + \beta s)\ ds.$$

Thus, we have $F(u) \le \alpha u + \beta u^2/2$ and

$$0 \le \frac{F(u)}{u^3} \le \frac{\alpha}{u^2} + \frac{\beta}{2}\frac{1}{u}.$$

Hence, $F(u)/u^3 \to 0$, as $u \to \infty$.

We claim that $\lim_{u\to 0^+} f(u)/u^2 = \infty$. Indeed, since $f'(0)$ exists, for any $\varepsilon \in (0, f'(0))$, there exists $\delta > 0$ such that

$$|f'(0) - f(u)/u| < \varepsilon, \quad \text{for any } |u| < \delta.$$

Thus, dividing by u^2, we obtain

$$\frac{u(f'(0)-\varepsilon)}{u^2} < \frac{f(u)}{u^2} < \frac{u(\varepsilon + f'(0))}{u^2}$$

and the result follows.

Since

$$f(u)/u^2 \to 0, \quad \text{as } u \to \infty,$$

and

$$f(u)/u^2 \to \infty, \quad \text{as } u \to 0^+,$$

for any $\varepsilon > 0$, there exists a first value $u_0 \in (0, \infty)$ where $f(u_0)/u_0^2 = \varepsilon$. Hence,

$$\frac{f(u)}{u^2} > \varepsilon,\ 0 < u < u_0.$$

From the above expression, we have

$$\int_0^u f(s)ds > \int_0^u \varepsilon s^2 ds$$

and

$$\varepsilon u^3/3 < F(u).$$

Then,

$$F(u)/u^3 > \varepsilon/3, \quad \text{if } 0 < u \leq u_0.$$

Since

$$F(u)/u^3 \to 0, \quad \text{as } u \to \infty,$$

we deduce that there exists a first $\overline{u} > u_0$ such that $F(\overline{u})/\overline{u}^3 = \varepsilon/3$. Hence, we have

$$\frac{F(u)}{u^3} > \frac{\varepsilon}{3},\ 0 < u < \overline{u},$$

with $F(\overline{u}) = \dfrac{\varepsilon}{3}\overline{u}^3$.

Now, computing τ_+ in $\overline{E} = F(\overline{u})$, we have

$$\tau_+(\overline{E}) = \int_0^{U^+(\overline{E})} \frac{1}{\sqrt{\overline{E} - F(u)}} du = \int_0^{\overline{u}} \frac{1}{\sqrt{\frac{\varepsilon}{3}\overline{u}^3 - F(u)}} du$$

$$\geq \int_0^{\overline{u}} \frac{1}{\sqrt{\frac{\varepsilon}{3}\overline{u}^3 - \frac{\varepsilon}{3}u^3}} du = \frac{\sqrt{3}}{\sqrt{\varepsilon}} \int_0^{\overline{u}} \frac{1}{\sqrt{\overline{u}^3 - u^3}} du$$

$$= \frac{\sqrt{3}}{\sqrt{\varepsilon}} \int_0^1 \frac{\overline{u}}{\sqrt{\overline{u}^3 - \overline{u}^3 t^3}} dt = \frac{\sqrt{3}}{\sqrt{\varepsilon}} \frac{\overline{u}}{\sqrt{\overline{u}^3}} \int_0^1 \left(1 - t^3\right)^{-\frac{1}{2}} dt$$

$$= \frac{\sqrt{3}}{\sqrt{\varepsilon}} \frac{\overline{u}}{\sqrt{\overline{u}^3}} \frac{1}{3} \int_0^1 s^{\frac{1}{3}-1} \left(1 - s\right)^{\frac{1}{2}-1} ds$$

$$= \frac{1}{\overline{u}^{\frac{1}{2}}} \frac{1}{\sqrt{\varepsilon}} \frac{\sqrt{3}}{3} \mathcal{B}\left(\frac{1}{2}, \frac{1}{3}\right).$$

Recall that $\varepsilon\overline{u}^3 = 3F(\overline{u})$. Then,

$$\varepsilon\overline{u} = 3\frac{F(\overline{u})}{\overline{u}^2}.$$

Taking $\varepsilon \to 0$, by construction $\overline{u} \to \infty$. Therefore, from condition (A6)(b) we have that $\lim_{u\to\infty} f(u)/u \leq 0$, so the last expression tends to 0 and $\tau_+(\overline{E}) \to \infty$. $\square$

Theorem 1.17. *Consider*

$$\lambda_n = n^2\pi^2.$$

Then, for each $n \geq 1$, there exist two continuous functions $E_n^{\pm} : [\lambda_n, \infty) \to [0, E_{\pm})$ with the following properties:

1. *For each integer $k \geq 1$ and for $f'(0) \in [\lambda_{2k-1}, \infty)$ the only solution of the equation (1.3.4) (resp. 1.3.5) is the value $E_{2k-1}^{+}(f'(0))$ (resp. $E_{2k-1}^{-}(f'(0))$);*
2. *For each integer $k \geq 1$ and for $f'(0) \in [\lambda_{2k}, \infty)$ the only solution of the equation (1.3.6) is the value $E_{2k}^{-}(f'(0)) = E_{2k}^{+}(f'(0)) = E_{2k}$;*
3. *For each integer $n \geq 1$, $E_n^{\pm}(f'(0)) = 0$, if $f'(0) = \lambda_n$.*

Proof. Let be $n \geq 1$. If n is odd, then $n = 2k - 1$ for $k \geq 1$. First, we prove that we can define the function

$$E_n^{\pm} : [\lambda_n, \infty) \longrightarrow [0, E_{\pm})$$

by putting $E_n^{\pm}(f'(0)) = E$, where E satisfies $k\tau_{\pm}(E) + (k-1)\tau_{\mp}(E) = 1/\sqrt{2}$.

Consider the function

$$h_{\pm}^{n} : (0, E_{\pm}) \longrightarrow (n\pi/\sqrt{2f'(0)}, \infty),$$

defined by $h_{\pm}^{n}(E) := k\tau_{\pm}(E) + (k-1)\tau_{\mp}(E)$. If $f'(0) > \lambda_n$ then, as $h_{\pm}$ is a strictly increasing function, there exists a unique $E_{2k-1}^{\pm} \in (0, E_{\pm})$ such that $h_{\pm}^{n}(E_{2k-1}^{\pm}) = 1/\sqrt{2}$.

Since $h_{\pm}$ has inverse, $E_{2k-1}^{\pm} = (h_{\pm}^{n})^{-1}(1/\sqrt{2})$ is the solution of the expressions (1.3.4) and (1.3.5). Moreover, $E_{2k-1}^{\pm}(\lambda_n) = 0$ by construction.

Second, if n is even, then $n = 2k$ for $k \geq 1$. As before, we consider $h_{\pm}^{n}(E) := k\tau_{\pm}(E) + k\tau_{\mp}(E)$. Since it is an increasing function, for $f'(0) > \lambda_n$, there exists a unique $E_{2k} \in (0, E_{\pm})$ such that $h_{\pm}^{n}(E_{2k}) = 1/\sqrt{2}$. Analogously, we obtain the solution of the expression (1.3.6), $E_{2k}^{\pm} = (h_{\pm}^{n})^{-1}(1/\sqrt{2})$, and $E_{2k-1}^{\pm}(\lambda_n) = 0$. $\square$

Theorem 1.18. *For each $n \geq 0$ and $f'(0) \in [\lambda_n, \infty)$, the equation (1.3.2) has two new more solutions $v_n^{\pm}$ with the following properties:*

1. *$a_- < u_n^{\pm}(x) < a_+$ for all $x \in [0, 1]$;*
2. *If $f'(0) = \lambda_n$, then $v_n^{\pm} = 0$;*
3. *For $f'(0) \in (\lambda_n, \infty)$, $v_n^{\pm}$ has $n + 1$ zeros in $[0, 1]$. Denoting these zeros by $x_q^{\pm}$, $q = 0, 1, \ldots, n$ with $0 = x_0^{\pm} < x_1^{\pm} < x_2^{\pm} < \ldots < x_n^{\pm} = 1$, we have $(-1)^q v_n^+(x) > 0$ for $x_q^+ < x < x_{q+1}^+, q = 0, 1, \ldots, n-1$ and $(-1)^q v_n^-(x) < 0$ for $x_q^- < x < x_{q+1}^-, q = 0, 1, \ldots, n-1$. Also, $v_n^+ = -v_n^-$, if f is odd;*

Proof. The first point follows from $F(u_n^{\pm}(x)) \leq E < E_{\pm}$.

The second point follows from the third one of Theorem 1.17. Indeed, for each $n \geq 1$ and $f'(0) \in [\lambda_n, \infty)$ we have the values $E_n^{\pm}(f'(0))$ by the above theorem. Also, we have a solution of the equation (1.3.2) which is denoted by $v_n^{\pm}$. If $f'(0) = \lambda_n$, then $E_n^{\pm}(\lambda_n) = 0$ and $v_0 = 0$, so $v_n^{\pm} = 0$.

The third point follows by Remark 1.13. If f is odd, then $-U^-(E) = U^+(E)$, $\tau_+(E) = \tau_-(E)$, so we have $v_n^+ = -v_n^-$. $\square$

Corollary 1.19. *If $n^2\pi^2 < f'(0) \leq (n+1)^2\pi^2, n \in \mathbb{N}$, then there are $2n+1$ fixed points: 0, $v_1^{\pm}$, ..., $v_n^{\pm}$, where $v_j^{\pm}$ possesses $j+1$ zeros in $[0, 1]$.*

1.3.2. Approximations

Once we have establish the properties of the fixed points, as it has been mentioned at the beginning of the section, we shall consider the following family of Chafee-Infante equations

$$\begin{cases} \dfrac{\partial u}{\partial t} - \dfrac{\partial^2 u}{\partial x^2} = f_\varepsilon(u), \ t > 0, \ x \in (0, 1)\,, \\ u(t, 0) = 0, \ u(t, 1) = 0, \\ u(0, x) = u_0(x), \end{cases} \tag{1.3.8}$$

where $\varepsilon \in (0, 1]$ is a small parameter and f_ε satisfies:

$(\widetilde{A1})$ $f_\varepsilon \in C(\mathbb{R})$ and is non-decreasing;

$(\widetilde{A2})$ $f_\varepsilon(0) = 0$;

$(\widetilde{A3})$ $f_\varepsilon'(0) > 0$ exists, is finite, monotone in ε and $f_\varepsilon'(0) \to \infty$, as $\varepsilon \to 0^+$;

$(\widetilde{A4})$ f_ε is strictly concave if $u > 0$ and strictly convex if $u < 0$;

$(\widetilde{A5})$ $-1 < f_\varepsilon(s) < 1$, for all s, and

$$|f_\varepsilon(s) - H_0(s)| < \varepsilon, \quad \text{if } |s| > \varepsilon, \tag{1.3.9}$$

where

$$H_0(u) = \begin{cases} -1, & \text{if} \quad u < 0, \\ [-1, 1], & \text{if} \quad u = 0, \\ 1, & \text{if} \quad u > 0, \end{cases}$$

is the Heaviside function.

Conditions (A1)-(A6) are satisfied with $p = 2$, so problem (1.3.8) is a particular case of (1.3.1).

Our aim now is to prove that for ε sufficiently small the multivalued semiflow G_ε generated by the weak solutions of problem (1.3.8) is dynamically gradient. Problem (1.3.8) is an approximation of the following problem, governed by a differential inclusion

$$\begin{cases} \dfrac{\partial u}{\partial t} - \dfrac{\partial^2 u}{\partial x^2} \in H_0(u), \text{ on } \Omega \times (0, T), \\ u|_{\partial\Omega} = 0, \\ u(0, x) = u_0(x). \end{cases} \tag{1.3.10}$$

This problem is widely studied in [7], where properties of fixed points, connections between them and structure of the attractor is obtained.

Definition 1.20. We say that the function $u \in C([0,T], L^2(\Omega))$ is a strong solution of (1.3.10) if

1. $u(0) = u_0$;

2. $u(\cdot)$ is absolutely continuous on $(0,T)$ and $u(t) \in H^2(\Omega) \cap H_0^1(\Omega)$ for a.e. $t \in (0,T)$;

3. There exists a function $g(\cdot)$ such that $g(t) \in L^2(\Omega)$, a.e. on $(0,T)$, $g(t,x) \in H_0(u(t,x))$, for a.e. $(t,x) \in (0,T) \times \Omega$, and

$$\frac{du}{dt} - \frac{\partial^2 u}{\partial x^2} - g(t) = 0, \text{ a.e. } t \in (0,T).$$

In this case we put $\mathcal{R}$ as the set of all strong solutions such that the map g belongs to $L^2(0,T;L^2(\Omega))$. Conditions (K1)-(K4) are satisfied (cf. [44]) and the map

$$G : \mathbb{R}_+ \times L^2(\Omega) \to P(L^2(\Omega))$$

defined by (3.3.1) is a strict multivalued semiflow possessing a global compact attractor $\mathcal{A}_0$ (cf. [83]) in $L^2(\Omega)$, which is connected (cf. [84]). The structure of this attractor is studied in [7]. It is shown that there exists an infinite (but countable) number of fixed points

$$v_0 = 0, v_1^+, v_1^-, \ldots, v_n^+, v_n^-, \ldots,$$

and that $\mathcal{A}_0$ consists of these fixed points and all bounded complete trajectories $\psi(\cdot)$, which always connect two fixed points, that is,

$$\begin{aligned} \psi(t) &\to z_1 \text{ as } t \to \infty, \\ \psi(t) &\to z_2 \text{ as } t \to -\infty, \end{aligned} \tag{1.3.11}$$

where $z_i = 0, z_i = v_n^+$ or $z_i = v_n^-$ for some $n \geq 1$. Moreover, if ψ is not a fixed point, then either $z_2 = 0$ and $z_1 = v_n^{\pm}$, for some $n \geq 1$, or $z_2 = v_k^{\pm}$, $z_1 = v_n^{\pm}$ with $k > n$.

We fix some $N_0 \in \mathbb{N}$. Denote

$$Z_{N_0} = \left(\cup_{k \geq N_0} \{v_k^{\pm}\}\right) \cup \{v_0\}$$

and define the sets

$$\Xi_k^0 = \{v_k^+, v_k^-\}, \quad 1 \leq k \leq N_0 - 1,$$

$$\Xi_{N_0}^0 = \left\{ \begin{array}{c} y : \exists \psi \in \mathbb{K} \text{ such that (1.3.11) holds with } z_j \in Z_{N_0}, \\ j = 1, 2 \text{ and } y = \psi(t) \text{ for some } t \in \mathbb{R} \end{array} \right\},$$

where as before $\mathbb{K}$ stands for the set of all bounded complete trajectories. We note that set $\Xi_{N_0}^0$ contains the fixed points in Z_{N_0} and all bounded complete trajectories connecting them.

Remark 1.21. It is known [44] that the family $\mathcal{M} = \{\Xi_1^0, \dots, \Xi_{N_0}^0\}$ is a disjoint family of isolated weakly invariant sets and that G_0 is dynamically gradient with respect to $\mathcal{M}$ in the sense of Remark 1.3. Hence, G_0 is dynamically gradient with respect to $\mathcal{M}$ in the sense of Definition 0.19.

Now our purpose is to adapt some lemmas from [7, p.2979] to problem (1.3.8). In view of Theorems 1.17 and 1.18 and the third condition on f_ε, there exists a sequence $\overline{\varepsilon}_k \to 0$, as $k \to \infty$, such that for every $\varepsilon \in (\overline{\varepsilon}_k, \overline{\varepsilon}_{k+1}]$ and any $k \geq 1$ problem (1.3.8) has exactly $2k+1$ fixed points $\{v_0^\varepsilon = 0, \{v_{\varepsilon,j}^+\}_{j=1}^k\}$ such that for each $1 \leq n \leq k$ $v_{\varepsilon,n}^\pm$ has $n+1$ zeros in $[0,1]$.

Let us consider a sequence $\{\varepsilon_m\}$ converging to zero.

Lemma 1.22. *Let $n \in \mathbb{N}$ be fixed. Then, $v_{\varepsilon_m,n}^+$ (resp. $v_{\varepsilon_m,n}^-$) do not converge to 0 in $H_0^1(0,1)$ as $\varepsilon_m \to 0$.*

Proof. Suppose that

$$v_{\varepsilon_m,n}^+ \to 0 \text{ in } H_0^1(0,1).$$

Then

$$v_{\varepsilon_m,n}^+ \to 0 \text{ in } C([0,1]).$$

By Remark 1.13, $v_{\varepsilon_m,n}^+$ has a unique maximum in $a \in (0, x_1^+)$ and by the properties of τ_+ described before $a = \frac{x_1^+}{2}$. We may assume that x_1^+ does not converge to 0.

Let $x_0(\varepsilon_m)$ be the first point where $v^+_{\varepsilon_m,n}(x_0) = \varepsilon_m$ or $x_0 = a$ if such a point does not exist. We claim that $x_0(\varepsilon_m) \to 0$, as $\varepsilon_m \to 0$. It is clear that

$$\partial^2 v^+_{\varepsilon_m,n}/\partial x^2 = -f_{\varepsilon_m}(v^+_{\varepsilon_m,n}) < 0 \quad \text{in } (0, x_1^+),$$

and then

$$\frac{v^+_{\varepsilon_m,n}(x_0)}{x_0} x \leq v^+_{\varepsilon_m,n}(x) \leq \varepsilon_m, \quad \forall x \in [0, x_0], \tag{1.3.12}$$

by concavity. Hence, integrating first on (s, a) and then on $(0, x)$ with $x \leq x_0$, we have

$$\frac{d}{dx} v^+_{\varepsilon_m,n}(s) = \int_s^a f_{\varepsilon_m}(v^+_{\varepsilon_m,n}(\tau)) d\tau, \tag{1.3.13}$$

$$v^+_{\varepsilon_m,n}(x) = \int_0^x \int_{x_0}^a f_{\varepsilon_m}(v^+_{\varepsilon_m,n}(\tau)) d\tau ds + \int_0^x \int_s^{x_0} f_{\varepsilon_m}(v^+_{\varepsilon_m,n}(\tau)) d\tau ds.$$

Since $f_\varepsilon(u)$ is concave, we have that $f_\varepsilon(u)/u \geq f_\varepsilon(\varepsilon)/\varepsilon$, for all $0 < u \leq \varepsilon$. Moreover, by assumption $(\widetilde{A5})$ of f_ε we get $f_\varepsilon(u) \geq \frac{1-\varepsilon}{\varepsilon} u$, for all $0 < u \leq \varepsilon$. Hence, using (1.3.12) we have

$$v^+_{\varepsilon_m,n}(x) \geq \int_0^x \int_s^{x_0} \frac{1-\varepsilon_m}{\varepsilon_m} v^+_{\varepsilon_m,n}(\tau) d\tau ds \geq \frac{1-\varepsilon_m}{\varepsilon_m} \frac{v^+_{\varepsilon_m,n}(x_0)}{x_0} \int_0^x \int_s^{x_0} \tau d\tau ds.$$

Thus,

$$1 \geq \frac{1-\varepsilon_m}{\varepsilon_m} \left(\frac{x x_0}{2} - \frac{x^3}{6x_0} \right),$$

so it follows that $x_0 \to 0$, as $\varepsilon_m \to 0$.

Let $\delta_1 < 0 < \delta_2$ be such that $x_0(\varepsilon_m) \leq \delta_1 < \delta_2 \leq a(\varepsilon_m)$. Since $v^+_{\varepsilon_m,n}(x) \geq \varepsilon_m$ $\forall x \in [x_0, a]$, if we intregate (1.3.13) over (δ_1, x) with $\delta_1 < x \leq \delta_2$, we have

$$v^+_{\varepsilon_m,n}(x) - v^+_{\varepsilon_m,n}(\delta_1) = \int_{\delta_1}^x \int_s^a f(v^+_{\varepsilon_m,n}(\tau)) d\tau ds \geq (1-\varepsilon_m) \int_{\delta_1}^x \int_s^a d\tau ds,$$

which implies a contradiction if $v^+_{\varepsilon_m,n} \to 0$ in $C([0,1])$.

The proof is similar for $v^-_{\varepsilon_m,n}$. $\square$

Lemma 1.23. *$v^+_{\varepsilon_m,k}$ (resp. $v^-_{\varepsilon_m,k}$) converges to v^+_k (resp. v^-_k) in $H^1_0(\Omega)$ as $m \to \infty$ for any $k \geq 1$.*

Proof. It is easy to see that $v^+_{\varepsilon_m k}$ is bounded in $H^2(\Omega) \cap H^1_0(\Omega)$, so $v^+_{\varepsilon_m k} \to v$ strongly in $H^1_0(\Omega)$ and $C^1([0,1])$ up to a subsequence. The proof will be finished if we prove that $v = v^+_k$. We observe that since in such a case every subsequence would have the same limit, the whole sequence would converge to v^+_k

It is clear that the functions $g_{\varepsilon_m} = f_{\varepsilon_n}(v^+_{\varepsilon_m k})$ are bounded in $L^\infty(0,1)$.

Passing to a subsequence we can then assume that g_{ε_n} converges to some g weakly in $L^2(0,1)$. It is clear that $-(\partial^2 v/\partial x^2) = g$ and v is a fixed point if we prove the inclusion

$$g(x) \in H_0(v(x)) \quad \text{for a.e. } x \in (0,1).$$

By Masur's theorem [92, p.120] there exist $z_m \in V_m = conv(\cup^\infty_{k\geq m} g_{\varepsilon_k})$ such that $z_m \to g$, as $m \to \infty$, strongly in $L^2(0,1)$. Taking a subsequence we have $z_m(x) \to g(x)$, a.e. in $(0,1)$. Since $z_m \in V_m$, we get $z_m = \sum_{i=1}^{N_m} \lambda_i g_{\varepsilon_{k_i}}$, where $\lambda_i \in [0,1]$, $\sum_{i=1}^{N_m} \lambda_i = 1$ and $k_i \geq m$, for all i.

Now (1.3.9) implies that $|g_{\varepsilon_k}(x) - H_0(v(x))| \to 0$, as $k \to \infty$, for a.e. x. Indeed, if $v(x) = 0$, then $g_{\varepsilon_k}(x) \in [-1,1] = H_0(v(x))$. If $v(x) > 0$, then $|g_{\varepsilon_k}(x) - H_0(v(x))| = |f_{\varepsilon_k}(v_{\varepsilon_k}(x)) - 1| \to 0$, as $k \to \infty$. If $v(x) < 0$, we apply a similar argument.

Thus, for any $\delta > 0$ and a.e. x there exists $m(x,\delta)$ such that $g_{\varepsilon_k}(x) \subset [a(x) - \delta, b(x) + \delta]$, for all $k \geq m$, where $[a(x), b(x)] = H_0(v(x))$. Hence, $z_m(x) \subset [a(x) - \delta, b(x) + \delta]$, as well. Passing to the limit we obtain $g(x) \in [a(x), b(x)]$, a.e. on $(0,1)$.

To conclude the proof, we have to prove that $v = v^+_k$. By Lemma 3.38 $v \neq 0$. Hence, as $v^+_{\varepsilon_m k}(x) > 0$ for all $x \in (0, x^+_1(\varepsilon_m))$, $v = v^+_n$ for some $n \in \mathbb{N}$. Since v^+_n has $n+1$ zeros, the convergence $v^+_{\varepsilon_m k} \to v^+_n$ implies that $v^+_{\varepsilon_m k}$ has $n+1$ zeros for $m \geq N$. But $v^+_{\varepsilon_m k}$ possesses $k+1$ zeros. Thus, $k = n$.

For the sequence $v^-_{\varepsilon_m k}$ the proof is analogous. $\square$

Lemma 1.24. *Let $\varepsilon_m \to 0$, $k_m \to \infty$ as $m \to \infty$. Then $v^+_{\varepsilon_m,k_m}$ (resp. $v^-_{\varepsilon_m,k_m}$) converges to 0 as $m \to \infty$.*

Proof. In the same way as in the proof of Lemma 1.23 we obtain that up to a subsequence $v^+_{\varepsilon_m,k_m} \to v$ in $H^1_0(\Omega)$ and $C^1([0,1])$, where v is a fixed point of problem (1.3.10). We will prove that $v = 0$ by contradiction. If not, then $v = v^\pm_n$ for some $n \in \mathbb{N}$. However, since $v^\pm_n$ has exactly $n+1$ zeros and $v^+_{\varepsilon_m,k_m} \to v$ in $C^1([0,1])$, we have that $v^+_{\varepsilon_m,k_m}$ has $n+1$ zeros for any $m \geq M$ with M big enough. This contradicts the fact that $v^+_{\varepsilon_m,k_m}$ possesses $k_m + 1$ zeros and $k_m \to \infty$. As the limit is 0 for every converging subsequence, the whole sequence converges to 0.

For the sequence $v^-_{\varepsilon_m k}$ the proof is analogous. □

Once we have described the preliminary properties, we are now ready to check that (1.3.8) satisfies the conditions given in Theorem 1.1 for certain families $\mathcal{M}_\varepsilon$. We recall that [86, Theorem 10] guarantees the existence of the global compact invariant attractors $\mathcal{A}_\varepsilon$, where each $\mathcal{A}_\varepsilon$ is the union of all bounded complete trajectories.

Let us check assumptions (H1)-(H5) of Theorem 1.1.

As we have seen before, condition (H2) follows from Remark 1.21. Therefore, we prove now condition (H1).

Multiplying the equation in (1.3.8) by u, we obtain

$$\frac{1}{2}\frac{d}{dt}\|u\|^2_{L^2} + \|u\|^2_{H^1_0} \leq \int_\Omega |u|dx \leq \frac{1}{2}\|u\|^2_{H^1_0} + C, \tag{1.3.14}$$

where we have used Poincaré's inequality. Denoting λ_1 the first eigenvalue of the operator $-\Delta$ in $H^1_0(\Omega)$, we have

$$\frac{d}{dt}\|u\|^2_{L^2} \leq -\lambda_1\|u\|^2_{L^2} + K.$$

Gronwall's lemma gives

$$\|u(t)\|^2_{L^2} \leq e^{-\lambda_1 t}\|u(0)\|^2_{L^2} + \frac{K}{\lambda_1}, \quad t \geq 0. \tag{1.3.15}$$

Integrating (1.3.14) over $(t, t+r)$ with $r > 0$ we have

$$\|u(t+r)\|_{L^2}^2 + \int_t^{t+r} \|u\|_{H_0^1}^2 ds \le \|u(t)\|_{L^2}^2 + rK$$

Then by (1.3.15),

$$\int_t^{t+r} \|u\|_{H_0^1}^2 ds \le \|u(0)\|_{L^2}^2 e^{-\lambda_1 t} + \left(\frac{1}{\lambda_1} + r\right) K. \tag{1.3.16}$$

On the other hand, multiplying (1.3.8) by $-\Delta u$ and using Young's inequality we obtain

$$\frac{d}{dt}\|u\|_{H_0^1}^2 + 2\|\Delta u\|_{L^2}^2 \le \|f_\varepsilon(u)\|_{L^2}^2 + \|\Delta u\|_{L^2}^2 \tag{1.3.17}$$

Since $f_\varepsilon(u(\cdot)) \in L^2(0,T;L^2(\Omega)), \forall T > 0$, we obtain by [11, p.189] that

$$u \in L^\infty(\eta, T; H_0^1(\Omega)),$$

$$\frac{du}{dt} \in L^2(\eta, T; L^2(\Omega)), \quad \forall\ 0 < \eta < T.$$

This regularity guarantees that the equality

$$\frac{1}{2}\frac{d}{dt}\|u\|_{H_0^1}^2 = \langle \frac{du}{dt}, -\Delta u\rangle, \text{ for a.e. } t, \tag{1.3.18}$$

is correct [77, p.102]. Then

$$\frac{d}{dt}\|u\|_{H_0^1}^2 \le \overline{K} + \|u\|_{H_0^1}^2.$$

We apply the uniform Gronwall lemma [79, p. 91] with $y(s) = \|u(s)\|_{H_0^1}^2$, $g(s) = 1$ and $w(s) = \overline{K}$. Also, using (1.3.16) we obtain

$$\|u(t+r)\|_{H_0^1}^2 \le \left(\frac{\|u(0)\|_{L^2}^2 e^{-\lambda_1 t} + (\frac{1}{\lambda_1} + r)K}{r} + \overline{K}r\right) e^r \tag{1.3.19}$$

It follows from (1.3.15) that $\|y\|_{L^2} \le \frac{K}{\lambda_1}$ for any $y \in \mathcal{A}_\varepsilon$, $0 < \varepsilon \le 1$. Hence,

$\cup_{0<\varepsilon\leq 1}\mathcal{A}_\varepsilon$ is bounded in $L^2(\Omega)$. Since $\mathcal{A}_\varepsilon \subset G_\varepsilon(t,\mathcal{A}_\varepsilon)$ for any $t\geq 0$, for any $y\in\mathcal{A}_\varepsilon$ there exists $z\in\mathcal{A}_\varepsilon$ such that $y\in G_\varepsilon(1,z)$. Then using (1.3.19) with $r=1$ and $t=0$ we obtain that

$$\|y\|^2_{H_0^1} \leq \left(\|z\|^2_{L^2} + \left(\frac{1}{\lambda_1}+1\right)K + \overline{K}\right)e,$$

so $\cup_{0<\varepsilon\leq 1}\mathcal{A}_\varepsilon$ is bounded in $H_0^1(\Omega)$. The compact embedding $H_0^1(\Omega)\subset L^2(\Omega)$ implies that $\cup_{0<\varepsilon\leq 1}\mathcal{A}_\varepsilon$ is relatively compact in $L^2(\Omega)$. As the global attractor A_0 of the differential inclusion (1.3.10) is compact, the set $\overline{\cup_{0\leq\varepsilon\leq 1}\mathcal{A}_\varepsilon}$ is compact in $L^2(\Omega)$.

In order to establish that (1.3.8) satisfies the rest of conditions given in Theorem 1.1, we need to proof two previous results related to the convergence of solutions of the approximations and the connections between fixed points.

Theorem 1.25. *If $u_{\varepsilon_n 0}\to u_0$ in $L^2(\Omega)$ as $\varepsilon_n\to 0$, then for any sequence of solutions of (1.3.8) $u_{\varepsilon_n}(\cdot)$ with $u_{\varepsilon_n}(0)=u_{\varepsilon_n 0}$ there exists a subsequence of ε_n such that u_{ε_n} converges to some strong solution u of (1.3.10) in the space $C([0,T],L^2(\Omega))$, for any $T>0$.*

Proof. We define $g_n(t)=f_{\varepsilon_n}(u_{\varepsilon_n}(t))$ and $u_n(t)=u_{\varepsilon_n}(t)$. From (1.3.15) we have that $\|u_n(t)\|_{L^2}\leq C_0$, for all $t\geq 0$, so that $\|g_n(t)\|_{L^2}\leq C_1$, for a.e. $t\geq 0$. Hence, there exists a subsequence such that $u_n\to u$ weakly in $L^2(0,T;L^2(\Omega))$. It follows from (1.3.17) and $\|g_n(t)\|_{L^2}\leq C_1$ that

$$\int_r^T \|\Delta u\|^2_{L^2}ds \leq C_1^2(T-r)+\|u_n(r)\|^2_{H_0^1}.$$

Using (1.3.19) we obtain that

$$\int_r^T \|\Delta u_n\|^2_{L^2}ds \leq C(r).$$

Hence, $\frac{du_n}{dt}$ is bounded in $L^2(r,T;L^2(\Omega))$ for any $0<r<T$, so passing to a subsequence $\frac{du_n}{dt}\to\frac{du}{dt}$ weakly in $L^2(r,T;L^2(\Omega))$.

Moreover, Ascoli-Arzelà theorem implies that for any fixed $r > 0$ we have $u_n \to u$ in $C([r,T], L^2(\Omega))$ and u is absolutely continuous on $[r,T]$.

Also, g_n converges to some $g \in L^\infty(0,T;L^2(\Omega))$ weakly star in $L^\infty(0,T;L^2(\Omega))$ and weakly in $L^2(0,T;L^2(\Omega))$. On the other hand, since $-\Delta u_n = -\frac{du_n}{dt} + g_n$, $-\Delta u_n$ converges to $l(t) = -(\frac{du}{dt}) + g$ weakly in $L^2(r,T;L^2(\Omega))$.
Hence, we find at once that u satisfies

$$\frac{du}{dt} - \Delta u(t) = g(t), \text{ a.e. on } (0,T).$$

We need to prove that $u(\cdot)$ is a strong solution of (1.3.10). Now, we show that $g(t) \in H_0(u(t))$, a.e. in $(0,T)$. For this, we shall prove first that for a.e. $x \in \Omega$ and $s \in (0,T)$

$$|g_n(s,x) - H_0(u(s,x))| \to 0, \text{ as } n \to \infty.$$

Indeed, if $u(s,x) = 0$, then

$$g_n(s,x) = f_{\varepsilon_n}(u_n(s,x)) \in [-1,1] = H_0(u(s,x)), \qquad \text{for all } n,$$

so that the result is evident. If $u(s,x) < 0$, then

$$|g_n(s,x) - H_0(u(s,x))| = |f_{\varepsilon_n}(u_n(s,x)) + 1| \to 0, \text{ as } n \to \infty.$$

Finally, if $u(s,x) > 0$, then

$$|g_n(s,x) - f_0(u(s,x))| = |f_{\varepsilon_n}(u_n(s,x)) - 1| \to 0, \text{ as } n \to \infty.$$

Now, by [81, Proposition 1.1] we have that for a.e. $t \in (0,T)$

$$g(t) \in \bigcap_{n \geq 0} \overline{co} \bigcup_{k \geq n} g_k(t).$$

Then $g(t) = \lim_{n\to\infty} y_n(t)$ strongly in $L^2(\Omega)$, where

$$y_n(t) = \sum_{i=1}^{M} \lambda_i g_{k_i}(t), \sum_{i=1}^{M} \lambda_i = 1, k_i \geq n.$$

We note that for any $t \in [0,T]$ and a.e. $x \in \Omega$ we can find $n(\varepsilon, x, t)$ such that if $k \geq n$, then $|g_k(t,x) - H_0(u(t,x))| \leq \varepsilon$. Therefore,

$$|y_n(t,x) - H_0(u(t,x))| \leq \sum_{i=1}^{M} \lambda_i |g_{k_i}(t,x) - H_0(u(t,x))| \leq \varepsilon.$$

Hence, since we can assume that

$$y_n(t,x) \to g(t,x), \qquad \text{for a.e. } (t,x) \in (0,T) \times \Omega,$$

it follows that $g(t,x) \in H_0(u(t,x))$.

It remains to check that u is continuous as $t \to 0^+$. Let $\hat{u}$ be the unique solution of

$$\begin{cases} \dfrac{du}{dt} - \Delta u = 0, \\ u|_{\partial\Omega} = 0, \\ u(0) = u_0, \end{cases}$$

and let $v_n(t) = u_n(t) - \hat{u}(t)$.
Multiplying by v_n the equation

$$\frac{dv_n}{dt} - \Delta v_n = f_{\varepsilon_n}(u_n),$$

we obtain

$$\frac{1}{2}\frac{d}{dt}\|v_n\|_{L^2}^2 + \|v_n\|_{H_0^1}^2 \leq (f_{\varepsilon_n}(u_n(t)), v_n) \leq \frac{1}{2}\|f_{\varepsilon_n}(u_n)\|_{L^2}^2 + \frac{1}{2}\|v_n\|_{L^2}^2,$$

so that

$$\|v_n(t)\|_{L^2}^2 \leq \|v_n(0)\|_{L^2}^2 + Kt.$$

Hence, $\|u(t) - \hat{u}(t)\|_{L^2}^2 = \lim_{n\to\infty} \|v_n(t)\|_{L^2}^2 \leq Kt$, for $t > 0$, and

$$\|u(t) - u_0\|_{L^2} \leq \|u(t) - \hat{u}(t)\|_{L^2} + \|\hat{u}(t) - u_0\|_{L^2} < \delta,$$

as soon as $t < \varepsilon(\delta)$. Therefore, $u(\cdot)$ is a strong solution.

Finally, if $t_n \to 0$, then

$$\begin{aligned}\|u_n(t_n) - u_0\|_{L^2} &\leq \|v_n(t_n)\|_{L^2} + \|\widehat{u}(t_n) - u_0\|_{L^2} \\ &\leq \sqrt{\|v_n(0)\|_{L^2}^2 + Kt_n} + \|\widehat{u}(t_n) - u_0\|_{L^2} \to 0.\end{aligned}$$

Hence, $u_n \to u$ in $C([0,T], L^2(\Omega))$. By a diagonal argument we obtain that the result is true for every $T > 0$. □

As a consequence of the last theorem, condition (H4) follows.

Remark 1.26. Let be $u_{\varepsilon_n}(\cdot)$ a bounded complete trajectory of (1.3.8). Fix $T > 0$. Since $\bigcup_{0<\varepsilon\leq\varepsilon_0} \mathcal{A}_\varepsilon$ is precompact in $L^2(\Omega)$, $u_{\varepsilon_n}(-T) \to y$ in L^2 up to a subsequence. Theorem 1.25 implies that u_{ε_n} converges in $C([0,T], L^2(\Omega))$ to some solution u of (1.3.10). If we choose successive subsequences for $-2T, -3T, \ldots,$ and apply the standard diagonal procedure, we obtain that a subsequence u_{ε_n} converges to a complete trajectory u of (1.3.10) in $C([-T,T], L^2(\Omega))$ for any $T > 0$. Since $\cup_{0<\varepsilon\leq 1}\mathcal{A}_\varepsilon$ is bounded in $L^2(\Omega)$ (in fact in $H_0^1(\Omega)$), it is clear that u is a bounded complete trajectory of problem (1.3.10).

Now, we need to prove a previous lemma to obtain the convergence of solutions of the approximations in the space $C([0,T], H_0^1)$.

Lemma 1.27. *Any sequence $\xi_n \in A_{\varepsilon_n}$ with $\varepsilon_n \to 0$ is relatively compact in $H_0^1(\Omega)$.*

Proof. There exists a bounded complete trajectory ψ_{ε_n} of (1.3.8) with $\psi_{\varepsilon_n}(0) = \xi_n$. Denote $u_n(\cdot) = \psi_{\varepsilon_n}(-T+\cdot)$ and choose some $T > 0$. Then $\xi_n = u_n(T)$, $u_n(0) = \psi_{\varepsilon_n}(t_0-T)$. In view of Remark 1.26 up to a subsequence $u_n \to u$ in $C([0,T], L^2(\Omega))$, where u is a strong solution of (1.3.10). On top of that, by (1.3.19) and the argument in the proof of Theorem 1.25 we obtain that for $r > 0$,

$$u_n \to u \text{ weakly star in } L^\infty(r, T; H_0^1(\Omega)),$$
$$\frac{du_n}{dt} \to \frac{du}{dt} \text{ weakly in } L^2(r, T; L^2(\Omega)),$$
$$u_n \to u \text{ weakly in } L^2(r, T; H^2(\Omega)).$$

Therefore, by the Compactness Theorem [65, p.58] we have

$$u_n \to u \text{ strongly in } L^2(r, T, H_0^1(\Omega)),$$
$$u_n(t) \to u\,(t) \ \text{ in } H_0^1(\Omega) \text{ for a.a. } t \in (r, T).$$

In addition, by standard results [77, p.102] we have that $u_n, u \in C([r, T], H_0^1(\Omega))$.

Multiplying (1.3.8) by $\frac{du_n}{dt}$ and using (1.3.18), we obtain

$$\left\| \frac{du_n}{dt} \right\|_{L^2}^2 + \frac{d}{dt} \|u_n\|_{H_0^1}^2 \leq \|f_\varepsilon(u_n)\|_{L^2}^2.$$

Thus,

$$\|u_n(t)\|_{H_0^1}^2 \leq \|u_n(s)\|_{H_0^1}^2 + C(t-s), \ C > 0, \ t \geq s \geq r.$$

The same inequality is valid for the limit function $u(\cdot)$. Hence, the functions

$$J_n(t) = \|u_n(t)\|_{H_0^1}^2 - Ct$$

and

$$J(t) = \|u(t)\|_{H_0^1}^2 - Ct,$$

are continuous and non-increasing in $[r, T]$.
Moreover, $J_n(t) \to J(t)$ for a.e. $t \in (r, T)$. Take $r < t_m < T$ such that $t_m \to T$ and $J_n(t_m) \to J(t_m)$ for all m. Then

$$J_n(T) - J(T) \leq J_n(t_m) - J(T) \leq |J_n(t_m) - J(t_m)| + |J(t_m) - J(T)|.$$

For any $\varepsilon > 0$ there exist $m(\varepsilon)$ and $N(\varepsilon)$ such that $J_n(T) - J(T) \leq \varepsilon$ if $n \geq N$. Then

$\limsup J_n(T) \leq J(T)$, so $\limsup \|u_n(T)\|^2_{H^1_0} \leq \|u(T)\|^2_{H^1_0}$. As $u_n(T) \to u(T)$ weakly in H^1_0 implies $\liminf \|u_n(T)\|^2_{H^1_0} \geq \|u(T)\|^2_{H^1_0}$, we obtain $\|u_n(T)\|^2_{H^1_0} \to \|u(T)\|^2_{H^1_0}$, so that $u_n(T) \to u(T)$ strongly in $H^1_0(\Omega)$. Hence, the result follows. □

Corollary 1.28. *If $u_{\varepsilon 0} \to u_0$ in $L^2(\Omega)$, where $u_{\varepsilon 0} \in \mathcal{A}_\varepsilon$, $u_0 \in \mathcal{A}_0$, then for any $T > 0$ there exists a subsequence ε_n such that u_{ε_n} converges to some strong solution u of (1.3.10) in $C([0,T], H^1_0(\Omega))$.*

Proof. We know from Theorem 1.25 that there exists a subsequence such that u_{ε_n} converges to some strong solution u of (1.3.10) in $C([0,T], L^2(\Omega))$. Then the statement follows from the invariance of $\mathcal{A}_\varepsilon$ and Lemma 1.27. □

Remark 1.29. Let $u_{\varepsilon_n}(\cdot)$ be a bounded complete trajectory of (1.3.8). Fix $T > 0$. By Lemma 1.27 $u_{\varepsilon_n}(-T) \to y$ in $H^1_0(\Omega)$ up to a subsequence. Corollary 1.28 implies then that u_{ε_n} converges in $C([0,T], H^1_0(\Omega))$ to some solution u of (1.3.10). If we choose successive subsequences for $-2T, -3T \ldots$ and apply the standard diagonal procedure we obtain that a subsequence u_{ε_n} converges to a complete trajectory u of (1.3.10) in $C([-T,T], H^1_0(\Omega))$ for any $T > 0$. By Remark 1.26 this trajectory is bounded.

Lemma 1.30. $dist_{H^1_0}(\mathcal{A}_\varepsilon, \mathcal{A}_0) \to 0$, *as* $\varepsilon \to 0$.

Proof. By contradiction let there exist $\delta > 0$ and a sequence $y_{\varepsilon_n} \in \mathcal{A}_{\varepsilon_n}$ such that

$$dist_{H^1_0}(y_{\varepsilon_n}, \mathcal{A}_0) > \delta.$$

Hence, as $y_{\varepsilon_n} = u_{\varepsilon_n}(0)$, where u_{ε_n} is a bounded complete trajectory of problem (1.3.8), using Remark 1.29 we obtain that up to a sequence, for every $T > 0$, u_{ε_n} converges to a bounded complete trajectory u of the problem (1.3.10) in the spaces $C([-T,T], H^1_0(\Omega))$. Thus, $u(t) \in \mathcal{A}_0$ for any $t \in \mathbb{R}$. We infer then that

$$y_{\varepsilon_n} = u_{\varepsilon_n}(0) \to u(0) \in \mathcal{A}_0,$$

which is a contradiction. □

We choose some $\delta > 0$ such that

$$\mathcal{O}_\delta(\Xi_i^0) \cap \mathcal{O}_\delta(\Xi_j^0) = \emptyset \text{ if } i \neq j$$

and Ξ_i^0 are maximal weakly invariant.

For problem (1.3.8) let us define the sets

$$M_i^\varepsilon = \{v_{\varepsilon,i}^+, v_{\varepsilon,i}^-\} \text{ for } 1 \leq i < N_0,$$

$$Z_{N_0}^\varepsilon = \left(\cup_{k \geq N_0} \{v_{\varepsilon,k}^\pm\}\right) \cup \{0\},$$

$$M_{N_0}^\varepsilon = \left\{ \begin{array}{c} y : \exists \psi \in \mathbb{K}^\varepsilon \text{ such that (1.3.11) holds with } z_j \in Z_{N_0}^\varepsilon, \\ j = 1, 2 \text{ and } y = \psi(t) \text{ for some } t \in \mathbb{R} \end{array} \right\}$$

where $\mathbb{K}^\varepsilon$ is the set of all bounded complete trajectories of (1.3.8).

In view of Lemma 1.23 we have

$$dist_{H_0^1}(M_i^\varepsilon, \Xi_i^0) \to 0, \text{ as } \varepsilon \to 0, \quad 1 \leq i < N_0$$

Lemma 1.31. *$dist_{H_0^1}(M_{N_0}^\varepsilon, \Xi_{N_0}^0) \to 0$, as $\varepsilon \to 0$.*

Proof. Suppose the opposite, that is, there exists $\delta > 0$ and a sequence $y_{\varepsilon_k} \in M_0^{\varepsilon_k}$ such that

$$dist_{H_0^1}(y_{\varepsilon_k}, \Xi_{N_0}^0) > \delta \text{ for all } k. \tag{1.3.20}$$

Let ξ_{ε_k} be a sequence of bounded complete trajectories of problem (1.3.8) such that $\xi_{\varepsilon_k}(0) = y_{\varepsilon_k}$ and

$$\xi_{\varepsilon_k}(t) \to z_{-1}^k \text{ as } t \to -\infty,$$

$$\xi_{\varepsilon_k}(t) \to z_0^k \text{ as } t \to \infty,$$

where $z_{-1}^k, z_0^k \in Z_{N_0}^{\varepsilon_k}$. By Lemmas 1.23 and 1.24, passing to a subsequence we have that

$$z_i^k \to z_i \in Z_{N_0}, i = -1, 0.$$

By Remark 1.29 we obtain that up to a subsequence ξ_{ε_k} converges to a complete trajectory ψ_0 of problem (1.3.10) in the spaces $C([-T,T],H_0^1(\Omega))$ for every $T>0$, so $y_{\varepsilon_k} \to \psi_0(0)$ in $H_0^1(\Omega)$. Thus, either ψ_0 is equal to a fixed point $\overline{z}_0 \neq 0$ or there exist two fixed points of problem (1.3.10), denoted by $\overline{z}_{-1}, \overline{z}_0$ such that

$$E(\overline{z}_{-1}) > E(\overline{z}_0),$$

$$\psi_0(t) \to \overline{z}_{-1} \text{ as } t \to -\infty,$$

$$\psi_0(t) \to \overline{z}_0 \text{ as } t \to \infty.$$

If $\overline{z}_0 = z_0$, then $\overline{z}_{-1}, \overline{z}_0 \in Z_{N_0}$, which means that $\psi_0(0) \in \Xi^0_{N_0}$. This would imply a contradiction with (1.3.20). Therefore, we assume that $\overline{z}_0 \neq z_0$. Also, it is clear that $\overline{z}_0 = v_m^{\pm} \neq 0$, for some $m \in \mathbb{N}$.

Let $r_0 > 0$ be such that $\mathcal{O}_{r_0}(\overline{z}_0) \cap \mathcal{O}_{r_0}(z_0) \neq \emptyset$ and $\mathcal{O}_{2r_0}(\overline{z}_0)$ does not contain any other fixed point of problem (1.3.10). The previous convergences imply that for each $r \leq r_0$ there exist a moment of time t_r and k_r such that $\xi_{\varepsilon_k}(t_r) \in \mathcal{O}_r(\overline{z}_0)$ for all $k \geq k_r$. On the other hand, since $\xi_{\varepsilon_k}(t) \to z_0^k$, as $t \to \infty$, and $z_0^k \to z_0$, there exists $t_r' > t_r$ such that

$$\xi_{\varepsilon_{k_r}}(t) \in \mathcal{O}_{r_0}(\overline{z}_0) \text{ for all } t \in [t_r, t_r'),$$

$$\|\xi_{\varepsilon_{k_r}}(t_r') - \overline{z}_0\|_{L^2} = r_0.$$

Let us consider two cases:

1. $t_r' - t_r \to \infty$,
2. $|t_r' - t_r| \leq C$.

We begin with the first case. We define the sequence of bounded complete trajectories of problem (1.3.8) given by

$$\xi^1_{k_r}(t) = \xi_{\varepsilon_{k_r}}(t + t_r').$$

By Remark 1.29 we can extract a subsequence of this sequence converging to a bounded complete trajectory ψ_1 of problem (1.3.10). Since $t_r' - t_r \to \infty$, we obtain that $\psi_1(t) \in \mathcal{O}_{r_0}(\overline{z}_0)$ for all $t \leq 0$. Since $\mathcal{O}_{2r_0}(\overline{z}_0)$ does not contain any other fixed point of problem (1.3.10), it follows that

$$\psi_1(t) \to \overline{z}_0, \qquad \text{as } t \to -\infty.$$

But $\|\psi_1(0) - \overline{z}_0\|_{L^2} = r_0$, so ψ_1 is not a fixed point. Therefore, $\psi_1(t) \to \overline{z}_1$ as $t \to \infty$, where $\overline{z}_1$ is a fixed point such that $E(\overline{z}_1) < E(\overline{z}_0)$.

In the second case we define the sequence

$$\xi_{k_r}^1(t) = \xi_{\varepsilon_{k_r}}(t + t_r).$$

Passing to a subsequence we have that

$$\xi_{k_r}^1(0) \to \overline{z}_0,$$

$$t_r' - t_r \to t'.$$

As $\xi_{k_r}^1$ converges to a solution ξ^1 of problem (1.3.10) uniformly in bounded subsets from $[0, \infty)$ such that $\xi^1(0) = \overline{z}_0$, $\xi_{k_r}^1(t_r' - t_r) \to \xi^1(t')$, so that

$$\|\xi^1(t') - \overline{z}_0\|_{L^2} = r_0.$$

We put

$$\psi_1(t) = \begin{cases} \overline{z}_0 \text{ if } t \leq 0, \\ \xi^1(t) \text{ if } t \geq 0. \end{cases}$$

Then ψ_1 is a bounded complete trajectory of problem (1.3.10) such that $\psi_1(t) \to \overline{z}_1$ as $t \to \infty$, where $\overline{z}_1$ is a fixed point satisfying $E(\overline{z}_1) < E(\overline{z}_0)$.

Now, if $\overline{z}_1 = z_0$, then we have the chain of connections

$$\psi_0(t) \to \overline{z}_{-1} \text{ as } t \to -\infty, \psi_0(t) \to \overline{z}_0 \text{ as } t \to +\infty,$$

$$\psi_1(t) \to \overline{z}_0 \text{ as } t \to -\infty, \psi_1(t) \to \overline{z}_1 \text{ as } t \to +\infty,$$

which implies that $\overline{z}_{-1}, \overline{z}_0, \overline{z}_1 \in Z_n$, an then $\psi_0(0) \in \Xi_n^0$. This would imply a contradiction with (1.3.20).

However, if $\overline{z}_1 \neq \overline{z}_0$, then we proceed in the same way and obtain a new connection from the point $\overline{z}_1$ to another fixed point with less energy. Since the number of fixed points with energy less than or equal to $E(\overline{z}_0)$ is finite, we will finally obtain a chain of connections of the form

$$\begin{aligned}
&\psi_0(t) \to \overline{z}_{-1} \text{ as } t \to -\infty,\ \psi_0(t) \to \overline{z}_0 \text{ as } t \to +\infty,\\
&\psi_1(t) \to \overline{z}_0 \text{ as } t \to -\infty,\ \psi_1(t) \to \overline{z}_1 \text{ as } t \to +\infty,\\
&\qquad \vdots\\
&\psi_n(t) \to \overline{z}_{m-1} \text{ as } t \to -\infty,\ \psi_n(t) \to \overline{z}_m = z_0 \text{ as } t \to +\infty.
\end{aligned}$$

And again, this implies a contradiction with (1.3.20). □

These convergences imply the existence of ε_0 such that if $\varepsilon \leq \varepsilon_0$, then

$$M_i^\varepsilon \subset \mathcal{O}_\delta(\Xi_i^0) \text{ for any } 1 \leq i \leq N_0.$$

Further, let

$$\Xi_i^\varepsilon = \left\{ \begin{array}{c} y : \exists \psi \in \mathbb{K}^\varepsilon \text{ such that } \psi(0) = y \\ \text{and } \psi(t) \in \mathcal{O}_\delta(\Xi_i^0) \text{ for all } t \in \mathbb{R} \end{array} \right\}.$$

These sets are clearly maximal weakly invariant for G_ε in $\mathcal{O}_\delta(\Xi_i^0)$, so condition (H5) is satisfied for $V_i = \mathcal{O}_\delta(\Xi_i^0)$. As a consequence of Lemmas 1.23, 1.31, Remark 1.26 and the definition of δ we have

$$dist_{L^2}(\Xi_i^\varepsilon, \Xi_i^0) \to 0, \text{ as } \varepsilon \to 0, \text{ for } 1 \leq i \leq N_0.$$

Therefore, condition (H3) is satisfied.

We also get by Remark 1.29 and the definition of δ that

$$dist_{H_0^1}(\Xi_i^\varepsilon, \Xi_i^0) \to 0, \text{ as } \varepsilon \to 0, \text{ for } 1 \leq i \leq N_0.$$

Moreover, $\mathcal{M}^{\varepsilon} = \{\Xi_1^{\varepsilon}, \dots, \Xi_{N_0}^{\varepsilon}\}$ is a disjoint family of isolated weakly invariant sets.

Applying Theorem 1.1 we obtain the following result.

Theorem 1.32. *There exists $\varepsilon_1 > 0$ such that for all $0 < \varepsilon \leq \varepsilon_1$ the multivalued semiflow G_{ε} is dynamically gradient with respect to the family $\mathcal{M}^{\varepsilon}$.*

Chapter 2

Existence and characterization of attractors for a nonlocal reaction-diffusion equation with an energy functional

Once robustness of multivalued semiflows is analyzed, we focus now on nonlocal reaction-diffusion equation in which the diffusion depends on the gradient of the solution.

Firstly, we prove the existence and uniqueness of regular and strong solutions. Thereupon, we obtain the existence of global attractors in both situations under rather weak assumptions by defining a multivalued semiflow.

Secondly, we characterize the attractor either as the unstable manifold of the set of stationary points or as the stable one when we consider solutions only in the set of bounded complete trajectories.

2.1. Existence of solutions

We consider the following nonlocal reaction-diffusion equation

$$\begin{cases} u_t - a(\|u\|^2_{H_0^1})\Delta u = f(u) + h(t), \text{ in } \Omega \times (0, \infty), \\ u = 0 \quad \text{on } \partial\Omega \times (0, \infty), \\ u(0, x) = u_0(x) \quad \text{in } \Omega, \end{cases} \tag{2.1.1}$$

where Ω is a bounded open set of $\mathbb{R}^n$ with smooth boundary $\partial\Omega$.

Let us consider the following conditions on the functions a, f, h :

$$h \in L^2(0, T; L^2(\Omega)) \; \forall T > 0, \tag{2.1.2}$$

$$a \in C(\mathbb{R}^+), \; f \in C(\mathbb{R}), \tag{2.1.3}$$

$$a\,(s) \geq m > 0, \tag{2.1.4}$$

$$-\kappa - \alpha_2 |s|^p \leq f(s)s \leq \kappa - \alpha_1 |s|^p, \tag{2.1.5}$$

where $m, \; \alpha_1, \; \alpha_2 > 0$ and $\kappa \geq 0$, $p \geq 2$. Observe that then there exists $C > 0$ such that

$$|f(s)| \leq C(1 + |s|^{p-1}) \quad \forall s \in \mathbb{R}, \tag{2.1.6}$$

and that the function $\mathcal{F}(s) := \int_0^s f(r)dr$ satisfies

$$-\widetilde{\alpha}_2 |s|^p - \widetilde{\kappa} \leq \mathcal{F}(s) \leq \widetilde{\kappa} - \widetilde{\alpha}_1 |s|^p \tag{2.1.7}$$

for certain positive constants $\widetilde{\alpha}_i$, $i = 1, 2$, and $\widetilde{\kappa} \geq 0$, and

$$|\mathcal{F}(s)| \leq \widetilde{C}(1 + |s|^p) \quad \forall s \in \mathbb{R}. \tag{2.1.8}$$

Conditions (2.1.2)-(2.1.5) will be always assumed throughout the chapter. Sometimes, some of the following additional assumptions will also be used:

$$f \in C^1(\mathbb{R}) \text{ be such that } f'(s) \leq \eta,\ \forall s \in \mathbb{R}, \tag{2.1.9}$$

$$p \leq \frac{2n-2}{n-2},\ \text{if } n \geq 3, \tag{2.1.10}$$

$$a\,(s) \leq M_1 + M_2 s,\ \forall s \geq 0, \tag{2.1.11}$$

$$s \mapsto a(s^2)s \text{ is non-decreasing}, \tag{2.1.12}$$

$$a\,(\cdot) \in C^1\left(\mathbb{R}^+\right) \text{ and } a'\,(s) \geq 0,\ \forall s \geq 0, \tag{2.1.13}$$

for some constants $M_1, M_2, \eta \geq 0$.

Remark 2.1. $a'\,(s) \geq 0$ implies that (2.1.12) holds, so condition (2.1.13) is stronger than (2.1.12). Assumption (2.1.12) is used to prove uniqueness of solutions. Assumption (2.1.13) is used to obtain the $H^2\,(\Omega)$ regularity of the global attractor.

Definition 2.2. A weak solution to (2.1.1) is a function $u\,(\cdot)$ such that $u \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega)) \cap L^p(0,T;L^p(\Omega))$ for any $T > 0$ and satisfies in the sense of scalar distributions the equality

$$\frac{d}{dt}(u,v) + a(\|u(t)\|^2_{H_0^1})(\nabla u(t), \nabla v) = (f(u(t)),v) + (h(t),v) \tag{2.1.14}$$

for all $v \in H_0^1(\Omega) \cap L^p(\Omega)$.

We need to guarantee that the initial condition of the problem makes sense for a weak solution. This can be achieved in a standard way assuming that the function a has an upper bound, that is, there exists $M > 0$ such that

$$a\,(s) \leq M \text{ for all } s \geq 0. \tag{2.1.15}$$

Indeed, if u is a weak solution to (2.1.1), taking into account (2.1.6) and (2.1.15) it follows that

$$u_t = a(\|u\|^2_{H_0^1})\Delta u + f(u) + h \in L^2(0,T;H^{-1}(\Omega)) + L^q(0,T;L^q(\Omega)). \quad (2.1.16)$$

Therefore, by [34, p.33] $u \in C([0,T], L^2(\Omega))$ and the initial condition makes sense when $u_0 \in L^2(\Omega)$.

For the operator $A = -\Delta$, thanks to the assumptions on the domain Ω, it is well known that $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$ [75, Proposition 6.19].

Definition 2.3. A regular solution to (2.1.1) is a weak solution with the extra regularity $u \in L^\infty(\varepsilon,T;H_0^1(\Omega))$ and $u \in L^2(\varepsilon,T;D(A))$ for any $0 < \varepsilon < T$.

Remark 2.4. Since $\dfrac{du}{dt} \in L^q(\varepsilon,T;L^q(\Omega))$ for any regular solution, in this case equality (2.1.14) is equivalent to the following one:

$$\begin{aligned} &\int_\varepsilon^T \int_\Omega \frac{du(t,x)}{dt}\xi(t,x)\,dxdt - \int_\varepsilon^T a(\|u(t)\|^2_{H_0^1})\int_\Omega \Delta u \xi dxdt \\ &= \int_\varepsilon^T \int_\Omega f(u(t,x))\,\xi(t,x)\,dxdt + \int_\varepsilon^T \int_\Omega h(t,x)\,\xi(t,x)\,dxdt, \end{aligned} \quad (2.1.17)$$

for all $0 < \varepsilon < T$ and $\xi \in L^p(0,T;L^p(\Omega))$.

Lemma 2.5. *Let $u \in L^p(\varepsilon,T;X)$, $\dfrac{du}{dt} \in L^q(\varepsilon,T;X')$ for all $0 < \varepsilon < T$, where X is a reflexive and separable Banach space and X' denotes its dual space. Assume that $\beta \in C(\mathbb{R}^+)$ is such that $\beta \in W^{1,\infty}(\varepsilon,T;[\beta(\varepsilon),\beta(T)])$ and $0 < \beta(\varepsilon) < \beta(T)$ for all $0 < \varepsilon < T$. Then $w(\cdot) = u(\beta(\cdot)) \in L^p(\varepsilon,T;X)$, $\dfrac{dw}{dt} \in L^q(\varepsilon,T;X')$, for all $0 < \varepsilon < T$, and*

$$\frac{dw}{dt}(t) = \frac{du}{dt}(\beta(t))\frac{d\beta}{dt}(t) \text{ for a.a. } t > 0. \quad (2.1.18)$$

Proof. We fix $0 < \varepsilon < T$. There exists a sequence $u_n \in C^1([\beta(\varepsilon),\beta(T)],X)$ such that $u_n \to u$ in $L^p(\beta(\varepsilon),\beta(T);X)$ and $\dfrac{du_n}{dt} \to \dfrac{du}{dt}$ in $L^q(\beta(\varepsilon),\beta(T);X')$ [49, Chapter IV]. We define $w_n(t) = u_n(\beta(t))$.

Following the same proof of [13, Corollary VIII.10] we obtain that

$$w_n(\cdot) \in W^{1,\infty}(\varepsilon, T; X)$$

and

$$\frac{dw_n}{dt}(t) = \frac{du_n}{dt}(\beta(t))\frac{d\beta}{dt}(t) \text{ for a.a. } t > 0.$$

It is clear that $w_n \to w$ in $L^p(\varepsilon, T; X)$ and $\frac{du_n}{dt}(\beta(\cdot)) \to \frac{du}{dt}(\beta(\cdot))$ in $L^q(\varepsilon, T; X')$. Passing to the limit we obtain that

$$\frac{dw}{dt}(\cdot) = \frac{du}{dt}(\beta(\cdot))\frac{d\beta}{dt}(\cdot)$$

in the sense of distributions $\mathcal{D}'(0, +\infty; X)$.
As $\frac{du}{dt}(\beta(\cdot))\frac{d\beta}{dt}(\cdot) \in L^q(\varepsilon, T; X')$, $\frac{dw}{dt} \in L^q(\varepsilon, T; X')$ and (2.1.18) holds true. □

We would like to avoid a being uniformly bounded by above (i.e. to relax assumption (2.1.15)). We can still prove the continuity in $L^2(\Omega)$ of u for regular solutions by assuming that a has at most linear growth.

Lemma 2.6. *Assume that conditions (2.1.2)-(2.1.5), (2.1.11) hold. Then any regular solution satisfies that $u \in C([0,T], L^2(\Omega))$ for all $T > 0$. Moreover, $w(t) = u(\alpha^{-1}(t))$, where $\alpha(t) = \int_0^t a(\|u(s)\|^2_{H^1_0})ds$, is a regular solution to the problem*

$$\begin{cases} w_t - \Delta w = \dfrac{f(w) + h(\alpha^{-1}(t))}{a(\|w\|^2_{H^1_0})}, & \text{in } \Omega \times (0,\infty), \\ w = 0 \quad \text{on } \partial\Omega \times (0,\infty), \\ w(0,x) = u_0(x) \quad \text{in } \Omega. \end{cases} \tag{2.1.19}$$

Proof. Condition (2.1.11) guarantees that if $u \in L^2(0,T; H^1_0(\Omega))$, then

$$a(\|u(\cdot)\|^2_{H^1_0}) \in L^1(0,T).$$

We make the following time rescaling

$$u(t,x) = w(\alpha(t), x).$$

As $a(\|u(\cdot)\|^2_{H^1_0}) \in L^1(0,T)$, the function $t \mapsto \alpha(t)$ is continuous and $\beta(\cdot) = \alpha^{-1}(\cdot)$ satisfies the conditions of Lemma 2.5. It is clear that the function

$$w(t,x) = u(\alpha^{-1}(t), x)$$

belongs to the space $L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1_0(\Omega)) \cap L^p(0,T;L^p(\Omega))$ and also to the spaces $L^\infty(\varepsilon,T;H^1_0(\Omega))$ and $L^2(\varepsilon,T;D(A))$ for any $0 < \varepsilon < T$.
Moreover, $\dfrac{du}{dt} \in L^q(\varepsilon,T;L^q(\Omega))$ and Lemma 2.5 give

$$\frac{dw}{dt} \in L^q(\varepsilon,T;L^q(\Omega))$$

and

$$\frac{dw}{dt}(t) = \frac{du}{dt}\left(\alpha^{-1}(t)\right)\frac{d}{dt}\alpha^{-1}(t) = \frac{du}{dt}\left(\alpha^{-1}(t)\right)\frac{1}{a\left(\|w(t))\|^2_{H^1_0}\right)}, \text{ for a.a. } t. \tag{2.1.20}$$

Equality (2.1.17) implies that

$$\frac{du}{dt}\left(\alpha^{-1}(t)\right) - a\left(\|u(\alpha^{-1}(t))\|^2_{H^1_0}\right)\Delta u\left(\alpha^{-1}(t)\right) = f\left(u\left(\alpha^{-1}(t)\right)\right) + h(\alpha^{-1}(t)),$$

for a.a. $t > 0$, so (2.1.20) gives

$$\frac{dw}{dt}(t) - \Delta w(t) = \frac{f(w(t))}{a(\|w(t)\|^2_{H^1_0})} + \frac{h(\alpha^{-1}(t))}{a(\|w(t)\|^2_{H^1_0})} \text{ for a.a. } t > 0.$$

Hence, w is a regular solution to problem (2.1.4119).
Since $0 < \dfrac{1}{a(s)} \leq \dfrac{1}{m}$, we obtain that

$$\frac{dw}{dt} \in L^2(0,T;H^{-1}(\Omega)) + L^q(0,T;L^q(\Omega)).$$

Therefore, $w \in C([0,T], L^2(\Omega))$, so that

$$u \in C([0,T], L^2(\Omega)).$$

$\square$

Remark 2.7. Under assumptions (2.1.2)-(2.1.5) any regular solution $u(\cdot)$ satisfies that $\frac{du}{dt} \in L^q(\varepsilon, T; L^q(\Omega))$ for all $0 < \varepsilon < T$. Then by [34, p.33] $u \in C([\varepsilon, T], L^2(\Omega))$, $t \mapsto \|u(t)\|^2$ is absolutely continuous on $[\varepsilon, T]$ and

$$\frac{d}{dt}\|u(t)\|_{L^2}^2 = 2\left(\frac{du}{dt}, u\right) \text{ for a.a. } t > \varepsilon.$$

If the initial condition belongs to $H_0^1(\Omega) \cap L^p(\Omega)$, we can define strong solutions as well.

Definition 2.8. A strong solution to (2.1.1) is a weak solution with the extra regularity $u \in L^\infty(0, T; H_0^1(\Omega) \cap L^p(\Omega))$, $u \in L^2(0, T; D(A))$ and $\frac{du}{dt} \in L^2(0, T; L^2(\Omega))$ for any $T > 0$.

We observe that if u is a strong solution, then $u \in C([0, T], H_0^1(\Omega))$ (see [77, p.102]). Also, $u \in L^\infty(0, T; L^p(\Omega))$ and $u \in C([0, T], L^2(\Omega))$ imply that $u \in C_w([0, T], L^p(\Omega))$ (see [79, p.263]). Thus, an initial condition in $H_0^1(\Omega) \cap L^p(\Omega)$ makes sense. Also, the equality $f(u) = u_t - a\left(\|u\|_{H_0^1}^2\right)\Delta u - h$ implies that $f(u) \in L^2(0, T; L^2(\Omega))$

In addition, if u is a regular solution such that $\frac{du}{dt} \in L^2(\varepsilon, T; L^2(\Omega))$ for all $0 < \varepsilon < T$, then $u \in C((0, T], H_0^1(\Omega))$.

The phase space for regular solutions will be $L^2(\Omega)$, whereas for strong solutions we will use the space $H^1(\Omega) \cap L^p(\Omega)$ (or just $H_0^1(\Omega)$ when $H_0^1(\Omega) \subset L^p(\Omega)$).

The following results will be proved in Theorems 2.9, 2.10, 2.11, 2.12, 2.14:

- Conditions (2.1.2)-(2.1.5), (2.1.9), (2.1.11) imply the existence of at least one regular solution for any $u_0 \in L^2(\Omega)$. If, in addition, (2.1.12) holds, then it is the unique regular solution.
- Conditions (2.1.2)-(2.1.5), (2.1.9) imply the existence of at least one strong solution for any $u_0 \in H_0^1(\Omega) \cap L^p(\Omega)$. If, in addition, (2.1.12) holds, then it is the unique strong solution.
- Conditions (2.1.2)-(2.1.5), (2.1.10) imply the existence of at least one strong solution for any $u_0 \in H_0^1(\Omega)$.

- Conditions (2.1.2)-(2.1.5), (2.1.10), (2.1.15) imply the existence of at least one regular solution for any $u_0 \in L^2(\Omega)$.

To start with we prove the existence of regular solutions for initial conditions in $L^2(\Omega)$.

Theorem 2.9. *Assume that conditions (2.1.2)-(2.1.5), (2.1.9) and (2.1.11) hold. Then, for any $u_0 \in L^2(\Omega)$ there exists at least one regular solution to (2.1.1).*

Proof. We will prove the result by compactness and using Faedo-Galerkin approximations.

Consider a fixed value $T > 0$. Let $\{w_j\}_{j\geq 1}$ be a sequence of eigenfunctions of $-\Delta$ in $H_0^1(\Omega)$ with homogeneous Dirichlet boundary conditions, which forms a special basis of $L^2(\Omega)$.

We need to ensure that the eigenfunctions are elements of $L^p(\Omega)$. Indeed, by the Sobolev embedding theorem, we have

$$H^s(\Omega) \subset L^p(\Omega) \quad \text{for } s \geq n(p-2)/2p.$$

Taking $A = -\Delta$, we define the domain of a fractional power of A as

$$D(A^{s/2}) = \{u \in L^2(\Omega) : \sum_{j=1}^{\infty} \lambda_j^s (u, w_j)^2 < \infty\},$$

where λ_j is the eigenvalue associated to w_j. Since the $w_j's$ are orthonormal, $\{w_j\} \in D(A^{s/2})$. If we assume Ω to be a bounded C^s domain (smoothness condition on the domain), by Theorem 6.18 in [75], we have that $D(A^{s/2}) \subset H^s(\Omega)$ and so $\{w_j\} \in L^p(\Omega)$.

Therefore, we can consider $\{w_j\} \subset H_0^1(\Omega) \cap L^p(\Omega)$ a basis of $L^2(\Omega)$, with $s \geq \max\{n(p-2)/2p, 1\}$. By this way,

$$H_0^s(\Omega) \subset H_0^1(\Omega) \cap L^p(\Omega)$$

and the set $\cup_{n\in\mathbb{N}} V_n$ is dense in $L^2(\Omega)$ and also in $H_0^1(\Omega) \cap L^p(\Omega)$ [62], where $V_n = span[w_1, \ldots, w_n]$.

As usual, P_n will be the orthogonal projection in $L^2(\Omega)$, that is

$$z_n := P_n z = \sum_{j=1}^{n} (z, w_j) w_j, \quad \forall z \in L^2(\Omega)$$

and λ_j will be the eigenvalues associated to the egienfunctions w_j.
For each integer $n \geq 1$, we consider the Galerkin approximations

$$u_n(t) = \sum_{j=1}^{n} \gamma_{nj}(t) w_j,$$

which satisfy the following nonlinear ODE system

$$\begin{cases} \dfrac{d}{dt}(u_n, w_i) + a(\|u_n\|^2_{H_0^1})(\nabla u_n, \nabla w_i) = (f(u_n), w_i) + (h, w_i) \quad \forall i = 1, \dots, n, \\ u_n(0) = P_n u_0. \end{cases} \tag{2.1.21}$$

where $P_n u_0 \to u_0$ in $L^2(\Omega)$.

Using the fact that the $w_j's$ are orthonormal, we obtain that (2.1.21) is equivalent to the Cauchy problem

$$\begin{aligned} \frac{du_{n_j}}{dt} &= -a(\|u_n\|^2_{H_0^1})\lambda_j u_{n_j} + (f(u_n), w_j) + (h(t), w_j), \\ (u_n(0), w_j) &= (u_0, w_j), \quad j = 1, \dots, n, \end{aligned} \tag{2.1.22}$$

where λ_j is the eigenvalue associated to the eigenfunction w_j and the vector $(u_{n_1}, \dots, u_{n_n})$ is the unknown.

Since the right hand side of (2.1.22) is continuous in $u_n(t)$ this Cauchy problem possesses a solution on some interval $[0, t_n), 0 < t_n < T$ [75, cf. p. 51].

We claim that for any $T > 0$ such a solution can be extended to the whole interval $[0, T]$, which follows from a priori estimates in the space $L^2(\Omega)$ of the sequence $\{u_n\}$.

Multiplying by $\gamma_{ni}(t)$ and summing from $i = 1$ to n, we obtain

$$\frac{1}{2}\frac{d}{dt}\|u_n(t)\|^2_{L^2} + a(\|u_n\|^2_{H_0^1})\|u_n(t)\|^2_{H_0^1} = (f(u_n(t)), u_n(t)) + (h, u_n(t)) \tag{2.1.23}$$

for a.e. $t \in (0, t_n)$.
Using (2.1.5) and the Young and Poincaré inequalities we deduce that

$$(f(u_n(t)), u_n(t)) \leq \kappa|\Omega| - \alpha_1 \|u_n(t)\|_{L^p}^p,$$

$$(h(t), u_n(t)) \leq \frac{m}{2}\|u_n(t)\|_{H_0^1}^2 + \frac{1}{2\lambda_1 m}\|h(t)\|_{L^2}^2.$$

Hence, from (2.1.23) it follows that

$$\frac{1}{2}\frac{d}{dt}\|u_n(t)\|_{L^2}^2 + \frac{m}{2}\|u_n(t)\|_{H_0^1}^2 + \alpha_1\|u_n(t)\|_{L^p}^p \leq \kappa|\Omega| + \frac{1}{2\lambda_1 m}\|h(t)\|_{L^2}^2 \tag{2.1.24}$$

for $a.e.$ $t \in (0, t_n)$.
Then, integrating (2.1.24) from 0 to $t \in (0, t_n)$ we deduce

$$\begin{aligned} \frac{1}{2}\|u_n(t)\|_{L^2}^2 + \frac{m}{2}\int_0^t \|u_n(s)\|_{H_0^1}^2 ds + \alpha_1 \int_0^t \|u_n(s)\|_{L^p}^p ds \\ \leq \kappa|\Omega|t + \frac{1}{2\lambda_1 m}\int_0^t \|h(s)\|_{L^2}^2 ds + \frac{1}{2}\|u_n(0)\|_{L^2}^2 \\ \leq TK_2 + K_3(T) + \frac{1}{2}\|u_n(0)\|_{L^2}^2. \end{aligned} \tag{2.1.25}$$

Therefore, the sequence $\{u_n\}$ is well defined and bounded in $L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega)) \cap L^p(0,T;L^p(\Omega))$. Also, $\{-\Delta u_n\}$ is bounded in $L^2(0,T;H^{-1}(\Omega))$.

On the other hand, by (2.1.6) it follows that

$$\int_0^T \int_\Omega |f(u(x,t))|^q dxdt \leq 2^{q-1}(C_1^q|\Omega|T + C_2^q \int_0^T \|u(t)\|_{L^p}^p dt),$$

with $\frac{1}{p} + \frac{1}{q} = 1$. Hence, since $\{u_n\}$ bounded in $L^p(0,T;L^p(\Omega))$, $f(u_n)$ is bounded in $L^q(0,T;L^q(\Omega))$.
On the other hand, multiplying (2.1.21) by $\lambda_i\gamma_{ni}(t)$ and summing from $i = 1$ to n, we obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|u_n\|_{H_0^1}^2 + m\|\Delta u_n\|_{L^2}^2 &\leq (f(u_n), -\Delta u_n) + (h(t), -\Delta u_n) \\ &\leq \eta\|u_n\|_{H_0^1}^2 + \frac{1}{2m}\|h(t)\|_{L^2}^2 + \frac{m}{2}\|\Delta u_n\|_{L^2}^2. \end{aligned}$$

In this estimate we have assumed that $f(0) = 0$ when integrating by parts. This can be done without loss of generality because $f(u) + h(t) = f(u) - f(0) + \widetilde{h}(t) = \widetilde{f}(u) + \widetilde{h}(t)$, and $\widetilde{f}, \widetilde{h}$ satisfy the same conditions as f, h.

Integrating the previous expression between s and t, with $0 < s \leq t \leq T$, and using (2.1.9) we have

$$\begin{aligned}&\frac{1}{2}\|u_n(t)\|^2_{H^1_0} + \frac{m}{2}\int_s^t \|\Delta u_n(r)\|^2_{L^2} dr \\ &\leq \eta \int_0^T \|u_n(r)\|^2_{H^1_0} dr + \frac{1}{2}\|u_n(s)\|^2_{H^1_0} + \frac{1}{2m}\int_s^t \|h(r)\|^2_{L^2} dr.\end{aligned} \tag{2.1.26}$$

Now, integrating in s between 0 and t, it follows that

$$t\|u_n(t)\|^2_{H^1_0} \leq (2\eta T + 1)\int_0^T \|u_n(r)\|^2_{H^1_0} dr + K_3(T)T.$$

Hence,

$$\|u_n(t)\|^2_{H^1_0} \leq \frac{2\eta T + 1}{\varepsilon}\int_0^T \|u_n(r)\|^2_{H^1_0} dr + \frac{K_3(T)T}{\varepsilon}. \tag{2.1.27}$$

for all $t \in [\varepsilon, T]$ with $\varepsilon \in (0, T)$.
From the last inequality and (2.1.25) we deduce that

$$\{\|u_n(t)\|_{H^1_0}\} \text{ is uniformly bounded in } [\varepsilon, T]$$

and by the continuity of the function a we get that

$$\{a(\|u_n(t)\|^2_{H^1_0})\} \text{ is bounded in } [\varepsilon, T].$$

Also, it follows that

$$\{u_n\} \text{ is bounded in } L^\infty(\varepsilon, T; H^1_0(\Omega)). \tag{2.1.28}$$

On the other hand, taking $s=\varepsilon$ and $t=T$ in (2.1.26), by (2.1.25) we obtain that

$$\{u_n\} \text{ is bounded in } L^2(\varepsilon,T;D(A)), \tag{2.1.29}$$

so $\{-\Delta u_n\}$ and $\{a(\|u_n\|^2_{H_0^1})\Delta u_n\}$ are bounded in $L^2(\varepsilon,T;L^2(\Omega))$.
Thus,

$$\left\{\frac{du_n}{dt}\right\} \text{ is bounded in } L^q(\varepsilon,T;L^q(\Omega)). \tag{2.1.30}$$

Therefore, there exists $u\in L^\infty(\varepsilon,T;H_0^1(\Omega))\cap L^2(0,T;H_0^1(\Omega))\cap L^\infty(0,T;L^2(\Omega))\cap L^2(\varepsilon,T;D(A))\cap L^p(0,T;L^p(\Omega))$ such that $\dfrac{du}{dt}\in L^q(\varepsilon,T;L^q(\Omega))$ and a subsequence $\{u_n\}$, relabelled the same, such that

$$\begin{aligned}
u_n &\overset{*}{\rightharpoonup} u \text{ in } L^\infty(\varepsilon,T;H_0^1(\Omega)),\\
u_n &\overset{*}{\rightharpoonup} u \text{ in } L^\infty(0,T;L^2(\Omega)),\\
u_n &\rightharpoonup u \text{ in } L^2(0,T;H_0^1(\Omega)),\\
u_n &\rightharpoonup u \text{ in } L^p(0,T;L^p(\Omega)),\\
u_n &\rightharpoonup u \text{ in } L^2(\varepsilon,T;D(A)),\\
\frac{du_n}{dt} &\rightharpoonup \frac{du}{dt} \text{ in } L^q(\varepsilon,T;L^q(\Omega)),\\
f(u_n) &\rightharpoonup \chi \text{ in } L^q(0,T;L^q(\Omega)),\\
a(\|u_n\|^2_{H_0^1}) &\overset{*}{\rightharpoonup} b \text{ in } L^\infty(\varepsilon,T),
\end{aligned} \tag{2.1.31}$$

for any $0<\varepsilon<T$, where $\rightharpoonup$ means weak convergence and $\overset{*}{\rightharpoonup}$ weak star convergence.

Moreover, by (2.1.29)-(2.1.30) the Aubin-Lions Compactness Lemma gives that $u_n\to u$ in $L^2(\varepsilon,T;H_0^1(\Omega))$, so $u_n(t)\to u(t)$ in $H_0^1(\Omega)$ a.e. on (ε,T) for any $\varepsilon>0$. Consequently, there exists a subsequence $\{u_n\}$, relabelled the same, such that $u_n(t,x)\to u(t,x)$ a.e. in $\Omega\times(0,T)$ [75, Corollary 1.12]. Also, we know that $P_nf(u_n)\rightharpoonup\chi$ (see [75, p.224]). Since f is continuous, it follows that $f(u_n(t,x))\to f(u(t,x))$ a.e. in $\Omega\times(0,T)$. Therefore, in view of (2.1.31), by [65, Lemma 1.3] we have that $\chi=f(u)$.

As a consequence, by the continuity of a, we get that

$$a(\|u_n(t)\|^2_{H^1_0}) \to a(\|u(t)\|^2_{H^1_0}) \quad \text{a.e. on } (\varepsilon, T).$$

Since the sequence is bounded, by the Lebesgue theorem this convergence takes place in $L^2(\varepsilon, T)$ and $b = a(\|u\|^2_{H^1_0})$ on (ε, T).
Thus,

$$a(\|u_n\|^2_{H^1_0})\Delta u_n \rightharpoonup a(\|u\|^2_{H^1_0})\Delta u, \quad \text{in } L^2(\varepsilon, T; L^2(\Omega)). \tag{2.1.32}$$

Finally, since $\{w_i\}$ is dense in $H^1_0(\Omega) \cap L^p(\Omega)$, in view of (2.1.31) and (2.1.32), we can pass to the limit in (2.1.21) and conclude that (2.1.14) holds for all $v \in H^1_0(\Omega) \cap L^p(\Omega)$.

To conclude the proof, we have to check that $u(0) = u_0$. Indeed, let be

$$\phi \in C^1([0, T]); H^1_0(\Omega) \cap L^p(\Omega)),$$

with $\phi(T) = 0$, $\phi(0) \neq 0$.
We consider the functions $w\left(t\right) = u(\alpha^{-1}\left(t\right))$, $w_n\left(t\right) = u_n\left(\alpha_n^{-1}\left(t\right)\right)$ (here $\alpha_n(t) = \int_0^t a(\|u_n\left(r\right)\|^2_{H^1_0} dr)$), which by Lemma 2.6 are regular solutions to problem (2.1.19) with initial conditions $w\left(0\right) = u_0$ and to the corresponding Galerkin approximations with initial condition $w_n\left(0\right) = u_n\left(0\right) = P_n u_0$, respectively.
Since

$$\frac{dw}{dt} \in L^2(0, T; H^{-1}(\Omega)) + L^q(0, T; L^q(\Omega)),$$

we can multiply the equation in (2.1.19) by ϕ and integrate by parts in the t variable to obtain that

$$\int_0^T \left(-\left(w\left(t\right), \phi'\left(t\right)\right) - \left\langle \Delta w\left(t\right), \phi\left(t\right)\right\rangle\right) dt = \int_0^T \left(\frac{f(w(t)) + h(\alpha^{-1}(t))}{a(\|w(t)\|^2_{H^1_0})}, \phi\left(t\right)\right) dt + \left(w\left(0\right), \phi\left(0\right)\right) \tag{2.1.33}$$

$$\int_0^T \left(-\left(w_n\left(t\right), \phi'\left(t\right)\right) - \left\langle \Delta w_n\left(t\right), \phi\left(t\right)\right\rangle\right) dt = \int_0^T \left(\frac{P_n f(w_n(t)) + P_n h(\alpha^{-1}(t))}{a(\|w_n(t)\|^2_{H^1_0})}, \phi\left(t\right)\right) dt + \left(w_n\left(0\right), \phi\left(0\right)\right) \tag{2.1.34}$$

We can easily obtain by the previous convergences and (2.1.4) that

$$\begin{aligned} w_n &\rightharpoonup w \text{ in } L^2\left(0,T;H_0^1\left(\Omega\right)\right), \\ \Delta w_n &\rightharpoonup \Delta w \text{ in } L^2\left(0,T;H^{-1}\left(\Omega\right)\right), \\ \frac{P_n f(w_n(t)) + P_n h\left(\alpha^{-1}(t)\right)}{a(\|w_n(t)\|_{H_0^1}^2)} &\rightharpoonup \frac{f(w(t)) + h\left(\alpha^{-1}(t)\right)}{a(\|w(t)\|_{H_0^1}^2)} \text{ in } L^q\left(0,T;L^q\left(\Omega\right)\right). \end{aligned}$$

Passing to the limit in (2.1.34), taking in to account (2.1.33) and bearing in mind $w_n(0) = P_n u_0 \to u_0$ we get

$$\left(w\left(0\right), \phi\left(0\right)\right) = \left(u_0, \phi\left(0\right)\right).$$

Since $\phi\left(0\right) \in H_0^1(\Omega) \cap L^p(\Omega)$ is arbitrary, we infer that $w(0) = u\left(0\right) = u_0$.

Hence, u is a regular solution to (2.1.1) satisfying $u\left(0\right) = u_0$. □

Second, we will prove the existence of strong solutions for initial conditions in $H_0^1(\Omega) \cap L^p(\Omega)$. In this case, we do not need to impose the upper bound (2.1.11) of the function a.

Theorem 2.10. *Suppose that conditions (2.1.2)-(2.1.5) and (2.1.9) are fulfilled. Then, for any $u_0 \in H_0^1(\Omega) \cap L^p(\Omega)$ there exists at least a strong solution to (2.1.1).*

Proof. We consider, as in Theorem 2.9, the Galerkin approximations $\{u_n\}$ and an element u for which (2.1.31) holds. Under the aforementioned conditions, we will obtain that u_n converges to a strong solution to (2.1.1). In this proof it is important to observe that

$$P_n u_0 \to u_0$$

in the spaces $H_0^1\left(\Omega\right)$ and $L^p\left(\Omega\right)$ [75, p.199 and 220]. Thus, the sequences $\|P_n u_0\|_{H_0^1}$ and $\|P_n u_0\|_{L^p}$ are bounded.

First, we multiply the equation in (2.1.1) by $\dfrac{du_n}{dt}$ to obtain

$$\|\frac{d}{dt}u_n(t)\|_{L^2}^2 + a(\|u_n\|_{H_0^1}^2)\frac{1}{2}\frac{d}{dt}\|u_n\|_{H_0^1}^2 = \frac{d}{dt}\int_\Omega \mathcal{F}(u_n)dx + (h(t), \frac{du_n}{dt}).$$

Introducing

$$A(s) = \int_0^s a(r)dr, \tag{2.1.35}$$

we have

$$\frac{1}{2}\|\frac{d}{dt}u_n(t)\|_{L^2}^2 + \frac{d}{dt}\left[\frac{1}{2}A(\|u_n\|_{H_0^1}^2) - \int_\Omega \mathcal{F}(u_n(t))dx\right] \le \frac{1}{2}\|h(t)\|_{L^2}^2. \tag{2.1.36}$$

Now, integrating (2.1.36) we have

$$\frac{1}{2}\int_0^t \|\frac{d}{ds}u_n(s)\|_{L^2}^2 ds + \frac{1}{2}A(\|u_n(t)\|_{H_0^1}^2) - \int_\Omega \mathcal{F}(u_n(t))dx$$

$$\le \frac{1}{2}A(\|u_n(0)\|_{H_0^1}^2) - \int_\Omega \mathcal{F}(u_n(0))dx + \frac{1}{2}\int_0^t \|h(s)\|_{L^2}^2 ds.$$

From (2.1.4) and (2.1.7) we get

$$\begin{aligned}&\frac{m}{2}\|u_n(t)\|_{H_0^1}^2 + \widetilde{\alpha}_1\|u_n(t)\|_{L^p}^p + \frac{1}{2}\int_0^t \|\frac{d}{ds}u_n(s)\|_{L^2}^2 ds\\ &\le \frac{1}{2}A(\|u_n(0)\|_{H_0^1}^2) + \widetilde{\alpha}_2\|u_n(0)\|_{L^p}^p + K(T).\end{aligned} \tag{2.1.37}$$

Now, from (2.1.37) we obtain that

$$\left\{\frac{du_n}{dt}\right\} \text{ is bounded in } L^2(0,T;L^2(\Omega)), \tag{2.1.38}$$

so

$$\frac{du_n}{dt} \rightharpoonup \frac{du}{dt} \text{ in } L^2(0,T;L^2(\Omega)). \tag{2.1.39}$$

On the other hand, the embedding $H_0^1(\Omega) \subset\subset L^2(\Omega)$ and the Aubin-Lion Compactness Lemma imply that

$$u_n \to u \text{ in } L^2(0,T;L^2(\Omega)).$$

Hence,

$$u_n \to u \text{ for a.e. } (x,t) \in \Omega \times (0,T).$$

Moreover, thanks to

$$\|u_n(t_2) - u_n(t_1)\|_{L^2}^2 = \left\| \int_{t_1}^{t_2} \frac{d}{dt} u_n(s) ds \right\|_{L^2}^2 \leq \|\frac{d}{dt} u_n\|_{L^2(0,T;L^2(\Omega))}^2 \, |t_2 - t_1|, \quad \forall t_1, t_2 \in [0,T],$$

(2.1.37), (2.1.38) and $H_0^1(\Omega) \subset\subset L^2(\Omega)$, the Ascoli-Arzelà theorem implies that $\{u_n\}$ converges strongly in the space $C([0,T]; L^2(\Omega))$ for all $T > 0$.
Therefore, we obtain from (2.1.37) that

$$u_n(t) \rightharpoonup u(t) \text{ in } H_0^1(\Omega) \cap L^p(\Omega),$$

for any $t \geq 0$, and

$$u_n \overset{*}{\rightharpoonup} u \text{ in } L^\infty(0,T; H_0^1(\Omega) \cap L^p(\Omega)). \tag{2.1.40}$$

Also, by the continuity of the function a, $\left\{ a(\|u_n(t)\|_{H_0^1}^2) \right\}$ is uniformly bounded in $[0,T]$. Multiplying (2.1.21) by $\lambda_i \gamma_{ni}(t)$ and summing from $i = 1$ to n, we obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|u_n\|_{H_0^1}^2 + m\| - \Delta u_n\|_{L^2}^2 &= (f(u_n), -\Delta u_n) + (h(t), -\Delta u_n) \\ &\leq \eta \|u_n\|_{H_0^1}^2 + \frac{1}{2m}\|h(t)\|_{L^2}^2 + \frac{m}{2}\| - \Delta u_n\|_{L^2}^2. \end{aligned}$$

Integrating the previous expression between 0 and T it follows that

$$\frac{1}{2}\|u_n(T)\|_{H_0^1}^2 + \frac{m}{2}\int_0^T \|\Delta u_n(s)\|_{L^2}^2 ds \leq \eta \int_0^T \|u_n(t)\|_{H_0^1}^2 dt + \frac{1}{2}\|u_n(0)\|_{H_0^1}^2 + K(T). \tag{2.1.41}$$

Finally, taking into account (2.1.25), from (2.1.41) we deduce that

$$u_n \text{ is uniformly bounded in } L^2(0,T; D(A)),$$

so

$$u_n \rightharpoonup u \text{ in } L^2(0,T; D(A)). \tag{2.1.42}$$

Arguing as in Theorem 2.9 we also obtain that

$$\begin{aligned} u_n &\to u \text{ in } L^2\left(0,T;H_0^1\left(\Omega\right)\right), \\ a\left(\|u_n\|_{H_0^1}^2\right) &\to a\left(\|u\|_{H_0^1}^2\right) \text{ in } L^2\left(0,T\right), \\ f\left(u_n\right) &\rightharpoonup f\left(u\right) \text{ in } L^q\left(0,T;L^q\left(\Omega\right)\right), \\ a\left(\|u_n\|_{H_0^1}^2\right)\Delta u_n &\rightharpoonup a\left(\|u\|_{H_0^1}^2\right)\Delta u \text{ in } L^2(0,T;L^2(\Omega)). \end{aligned} \tag{2.1.43}$$

Therefore, we can pass to the limit to conclude that u is a strong solution.

It remains to show that $u\left(0\right) = u_0$. This can be done, in a similar way as in Theorem 2.9, by multiplying the equation in (2.1.1) by a function $\phi \in C^1([0,T]); H_0^1(\Omega) \cap L^p(\Omega))$, with $\phi(T) = 0$, $\phi(0) \neq 0$ for the Galerkin approximations u_n and the limit function u and integrating by parts. Then taking into account the above convergences and $P_n u_0 \to u_0$ in $L^2\left(\Omega\right)$ we obtain that $u\left(0\right) = u_0$. □

We can still ensure the existence of strong solutions without using condition (2.1.9) by imposing extra assumptions on the parameter p. Indeed, if (2.1.10) is satisfied, then the embedding $H_0^1\left(\Omega\right) \subset L^{2(p-1)}\left(\Omega\right) \subset L^p\left(\Omega\right)$ and (2.1.6) imply that

$$||f(u(t))||_{L^2}^2 \leq 2C(1 + \int_\Omega |u(t,x)|^{2(p-1)}dx) \leq \widetilde{C}\left(1 + \|u\left(t\right)\|_{H_0^1}^{2(p-1)}\right), \tag{2.1.44}$$

so

$$f(u) \in L^2(0,T;L^2(\Omega)) \tag{2.1.45}$$

provided that $u \in L^\infty(0,T;H_0^1(\Omega))$. Moreover, $f\left(A\right)$ is bounded in $L^2(0,T;L^2(\Omega)$ if A is a bounded set of $L^\infty(0,T;H_0^1(\Omega))$.

Theorem 2.11. *Assume that (2.1.2)-(2.1.5) and (2.1.10) hold. Then for any* $u_0 \in H_0^1(\Omega)$ *there exists at least one strong solution to (2.1.1).*

Proof. Reasoning as in Theorem 2.10 and considering as well the Galerkin scheme, (2.1.31), (2.1.39) and (2.1.40) hold. We just need to check that (2.1.42) is also true and then repeat the same lines of Theorem 2.10.

Multiplying (2.1.21) by $\lambda_i \gamma_{ni}(t)$ and summing from $i = 1$ to n, we obtain

$$\frac{1}{2}\frac{d}{dt}\|u_n\|^2_{H_0^1} + m\|\Delta u_n\|^2_{L^2} = (f(u_n), -\Delta u_n) + (h(t), -\Delta u)$$
$$\leq \frac{1}{2m}\|f(u_n)\|^2_{L^2} + \frac{m}{2}\| - \Delta u_n\|^2_{L^2} + \frac{1}{m}\|h(t)\|^2_{L^2} + \frac{m}{4}\|\Delta u_n\|^2_{L^2}.$$

Integrating between 0 and T it follows that

$$\frac{1}{2}\|u_n(T)\|^2_{H_0^1} + \frac{m}{4}\int_0^T \|\Delta u_n(s)\|^2_{L^2} ds \tag{2.1.46}$$
$$\leq \frac{1}{2m}\int_0^T \|f(u_n(t))\|^2_{L^2} dt + \frac{1}{2}\|u_n(0)\|^2_{H_0^1} + \frac{1}{m}\int_0^T \|h(t)\|^2_{L^2} dt.$$

In view of (2.1.40) and (2.1.44), we have that $f(u)$ is bounded in $L^2(0,T;L^2(\Omega))$, so from (2.1) we get that $\{u_n\}$ is bounded in $L^2(0,T;D(A))$. Therefore,

$$u_n \rightharpoonup u \text{ in } L^2(0,T;D(A)), \tag{2.1.47}$$

as required. □

Actually, in the case of regular solutions, we can get rid of the condition (2.1.9) as well by imposing the extra assumption (2.1.10) on the constant p.

Theorem 2.12. *Assume that (2.1.2)-(2.1.5), (2.1.10) and (2.1.15) hold. Then, for any $u_0 \in L^2(\Omega)$ there exists at least one regular solution to (2.1.1).*

Proof. Let $u_0^n \in H_0^1(\Omega)$ be a sequence such that $u_0^n \to u_0$ in $L^2(\Omega)$. By Theorem 2.11 there exists a strong solution $u^n(\cdot)$ of (2.1.1) with $u^n(0) = u_0^n$. Since $u^n \in L^2(0,T;D(A))$ and $\frac{du^n}{dt} \in L^2(0,T;L^2(\Omega))$, from [77, p.102] the equality

$$\frac{d}{dt}\|u^n\|^2_{H_0^1} = 2(-\Delta u^n, u_t^n)$$

holds true for a.a. $t > 0$.

Now, multiplying (2.1.1) by u^n and using (2.1.5) it follows that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|u^n(t)\|_{L^2}^2+m\|u^n\|_{H_0^1}^2+\alpha_1\|u^n(t)\|_{L^p}^p\\&\leq\kappa|\Omega|+\|h(t)\|_{L^2}\|u^n(t)\|_{L^2}\leq\kappa|\Omega|+\frac{1}{2m\lambda_1}\|h(t)\|_{L^2}^2+\frac{m}{2}\|u^n(t)\|_{H_0^1}^2,\end{aligned}\tag{2.1.48}$$

so

$$\|u^n(t)\|_{L^2}^2\leq\|u^n(0)\|_{L^2}^2+K_1(T).\tag{2.1.49}$$

Thus, integrating in (2.1.48) between t and $t+r$ we get

$$\begin{aligned}&\|u^n(t+r)\|_{L^2}^2+m\int_t^{t+r}\|u^n(s)\|_{H_0^1}^2ds+2\alpha_1\int_t^{t+r}\|u^n(s)\|_{L^p}^pds\\&\leq2\kappa|\Omega|r+\frac{1}{m\lambda_1}\int_t^{t+r}\|h(s)\|_{L^2}^2ds+\|u^n(t)\|_{L^2}^2\leq\|u^n(0)\|_{L^2}^2+K_2(T).\end{aligned}\tag{2.1.50}$$

Also, by (2.1.7) and (2.1.15) we deduce that

$$\begin{aligned}&\int_t^{t+r}\left(\frac{1}{2}A(\|u^n(s)\|_{H_0^1}^2)-\int_\Omega\mathcal{F}(u^n(s))dx\right)ds\\&\leq\int_t^{t+r}\left(\frac{M}{2}\|u^n(s)\|_{H_0^1}^2\right)ds+\widetilde{\kappa}\,|\Omega|\,r+\widetilde{\alpha}_2\int_t^{t+r}\|u^n(s)\|_{L^p}^pds\\&\leq K_3(T)\left(1+\|u^n(0)\|_{L^2}^2\right),\end{aligned}\tag{2.1.51}$$

for all $n>0$ and $t\geq0$. On the other hand, multiplying (2.1.1) by u_t^n we have

$$\frac{1}{2}\|u_t^n(t)\|_{L^2}^2+\frac{d}{dt}\left(\frac{1}{2}A(\|u^n(t)\|_{H_0^1}^2)-\int_\Omega\mathcal{F}(u^n(t))dx\right)\leq\frac{1}{2}\|h(t)\|_{L^2}^2,\tag{2.1.52}$$

where the fact that $t\mapsto\int_\Omega\mathcal{F}(u^n(t))dx$ is absolutely continuous on $[0,T]$ and

$$\frac{d}{dt}\int_\Omega\mathcal{F}(u^n(t))dx=\left(f\left(u^n\left(t\right)\right),\frac{du^n}{dt}\left(t\right)\right),\ \text{for a.a. } t>0,$$

is proved by regularization using the regularity of strong solutions and (2.1.44) (see Lemma 2.20). By the Uniform Gronwall Lemma [80] we obtain

$$\frac{1}{2}A(\|u^n(t+r)\|^2_{H^1_0}) - \int_\Omega \mathcal{F}(u^n(t+r))dx \leq \frac{K_3(T)(1+\|u^n(0)\|^2_{L^2})}{r} + K_4(T), \quad (2.1.53)$$

for all $0 \leq t \leq t+r$, so that by (2.1.4) and (2.1.7) we have that

$$\|u^n(t+r)\|^2_{H^1_0} + \|u^n\,(t+r)\|^p_{L^p} \leq \frac{K_5(T)(1+\|u^n(0)\|^2_{L^2})}{r} + K_6(T), \qquad (2.1.54)$$

for all $t \geq 0$.

Therefore, the sequence $u^n(\cdot)$ is bounded in $L^\infty(r,T;H^1_0(\Omega))$ for all $0 < r < T$. Consequently, $a(\|u^n\,(\cdot)\,\|^2_{H^1_0})$ is bounded in $[r,T]$.

Integrating (2.1.52) over (r,T), from (2.1.4), (2.1.7) and (2.1.53) it follows that

$$\begin{aligned}
&\frac{1}{2}\int_r^T \|\frac{d}{dt}u^n(t)\|^2_{L^2}dt + \frac{m}{2}\|u^n(T)\|^2_{H^1_0} + \widetilde{\alpha}_1\|u^n(T)\|^p_{L^p} - \kappa|\Omega| \\
&\leq \frac{1}{2}\int_r^T \|\frac{d}{dt}u^n(t)\|^2_{L^2}dt + \frac{1}{2}A(\|u^n(T)\|^2_{H^1_0}) - \int_\Omega \mathcal{F}(u^n(T))dx \\
&\leq \frac{1}{2}\int_r^T \|h(t)\|^2_{L^2}dt + \frac{1}{2}A(\|u^n(r)\|^2_{H^1_0}) - \int_\Omega \mathcal{F}(u^n(r))dx \\
&\leq \frac{1}{2}\int_r^T \|h(t)\|^2_{L^2}dt + \frac{K_3(T)(1+\|u^n(0)\|^2_{L^2})}{r} + K_4(T).
\end{aligned} \qquad (2.1.55)$$

Thus $\dfrac{du^n}{dt}$ is bounded in $L^2(r,T;L^2(\Omega))$ for all $0 < r < T$.

Taking into account (2.1.44) and (2.1.54) we infer that $f\,(u^n)$ is bounded in $L^2\,(r,T;L^2\,(\Omega))$. By this way, the equality

$$a(\|u^n\|^2_{H^1_0})\Delta u^n = u^n_t - f(u^n) + h(t)$$

implies that

$$\begin{aligned}
\{u^n\} &\text{ is bounded in } L^2(r,T;D(A)), \\
a(\|u^n\|^2_{H^1_0})\Delta u^n &\text{ is bounded in } L^2(r,T;L^2(\Omega)),
\end{aligned}$$

for all $0 < r < T$.

By the compact embedding $H_0^1(\Omega) \subset L^2(\Omega)$, we can apply the Ascoli-Arzelà theorem and obtain that, up to a sequence, there exists a function u such that

$$\begin{aligned} u^n &\overset{*}{\rightharpoonup} u \text{ in } L^\infty(r,T;H_0^1(\Omega)), \\ u^n &\to u \text{ in } C([r,T],L^2(\Omega)), \\ u^n &\rightharpoonup u \text{ in } L^2(r,T;D(A)), \\ \frac{du^n}{dt} &\rightharpoonup \frac{du}{dt} \text{ in } L^2(r,T;L^2(\Omega)), \end{aligned} \tag{2.1.56}$$

for all $0 < r < T$.

On the other hand, from (2.1.50) we infer that

$$u^n \text{ is bounded in } L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega)) \cap L^p(0,T;L^p(\Omega)),$$

for all $T > 0$.

Therefore, there exists a subsequence u^n, relabelled the same, such that

$$\begin{aligned} u^n &\overset{*}{\rightharpoonup} u \text{ in } L^\infty(0,T;L^2(\Omega)), \\ u^n &\rightharpoonup u \text{ in } L^2(0,T;H_0^1(\Omega)), \\ u^n &\rightharpoonup u \text{ in } L^p(0,T;L^p(\Omega)), \end{aligned} \tag{2.1.57}$$

for all $T > 0$.

Moreover, arguing as in the proof of Theorem 2.9 we obtain that

$$\begin{aligned} f(u^n) &\rightharpoonup f(u) \text{ in } L^q(0,T;L^q(\Omega)), \\ u^n &\to u \text{ in } L^2(r,T;H_0^1(\Omega)), \\ a(\|u^n\|^2_{H_0^1}) &\to a(\|u\|^2_{H_0^1}) \text{ in } L^2\left(0,T\right), \\ a(\|u^n(t)\|^2_{H_0^1})\Delta u^n &\rightharpoonup a(\|u(t)\|^2_{H_0^1})\Delta u \text{ in } L^2(r,T;L^2(\Omega)). \end{aligned}$$

Passing to the limit we obtain that $u\left(\cdot\right)$ is a regular solution.

Finally, by a similar argument as in the proof of Theorem 2.9 we establish that $u\left(0\right) = u_0$. □

Remark 2.13. Under the conditions of Theorem 2.12 any regular solution $u(\cdot)$ satisfies from (2.1.44) that $f(u) \in L^2(\varepsilon, T; L^2(\Omega))$ for all $0 < \varepsilon < T$, and then $\frac{du}{dt} \in L^2(\varepsilon, T; L^2(\Omega))$ as well. Hence, $u \in C((0,T], H_0^1(\Omega))$ for all $T > 0$.

We finish this section by giving a sufficient condition ensuring the uniqueness of solutions.

Theorem 2.14. *Assume the conditions of Theorem 2.9 and additionally that (2.1.12) is satisfied. Then there can exists at most one regular solution to the Cauchy problem (2.1.1) for $u_0 \in L^2(\Omega)$.*

If, moreover, $M_2 = 0$ in condition (2.1.11), then there can be at most one weak solution.

Under the conditions of Theorem 2.10 and (2.1.12), there can exists at most one strong solution to the Cauchy problem (2.1.1) for $u_0 \in H_0^1(\Omega) \cap L^p(\Omega)$.

Proof. Suppose that u and v are two regular solutions to (2.1.1) with the same initial condition $u_0 = v_0$. Then by subtraction and multiplying by $u - v$ we get by Remark 2.7 that

$$\frac{1}{2}\frac{d}{dt}\|u-v\|_{L^2}^2 + \langle -a(\|u(t)\|_{H_0^1}^2)\Delta u + a(\|v(t)\|_{H_0^1}^2)\Delta v, u-v\rangle = (f(u)-f(v), u-v).$$

Let us consider

$$I = \langle -a(\|u(t)\|_{H_0^1}^2)\Delta u + a(\|v(t)\|_{H_0^1}^2)\Delta v, u-v\rangle.$$

After integrating by parts, we obtain

$$\begin{aligned} I &= \int_\Omega (a(\|u(t)\|_{H_0^1}^2)|\nabla u|^2 - a(\|u(t)\|_{H_0^1}^2)\nabla u \nabla v - a(\|v(t)\|_{H_0^1}^2)\nabla u\nabla v + a(\|v(t)\|_{H_0^1}^2)|\nabla v|^2)dx \\ &\geq a(\|u(t)\|_{H_0^1}^2)\|u(t)\|_{H_0^1}^2 - \left(a(\|u(t)\|_{H_0^1}^2) + a(\|v(t)\|_{H_0^1}^2)\right)\|u(t)\|_{H_0^1}\|v(t)\|_{H_0^1} + a(\|v(t)\|_{H_0^1}^2)\|v(t)\|_{H_0^1}^2 \\ &= \left(a(\|u(t)\|_{H_0^1}^2)\|u(t)\|_{H_0^1} - a(\|v(t)\|_{H_0^1}^2)\|v(t)\|_{H_0^1}\right)\left(\|u(t)\|_{H_0^1} - \|v(t)\|_{H_0^1}\right) \geq 0, \end{aligned} \tag{2.1.58}$$

where we have used (2.1.12) in the last inequality.

Hence, from (2.1.58) and $f'(s) \leq \eta$, we infer

$$\frac{1}{2}\frac{d}{dt}\|u-v\|_{L^2}^2 \leq \int_\Omega \left(f(u)-f(v)\right)(u-v)dx$$
$$= \int_\Omega \left(\int_v^u f'(s)ds\right)(u-v)dx \leq \eta\|u-v\|_{L^2}^2.$$

By Remark 2.7 it is correct to apply the Gronwall lemma over an arbitrary interval (ε, t), so

$$\|u(t)-v(t)\|_{L^2}^2 \leq \|u(\varepsilon)-v(\varepsilon)\|_{L^2}^2 \, e^{2\eta(t-\varepsilon)}, \quad t \geq 0.$$

Since Lemma 2.6 implies that $u, v \in C([0,T], L^2(\Omega))$, we pass to the limit as $\varepsilon \to 0$ to get

$$\|u(t)-v(t)\|_{L^2}^2 \leq \|u(0)-v(0)\|_{L^2}^2 \, e^{2\eta t}, \quad t \geq 0.$$

Hence, the uniqueness follows.

If $M_2 = 0$ in (2.1.11), then by (2.1.16) the above argument is valid for weak solutions as well.

The proof of the last statement is the same with the only difference that condition (2.1.11) is not needed.

□

2.2. Existence and structure of attractors

In this section we will prove the existence of global attractors for the semiflows generated by regular and strong solutions under different assumptions in the autonomous case, that is, when the function h does not depend on t. We will also establish that the attractor is equal to the unstable set of the stationary points or to the stable one when we only consider solutions in the set of bounded complete trajectories.

We consider the following condition instead of (2.1.2):

$$h \in L^2(\Omega). \tag{2.2.1}$$

Throughout this section, for a metric space X with metric ρ we will denote by $dist_X(C,D)$ the Hausdorff semidistance from C to D, that is, $dist_X(C,D) = \sup_{c\in C}\inf_{d\in D}\rho(c,d)$.

It is important to observe that in the theorems of existence of solutions of the previous section we have used either assumption (2.1.9) or (2.1.10). Now, when we use condition (2.1.9) in some cases it is necessary to add a restriction on the constant p given below in (2.2.23).

We summarize the main results of this section:

- Conditions (2.1.3)-(2.1.5), (2.1.9), (2.1.12), (2.1.15) and (2.2.1) imply that the regular solutions generate a semigroup in the phase space $L^2(\Omega)$ possessing a global attractor, which is compact in $H_0^1(\Omega)$ and bounded in $L^p(\Omega)$ (Theorem 2.17 and Lemma 2.31). If, in addition, either $h \in L^\infty(\Omega)$ or $p \leq 2n/(n-2)$ for $n \geq 3$, then it is characterized by the unstable set of the stationary points (Proposition 2.32). Moreover, condition (2.1.13) implies that the attractor is bounded in $H^2(\Omega)$ (Proposition 2.19).

- Conditions (2.1.3)-(2.1.5), (2.1.15), (2.2.1) and either (2.1.10) or (2.1.9), (2.2.23) imply that the regular solutions generate a (possibly) multivalued semiflow in the phase space $L^2(\Omega)$ possessing a global attractor, which is compact in $H_0^1(\Omega)$ and $L^p(\Omega)$ and is equal to the unstable set of the stationary points (Theorems 2.25, 2.29).

- Conditions (2.1.3)-(2.1.5), (2.1.15), (2.2.1) and either (2.1.10) or (2.1.9), (2.2.23) imply that the strong solutions generate a (possibly) multivalued semiflow in the phase space $H_0^1(\Omega)$ possessing a global attractor, which is compact in $H_0^1(\Omega)$ and $L^p(\Omega)$ and is equal to the unstable set of the stationary points (Theorems 2.37, 2.40).

- Conditions (2.1.3)-(2.1.5), (2.1.9), (2.1.12), (2.1.15), (2.2.1) and (2.2.23) imply that the strong solutions generate a semigroup in the phase space $H_0^1(\Omega)$ possessing a global attractor, which is compact in $H_0^1(\Omega)$ and $L^p(\Omega)$ and is equal to the unstable set of the stationary points (Theorems 2.42, 2.45). Moreover, condition (2.1.13) implies that the attractor is bounded in $H^2(\Omega)$ (Proposition 2.46).

- Conditions (2.1.3)-(2.1.5), (2.1.9), (2.1.12), (2.1.15) and (2.2.1) imply that the strong solutions generate a semigroup in the phase space $H_0^1(\Omega) \cap L^p(\Omega)$ (endowed with the induced topology of $H_0^1(\Omega)$) possessing a global attractor, which is compact in $H_0^1(\Omega)$ and bounded in $L^p(\Omega)$ (Theorem 2.49). If, in addition, either $h \in L^\infty(\Omega)$ or $p \leq 2n/(n-2)$ for $n \geq 3$, then it is characterized by the unstable set of the stationary points (Theorem 2.52). Moreover, condition (2.1.13) implies that the attractor is bounded in $H^2(\Omega)$ (Proposition 2.53).

- In all the above situations $h \in L^\infty(\Omega)$ implies that the global attractor is bounded in $L^\infty(\Omega)$ (Theorems 2.18, 2.28, 2.39, 2.51).

2.2.1. Regular solutions

We split this part into three subsections.

2.2.2. The case of uniqueness

If we assume conditions (2.1.3)-(2.1.5), (2.1.9), (2.1.11), (2.1.12), (2.2.1), then by Theorems 2.9 and 2.14 we can define the following continuous semigroup $T_r : \mathbb{R}^+ \times L^2(\Omega) \to L^2(\Omega)$:

$$T_r(t, u_0) = u(t), \tag{2.2.2}$$

where $u(\cdot)$ is the unique regular solution to (2.1.1). We denote by $\mathfrak{R}$ the set of fixed points of T_r, that is, the points z such that $T_r(t, z) = z$ for any $t \geq 0$.

We also observe that if we assume (2.1.15), then using the calculations in (2.1.51)-(2.1.54) for the Galerkin approximations of any regular solution $u(\cdot)$ one can obtain that $u \in L^\infty(\varepsilon, T; L^p(\Omega))$, for all $0 < \varepsilon < T$, and then $u \in C_w((0, +\infty), L^p(\Omega))$.

Our first purpose is to obtain a global attractor. We recall that the set $\mathcal{A}$ is a global compact attractor for T_r if it is compact, invariant (which means $T_r(t, \mathcal{A}) = \mathcal{A}$ for any $t \geq 0$) and it attracts any bounded set B, that is,

$$dist_{L^2}(T_r(t, B), \mathcal{A}) \to 0 \text{ as } t \to +\infty.$$

For this aim, we follow the classical procedure and we start with the existence of an absorbing set.

Proposition 2.15. *Let (2.1.3)-(2.1.5), (2.1.9), (2.1.11), (2.1.12) and (2.2.1) hold. Then the semigroup T_r has a bounded absorbing set in L^2; that is, there exists a constant K such that for any $R>0$ there is a time $t_0 = t_0(R)$ such that*

$$\|u(t)\|_{L^2} \le K \quad \textit{for all} \quad t \ge t_0, \tag{2.2.3}$$

where $\|u_0\|_{L^2} \le R$, $u(t) = T_r(t, u_0)$. Moreover, there is a constant L such that

$$\int_t^{t+1} \|u(s)\|_{H_0^1}^2 ds \le L \quad \textit{for all} \quad t \ge t_0. \tag{2.2.4}$$

Proof. Multiplying equation (2.1.1) by u and using (2.1.5) and Remark 2.7 we have

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|_{L^2}^2 + \frac{m}{2}\|u(t)\|_{H_0^1}^2 + \alpha_1\|u(t)\|_{L^p}^p \le \kappa|\Omega| + \frac{1}{2\lambda_1 m}\|h\|_{L^2}^2 = \frac{\kappa_1}{2}. \tag{2.2.5}$$

The Gronwall lemma and the inequality $\|u(t)\|_{H_0^1}^2 \ge \lambda_1\|u(t)\|_{L^2}^2$ give

$$\|u(t)\|_{L^2}^2 \le \|u(\varepsilon)\|_{L^2}^2 e^{-\lambda_1 m(t-\varepsilon)} + \frac{\kappa_1}{\lambda_1 m}, \text{ for any } \varepsilon > 0.$$

As $u \in C([0,T], L^2(\Omega)$ by Lemma 2.6, passing to the limit we have

$$\|u(t)\|_{L^2}^2 \le \|u(0)\|_{L^2}^2 e^{-\lambda_1 m t} + \frac{\kappa_1}{\lambda_1 m}. \tag{2.2.6}$$

Hence, taking

$$t \ge t_0 \equiv \frac{1}{\lambda_1 m}\ln\left(\frac{\lambda_1 m R^2}{\kappa_1}\right)$$

we get (3.3.6) for $K = \frac{2\kappa_1}{\lambda_1 m}$. On the other hand, integrating (3.3.8) between t and $t+1$ and using (2.2.6) we obtain

$$m\int_t^{t+1} \|u(s)\|_{H_0^1}^2 ds \le \|u(t)\|_{L^2}^2 + \kappa_1$$

and using the previous bound we get

$$\int_{t}^{t+1} \|u(s)\|_{H_0^1}^2 ds \leq \frac{\kappa_1}{m} + \frac{2\kappa_1}{\lambda_1 m^2}, \qquad \text{for all } t \geq t_0,$$

so that (2.2.4) follows. □

Proposition 2.16. *Let (2.1.3)-(2.1.5), (2.1.9), (2.1.12), (2.1.15) and (2.2.1) hold. Then there exists a bounded absorbing set in $H_0^1(\Omega) \cap L^p(\Omega)$; that is, there is a constant M such that for any $R > 0$ there is a time $t_1 = t_1(R)$ such that*

$$\|u(t)\|_{H_0^1} + \|u(t)\|_{L^p} \leq M \quad \textit{for all } t \geq t_1,$$

where $\|u_0\|_{L^2} \leq R$, $u(t) = T_r(t, u_0)$.

Proof. The following calculations are formal but can be justified by the Galerkin approximations. Arguing as in (2.1.51)-(2.1.54) we obtain the existence of a constant C such that

$$\|T_r(1, u(0))\|_{H_0^1}^2 + \|T_r(1, u(0))\|_{L^p}^p \leq C(1 + \|u(0)\|_{L^2}^2).$$

Hence, the semigroup property $T_r(t+1, u_0) = T_r(1, T_r(t, u_0))$ and (3.3.6) imply that

$$\|T_r(t+1, u_0)\|_{H_0^1}^2 + \|T_r(t+1, u_0)\|_{L^p}^p \leq C(1 + K^2) \ \forall t \geq t_0(R),$$

if $\|u_0\|_{L^2} \leq R$, which proves the statement. □

Theorem 2.17. *Let (2.1.3)-(2.1.5), (2.1.9), (2.1.12), (2.1.15) and (2.2.1). Then equation (2.1.1) has a connected global compact attractor $\mathcal{A}_r$, which is bounded in $H_0^1(\Omega) \cap L^p(\Omega)$.*

Proof. Since a bounded set in $H_0^1(\Omega)$ is relatively compact in $L^2(\Omega)$ which is a connected space, the result follows from Theorem 10.5 in [75] and Proposition 2.16.

□

We will also obtain the boundedness of the attractor in the spaces $L^{\infty}(\Omega)$ and $H^{2}(\Omega)$.

First, we recall that a function $\phi : \mathbb{R} \to L^{2}(\Omega)$ is a complete trajectory of the semigroup T_r if

$$\phi(t) = T_r(t-s, \phi(s))$$

for any $t \geq s$.
We say that ϕ is bounded if the set $\cup_{s\in\mathbb{R}}\phi(s)$ is bounded. It is well known [61] that the global attractor is characterized by

$$\mathcal{A}_r = \{\phi(0) : \phi \text{ is a bounded complete trajectory}\}. \tag{2.2.7}$$

Theorem 2.18. *Let (2.1.3)-(2.1.5), (2.1.9), (2.1.12), (2.1.15) and (2.2.1) hold. Then the global attractor $\mathcal{A}_r$ is bounded in $L^{\infty}(\Omega)$, provided that $h \in L^{\infty}(\Omega)$.*

Proof. We define $v_+ = \max\{v, 0\}$, $v_- = -\max\{-v, 0\}$. We multiply equation (2.1.1) by $(u-M)_+$ for some appropriate constant M and integrate over Ω to obtain

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega}|(u-M)_+|^2dx + a(\|u(t)\|^2_{H_0^1})\int_{\Omega}|\nabla(u-M)_+|^2dx = \int_{\Omega}(f(u(t))+h)(u-M)_+dx,$$

where we have used the equality $\frac{1}{2}\frac{d}{dt}\int_{\Omega}|(u-M)_+|^2dx = (u_t, (u-M)_+)$, which is proved by regularization.

Since $h \in L^{\infty}(\Omega)$, by (2.1.5) we deduce that

$$(f(u)+h)u \leq \widetilde{\kappa} - \widetilde{\alpha}|u|^p.$$

It follows that

$$f(u)+h \leq 0 \quad \text{when} \quad u \geq (\frac{\widetilde{\kappa}}{\widetilde{\alpha}})^{1/p} = M.$$

Therefore, we have

$$(f(u)+h)(u-M)_+ \leq 0.$$

Thus, by (2.1.4) and the the Poincaré inequality, we deduce that

$$\frac{d}{dt}\int_{\Omega}|(u-M)_{+}|^{2}dx\leq-2m\lambda_{1}\int_{\Omega}|(u-M)_{+}|^{2}dx.$$

Using the Gronwall inequality, we have

$$\int_{\Omega}|(u(t)-M)_{+}|^{2}dx\leq e^{-2m\lambda(t-\tau)}\int_{\Omega}|(u(\tau)-M)_{+}|^{2}dx.$$

For any $y\in\mathcal{A}_r$ there is by (2.2.7) a bounded complete trajectory ϕ such that $\phi(0)=y$. Then taking $t=0$ and $\tau\to-\infty$ in the last inequality, we obtain $y(x)=\phi(0,x)\leq M$, for a.a. $x\in\Omega$. The same arguments can be applied to $(u-M)_{-}$, which shows that

$$\|y\|_{L^{\infty}}\leq M,\quad\forall y\in\mathcal{A}_r.$$

□

If we assume (2.1.13), then it is possible to show that the global attractor is more regular.

Proposition 2.19. *Let (2.1.3)-(2.1.5), (2.1.9), (2.1.15) and (2.2.1) hold. If, additionally, (2.1.13) is satisfied, then there exists an absorbing set in $H^2(\Omega)$ and the global attractor is bounded in $H^2(\Omega)$.*

Proof. We will prove the existence of an absorbing set in $H^2(\Omega)$. The boundedness of the global attractor in this space follows then immediately. We proceed formally, but the estimates can be justified via Galerkin approximations.

Let $u(t)=T_r(t,u_0)$ with $\|u_0\|_{L^2}\leq R$. First, we differentiate the equation with respect to t

$$u_{tt}-a'(\|u\|^2_{H^1_0})\frac{d}{dt}\|u\|^2_{H^1_0}\Delta u-a(\|u\|^2_{H^1_0})\Delta u_t=f'(u)u_t.$$

Multiplying by u_t we get

$$\frac{1}{2}\frac{d}{dt}\|u_t\|^2_{L^2}+\frac{1}{2}a'(\|u\|^2_{H^1_0})(\frac{d}{dt}\|u\|^2_{H^1_0})^2+a(\|u\|^2_{H^1_0})\|u_t\|^2_{H^1_0}=\int_{\Omega}f'(u)(u_t)^2dx.\quad(2.2.8)$$

By (2.1.4), $a'(s) \geq 0$ and $f'(s) \leq \eta$ we obtain

$$\frac{1}{2}\frac{d}{dt}\|u_t\|_{L^2}^2 + m\|u_t\|_{H_0^1}^2 \leq \eta\|u_t\|_{L^2}^2. \tag{2.2.9}$$

Second, multiplying (2.1.1) by u_t and reordering terms, it follows that

$$\frac{d}{dt}\left(\frac{a(\|u\|_{H_0^1}^2)}{2}\|u\|_{H_0^1}^2 - \int_\Omega \mathcal{F}(u)dx - \int_\Omega h(x)udx\right) + \|u_t\|_{L^2}^2 = \frac{a'(\|u\|_{H_0^1}^2)}{2}\|u\|_{H_0^1}^2\frac{d}{dt}\|u\|_{H_0^1}^2. \tag{2.2.10}$$

Proposition 2.16 implies that

$$a'(\|z\|_{H_0^1}^2) \leq \gamma := sup_{|s|\leq M} a'(s^2)$$

if z belongs to the absorbing set in $H_0^1(\Omega)$. On the other hand, multiplying the equation by $-\Delta u$ and using Proposition 2.16, we obtain

$$\frac{d}{dt}\|u\|_{H_0^1}^2 + m\|\Delta u(t)\|_{L^2}^2 \leq 2\eta\|u(t)\|_{H_0^1}^2 + \frac{1}{m}\|h\|_{L^2}^2 \leq K_1 \quad \forall t \geq t_1(R).$$

Hence, by (2.2.10) and Proposition 2.16, it follows that

$$\frac{d}{dt}\left(\frac{a(\|u\|_{H_0^1}^2)}{2}\|u\|_{H_0^1}^2 - \int_\Omega \mathcal{F}(u)dx - \int_\Omega h(x)udx\right) + \|u_t\|_{L^2}^2 \leq \frac{\gamma}{2}K_1M^2, \quad \forall t \geq t_1(R). \tag{2.2.11}$$

Multiplying both sides of the inequality $f'(s) \leq \eta$ by s and integrating between 0 and s, we obtain

$$sf(s) \leq \mathcal{F}(s) + \frac{s^2}{2}\eta, \quad \forall s \in \mathbb{R}. \tag{2.2.12}$$

Moreover, integrating $f'(s) \leq \eta$ twice between 0 and s, we infer

$$\mathcal{F}(s) \leq \frac{\eta}{2}s^2 + Cs, \quad \forall s \in \mathbb{R}. \tag{2.2.13}$$

Now, we multiply (2.1.1) by u and integrate between t and $t+1$ to obtain

$$\frac{1}{2}\|u(t+1)\|_{L^2}^2 + \int_t^{t+1}\left(a(\|u\|_{H_0^1}^2)\|u(s)\|_{H_0^1}^2 - \int_\Omega f(u)udx - \int_\Omega h(x)udx\right)ds = \frac{1}{2}\|u(t)\|_{L^2}^2. \tag{2.2.14}$$

From (2.2.12), (2.2.14) and Proposition 2.15 it follows

$$\int_t^{t+1}\left(\frac{a(\|u\|^2_{H_0^1})}{2}\|u\|^2_{H_0^1}-\int_\Omega \mathcal{F}(u)dx-\int_\Omega h(x)udx\right)ds$$
$$\leq \frac{1}{2}\|u(t)\|^2_{L^2}+\frac{\eta}{2}\int_t^{t+1}\|u\|^2_{L^2}ds\leq \widetilde{L},\quad \forall t\geq t_0.$$

The last inequality allows us to apply the Uniform Gronwall Lemma [80] to (2.2.11) in order to obtain

$$\frac{a(\|u\|^2_{H_0^1})}{2}\|u\|^2_{H_0^1}-\int_\Omega \mathcal{F}(u)dx-\int_\Omega h(x)udx\leq \widetilde{L}+\frac{\gamma}{2}K_1M^2,\quad \forall t\geq t_1+1.\tag{2.2.15}$$

Using (2.1.4) and (2.2.13) we get

$$\frac{a(\|u\|^2_{H_0^1})}{2}\|u\|^2_{H_0^1}-\int_\Omega \mathcal{F}(u)dx-\int_\Omega h(x)udx\geq -\frac{\eta}{2}\|u\|^2_{L^2}-\widetilde{C}\|u\|_{L^2}.\tag{2.2.16}$$

Now, integrating (2.2.11) from t to $t+1$, using (2.2.15), (2.2.16), by Proposition 2.15 we have

$$\int_t^{t+1}\|u_s\|^2_{L^2}ds\leq \widetilde{L}+\gamma K_1M^2+\frac{\eta}{2}K^2+\widetilde{C}K=\rho_1,\quad \forall t\geq t_1+1.\tag{2.2.17}$$

Hence, the last equation allow us to apply to (2.2.9) the Uniform Gronwall Lemma to obtain

$$\|\frac{du}{dt}(t)\|^2_{L^2}\leq \rho_2,\quad \forall t\geq t_1+2.\tag{2.2.18}$$

Finally, we multiply (2.1.1) by $-\Delta u$ and use (2.1.4) to obtain

$$\frac{m}{2}\|\Delta u\|^2_{L^2}\leq \eta\|u\|^2_{H_0^1}+\frac{1}{m}\|h\|^2_{L^2}+\frac{1}{m}\|u_t\|^2_{L^2}.$$

Thus, by Proposition 2.16 and (2.2.18), we deduce that

$$\|u(t)\|^2_{H^2}\leq \rho_3\quad \forall t\geq t_1+2.$$

□

2.2.3. The case of non-uniqueness

We recall that the multivalued map $G : \mathbb{R}_+ \times X \to P(X)$ associated with the family $\mathcal{R}$ is defined as follows

$$G(t, u_0) = \{u(t) : u(\cdot) \in \mathcal{R}, u(0) = u_0\}, \tag{2.2.19}$$

where $\mathcal{R} \subset \mathcal{C}(\mathbb{R}_+; X)$ is a family of functions satisfiying the set of axomatic properties described in Chapter 0. The set of all fixed points will be denoted by $\mathfrak{R}_{\mathcal{R}}$.

In this section we will show that regular solutions genrate a multivalued semiflow possessing a global attractor.

If we do not assume the additional assumptions on the function $a(\cdot)$ of Section 2.2.2 ensuring uniqueness of the Cauchy problem, we have to define a multivalued semiflow.

We have two possibilities: either to consider the conditions of Theorem 2.9 with an extra growth assumption or to use the conditions of Theorem 2.12.

If we assume conditions (2.1.3)-(2.1.5), (2.1.10), (2.1.15) and (2.2.1), then by Theorem 2.12 for any $u_0 \in L^2(\Omega)$ there exists at least one regular solution and (2.1.44) implies that $f(u) \in L^2(\varepsilon, T; L^2(\Omega))$ for any regular solution, so $\frac{du}{dt} \in L^2(\varepsilon, T; L^2(\Omega))$ as well. In this case, as $H_0^1(\Omega) \subset L^p(\Omega)$, we have that $u \in C((0, +\infty), H_0^1(\Omega)) \subset C((0, +\infty), L^p(\Omega))$.

If we assume conditions (2.1.3)-(2.1.5), (2.1.9), (2.1.11) and (2.2.1) as well, then we known by Theorem 2.9 that for any $u_0 \in L^2(\Omega)$ there exists at least one regular solution.

In order to obtain the necessary estimates leading to the existence of a global attractor, we need to ensure that

$$\frac{du}{dt} \in L^2(\varepsilon, T; L^2(\Omega)), \text{ for all } 0 < \varepsilon < T, \tag{2.2.20}$$

holds, as by [77, p.102] we obtain that

$$\frac{d}{dt}\|u\|^2_{H_0^1} = 2(-\Delta u, u_t) \text{ for a.a. } t. \tag{2.2.21}$$

and $u \in C((0, +\infty), H_0^1(\Omega))$.

We note that the set of regular solutions of that kind is non-empty if we assume (2.1.15), as using inequalities (2.1.51)-(2.1.55) in the proof of Theorem 2.9 we prove that the regular solution satisfies (2.2.20).

We also observe that we can force all the regular solutions to satisfy

$$\frac{du}{dt} \in L^2(\varepsilon, T; L^2(\Omega))$$

with an additional assumption on the constant p, which is weaker than (2.1.10). This is achieved by obtaining that $f(u) \in L^2(\varepsilon, T; L^2(\Omega))$, which can be done by using an interpolation inequality. Indeed, for

$$u \in L^\infty(\varepsilon, T; H_0^1(\Omega)) \cap L^2(\varepsilon, T; D(A))$$

we have the interpolation inequality

$$\|u\|^{2(\gamma+1)}_{L^{2(\gamma+1)}(\varepsilon,T;L^{2(\gamma+1)}(\Omega))} \leq \|u\|^{2\gamma}_{L^\infty(\varepsilon,T;L^{p_1}(\Omega))} \|u\|^2_{L^2(\varepsilon,T;L^{p_2}(\Omega))}, \tag{2.2.22}$$

where $\gamma = \frac{4}{n-2}$, $p_1 = \frac{2n}{n-2}$, $p_2 = \frac{2n}{n-4}$, provided that $n > 4$; $\gamma < 2$, $p_1 = 4$, $p_2 = \frac{4}{2-\gamma}$ if $n = 4$; $\gamma = 3$, $p_1 = 6$, $p_2 = +\infty$ if $n = 3$; and $\gamma \geq 0$ is arbitrary for $n = 1, 2$. We have used the embeddings $H_0^1(\Omega) \subset L^{p_1}(\Omega)$, $H^2(\Omega) \subset L^{p_2}(\Omega)$ and [90, Lemma II.4.1, p. 72]. Thus, (2.1.6) implies that $f(u) \in L^2(\varepsilon, T; L^2(\Omega))$ if

$$p \leq \gamma + 2 \tag{2.2.23}$$

and also that

$$\|f(u)\|^2_{L^2(\varepsilon,T;L^2(\Omega))} = \int_\varepsilon^T \int_\Omega |f(u(x,t))|^2 dxdt \leq C_1 + C_2 \int_\varepsilon^T \int_\Omega |u(x,t)|^{2(\gamma+1)} dxdt. \tag{2.2.24}$$

Condition (2.2.23) also implies $H_0^1(\Omega) \subset L^p(\Omega)$, so $u \in C((0, +\infty), L^p(\Omega))$.

Using the regularity of regular solutions and either (2.1.44) or (2.2.24), another necessary property to obtain estimates is proved by regularization.

Lemma 2.20. *Assume condition (2.1.5) for $f \in (\mathbb{R})$ and one of the following assumptions:*

1. *$p \leq \frac{2n-2}{n-2}$, for $n \geq 3$, $u \in L^p(0,T;L^p(\Omega)) \cap L^\infty(\varepsilon,T;H_0^1(\Omega))$ and $\dfrac{du}{dt} \in L^2(\varepsilon,T;L^2(\Omega))$, for all $0<\varepsilon<T$.*

2. *$p \leq \gamma+2$ (where γ comes from the interpolation), $u \in L^p(0,T;L^p(\Omega)) \cap L^\infty(\varepsilon,T;H_0^1(\Omega)) \cap L^2(\varepsilon,T;D(A))$ and $\dfrac{du}{dt} \in L^2(\varepsilon,T;L^2(\Omega))$, for all $0<\varepsilon<T$.*

3. *$u \in L^\infty(0,T;L^\infty(\Omega))$ and $\frac{du}{dt} \in L^2(\varepsilon,T;L^2(\Omega))$, for all $0<\varepsilon<T$.*

Then the map $t \mapsto \int_\Omega \mathcal{F}(u(t))dx$ is absolutely continuous on $[\varepsilon,T]$ for all $0<\varepsilon<T$ and the equality

$$\frac{d}{dt}\int_\Omega \mathcal{F}(u(t))dx = \left(f\left(u\left(t\right)\right), \frac{du}{dt}\left(t\right)\right), \text{ for a.a. } t>0, \tag{2.2.25}$$

is true for any regular solution.

Proof. Let first assume condition 1. Arguing as in [49, pp. 173-175] one can show that there exists a sequence $\{u^n\}$ such that

$$u^n \in C^1([0,\infty);L^p(\Omega)), \tag{2.2.26}$$
$$u^n \to u \text{ in } L^p(0,T;L^p(\Omega)), \tag{2.2.27}$$
$$\frac{du^n}{dt} \to \frac{du}{dt} \text{ in } L^2(\varepsilon,T;L^2(\Omega)), \tag{2.2.28}$$
$$\{u^n\} \text{ is bounded in } L^\infty(\varepsilon,T;H_0^1(\Omega)), \tag{2.2.29}$$

for all $0<\varepsilon<T$. We can deduce that

$$\frac{d}{dt}\int_\Omega \mathcal{F}(u^n(t,x))dx = \left(f(u^n(t)), \frac{du^n}{dt}(t)\right), \quad \text{for all } t \geq 0. \tag{2.2.30}$$

By (2.2.27) we have passing to a subsequence that $u^n(t,x) \to u(t,x)$ for a.a. (t,x).

Hence, by (2.1.5) and the Lebesgue theorem we infer that

$$\mathcal{F}(u^n) \to \mathcal{F}(u) \text{ in } L^1(0,T;L^1(\Omega)), \tag{2.2.31}$$

so

$$\frac{d}{dt}\int_\Omega \mathcal{F}(u^n(t,x))dx \to \frac{d}{dt}\int_\Omega \mathcal{F}(u(t,x))dx, \tag{2.2.32}$$

in the sense of distributions.

Finally, the inequality

$$\|f(u^n(t))\|_{L^2}^2 \leq 2C\left(1+\int_\Omega |u^n(t,x)|^{2(p-1)}dx\right) \leq \widetilde{C}\left(1+\|u^n(t)\|_{H_0^1}^{2(p-1)}\right)$$

implies that

$$\{f(u^n)\} \text{ is bounded in } L^2(\varepsilon,T;L^2(\Omega)), \tag{2.2.33}$$

which gives by a standard argument that

$$f(u^n) \rightharpoonup f(u) \text{ in } L^2(\varepsilon,T;L^2(\Omega)), \tag{2.2.34}$$

so

$$\left(f(u^n(\cdot)),\frac{du^n}{dt}(\cdot)\right) \to \left(f(u(\cdot)),\frac{du}{dt}(\cdot)\right) \text{ in } L^1(\varepsilon,T). \tag{2.2.35}$$

Passing to the limit we have

$$\frac{d}{dt}\int_\Omega \mathcal{F}(u(\cdot,x))dx = \left(f(u(\cdot)),\frac{du}{dt}(\cdot)\right) \tag{2.2.36}$$

in the sense of distributions.

As $\left(f(u(\cdot)),\frac{du}{dt}(\cdot)\right) \in L^1(\varepsilon,T)$ for all $0<\varepsilon<T$, we obtain that $\int_\Omega \mathcal{F}(u(t,x))dx$ is absolutely continuous on $[\varepsilon,T]$ and (2.2.25) holds true.

Assuming now condition 2, as before, we obtain (2.2.26)-(2.2.32) and additionally that

$$u^n \to u \text{ in } L^2(\varepsilon, T; D(A)).$$

Hence, the inequalities (2.2.22), (2.2.24) and the embedding $H_0^1(\Omega) \subset L^{p_1}(\Omega)$, $H^2(\Omega) \subset L^{p_1}(\Omega)$ imply that (2.2.33)-(2.2.36) hold and we finish the proof in the same way.

Finally, for condition 3 we obtain as before that (2.2.26)-(2.2.28), (2.2.30)-(2.2.32) hold. In addition,

$$\{u^n\} \text{ is bounded in } L^\infty(0,T; L^\infty(\Omega)),$$

so

$$\{f(u^n)\} \text{ is bounded in } L^\infty(0,T; L^\infty(\Omega)).$$

The rest of the proof follows the same lines. □

Therefore, under either the conditions of Theorem 2.9 with the extra assumption (2.2.23) or the conditions of Theorem 2.12 we define the set

$$\mathcal{R} = K_r^+ := \{u(\cdot) : u \text{ is a regular solution of } (2.1.1)\}.$$

We define the (possibly multivalued) map $G_r : \mathbb{R}^+ \times L^2(\Omega) \to P(L^2(\Omega))$ by

$$G_r(t, u_0) = \{u(t) : u \in K_r^+ \text{ and } u(0) = u_0\}.$$

With respect to the axiomatic properties $(K1) - (K4)$ given in Chapter 0, we observe that obviously $(K1)$ is true, and $(K2)$ can be proved easily using equality (2.1.17). Therefore, G_r is a multivalued semiflow by the results of the previous section. In this case we are not able to prove $(K3)$, so G_r could be non-strict. Further we will prove that $(K4)$ holds true.

Lemma 2.21. *Let us assume (2.1.3)-(2.1.5), (2.1.15) and (2.2.1). Additionally, assume one of the following assumptions:*

1. *(2.1.9) and (2.2.23) hold;*

2. *(2.1.10) is true.*

Given a sequence $\{u^n\} \subset K_r^+$ *such that* $u^n(0) \to u_0$ *weakly in* $L^2(\Omega)$*, there exists a subsequence of* $\{u^n\}$ *(relabeled the same) and* $u \in K_r^+$*, satisfying* $u(0) = u_0$*, such that*

$$u^n(t) \to u(t) \text{ strongly in } H_0^1(\Omega) \quad \forall t > 0.$$

Proof. We take an arbitrary $T > 0$. Arguing as in the proof of Theorem 2.9 we obtain the existence of a subsequence of u^n such that

$$\begin{aligned} \{u^n\} &\text{ is bounded in } L^\infty(0,T;L^2(\Omega)), \\ \{u^n\} &\text{ is bounded in } L^p(0,T;L^p(\Omega)), \\ \{f(u^n)\} &\text{ is bounded in } L^q(0,T;L^q(\Omega)). \end{aligned} \tag{2.2.37}$$

The only difference is that we obtain inequality (2.1.25) in an arbitrary interval $[\varepsilon, T]$ and then pass to the limit as $\varepsilon \to 0$ (see the proof of Proposition 2.15).

Since $\dfrac{du^n}{dt} \in L^2(\varepsilon,T;L^2(\Omega))$, for any $\varepsilon > 0$, we have that $u \in C((0,T], H_0^1(\Omega))$ and we know that (2.2.21), (2.2.25) are true. Therefore, arguing as in the proofs of Theorems 2.9 and 2.12 and using (2.2.24) and (2.1.44) there exists $u \in L^\infty(\varepsilon,T;L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega))$ and a subsequence $\{u^n\}$, relabelled the same, such that

$$\begin{aligned} u_n &\overset{*}{\rightharpoonup} u \text{ in } L^\infty(0,T;L^2(\Omega)) \\ u_n &\overset{*}{\rightharpoonup} u \text{ in } L^\infty(\varepsilon,T;H_0^1(\Omega)) \\ u_n &\rightharpoonup u \text{ in } L^2(0,T;H_0^1(\Omega)) \\ u_n &\rightharpoonup u \text{ in } L^p(0,T;L^p(\Omega)) \\ u_n &\rightharpoonup u \text{ in } L^2(\varepsilon,T;D(A)), \\ \frac{du_n}{dt} &\rightharpoonup \frac{du}{dt} \text{ in } L^2(\varepsilon,T;L^2(\Omega)) \\ f(u_n) &\rightharpoonup f(u) \text{ in } L^q(0,T;L^q(\Omega)), \\ f(u_n) &\rightharpoonup f(u) \text{ in } L^2(\varepsilon,T;L^2(\Omega)), \\ a(\|u_n\|^2_{H_0^1})\Delta u_n &\rightharpoonup a(\|u\|^2_{H_0^1})\Delta u \text{ in } L^2(\varepsilon,T;L^2(\Omega)). \end{aligned} \tag{2.2.38}$$

In view of (2.2.38), the Aubin-Lions Compactness Lemma gives

$$u_n \to u \text{ in } L^2(\varepsilon, T; H_0^1(\Omega)). \tag{2.2.39}$$

Since the sequence $\{u^n\}$ is equicontinuous in $L^2(\Omega)$ on $[\varepsilon, T]$ and bounded in $C([\varepsilon, T], H_0^1(\Omega))$, by the compact embedding $H_0^1(\Omega) \subset L^2(\Omega)$ and the Ascoli-Arzelà theorem, a subsequence fulfills

$$u^n \to u \text{ in } C([\varepsilon, T], L^2(\Omega)),$$

$$u^n(t) \rightharpoonup u(t) \text{ in } H_0^1(\Omega) \quad \forall t \in [\varepsilon, T].$$

By a similar argument as in the proof of Theorem 2.9 we establish that $u \in K_r^+$, $u\,(0) = u_0$.

Finally, we shall prove that $u^n(t) \to u(t)$ in $H_0^1(\Omega)$ for all $t \in [\varepsilon, T]$.
Multiplying (2.1.1) by u_t^n and using (2.1.35), (2.2.21), and (2.2.25) we obtain

$$\frac{1}{2}\left\|\frac{du^n}{dt}\right\|_{L^2}^2 + \frac{d}{dt}\left(\frac{1}{2}A(\|u^n(t)\|_{H_0^1}^2 - \int_\Omega \mathcal{F}(u^n(t))dx\right) \leq \frac{1}{2}\|h\|_{L^2}^2 = D.$$

Thus,

$$\frac{1}{2}A(\|u^n(t)\|_{H_0^1}^2) - \int_\Omega \mathcal{F}(u^n(t))dx \leq \frac{1}{2}A(\|u^n(s)\|_{H_0^1}^2) - \int_\Omega \mathcal{F}(u^n(s))dx + D(t-s),$$

where $t \geq s \geq \varepsilon > 0$.
The same inequality is valid for the limit function $u(\cdot)$. We observe that the map

$$y \longmapsto \int_\Omega \mathcal{F}(y\,(x))dx$$

is continuous in the topology of $H_0^1\,(\Omega)$, which follows easily from $H_0^1\,(\Omega) \subset L^p\,(\Omega)$ and (2.1.8) using Lebesgue's theorem.
Hence, the functions

$$J_n(t) = \frac{1}{2}A(\|u^n(t)\|_{H_0^1}^2) - \int_\Omega \mathcal{F}(u^n(t))dx - Dt$$

and

$$J(t) = \frac{1}{2}A(\|u(t)\|^2_{H_0^1}) - \int_\Omega \mathcal{F}(u(t))dx - Dt$$

are continuous and non-increasing in $[\varepsilon, T]$.
Moreover, from (2.2.39) we deduce that $J_n(t) \to J(t)$ for a.e. $t \in (\varepsilon, T)$. Take $\varepsilon < t_m < T$ such that $t_m \to T$ and $J_n(t_m) \to J(t_m)$ for all m. Then

$$J_n(T) - J(T) \leq J_n(t_m) - J(T) \leq |J_n(t_m) - J(t_m)| + |J(t_m) - J(T)|.$$

For any $\delta > 0$ there exist $m(\delta)$ and $N(m(\delta))$ such that $J^n(T) - J(T) \leq \delta$ if $n \geq N$. Then $\limsup J_n(T) \leq J(T)$, so $\limsup \|u^n(T)\|^2_{H_0^1} \leq \|u(T)\|^2_{H_0^1}$ (see the explanation below).
As $u^n(T) \to u(T)$ weakly in $H_0^1(\Omega)$ implies $\liminf \|u^n(T)\|^2_{H_0^1} \geq \|u(T)\|^2_{H_0^1}$, we obtain

$$\|u^n(T)\|^2_{H_0^1} \to \|u(T)\|^2_{H_0^1},$$

so that $u^n(T) \to u(T)$ strongly in $H_0^1(\Omega)$.

In order to finish the proof rigorously, we have to justify that $\limsup J_n(T) \leq J(T)$ implies the inequality $\limsup \|u^n(T)\|^2_{H_0^1} \leq \|u(T)\|^2_{H_0^1}$. First, we observe that by (2.1.8) we have

$$\left| \int_\Omega \mathcal{F}(u_n(T,x))dx \right| \leq C \int_\Omega (1 + |u_n(T,x)|^p)\, dx,$$

so the boundedness of $u_n(T)$ in $L^p(\Omega)$ implies that $-\int_\Omega \mathcal{F}(u_n(T,x))dx < \infty$. Also, (2.1.7) gives $-\mathcal{F}(u_n(T,x)) \geq -\widetilde{\kappa}$, so by Fatou's lemma we obtain

$$\begin{aligned} \liminf \left(-\int_\Omega \mathcal{F}(u_n(T,x))dx \right) &\geq \int_\Omega \liminf \left(-\mathcal{F}(u_n(T,x))\right) dx \\ &= -\int_\Omega \mathcal{F}(u(T,x))dx, \end{aligned}$$

where we have used that $\mathcal{F}(u_n(T,x) \to \mathcal{F}(u(T,x))$ for a.a. $x \in \Omega$. By contradiction let us assume that $\limsup \|u_n(T)\|_{H_0^1} > \|u(T)\|$.

Then using the continuity of the function $A(s)$ we have

$$
\begin{aligned}
&\limsup \left(\frac{1}{2} A\left(\|u^n(T)\|^2_{H_0^1} \right) - \int_\Omega \mathcal{F}(u_n(T,x))dx \right) \\
\geq &\limsup \frac{1}{2} \int_0^{\|u^n(T)\|^2_{H_0^1}} a(s)\, ds + \liminf \left(-\int_\Omega \mathcal{F}(u_n(T,x))dx \right) \\
\geq &\frac{1}{2} \int_0^{\limsup \|u^n(T)\|^2_{H_0^1}} a(s)\, ds - \int_\Omega \mathcal{F}(u_n(T,x))dx \\
> &\frac{1}{2} \int_0^{\|u(T)\|^2_{H_0^1}} a(s)\, ds - \int_\Omega \mathcal{F}(u_n(T,x))dx,
\end{aligned}
$$

which is a contradiction with $\limsup J_n(T) \leq J(T)$. □

Corollary 2.22. *Assume the conditions of Lemma 2.21. Then the set K_r^+ satisfies condition $(K4)$.*

Proposition 2.23. *Assume the conditions of Lemma 2.21. The multivalued semi-flow G_r is upper semicontinuous for all $t \geq 0$, that is, for any neighborhood $O(G_r(t,u_0))$ in $L^2(\Omega)$ there exists $\delta > 0$ such that if $\|u_0 - v_0\| < \delta$, then $G_r(t,v_0) \subset O$. Also, it has compact values.*

Proof. We argue by contradiction. Assume that there exists $t \geq 0, u_0 \in L^2(\Omega)$, a neighbourhood $O(G_r(t,u_0))$ and a sequence $\{y_n\}$ which fulfills that each $y_n \in G_r(t,u_0^n)$, where u_0^n converges strongly to u_0 in $L^2(\Omega)$, and $y_n \notin O(G_r(t,u_n))$ for all $n \in \mathbb{N}$. Since $y_n \in G_r(t,u_0^n)$ for all n, there exists $u^n \in K_r^+$, $u^n(0) = u_0^n$, such that $y_n = u^n(t)$.

Now, since $\{u_0^n\}$ is a convergent sequence of initial data, making use of Lemma 2.21 there exists a subsequence of $\{u^n\}$ which converges to a function $u \in K_r^+$. Hence,

$$y_n \to y \in G_r(t,u_0).$$

This is a contradiction because $y_n \notin O(G_r(t,u_0))$ for any $n \in \mathbb{N}$. □

Proposition 2.24. *Assume the conditions of Lemma 2.21. Then there exists an absorbing set B_1 for G_r, which is compact in $H_0^1(\Omega)$ and $L^p(\Omega)$.*

Proof. Reasoning as in Proposition 2.15, we obtain an absorbing set B_0 in $L^2(\Omega)$.

Let $K>0$ be such that $\|y\| \leq K$ for all $y \in B_0$. Since $\dfrac{du}{dt} \in L(\varepsilon, T; L^2(\Omega))$ and (2.2.25) holds, we are allowed to multiply (2.1.1) by u_t, use (2.2.21) and argue as in (2.1.51)-(2.1.54) to obtain the existence of a constant C such that

$$\|u(1)\|_{H_0^1}^2 + \|u(1)\|_{L^p}^p \leq C(1+\|u(0)\|_{L^2}^2), \tag{2.2.40}$$

for any regular solution $u(\cdot)$ with initial condition $u(0)$.

For any $u_0 \in L^2(\Omega)$ with $\|u_0\|_{L^2} \leq R$ and any $u \in K_r^+$ such that $u(0) = u_0$, the semiflow property $G_r(t+1,u_0) \subset G_r(1, G_r(t,u_0))$ and $G_r(t,u_0) \subset B_0$, if $t \geq t_0(R)$, imply that

$$\|u(t+1)\|_{H_0^1}^2 + \|u(t+1)\|_{L^p}^p \leq C(1+K^2) \ \forall t \geq t_0(R).$$

Then there exists $M>0$ such that the closed ball B_M in $H_0^1(\Omega)$ centered at 0 with radius M is absorbing for G_r.

By Lemma 2.21 the set $B_1 = \overline{G_r(1,B_M)}$ is an absorbing set which is compact in $H_0^1(\Omega)$. The embedding $H_0^1(\Omega) \subset L^p(\Omega)$ implies that it is compact in $L^p(\Omega)$ as well. □

Theorem 2.25. *Assume the conditions of Lemma 2.21. Then the multivalued semiflow G_r possesses a global compact attractor $\mathcal{A}_r$. Moreover, for any set B bounded in $L^2(\Omega)$ we have*

$$dist_{H_0^1}(G_r(t,B), \mathcal{A}_r) \to 0 \quad \text{as } t \to \infty. \tag{2.2.41}$$

Also $\mathcal{A}_r$ is compact in $H_0^1(\Omega)$ and $L^p(\Omega)$.

Proof. From Propositions 2.23 and 2.24 we deduce that the multivalued semiflow G_r is upper semicontinuous with closed values and the existence of an absorbing which is compact in $H_0^1(\Omega)$ and $L^p(\Omega)$. Therefore, by Theorem 0.17 the existence of the global attractor and its compactness in $H_0^1(\Omega)$ and $L^p(\Omega)$ follow.

The proof of (2.2.41) is analogous to that in Theorem 29 in [57]. □

The set of all complete trajectories of K_r^+ (see Definition 0.2) will be denoted by $\mathbb{F}_r$. Moreover, we write $\mathbb{K}_r$ as the set of all complete trajectories which are bounded in $L^2(\Omega)$, and $\mathbb{K}_r^1$ as the ones bounded in $H_0^1(\Omega)$.

Lemma 2.26. *Assume the conditions of Lemma 2.21. Then the sets defined above coincide, that is,* $\mathbb{K}_r = \mathbb{K}_r^1$.

Proof. Let $\gamma(\cdot) \in \mathbb{K}_r$. Then there is C such that $\|\gamma(t)\|_{L^2} \leq C$ for any $t \in \mathbb{R}$. Let $u_\tau(\cdot) = \gamma(\cdot + \tau)$ for any τ, which is a regular solution. Since

$$\frac{du}{dt} \in L^2(\varepsilon, T; L^2(\Omega)),$$

for any $\varepsilon > 0$, the equality (2.2.21) holds true. Also, (2.2.25) is satisfied. Therefore, we can multiply the equation in (2.1.1) by u_t and apply again similar arguments as in Theorem 2.12 to deduce that

$$\|u(t+r)\|_{H_0^1}^2 \leq \frac{K_1(T)\left(1 + \|u(0)\|_{L^2}^2\right)}{r} + K_2(T) \text{ for any } 0 < r < T. \tag{2.2.42}$$

Denote $B_\gamma = \cup_{t\in\mathbb{R}}\gamma(t)$. Therefore,

$$B_\gamma \subset G_r(1, B_\gamma)$$

and (2.2.42) implies that B_γ is bounded in $H_0^1(\Omega)$, so $\gamma(\cdot) \in \mathbb{K}_r^1$.
The other inclusion is obvious. □

In view of Corollary 2.22 and Theorem 0.15, the global attractor is characterized in terms of bounded complete trajectories:

$$\begin{aligned} \mathcal{A}_r &= \{\gamma(0) : \gamma(\cdot) \in \mathbb{K}_r\} = \{\gamma(0) : \gamma(\cdot) \in \mathbb{K}_r^1\} \\ &= \bigcup_{t\in\mathbb{R}}\{\gamma(t) : \gamma(\cdot) \in \mathbb{K}_r\} = \bigcup_{t\in\mathbb{R}}\{\gamma(t) : \gamma(\cdot) \in \mathbb{K}_r^1\}. \end{aligned} \tag{2.2.43}$$

The set $\mathfrak{R}_{K_r^+}$ was defined at the beginning of this section as the set of fixed points of K_r^+, which means that $z \in \mathfrak{R}_{K_r^+}$ if the function $u(\cdot)$ defined by $u(t) = z$, for all $t \geq 0$, belongs to K_r^+. This set can be characterized as follows.

Lemma 2.27. *Assume the conditions of Lemma 2.21. Let* $\mathfrak{R}$ *be the set of* $z \in H^2(\Omega) \cap H_0^1(\Omega)$ *such that*

$$-a(\|z\|_{H_0^1}^2)\Delta z = f(z) + h \text{ in } L^2(\Omega). \tag{2.2.44}$$

Then $\mathfrak{R}_{K_r^+} = \mathfrak{R}$.

Proof. If $z \in \mathfrak{R}_{K_r^+}$, then $u(t) \equiv z \in K_r^+$. Thus, $u(\cdot)$ satisfies (2.1.17) and

$$\frac{du}{dt} = 0 \text{ in } L^2\left(0, T; L^2(\Omega)\right),$$

so (2.2.44) is satisfied.

Let $z \in \mathfrak{R}$. Then the map $u(t) \equiv z$ satisfies (2.2.44) for any $t \geq 0$ and

$$\frac{du}{dt} = 0 \text{ in } L^2\left(0, T; L^2(\Omega)\right),$$

so (2.1.17) holds true. □

The following result is proved exactly as Theorem 2.18.

Theorem 2.28. *Assume the conditions of Lemma 2.21. Then the global attractor* $\mathcal{A}$ *is bounded in* $L^\infty(\Omega)$*, provided that* $h \in L^\infty(\Omega)$.

We are now ready to obtain the characterization of the global attractor.

Theorem 2.29. *Assume the conditions of Lemma 2.21. Then it holds that*

$$\mathcal{A}_r = M_r^u(\mathfrak{R}) = M_r^s(\mathfrak{R}),$$

where

$$M_r^s(\mathfrak{R}) = \{z : \exists \gamma(\cdot) \in \mathbb{K}_r,\ \gamma(0) = z,\ dist_{L^2(\Omega)}(\gamma(t), \mathfrak{R}) \to 0,\ t \to +\infty\}, \tag{2.2.45}$$

$$M_r^u(\mathfrak{R}) = \{z : \exists \gamma(\cdot) \in \mathbb{F}_r,\ \gamma(0) = z,\ dist_{L^2(\Omega)}(\gamma(t), \mathfrak{R}) \to 0,\ t \to -\infty\}. \tag{2.2.46}$$

Remark 2.30. In the definition of $M_r^u(\mathfrak{R})$ we can replace $\mathbb{F}_r$ by $\mathbb{K}_r$. Also, as the global attractor $\mathcal{A}$ is compact in $H_0^1(\Omega)$, in the definitions of $M_r^s(\mathfrak{R})$ and $M_r^u(\mathfrak{R})$, it is equivalent to write $H_0^1(\Omega)$ instead of $L^2(\Omega)$.

Proof. We consider the function $E : \mathcal{A}_r \to \mathbb{R}$

$$E(y) = \frac{1}{2}A(\|y\|_{H_0^1}^2) - \int_\Omega \mathcal{F}(y(x))dx - \int_\Omega h(x)\, y(x)\, dx, \tag{2.2.47}$$

where $A(r) = \int_0^r a(s)ds$.

We observe that $E(y)$ is continuous in $H_0^1(\Omega)$. Indeed, the maps

$$y \mapsto \frac{1}{2}A(\|y\|_{H_0^1}^2)$$

and

$$y \mapsto \int_\Omega h(x)\, y(x)\, dx$$

are obviously continuous in $H_0^1(\Omega)$.

On the other hand, both conditions (2.1.10) and (2.2.23) imply that

$$H_0^1(\Omega) \subset L^p(\Omega),$$

so making use of the Lebesgue theorem the continuity of

$$y \mapsto \int_\Omega \mathcal{F}(y(x))dx$$

follows as well.

Since $\frac{du}{dt} \in L^2(\varepsilon, T; L^2(\Omega))$ and (2.2.25) holds for any $u \in K_r^+$ and $0 < \varepsilon < T$, we obtain the energy equality

$$\int_s^t \|\frac{d}{dr}u(r)\|_{L^2}^2 dr + E(u(t)) = E(u(s)) \quad \text{for all } t \geq s > 0. \tag{2.2.48}$$

Hence, $E(u(t))$ is non-increasing and, by (2.1.4) and (2.1.7), bounded from below. Thus, $E(u(t)) \to l$, as $t \to +\infty$, for some $l \in \mathbb{R}$.

Let $x \in \mathcal{A}_r$ and $\gamma(0) = x$, where $\gamma \in \mathbb{K}_r$. We reason by contradiction, so let suppose that there exists $\varepsilon > 0$ and a sequence $\gamma(t_n)$, $t_n \to +\infty$, such that

$$dist_{\,L^2(\Omega)}(\gamma(t_n), \mathfrak{R}) > \varepsilon.$$

In view of Theorem 2.25, $\mathcal{A}_r$ is compact in $H_0^1(\Omega)$, so we can take a converging subsequence (relabeled the same) such that $\gamma(t_n) \to y$ in $H_0^1(\Omega)$, where $t_n \to +\infty$. Since the function $E : H_0^1(\Omega) \to \mathbb{R}$ is continuous, it follows that $E(y) = l$. We obtain a contradiction by proving that $y \in \mathfrak{R}$. In view of Lemma 2.21, there exists $v \in K_r^+$ and a subsequence $v_n(\cdot) = \gamma(\cdot + t_n)$ such that $v(0) = y$ and $v_n(t) \to v(t) = z$ in $H_0^1(\Omega)$ for $t > 0$. Thus, $E(v_n(t)) \to E(z)$ implies that $E(z) = l$. Also, $v(\cdot)$ satisfies the energy equality for all $0 \leq s \leq t$, so that

$$l + \int_0^t \|v_r\|_{L^2}^2 dr = E(z) + \int_0^t \|v_r\|_{L^2}^2 dr = E(v(0)) = E(y) = l.$$

Therefore, $\dfrac{dv}{dt}(t) = 0$ for a.a. t, and then by Lemma 2.27 we have $y \in \mathfrak{R}_{K_r^+} = \mathfrak{R}$. As a consequence, $\mathcal{A}_r \subset M_r^s(\mathfrak{R})$. The converse inclusion follows from (2.2.43).

For the second equality we observe that for any $\gamma \in \mathbb{F}_r$ the energy equality (2.2.48) is satisfied for all $-\infty < s \leq t$. Let $x \in \mathcal{A}_r$ and let $\gamma \in \mathbb{K}_r = \mathbb{K}_r^1$ (cf. Lemma 2.26) be such that $\gamma(0) = x$. Since the second term of the energy function is bounded from above by (2.1.7), $E(\gamma(t)) \to l$, as $t \to -\infty$, for some $l \in \mathbb{R}$. We reason as before, so let suppose that there exists $\varepsilon > 0$ and a sequence $\gamma(-t_n)$, $t_n \to \infty$, such that

$$dist_{\,L^2(\Omega)}(\gamma(-t_n), \mathfrak{R}) > \varepsilon,$$

and we have that $\gamma(-t_n) \to y$ in $H_0^1(\Omega)$, $E(y) = l$. Moreover, for a fixed $t > 0$, there exists $v \in K_r^+$ and a subsequence of $v_n(\cdot) = \gamma(\cdot - t_n)$ (relabeled the same) such that $v(0) = y$ and $v_n(t) \to v(t) = z$ in $H_0^1(\Omega)$. Therefore, $E(v_n(t)) \to E(z)$ implies that $E(z) = l$ and reasoning as before we get a contradiction since it follows that $y \in \mathfrak{R}$. Hence, $\mathcal{A}_r \subset M_r^u(\mathfrak{R})$ and the converse inclusion follows from (2.2.43). $\square$

We can improve the regularity of the global attractor of the semigroup T_r of Section 2.2.2 and obtain its characterization

Lemma 2.31. *Let the conditions of Theorem 2.17 hold. Then the global attractor $\mathcal{A}_r$ of the semigroup T_r is compact in $H_0^1(\Omega)$, bounded in $L^p(\Omega)$ and the convergence takes place in the topology of $H_0^1(\Omega)$, that is,*

$$dist_{H_0^1(\Omega)}(T_r(t,B),\mathcal{A}) \to 0, \text{ as } t \to +\infty,$$

for any set B bounded in $L^2(\Omega)$.

Proof. The estimates of Lemma 2.21 can be justified for T_r via Galerkin approximations, so in this case we do not need to impose assumption (2.2.23) in order to use (2.2.25). Thus, the proof follows the same lines as in Proposition 2.24 and Theorem 2.25. □

Proposition 2.32. *Let the conditions of Theorem 2.17 hold. Also, assume one of the following conditions:*

1. $h \in L^\infty(\Omega)$;
2. $p \le \frac{2n}{n-2}$ *if* $n \ge 3$.

Then the global attractor $\mathcal{A}_r$ can be characterized as follows:

$$\mathcal{A}_r = M_r^u(\mathfrak{R}) = M_r^s(\mathfrak{R}),$$

where $M_r^s(\mathfrak{R})$, $M_r^u(\mathfrak{R})$ are defined in (2.2.45)-(2.2.46).

Proof. We recall that a function $E : \mathcal{A} \to \mathbb{R}$ is a Lyapunov functional if E is continuous (with respect to the topology of $H_0^1(\Omega)$), for any $u_0 \in \mathcal{A}$ the map $t \mapsto E(T_r(t,u_0))$ is non-increasing and $E(T_r(\tau,u_0)) = E(u_0)$, for some $\tau > 0$, implies that $u(\cdot)$ is a fixed point. We estate that the function E given in (2.2.47) is a Lyapunov functional for the semigroup T_r.

We prove that $E(y)$ is continuous. First, the maps

$$y \mapsto \frac{1}{2} A(\|y\|^2_{H^1_0}),$$

and

$$y \mapsto \int_\Omega h(x)\, y(x)\, dx$$

are obviously continuous in $H^1_0(\Omega)$. Second, if $h \in L^\infty(\Omega)$, taking into account that $\mathcal{A}$ is bounded in $L^\infty(\Omega)$ by Theorem 2.18, it follows that

$$\begin{aligned} \left| \int_\Omega \mathcal{F}(y_1) - \mathcal{F}(y_2) dx \right| &= \left| \int_\Omega \int_{y_2(x)}^{y_1(x)} f(s) ds dx \right| \\ \leq \int_\Omega C_1 |y_1(x) - y_2(x)| dx &\leq C_2 \|y_1 - y_2\|_{L^2}, \end{aligned}$$

so

$$y \mapsto \int_\Omega \mathcal{F}(y(x)) dx$$

is continuous as well. In the case of the second condition, this result follows from the embedding $H^1_0(\Omega) \subset L^p(\Omega)$ and the Lebesgue theorem.

Multiplying the equation in (2.1.1) by u_t we obtain the energy inequality

$$\int_s^t \|\frac{d}{dr} u(r)\|^2_{L^2} dr + E(u(t)) \leq E(u(s)), \qquad \text{for all } t \geq s,$$

if $u(\cdot)$ is a bounded complete trajectory of T_r. This calculation is rigorous when $h \in L^\infty(\Omega)$ as the boundedness of the solutions in $L^\infty(\mathbb{R}; L^\infty(\Omega))$ implies by Lemma 2.20 that (2.2.25) is true. Under the second condition, the calculations are formal but can be justified via Galerkin approximations. Hence, $E(u(t))$ is non-increasing as a function of t. Also, if $E(u(\tau)) = E(u_0)$, then

$$\|\frac{du}{dt}(t)\|^2_{L^2} = 0$$

for a.a. $0 < t < \tau$, so u must be a fixed point.

The result follows then from [8, p.160]. $\square$

2.2.4. Strong solutions

We split this part into two cases.

2.2.5. Attractor in the phase space $H_0^1(\Omega)$

If we assume conditions (2.1.3)-(2.1.5), (2.2.1) and that either p satisfies (2.1.10) or that (2.1.9) is satisfied, then we know by Theorems 2.10 and 2.11 that for any $u_0 \in H_0^1(\Omega) \cap L^p(\Omega)$ there exists at least one strong solution $u(\cdot)$.

In the first case, $H_0^1(\Omega) \subset L^p(\Omega)$ implies that $H_0^1(\Omega) \cap L^p(\Omega) = H_0^1(\Omega)$. This is also true in the second case if we assume additionally that (2.2.23) holds true. Under such assumptions we define then the set

$$\mathcal{R} = K_s^+ := \{u(\cdot) : u \text{ is a strong solution of (2.1.1) with } u(0) \in H_0^1(\Omega)\}.$$

We define the (possibly multivalued) map $G_s : \mathbb{R}^+ \times H_0^1(\Omega) \to P(H_0^1(\Omega))$ by

$$G_s(t, u_0) = \{u(t) : u \in K_s^+ \text{ and } u(0) = u_0\}.$$

With respect to the axiomatic properties $(K1)-(K4)$ given in Chapter 0, property $(K1)$ is obviously true, and $(K2)-(K3)$ can be proved easily using equality (2.1.17). Therefore, G_s is a strict multivalued semiflow by the results of Chapter 0.

We shall obtain a similar result as in Lemma 2.21.

Lemma 2.33. *Let assume conditions (2.1.3)-(2.1.5), (2.2.1). Additionally, assume one of the following assumptions:*

1. *(2.1.9) and (2.2.23) hold;*

2. *(2.1.10) is true.*

Given a sequence $\{u^n\} \subset K_s^+$ such that $u^n(0) \to u_0$ weakly in $H_0^1(\Omega)$, there exists a subsequence of $\{u^n\}$ (relabeled the same) and $u \in K_s^+$, satisfying $u(0) = u_0$, such that

$$u^n(t) \to u(t) \quad \text{in } H_0^1(\Omega), \ \forall t > 0.$$

Proof. Since $\frac{du^n}{dt} \in L^2(0,T;L^2(\Omega))$ and (2.2.25) hold, we can use (2.2.21) and multiplying (2.1.1) by u_t and integrating between s and t we obtain

$$\int_s^t \|\frac{d}{dr}\|u(r)\|_{L^2}^2 dr + E(u(t)) = E(u(s)) \quad \text{for all } t \geq s \geq 0,$$

where E was defined in (2.2.47). Therefore, by (2.1.4) and (2.1.7) we have that

$$\begin{aligned}&\int_0^t \|\frac{d}{dr}u(r)\|_{L^2}^2 dr + \frac{m}{4}\|u(t)\|_{H_0^1}^2 + \widetilde{\alpha}_1\|u(t)\|_{L^p}^p \\ &\leq \frac{1}{2}A(\|u(0)\|_{H_0^1}^2) + \widetilde{\alpha}_2\|u(0)\|_{L^p}^p + K_1\|u(0)\|_{L^2}^2 + K_2\end{aligned} \tag{2.2.49}$$

holds for all $t > 0$.

In the first case, multiplying by $-\Delta u$, integrating over $(0,T)$ and using (2.2.49) it follows that

$$\begin{aligned}&\frac{1}{2}\|u(T)\|_{H_0^1}^2 + \frac{m}{2}\int_0^T \|\Delta u(s)\|_{L^2}^2 ds \\ &\leq \eta \int_0^T \|u(s)\|_{H_0^1}^2 ds + \frac{1}{2}\|u(0)\|_{H_0^1}^2 + K_3 \leq K_4(T),\end{aligned} \tag{2.2.50}$$

for all $T > 0$. In the second case, combining (2.2.49) with (2.1.44) the boundedness of $f(u^n)$ in $L^2(0,T;L^2(\Omega))$ follows for any $T > 0$. Hence, the equality

$$a\left(\|u\|_{H_0^1}^2\right)\Delta u = \frac{du^n}{dt} - f(u^n) - h$$

and (2.1.4) imply that u^n is bounded in $L^2(0,T;D(A))$.

Thus, the sequence

$$u^n \text{ is bounded in } L^\infty(0,T;H_0^1(\Omega)) \cap L^2(0,T;D(A))$$

and

$$\frac{du^n}{dt},\ f(u^n) \text{ are bounded in } L^2(0,T;L^2(\Omega)),$$

for all $T > 0$.

Therefore, there is u such that

$$u^n \overset{*}{\rightharpoonup} u \text{ in } L^\infty(0,T;H_0^1(\Omega)),$$
$$u^n \rightharpoonup u \text{ in } L^2(0,T;D(A)),$$
$$u_t^n \rightharpoonup u_t \text{ in } L^2(0,T;L^2(\Omega)).$$

Arguing in a similar way as in Theorem 2.9 we have

$$u_n \to u \text{ in } L^2(0,T;H_0^1(\Omega)),$$
$$u_n(t,x) \to u(t,x) \text{ a.e. on } (0,T)\times\Omega,$$
$$f\left(u^n\right) \rightharpoonup f\left(u\right) \text{ in } L^2(0,T;L^2(\Omega)),$$
$$a(\|u_n\|^2_{H_0^1})\Delta u_n \rightharpoonup a(\|u\|^2_{H_0^1})\Delta u \text{ in } L^2(0,T;L^2(\Omega)).$$

Hence, we can pass to the limit and obtain that $u \in K_s^+$. Following the same lines of Theorem 2.10 we check that $u\left(0\right) = u_0$.

Moreover, arguing as in Lemma 2.21 we obtain

$$u^n(t) \to u(t) \text{ in } H_0^1(\Omega) \text{ for all } t > 0.$$

□

Corollary 2.34. *Assume the conditions of Lemma 2.33. Then the set K_s^+ satisfies condition $(K4)$.*

Using Lemma 2.33 and reasoning as before the following result holds.

Proposition 2.35. *Assume the conditions of Lemma 2.33. Then the map $G_s\left(t,\cdot\right)$ is upper semicontinuous for all $t \geq 0$ with compact values.*

Proposition 2.36. *Assume the conditions of Lemma 2.33 and (2.1.15). Then there exists an absorbing set B_1 for G_s, which is compact in $H_0^1\left(\Omega\right)$ and $L^p\left(\Omega\right)$.*

Proof. The proof follows the same lines of that in Proposition 2.24 but using Lemma 2.33. □

From these results and Theorem 0.17 we obtain the existence of the global attractor.

Theorem 2.37. *Assume the conditions of Lemma 2.33 and (2.1.15). Then the multivalued semiflow G_s possesses a global compact invariant attractor $\mathcal{A}_s$, which is compact in $L^p(\Omega)$.*

Lemma 2.38. *Assume the conditions of Lemma 2.33 and (2.1.15). Then $\mathcal{A}_s = \mathcal{A}_r$, where $\mathcal{A}_r$ is the global attractor in Theorem 2.25.*

Proof. Since $G_s(t, u_0) \subset G_r(t, u_0)$ for all $u_0 \in H_0^1(\Omega)$, it is clear that $\mathcal{A}_r$ is a compact attracting set. Hence, the minimality of the global attractor gives $\mathcal{A}_s \subset \mathcal{A}_r$.

Let $z \in \mathcal{A}_r$. Since $z = \gamma(0)$, where $\gamma \in \mathbb{K}_r^1$, and $\gamma\mid_{[s,+\infty)}$ is a strong solution of (2.1.1) for any $s \in \mathbb{R}$, we get that $z \in G_s(t_n, \gamma(-t_n))$ for $t_n \to +\infty$. Hence,

$$dist(z, \mathcal{A}_s) \leq dist(G_s(t_n, \gamma(-t_n)), \mathcal{A}_s) \to 0 \text{ as } n \to \infty,$$

so $z \in \mathcal{A}_s$. □

The set of all complete trajectories of K_s^+ (see Definition 0.2) will be denoted by $\mathbb{F}_s$. Let $\mathbb{K}_s$ be the set of all complete trajectories which are bounded in $H_0^1(\Omega)$.

In view of Theorem 0.15, the global attractor is characterized in terms of bounded complete trajectories:

$$\mathcal{A}_s = \{\gamma(0) : \gamma(\cdot) \in \mathbb{K}_s\} = \bigcup_{t\in\mathbb{R}} \{\gamma(t) : \gamma(\cdot) \in \mathbb{K}_s\}. \tag{2.2.51}$$

In the same way as in Lemma 2.27 we obtain that $\mathfrak{R}_{K_s^+} = \mathfrak{R}$.

Reasoning as in Theorem 2.18 we obtain the following result.

Theorem 2.39. *Assume the conditions of Lemma 2.33 and (2.1.15)). Then the global attractor $\mathcal{A}_s$ is bounded in $L^\infty(\Omega)$, provided that $h \in L^\infty(\Omega)$.*

Following the same procedure of Theorem 2.29 we can prove an analogous characterization of the global attractor.

Theorem 2.40. *Assume the conditions of Lemma 2.33 and (2.1.15). Then it holds that*

$$\mathcal{A}_s = M_s^u(\mathfrak{R}) = M_s^s(\mathfrak{R}),$$

where

$$M_s^s(\mathfrak{R}) = \{z : \exists\gamma(\cdot) \in \mathbb{K}_s,\ \gamma(0) = z,\quad dist\ _{H_0^1(\Omega}(\gamma(t), \mathfrak{R}) \to 0,\ t \to +\infty\}, \tag{2.2.52}$$

$$M_s^u(\mathfrak{R}) = \{z : \exists\gamma(\cdot) \in \mathbb{F}_s,\ \gamma(0) = z,\quad dist_{H_0^1(\Omega}(\gamma(t), \mathfrak{R}) \to 0,\ t \to -\infty\}. \tag{2.2.53}$$

Remark 2.41. In the definition of $M_s^u(\mathfrak{R})$ we can replace $\mathbb{F}_r$ by $\mathbb{K}_r$.

Let us consider now the particular situation when G_s is single-valued semigroup. Under the conditions (2.1.3)-(2.1.5), (2.1.9), (2.2.1), (2.2.23), if we assume additionally that (2.1.12) is satisfied, then by Theorem 2.14 for any $u_0 \in H_0^1(\Omega)$ there exists a unique strong solution $u(\cdot)$. Then we can define the following semigroup $T_s : \mathbb{R}^+ \times H_0^1(\Omega) \to H_0^1(\Omega)$:

$$T_s(t, u_0) = u(t),$$

where $u(\cdot)$ is the unique strong solution to (2.1.1). We recall also that $u \in C([0,T], H_0^1(\Omega))$ for any $T > 0$. Also, by Lemma 2.33 if $u_0^n \to u_0$ weakly in $H_0^1(\Omega)$, then $T_s(t, u_0^n) \to T(t, u_0)$ in $H_0^1(\Omega)$ for all $t > 0$.

Since $T_s = G_s$, by Theorems 2.37, 2.39, 2.40 and Lemma 2.38 we obtain the following results.

Theorem 2.42. *Assume the conditions (2.1.3)-(2.1.5), (2.1.9), (2.1.12), (2.1.15), (2.2.1) and (2.2.23). Then the semigroup T_s possesses a global invariant attractor $\mathcal{A}_s$, which is compact in $H_0^1(\Omega)$ and $L^p(\Omega)$.*

Lemma 2.43. *Under the conditions of Theorem 2.42, $\mathcal{A}_s = \mathcal{A}_r$, where $\mathcal{A}_r$ is the attractor of Theorem 2.17.*

Theorem 2.44. *Assume the conditions of Theorem 2.42. Then the global attractor $\mathcal{A}_s$ is bounded in $L^\infty(\Omega)$ provided that $h \in L^\infty(\Omega)$.*

As before, we denote by $\mathfrak{R}$ the set of fixed points of T_s. Also, the global attractor is the union of all bounded complete trajectories

$$\mathcal{A}_s = \{\phi(0) : \phi \text{ is a bounded complete trajectory of } T_s\}.$$

Theorem 2.45. *Assume the conditions of Theorem 2.42. Then the global attractor $\mathcal{A}_s$ can be characterized as follows*

$$\mathcal{A}_s = M_s^u(\mathfrak{R}) = M_s^s(\mathfrak{R}),$$

where the sets $M_s^u(\mathfrak{R})$, $M_s^s(\mathfrak{R})$ are defined in (2.2.52)-(2.2.53).

In this case we can obtain additionally that the attractor is bounded in $H^2(\Omega)$.

Proposition 2.46. *Assume the conditions of Theorem 2.42 and also that (2.1.13) holds true. Then $\mathcal{A}_s$ is bounded in $H^2(\Omega)$.*

Proof. The proof follows the same lines as in Proposition 2.19, so we omit it. □

2.2.6. Attractor in the phase space $H_0^1(\Omega) \cap L^p(\Omega)$

We consider the metric space $X = H_0^1(\Omega) \cap L^p(\Omega)$ endowed with the induced topology of the space $H_0^1(\Omega)$.

If we assume conditions (2.1.3)-(2.1.5), (2.1.9), (2.1.12) and (2.2.1), then by Theorems 2.10 and 2.14 for any $u_0 \in H_0^1(\Omega) \cap L^p(\Omega)$ there exists a unique strong solution $u(\cdot)$. Then we can define the following semigroup $T_s : \mathbb{R}^+ \times X \to X$:

$$T_s(t, u_0) = u(t),$$

where $u(\cdot)$ is the unique strong solution to (2.1.1). We recall also that $u \in C([0,T], H_0^1(\Omega)) \cap C_w([0,T], L^p(\Omega))$ for any $T > 0$.

Lemma 2.47. *Assume conditions (2.1.3)-(2.1.5), (2.1.9), (2.1.12) and (2.2.1). If $u_0^n \to u_0$ weakly in $H_0^1(\Omega) \cap L^p(\Omega)$, then $T_s(t, u_0^n) \to T_s(t, u_0)$ strongly in $H_0^1(\Omega)$ and weakly in $L^p(\Omega)$ for any $t > 0$.*

Proof. Repeating the same proof of Lemma 2.33 we obtain that $T_s(t, u_0^n) \to T_s(t, u_0)$ strongly in $H_0^1(\Omega)$ for all $t > 0$. We observe that in this case the estimates are justified via Galerkin approximations, so we do not need condition (2.2.23) in order to provide property (2.2.25).

Finally, by the Ascoli-Arzelà theorem we deduce

$$u^n \to u \text{ in } C([0,T], L^2(\Omega))$$

and combining this with (2.2.49) we infer that

$$u^n(t) \rightharpoonup u(t) \text{ in } L^p(\Omega) \ \forall t \geq 0.$$

□

Proposition 2.48. *Assume the conditions of Lemma 2.47 and (2.1.15). Then there exists an absorbing set B_1 for T_s, which is compact in $H_0^1(\Omega)$ and bounded $L^p(\Omega)$.*

Proof. Following the same lines of that in Proposition 2.24 (and justifying the estimates via Galerkin approximations), we obtain that there exists $M > 0$ such that the closed ball B_M in $H_0^1(\Omega) \cap L^p(\Omega)$ centered at 0 with radius M is absorbing for T_s. By Lemma 2.47 the set $B_1 = \overline{T_s(1, B_M)}$ is an absorbing set which is compact in $H_0^1(\Omega)$ and bounded in $L^p(\Omega)$. □

Theorem 2.49. *We assume the conditions of Lemma 2.47 and (2.1.15). Then the semigroup T_s possesses a global attractor $\mathcal{A}_s$, which is compact in X and bounded in $L^p(\Omega)$.*

Proof. We cannot apply directly the general theory of attractors for semigroup because we do not know whether the semigroup T_s is continuous with respect to the initial datum in X.

We state that

$$\mathcal{A}_s = \omega(B_1) = \{y : \exists t_n \to +\infty,\ y_n \in T_s(t_n, B_1) \text{ such that } y_n \to y \text{ in } X\}$$

is a global compact attractor. The fact that set $\omega(B_1)$ is non-empty, compact and the minimal closed set attracting B_1 can be proved in a standard way (see for example Theorem 10.5 in [75]). Since B_1 is absorbing, $\omega(B_1)$ attracts any bounded set B. As $\omega(B_1) \subset B_1$, $\mathcal{A}_s$ is bounded in $L^p(\Omega)$.

We need to prove that it is invariant.

First, we prove that it is negatively invariant. Let $y \in \mathcal{A}_s$ and $t > 0$ be arbitrary. We take a sequence $y_n \in T_s(t_n, B_1)$ such that

$$y_n \to y, \quad t_n \to +\infty.$$

Since $T_s(t_n, B_1) = T_s(t, T_s(t_n - t, B_1))$, there are $x_n \in T_s(t_n - t, B_1)$ such that $y_n = T_s(t, x_n)$. As for n large $T_s(t_n - t, B_1) \subset B_1$, the sequence $\{x_n\}$ is bounded in $L^p(\Omega)$ and relatively compact in $H_0^1(\Omega)$. Hence, there exists $x \in \mathcal{A}_s$ such that up to a subsequence

$$x_n \rightharpoonup x \text{ in } L^p(\Omega),$$
$$x_n \to x \text{ in } H_0^1(\Omega).$$

We deduce by Lemma 2.47 that

$$T_s(t, x_n) \rightharpoonup T_s(t, x) \text{ in } L^p(\Omega),$$
$$T_s(t, x_n) \to T_s(t, x) \text{ in } H_0^1(\Omega).$$

Thus, $y = T_s(t, x) \subset T_s(t, \mathcal{A}_s)$.

Second, we prove that it is positively invariant. As $\mathcal{A}_s = T_s(\tau, \mathcal{A}_s)$ for any

$\tau \geq 0$, this follows from

$$dist_X\left(T_s\left(t,\mathcal{A}_s\right),\mathcal{A}_s\right) = dist_X\left(T_s\left(t,T_s(\tau,\mathcal{A}_s)\right),\mathcal{A}_s\right) = dist_X\left(T_s\left(t+\tau,\mathcal{A}_s\right),\mathcal{A}_s\right) \underset{\tau\to+\infty}{\to} 0.$$

□

Lemma 2.50. *Under the conditions of Theorem 2.49, $\mathcal{A}_s = \mathcal{A}_r$, where $\mathcal{A}_r$ is the attractor of Theorem 2.17.*

Proof. Since $T_r\left(t,u_0\right) = T_s\left(t,u_0\right)$ for any $u_0 \in X$, we have

$$dist_{L^2}\left(\mathcal{A}_s,\mathcal{A}_r\right) = dist_{L^2}\left(T_s(t,\mathcal{A}_s),\mathcal{A}_r\right) = dist_{L^2}\left(T_r(t,\mathcal{A}_s),\mathcal{A}_r\right) \underset{t\to+\infty}{\to} 0,$$

so $\mathcal{A}_s \subset \mathcal{A}_r$. In the same way,

$$dist_X\left(\mathcal{A}_r,\mathcal{A}_s\right) = dist_X\left(T_r(t,\mathcal{A}_r),\mathcal{A}_s\right) = dist_X\left(T_s(t,\mathcal{A}_r),\mathcal{A}_s\right) \underset{t\to+\infty}{\to} 0,$$

and then $\mathcal{A}_r \subset \mathcal{A}_s$. □

The following two theorems are proved in the same way as Theorem 2.18 and Proposition 2.32

Theorem 2.51. *Assume the conditions of Theorem 2.49. Then the global attractor $\mathcal{A}_s$ is bounded in $L^\infty(\Omega)$ provided that $h \in L^\infty(\Omega)$.*

As before, we denote by $\mathfrak{R}$ the set of fixed points of T_s. Also, the global attractor is the union of all bounded complete trajectories

$$\mathcal{A}_s = \{\phi\left(0\right) : \phi \text{ is a bounded complete trajectory of } T_s\}.$$

Theorem 2.52. *We assume the conditions of Theorem 2.49 and one of the following assumptions:*

1. $h \in L^\infty\left(\Omega\right)$;
2. $p \leq \frac{2n}{n-2}$ *if* $n \geq 3$.

Then the global attractor $\mathcal{A}_s$ can be characterized as follows

$$\mathcal{A}_s = M_s^u(\mathfrak{R}) = M_s^s(\mathfrak{R}),$$

where the sets $M_s^u(\mathfrak{R})$, $M_s^s(\mathfrak{R})$ are defined in (2.2.45)-(2.2.46).

We obtain additionally that the attractor is bounded in $H^2(\Omega)$.

Proposition 2.53. *Assume the conditions of Theorem 2.49 and also that (2.1.13) is satisfied. Then $\mathcal{A}_s$ is bounded in $H^2(\Omega)$.*

Proof. The proof follows the same lines as in Proposition 2.19, so we omit it. $\square$

Chapter 3

Structure of attractors for a nonlocal Chafee-Infante problem

In this chapter, we study the structure of the global attractor for the multi-valued semiflow generated by a nonlocal reaction-diffusion equation in which we cannot guarantee uniqueness of the Cauchy problem.

The main aim consists in describing in as much detail as possible the internal structure of the global attractor in a similar way as for the classical Chafee-Infante equation.

First, we analyse the existence and properties of stationary points, showing that the problem undergoes the same cascade of bifurcations as in the Chafee-Infante equation. Second, we study the stability of the fixed points and establish that the semiflow is dynamically gradient. We prove that the attractor consists of the stationary points and their heteroclinic connections and analyse some of the possible connections.

3.1. Setting of the problem

Let us consider the following problem

$$\begin{cases} \dfrac{\partial u}{\partial t} - a(\|u\|^2_{H^1_0})\dfrac{\partial^2 u}{\partial x^2} = \lambda f(u) + h(t), \quad t > 0, x \in \Omega, \\ u(t,0) = u(t,1) = 0, \\ u(0,x) = u_0(x), \end{cases} \tag{3.1.1}$$

where $\Omega = (0,1)$ and $\lambda > 0$.
Throughout the chapter we will use the following conditions (but not all of them at the same time):

(A1) $f \in C(\mathbb{R})$.

(A2) $f(0) = 0$.

(A3) $f'(0)$ exists and $f'(0) = 1$.

(A4) f is strictly concave if $u > 0$ and strictly convex if $u < 0$.

(A5) Growth and dissipation conditions: for $p \geq 2,\ C_i > 0,\ i = 1, .., 4$, we have

$$|f(u)| \leq C_1 + C_2|u|^{p-1}, \tag{3.1.2}$$

$$f(u)u \leq C_3 - C_4|u|^p, \text{ if } p > 2, \tag{3.1.3}$$

$$\limsup_{u\to\pm\infty} \frac{f(u)}{u} \leq 0, \text{ if } p = 2. \tag{3.1.4}$$

(A6) The function $a \in C(\mathbb{R}^+)$ satisfies:

$$a(s) \geq m > 0.$$

(A7) The function $a \in C(\mathbb{R}^+)$ satisfies:

$$a(s) \leq M_1, \quad \forall s \geq 0,$$

where $M_1 > 0$.

(A8) The function $a \in C(\mathbb{R}^+)$ is non-decreasing.

(A9) $h \in L^2_{loc}\left(0, +\infty; L^2\left(\Omega\right)\right)$.

(A10) h does not depend on time and $h \in L^2\left(\Omega\right)$.

As usual, defining the function $\mathcal{F}(u) = \int_0^u f(s)ds$, we observe that from (3.1.2) we have

$$|\mathcal{F}(s)| \leq \widetilde{C}(1 + |s|^p) \quad \forall s \in \mathbb{R}, \tag{3.1.5}$$

whereas (3.1.3) implies

$$\mathcal{F}(s) \leq \widetilde{\kappa} - \widetilde{\alpha}_1 |s|^p. \tag{3.1.6}$$

Also, from condition (3.1.4) it follows that for all $\varepsilon > 0$, there exists a constant $M > 0$ such that $\frac{f(u)}{u} \leq \varepsilon$, for all $|u| \geq M$. Hence, there exists $m_\varepsilon > 0$ such that

$$f(u)u \leq m_\varepsilon + \varepsilon u^2, \quad \forall u \in \mathbb{R}. \tag{3.1.7}$$

In addition, it follows that

$$\mathcal{F}(u) \leq \varepsilon u^2 + C_\varepsilon, \tag{3.1.8}$$

where $C_\varepsilon > 0$. These two inequaities are also true under condition (3.1.3).

Some of these conditions will be used all the time, whereas other ones will be used only in certain results. In particular, the function h will be considered as a time-dependent function satisfying (A9) only for establishing the existence of solution for problem (3.1.1). However, since we will study the asymptotic behaviour of solutions in the autonomous situation, for the second part concerning the existence and properties of global attractors the function h will be time-independent, so assumption (A10) will be used instead. Finally, in order to study the structure of the global attractors in terms of the stationary points and their possible heteroclinic connections we will assume that $h \equiv 0$.

3.2. Existence of solutions

In this section we will establish the existence of strong solutions for problem (3.1.1) with initial condition in the phase space $H_0^1(\Omega)$. Although we will follow the same lines of a similar result given in Chapter 2, we would like to point out that in the present case, as we are working in a one-dimensional problem, the assumptions on the function f are much weaker. In particular, we do not need to impose a growth assumption of any kind.

Definition 3.1. For $u_0 \in L^2(\Omega)$, a weak solution to (3.1.1) is an element $u \in$

$L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega))$, for any $T>0$, such that

$$\frac{d}{dt}(u,v) + a(\|u\|^2_{H_0^1})(\nabla u, \nabla v) = \lambda(f(u),v) + (h(t),v) \quad \forall v \in H_0^1(\Omega), \qquad (3.2.1)$$

where the equation is understood in the sense of distributions.

As before, let $A : D(A) \to H$, $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$, be the operator $A = -\dfrac{d^2}{dx^2}$ with Dirichlet boundary conditions. This operator is the generator of a C_0-semigroup $T(t) = e^{-At}$.

Definition 3.2. For $u_0 \in H_0^1(\Omega)$, a strong solution to (3.1.1) is a weak solution with the extra regularity $u \in L^\infty(0,T;H_0^1(\Omega))$, $u \in L^2(0,T;D(A))$ and $\dfrac{du}{dt} \in L^2(0,T;L^2(\Omega))$ for any $T>0$.

Remark 3.3. We observe that if u is a strong solution, then $u \in C([0,T];H_0^1(\Omega))$ (see [77, p.102]). By this way, the initial condition makes sense.

Remark 3.4. Since $\dfrac{du}{dt} \in L^2(0,T;L^2(\Omega))$ for any strong solution, in this case equality (3.2.1) is equivalent to the following one:

$$\begin{aligned}&\int_0^T \int_\Omega \frac{du(t,x)}{dt}\xi(t,x)\,dxdt - \int_0^T a(\|u(t)\|^2_{H_0^1})\int_\Omega \frac{\partial^2 u}{\partial x^2}\xi dxdt \qquad (3.2.2)\\ &= \int_0^T \int_\Omega \lambda f(u(t,x))\,\xi(t,x)\,dxdt + \int_0^T \int_\Omega h(t,x)\,\xi(t,x)\,dxdt,\end{aligned}$$

for all $\xi \in L^2(0,T;L^2(\Omega))$.

Theorem 3.5. *Assume conditions (A1), (A6) and (A9). Assume also the existence of constants $\beta, \gamma > 0$ such that*

$$f(u)\,u \le \gamma + \beta u^2 \text{ for all } u \in \mathbb{R}. \qquad (3.2.3)$$

Then, for any $u_0 \in H_0^1(\Omega)$ problem (3.1.1) has at least one strong solution.

Remark 3.6. Assumption (3.2.3) is weaker than the dissipative property (3.1.7) as the constant ε is arbitrarily small. Due to the fact that we are working in a one-dimensional domain, no growth condition of the type given in (A5) is necessary in order to prove existence of solutions. Also, (3.2.3) implies that

$$F\left(u\right) \leq \widetilde{\gamma} + \widetilde{\beta} u^{2} \tag{3.2.4}$$

for some constants $\widetilde{\gamma}, \widetilde{\beta} > 0$.

Proof. Consider a fixed value $T > 0$. In order to use the Faedo-Galerkin method let $\{w_j\}_{j\geq 1}$ be the sequence of eigenfunctions of $-\Delta$ in $H_0^1(\Omega)$ with homogeneous Dirichlet boundary conditions, which forms a special basis of $L^2(\Omega)$. Since Ω is a bounded regular domain, it is known that $\{w_j\} \subset H_0^1(\Omega)$ and that $\cup_{n\in\mathbb{N}} V_n$ is dense in the spaces $L^2(\Omega)$ and $H_0^1(\Omega)$, where $V_n = span[w_1, \ldots, w_n]$.

As usual, P_n will be the orthogonal projection in $L^2\left(\Omega\right)$, that is

$$z_n := P_n z = \sum_{j=1}^{n} (z, w_j) w_j,$$

and λ_j will be the eigenvalues associated to the eigenfunctions w_j.

For each integer $n \geq 1$, we consider the Galerkin approximations

$$u_n(t) = \sum_{j=1}^{n} \gamma_{nj}(t) w_j,$$

which are given by the following nonlinear ODE system

$$\begin{cases} \dfrac{d}{dt}(u_n, w_i) + a(\|u_n\|^2_{H_0^1})(\nabla u_n, \nabla w_i) = \lambda(f(u_n), w_i) + (h, w_i) \quad \forall i = 1, \ldots, n, \\ u_n(0) = P_n u_0. \end{cases} \tag{3.2.5}$$

We observe that $P_n u_0 \to u_0$ in $H_0^1(\Omega)$.
This Cauchy problem possesses a solution on some interval $[0, t_n)$ and by the estimates in the space $L^2(\Omega)$ of the sequence $\{u_n\}$ given below for any $T > 0$ such a solution can be extended to the whole interval $[0, T]$ (cf. Theorem 2.9).

Firstly, multiplying the equation in (3.2.5) by $\gamma_{ni}(t)$ and summing from $i = 1$

to n, we obtain

$$\frac{1}{2}\frac{d}{dt}\|u_n(t)\|_{L^2}^2 + a(\|u_n\|_{H_0^1}^2)\|u_n(t)\|_{H_0^1}^2 = \lambda(f(u_n(t), u_n(t)) + (h(t), u_n(t)), \quad (3.2.6)$$

for a.e. $t \in (0, t_n)$.
Using the Young and Poincaré inequalities we deduce that

$$(h(t), u_n(t)) \leq \frac{m}{2}\|u_n(t)\|_{H_0^1}^2 + \frac{1}{2\lambda_1 m}\|h(t)\|_{L^2}^2,$$

where m is the constant from (A6).
Hence, from (A6), (3.2.3) and (3.2.6) it follows that

$$\frac{1}{2}\frac{d}{dt}\|u_n(t)\|_{L^2}^2 + \frac{m}{2}\|u_n(t)\|_{H_0^1}^2 \leq \lambda\gamma|\Omega| + \beta\lambda\|u_n(t)\|_{L^2}^2 + \frac{1}{2\lambda_1 m}\|h(t)\|_{L^2}^2.$$

We infer that

$$\begin{aligned}\|u_n(t)\|_{L^2}^2 \leq \|u_n(0)\|_{L^2}^2 e^{2\beta\lambda t} + \int_0^t e^{2\beta\lambda(t-s)}\left(2\lambda\gamma|\Omega| + \frac{1}{\lambda_1 m}\|h(s)\|_{L^2}^2\right) ds \\ \leq \|u_n(0)\|_{L^2}^2 e^{2\beta\lambda T} + K_1(T).\end{aligned} \quad (3.2.7)$$

Therefore, the solution exists on any given interval $[0, T]$ and

$$\{u_n\} \text{ is bounded in } L^\infty(0, T; L^2(\Omega)). \quad (3.2.8)$$

Now, we multiply the equation (3.1.1) by $\frac{du_n}{dt}$ to obtain

$$\|\frac{du_n}{dt}(t)\|_{L^2}^2 + a(\|u_n\|_{H_0^1}^2)\frac{1}{2}\frac{d}{dt}\|u_n\|_{H_0^1}^2 = \frac{d}{dt}\int_\Omega \lambda\mathcal{F}(u_n)dx + (h(t), \frac{du_n}{dt}).$$

Introducing

$$A(s) = \int_0^s a(r)dr \quad (3.2.9)$$

we have

$$\frac{1}{2}\|\frac{du_n}{dt}(t)\|_{L^2}^2+\frac{d}{dt}\left(\frac{1}{2}A(\|u_n\|_{H_0^1}^2)-\int_\Omega \lambda\mathcal{F}(u_n)dx\right)\leq\frac{1}{2}\|h(t)\|_{L^2}^2.$$

Integrating the previous expression between 0 and t we get

$$\begin{aligned}&\frac{1}{2}A(\|u_n(t)\|_{H_0^1}^2)+\lambda\int_\Omega \mathcal{F}(u_n(0))dx+\frac{1}{2}\int_0^t\|\frac{d}{ds}u_n(s)\|_{L^2}^2ds\\&\leq\frac{1}{2}A(\|u_n(0)\|_{H_0^1}^2)+\lambda\int_\Omega\mathcal{F}(u_n(t))dx+\frac{1}{2}\int_0^t\|h(s)\|_{L^2}^2ds.\end{aligned}\tag{3.2.10}$$

By (A6), (3.2.4) and (3.2.7) it follows that

$$\begin{aligned}&\frac{m}{2}\|u_n(t)\|_{H_0^1}^2+\lambda\int_\Omega\mathcal{F}(u_n(0))dx+\frac{1}{2}\int_0^t\|\frac{d}{ds}u_n(s)\|_{L^2}^2ds\\&\leq\frac{1}{2}A(\|u_n(0)\|_{H_0^1}^2)+\lambda\widetilde{\beta}\|u_n(t)\|_{L^2}^2+\lambda\widetilde{\gamma}|\Omega|+K_2(T)\\&\leq\frac{1}{2}A(\|u_n(0)\|_{H_0^1}^2)+\lambda\widetilde{\beta}e^{2\beta\lambda T}\|u_n(0)\|_{L^2}^2+K_3(T).\end{aligned}\tag{3.2.11}$$

Since $dim(\Omega)=1$, $H_0^1(\Omega)\subset L^\infty(\Omega)$, so $u_n(0)$ is bounded in $L^\infty(\Omega)$. Thus, as f maps bounded sets of $\mathbb{R}$ into bounded ones, $\mathcal{F}(u_n(0))$ is bounded in $L^\infty(\Omega)$ as well. Therefore, we deduce that

$$\{u_n\}\text{ is bounded in }L^\infty(0,T;H_0^1(\Omega))$$

and

$$\frac{du_n}{dt}\text{ is bounded in }L^2(0,T;L^2(\Omega)).\tag{3.2.12}$$

Using again the embedding $H_0^1(\Omega)\subset L^\infty(\Omega)$ we obtain that u_n is bounded in the space $L^\infty(0,T;L^\infty(\Omega))$. Thus,

$$f(u_n)\text{ is bounded in }L^\infty(0,T;L^\infty(\Omega)).\tag{3.2.13}$$

Also, we deduce that $\|u_n(t)\|^2_{H^1_0}$ is uniformly bounded in $[0,T]$ and then by the continuity of the function $a(\cdot)$ we get that the sequence $a\left(\|u_n(t)\|^2_{H^1_0}\right)$ is also uniformly bounded in $[0,T]$.

Finally, multiplying (3.2.5) by $\lambda_j\gamma_{ni}(t)$ and summing from $i=1$ to n we obtain

$$\frac{1}{2}\frac{d}{dt}\|u_n\|^2_{H^1_0}+m\|\Delta u_n\|^2_{L^2}\leq \lambda(f(u_n),-\Delta u_n)+(h(t),-\Delta u).$$

By (3.2.13) and applying the Young inequality, we get

$$\frac{1}{2}\frac{d}{dt}\|u_n\|^2_{H^1_0}+m\|\Delta u_n\|^2_{L^2}\leq \frac{\lambda^2}{m}\|f(u_n)\|^2_{L^2}+\frac{m}{4}\|\Delta u_n\|^2_{L^2}+\frac{1}{m}\|h(t)\|^2_{L^2}+\frac{m}{4}\|\Delta u\|^2_{L^2}.$$

Integrating the previous expression between 0 and t, it follows that

$$\begin{aligned}&\|u_n(t)\|^2_{H^1_0}+m\int_0^t\|\Delta u_n(s)\|^2_{L^2}ds\\ \leq\ &\|u_n(0)\|^2_{H^1_0}+\frac{2\lambda^2}{m}\int_0^t\|f(u_n(s))\|^2_{L^2}ds+\frac{2}{m}\int_0^t\|h(s)\|^2_{L^2}ds.\end{aligned}$$

Taking into account (3.2.13), the last inequality implies that

$$u_n \text{ is bounded in } L^2(0,T;D(A)), \tag{3.2.14}$$

so $\{-\Delta u_n\}$ and $\{a(\|u_n\|^2_{H^1_0})\Delta u_n\}$ are bounded in $L^2(0,T;L^2(\Omega))$.
As a consequence, there exists $u\in L^\infty(0,T;H^1_0(\Omega))$ and a subsequence u_n (relabeled the same) such that

$$\begin{aligned}u_n &\overset{*}{\rightharpoonup} u \text{ in } L^\infty(0,T;H^1_0(\Omega)),\\ u_n &\rightharpoonup u \text{ in } L^2(0,T;D(A)),\\ f(u_n) &\overset{*}{\rightharpoonup} \chi \text{ in } L^\infty(0,T;L^\infty(\Omega)),\\ a(\|u_n\|^2_{H^1_0}) &\overset{*}{\rightharpoonup} b \text{ in } L^\infty(0,T),\end{aligned} \tag{3.2.15}$$

where $\rightharpoonup$ ($\overset{*}{\rightharpoonup}$) stands for the weak (weak star) convergence.
By (3.2.12) and (3.2.14) the Aubin-Lions Compactness Lemma gives that

$$u_n \to u \text{ in } L^2(0,T;H_0^1(\Omega)),$$

so

$$u_n(t) \to u(t) \text{ in } H_0^1(\Omega), \quad \text{a.e. on } (0,T).$$

Consequently, there exists a subsequence u_n, relabelled the same, such that

$$u_n(t,x) \to u(t,x) \quad \text{a.e. in } \Omega \times (0,T).$$

Moreover, thanks to the inequality

$$\|u_n(t_2)-u_n(t_1)\|_{L^2}^2 = \left\| \int_{t_1}^{t_2} \frac{d}{dt}u_n(s)ds \right\|_{L^2}^2 \le \|\frac{d}{dt}u_n\|_{L^2(0,T;L^2(\Omega))}^2 \, |t_2-t_1|, \quad \forall t_1,t_2 \in [0,T],$$

(3.2.11), (3.2.12) and $H_0^1(\Omega) \subset\subset L^2(\Omega)$, the Ascoli-Arzelà theorem implies that

$$\{u_n\} \to u \text{ in } C([0,T];L^2(\Omega)),$$

for all $T>0$.
Therefore, we obtain from (3.2.11) that

$$u_n(t) \rightharpoonup u(t) \text{ in } H_0^1(\Omega),$$

for any $t \ge 0$.

Also, by (3.2.15) we have that

$$P_n f(u_n)) \rightharpoonup \chi \text{ in } L^q(0,T;L^q(\Omega)),$$

for any $q \ge 1$ (see [75, p.224]).

Since f is continuous, it follows that

$$f(u_n(t,x)) \to f(u(t,x)) \quad \text{a.e. in } \Omega \times (0,T).$$

Therefore, in view of (3.2.15), by [65, Lemma 1.3] we have that $\chi = f(u)$.

As a consequence, by the continuity of a we get that

$$a(\|u_n(t)\|^2_{H^1_0}) \to a(\|u(t)\|^2_{H^1_0}) \quad \text{a.e. on } (0,T).$$

Since the sequence is uniformly bounded, by Lebesgue's theorem this convergence takes place in $L^2(0,T)$, so $b = a(\|u\|^2_{H^1_0})$. Thus,

$$a(\|u_n\|^2_{H^1_0})\Delta u_n \rightharpoonup a(\|u\|^2_{H^1_0})\Delta u \text{ in } L^2(0,T;L^2(\Omega)).$$

Therefore, we can pass to the limit to conclude that u is a strong solution.

It remains to show that $u(0) = u_0$ which makes sense since $u \in C([0,T];H^1_0(\Omega))$ (see Remark 4). Indeed, let be $\phi \in C^1([0,T];H^1_0(\Omega))$ with $\phi(T) = 0,\ \phi(0) \neq 0$. We multiply the equation in (3.1.1) and (3.2.5) by ϕ and integrate by parts in the t variable to obtain that

$$\begin{aligned}&\int_0^T \Big(-(u(t),\phi'(t)) - a(\|u(t)\|^2_{H^1_0})(\Delta u(t),\phi(t))\Big)\,dt \\ &= \int_0^T (\lambda f(u(t)) + h(t),\phi(t))dt + (u(0),\phi(0)),\end{aligned} \tag{3.2.17}$$

$$\begin{aligned}&\int_0^T \Big(-(u_n(t),\phi'(t)) - a(\|u_n(t)\|^2_{H^1_0})(\Delta u_n(t),\phi(t))\Big)\,dt \\ &= \int_0^T (\lambda f(u_n(t)) + h(t),\phi(t))dt + (u_n(0),\phi(0)).\end{aligned} \tag{3.2.18}$$

In view of the previous convergences, we can pass to the limit in (3.2.18). Taking into account (3.2.17) and bearing in mind $u_n(0) = P_n u_0 \to u_0$, since $\phi\,(0) \in H^1_0(\Omega)$ is arbitrary, we infer that $u(0) = u_0$. □

3.3. Existence and structure of attractors

In this section, we will prove the existence of a global attractor for the semiflow generated by strong solutions in the autonomous case. Thus, the function h will be an independent of time function satisfying (A10) instead of (A9). Also, we will establish that the attractor is equal to the unstable set of the stationary points (see the definition in (3.3.18)).

Recall that for a metric space X with metric d, we denote by $dist_X(C, D)$ the Hausdorff semidistance from C to D, that is,

$$dist_X(C, D) = \sup_{c \in C} \inf_{d \in D} \rho(c, d).$$

Let us consider the phase space $X = H_0^1(\Omega)$ and the sets

$$K(u_0) = \{u(\cdot) : u \text{ is a strong solution of (3.1.1) such that } u(0) = u_0\},$$

$$\mathcal{R} = \bigcup_{u_0 \in X} K(u_0).$$

Denote by $P(X)$ the class of nonempty subsets of X. We define the (possibly multivalued) map $G : \mathbb{R}^+ \times X \to P(X)$ by

$$G(t, u_0) = \{u(t) : u \in \mathcal{R} \text{ and } u(0) = u_0\}. \tag{3.3.1}$$

In order to study the map G and for the convenience of the reader, let us recall the axiomatic properties of the set $\mathcal{R}$ described in Chapter 0:

(K1) For every $x \in X$ there is $\phi \in \mathcal{R}$ satisfying $\phi(0) = x$.

(K2) $\phi_\tau(\cdot) := \phi(\cdot + \tau) \in \mathcal{R}$ for every $\tau \geq 0$ and $\phi \in \mathcal{R}$ (translation property).

(K3) Let $\phi_1, \phi_2 \in \mathcal{R}$ be such that $\phi_2(0) = \phi_1(s)$ for some $s > 0$. Then, the function

ϕ defined by

$$\phi(t) = \begin{cases} \phi_1(t) & 0 \leq t \leq s, \\ \phi_2(t-s) & s \leq t, \end{cases}$$

belongs to $\mathcal{R}$ (concatenation property).

(K4) For every sequence $\{\phi^n\} \subset \mathcal{R}$ satisfying $\phi^n(0) \to x_0$ in X, there is a subsequence $\{\phi^{n_k}\}$ and $\phi \in \mathcal{R}$ such that $\phi^{n_k}(t) \to \phi(t)$ for every $t \geq 0$.

We shall obtain that the set $\mathcal{R}$ defined above satisfies properties $(K1)-(K4)$. Firstly, assuming conditions (A1), (A6), (A10) and (3.2.3) property (K1) follows from Theorem 3.5, whereas (K2)-(K3) can be proved easily using equality (3.2.2). As we have seen in Chapter 0, we know that $\mathcal{R}$ fulfilling (K1) and (K2) gives rise to a multivalued semiflow G through (3.3.1) (m-semiflow for short), which means that:

- $G(0, x) = x$ for all $x \in X$;
- $G(t+s, x) \subset G(t, G(s, x))$ for all $t, s \geq 0$ and $x \in X$.

Moreover, (K3) implies that the m-semiflow is strict, that is,

$$G(t+s, x) = G(t, G(s, x))$$

for all $t, s \geq 0$ and $x \in X$.

Finally, in order to show that property $(K4)$ is satisfied, we will show first that the m-semiflow G possesses a bounded absorbing set in the space $L^2(\Omega)$.

Lemma 3.7. *Assume conditions (A1), (A6), (A10) and (3.2.3). Given $\{u^n\} \subset \mathcal{R}$, $u^n(0) \to u_0$ weakly in $H_0^1(\Omega)$, there exists a subsequence of $\{u^n\}$ (relabeled the same) and $u \in K(u_0)$ such that*

$$u^n(t) \to u(t) \quad \text{in } H_0^1(\Omega), \ \forall t > 0.$$

Also, if $u^n(0) \to u_0$ *strongly in* $H_0^1(\Omega)$*, then for* $t_n \to 0$ *we get* $u^n(t_n) \to u_0$ *strongly in* $H_0^1(\Omega)$*.*

Proof. Since $\dfrac{du^n}{dt} \in L^2(0,T;L^2(\Omega))$ and $u^n \in L^2(0,T;H_0^1(\Omega))$, we have by [77, pg. 102] that

$$\frac{d}{dt}\|u^n\|^2_{H_0^1} = 2(-\Delta u^n, u^n_t) \quad \text{for a.a. } t \tag{3.3.2}$$

and $u^n \in C([0,T];H_0^1(\Omega))$.
Also, by Lemma 2.20 $(F(u^n(t)),1)$ is an absolutely continuous function on $[0,T]$ and

$$\frac{d}{dt}(F(u^n(t)),1) = (f(u^n(t)),\frac{du^n}{dt}) \text{ for a.a. } t > 0. \tag{3.3.3}$$

By a similar argument as in Theorem 3.5, there is a subsequence of u^n such that

$$\begin{aligned} u^n &\text{ is bounded in } L^\infty(0,T;L^\infty(\Omega)),\\ u^n &\text{ is bounded in } L^\infty(0,T;H_0^1(\Omega)),\\ f(u^n) &\text{ is bounded in } L^\infty(0,T;L^\infty(\Omega)),\\ u^n &\text{ is bounded in } L^2(0,T;D(A)). \end{aligned} \tag{3.3.4}$$

Therefore, arguing as in the proof of Theorem 3.5, there exists $u \in K(u_0)$ and a subsequence u^n, relabelled the same, such that

$$\begin{aligned}
u^n &\overset{*}{\rightharpoonup} u \text{ in } L^\infty(0,T;H_0^1(\Omega)) \\
u_n &\rightharpoonup u \text{ in } L^2(0,T;D(A)) \\
f(u^n) &\overset{*}{\rightharpoonup} f(u) \text{ in } L^\infty(0,T;L^\infty(\Omega)) \\
\frac{du^n}{dt} &\rightharpoonup \frac{du}{dt} \text{ in } L^2(0,T;L^2(\Omega)) \\
a(\|u^n\|^2_{H_0^1})\Delta u^n &\rightharpoonup a(\|u\|^2_{H_0^1})\Delta u \text{ in } L^2(0,T;L^2(\Omega)), \\
u^n &\to u \text{ in } L^2(0,T;H_0^1(\Omega)), \\
u^n &\to u \text{ in } C([0,T],L^2(\Omega)), \\
u^n(t) &\rightharpoonup u(t) \text{ in } H_0^1(\Omega) \quad \forall t \in (0,T].
\end{aligned} \tag{3.3.5}$$

We also need to prove that

$$u^n(t) \to u(t) \text{ in } H_0^1(\Omega)$$

for all $t \in (0,T]$.

For this end, we multiply (3.1.1) by u_t^n and using (A10), (3.3.2) and (3.3.4) we have

$$\frac{1}{2}\left\|\frac{du^n}{dt}\right\|^2_{L^2} + \frac{d}{dt}\left(\frac{1}{2}A(\|u^n(t)\|^2_{H_0^1})\right) \leq C.$$

Thus, we obtain

$$A(\|u^n(t)\|^2_{H_0^1}) \leq A(\|u^n(s)\|^2_{H_0^1}) + 2C(t-s), \quad t \geq s \geq 0.$$

Since this inequality is also true for $u(\cdot)$, the functions

$$Q_n(t) = A(\|u^n(t)\|^2_{H_0^1}) - 2Ct$$

and

$$Q(t) = A(\|u(t)\|^2_{H_0^1}) - 2Ct$$

are continuous and non-increasing in $[0, T]$. Moreover, from (3.3.5) we deduce that

$$Q_n(t) \to Q(t) \quad \text{for a.e. } t \in (0, T).$$

Take $0 < t \leq T$ and $0 < t_j < t$ such that $t_j \to t$ and $Q_n(t_j) \to Q(t_j)$ for all j. Then

$$Q_n(t) - Q(t) \leq Q_n(t_j) - Q(t) \leq |Q_n(t_j) - Q(t_j)| + |Q(t_j) - Q(t)|.$$

For any $\delta > 0$ there exist $j(\delta)$ and $N(j(\delta))$ such that $Q_n(t) - Q(t) \leq \delta$ if $n \geq N$. Then $\limsup Q_n(t) \leq Q(t)$, so $\limsup \|u^n(t)\|^2_{H_0^1} \leq \|u(t)\|^2_{H_0^1}$, which follows by contradiction using the continuity of the function $A(s)$. As $u^n(t) \to u(t)$ weakly in $H_0^1(\Omega)$ implies that $\liminf \|u^n(t)\|^2_{H_0^1} \geq \|u(t)\|^2_{H_0^1}$, we obtain

$$\|u^n(t)\|^2_{H_0^1} \to \|u(t)\|^2_{H_0^1},$$

so that $u^n(t) \to u(t)$ strongly in $H_0^1(\Omega)$.

Finally, if $u^n(0) \to u_0$ strongly in $H_0^1(\Omega)$ and we take $t_n \to 0$, then

$$Q_n(t_n) - Q(0) \leq Q_n(0) - Q(0) = A(\|u^n\,(0)\,\|^2_{H_0^1}) - A(\|u_0\|^2_{H_0^1}) \to 0,$$

so $\limsup Q_n(t_n) \leq Q(0)$. Repeating the above argument, we infer that $u^n(t_n) \to u_0$ strongly in $H_0^1(\Omega)$. □

Corollary 3.8. *Assume the conditions of Lemma 3.7. Then the set $\mathcal{R}$ satisfies condition $(K4)$.*

The map $t \mapsto G(t, x)$ satisfies the important property of being upper semicontinuous, which is fundamental to establish the existence of a global attractor.

Proposition 3.9. *Assume the conditions of Lemma 3.7. The multivalued semiflow G is upper semicontinuous for all $t \geq 0$. Also, it has compact values.*

Proof. By contradiction let us assume that there exist $t \geq 0$, $u_0 \in H_0^1(\Omega)$, a neighbourhood $O(G(t, u_0))$ and sequences $\{y_n\}$, $\{u_0^n\}$ such that $y_n \in G(t, u_0^n)$, u_0^n

converges strongly to u_0 in $H_0^1(\Omega)$ and $y_n \notin O(G(t,u_n))$ for all $n \in \mathbb{N}$. Thus, there exist $u^n \in K(u_0^n)$ such that $y_n = u^n(t)$. From Lemma 3.7 there exists a subsequence of y_n which converges to some $y \in G(t,u_0)$. This contradicts $y_n \notin O(G(t,u_0))$ for any $n \in \mathbb{N}$. □

In order to prove the existence of an absorbing set in the space $L^2(\Omega)$ we need to use the stronger condition (A5) instead of (3.2.3).

Proposition 3.10. *Assume that conditions (A1), (A5), (A6) and (A10) hold. Then the m-semiflow G has a bounded absorbing set in $L^2(\Omega)$, that is, there exists a constant $K > 0$ such that for any $R > 0$ there is a time $t_0 = t_0(R)$ such that*

$$\|y\|_{L^2} \leq K \quad \textit{for all} \quad t \geq t_0,\ y \in G(t,u_0), \tag{3.3.6}$$

where $\|u_0\|_{L^2} \leq R$. Moreover, there is $L > 0$ such that

$$\int_t^{t+1} \|u(s)\|_{H_0^1}^2 ds \leq L \quad \textit{for all} \quad t \geq t_0,\ u \in K(u_0). \tag{3.3.7}$$

Proof. Multiplying equation (3.1.1) by u and using (A6) and (3.1.7) we get

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|u(t)\|_{L^2}^2 + m\|u(t)\|_{H_0^1}^2 \leq (f(u),u)) + (h,u) \\ &\leq m_\varepsilon|\Omega| + \varepsilon\|u(t)\|_{L^2}^2 + \frac{1}{2\lambda_1 m}\|h\|_{L^2}^2 + \frac{\lambda_1 m}{2}\|u\|_{L^2}^2. \end{aligned} \tag{3.3.8}$$

Using the Poincaré inequality it follows that

$$\frac{d}{dt}\|u\|_{L^2}^2 \leq 2m_\varepsilon|\Omega| + 2(\varepsilon - \frac{m}{2}\lambda_1)\|u(t)\|_{L^2}^2 + \frac{1}{\lambda_1 m}\|h\|_{L^2}^2 = -\delta\|u(t)\|_{L^2}^2 + \kappa,$$

where $\delta = m\lambda_1 - 2\varepsilon$, $\kappa = 2m_\varepsilon|\Omega| + \frac{1}{\lambda_1 m}\|h\|_{L^2}^2$. We take $\varepsilon > 0$ small enough, so $\delta > 0$. Then Gronwall's lemma gives

$$\|u(t)\|_{L^2}^2 \leq \|u(0)\|_{L^2}^2 e^{-\delta t} + \frac{\kappa}{\delta}. \tag{3.3.9}$$

Hence, taking

$$t \geq t_0 = \frac{1}{\delta} \ln \left(\frac{\delta R^2}{\kappa} \right)$$

we get (3.3.6) for $K = \sqrt{\frac{2\kappa}{\delta}}$.

On the other hand, using again the Poincaré inequality from (3.3.8) we get

$$\frac{d}{dt} \|u(t)\|_{L^2}^2 + \left(\frac{m\lambda_1 - 2\varepsilon}{\lambda_1} \right) \|u(t)\|_{H_0^1}^2 \leq \kappa$$

and integrating from t to $t+1$ we obtain

$$\left(\frac{m\lambda_1 - 2\varepsilon}{\lambda_1} \right) \int_t^{t+1} \|u(s)\|_{H_0^1}^2 ds \leq \|u(t)\|_{L^2}^2 + \kappa.$$

Therefore, applying (3.3.6), (3.3.7) follows. □

Further, in order to obtain an absorbing set in $H_0^1(\Omega)$ we need to assume additionally that either the function $a(\cdot)$ is bounded above or that it is non-decreasing.

Proposition 3.11. *Assume the conditions in Proposition 3.10 and that either (A7) or (A8) holds true. Then there exists an absorbing set B_1 for G, which is compact in $H_0^1(\Omega)$.*

Proof. In view of Proposition 3.10 we have an absorbing set B_0 in $L^2(\Omega)$. Let $K > 0$ be such that $\|y\| \leq K$ for all $y \in B_0$.

Multiplying (3.1.1) by u and using (3.1.7) and (3.3.9) we get

$$\frac{d}{dt}\|u(t)\|_{L^2}^2 + a\left(\|u(t)\|_{H_0^1}^2\right)\|u(t)\|_{H_0^1}^2 \leq 2m_\varepsilon|\Omega| + 2\varepsilon\|u(t)\|_{L^2}^2 + \frac{1}{\lambda_1 m}\|h\|_{L^2}^2 \leq K_1 + K_2\|u(0)\|_{L^2}^2.$$

Thus, integrating between t and $t+r$, $0 < r \leq 1$, we deduce by using (3.3.9) again that

$$\begin{aligned} &\|u(t+r)\|_{L^2}^2 + \int_t^{t+r} a\left(\|u(s)\|_{H_0^1}^2\right)\|u(s)\|_{H_0^1}^2 ds \\ &\leq K_1 + K_2\|u(0)\|_{L^2}^2 + \|u(t)\|_{L^2}^2 \leq K_3\|u(0)\|_{L^2}^2 + K_4. \end{aligned} \tag{3.3.10}$$

Also, if $p>2$ in (A5), we multiply again by (3.1.1) by u and use (3.1.3) and (A6) to obtain

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|_{L^2}^2+\frac{m}{2}\|u(t)\|_{H_0^1}^2+C_4\left\|u\left(t\right)\right\|_{L^p}^p\leq C_3+\frac{1}{2\lambda_1 m}\|h\|_{L^2}^2.$$

Integrating over $(t,t+r)$ we have

$$\|u(t+r)\|_{L^2}^2+2C_4\int_t^{t+r}\left\|u\left(s\right)\right\|_{L^p}^p ds\leq K_5+\|u(t)\|_{L^2}^2\leq K_6+\|u(0)\|_{L^2}^2. \quad (3.3.11)$$

If we assume (A7), by (3.3.10) and (A6) we have that

$$\int_t^{t+r}A(\|u(s)\|_{H_0^1}^2)ds\leq\int_t^{t+r}M_1\|u(s)\|_{H_0^1}^2 ds\leq K_7(1+\|u(0)\|_{L^2}^2). \quad (3.3.12)$$

If we assume (A8), by (3.3.10) we obtain

$$\begin{aligned}&\int_t^{t+r}A(\|u(s)\|_{H_0^1}^2)ds=\int_t^{t+r}\int_0^{\|u(s)\|_{H_0^1}^2}a\left(r\right)drds\\&\leq\int_t^{t+r}a\left(\|u(s)\|_{H_0^1}^2\right)\|u(s)\|_{H_0^1}^2 ds\leq K_3\|u(0)\|_{L^2}^2+K_4.\end{aligned} \quad (3.3.13)$$

On the other hand, by (3.1.5) we get

$$-\int_\Omega F\left(u\left(t\right)\right)dx\geq-\widetilde{C}\int_\Omega(1+\left|u\left(t\right)\right|^p)dx. \quad (3.3.14)$$

Using (3.3.2) and (3.3.3) we can argue as in Theorem 3.5 to obtain

$$\frac{1}{2}\|u_t\|_{L^2}^2+\frac{d}{dt}\left(\frac{1}{2}A(\|u(t)\|_{H_0^1}^2-\int_\Omega\lambda\mathcal{F}(u_n)dx\right)\leq\frac{1}{2}\|h\|_{L^2}^2.$$

Since (3.3.11)-(3.3.14) imply that

$$\int_t^{t+r}\left(\frac{1}{2}A(\|u(s)\|_{H_0^1}^2-\int_\Omega\lambda\mathcal{F}(u\left(s\right))dx\right)ds\leq K_8+K_9\|u(0)\|_{L^2}^2,$$

we can apply the Uniform Gronwall Lemma to get

$$\frac{1}{2}A(\|u(t+r)\|^2_{H^1_0}) - \int_\Omega \lambda\mathcal{F}(u\,(t+r))dx \leq \frac{K_8 + K_9\|u(0)\|^2_{L^2}}{r} + K_{10}, \quad \text{for all } t \geq 0,$$

so by condition $(A6)$, (3.1.8) and (3.3.9) it follows that

$$\|u(t+1)\|^2_{H^1_0} \leq K_{11} + K_{12}\|u(0)\|^2_{L^2},$$

for all $t \geq 0$.
In particular,

$$\|u(1)\|^2_{H^1_0} \leq K_{11} + K_{12}\|u(0)\|^2_{L^2},$$

for any strong solution $u(\cdot)$ with initial condition $u(0)$.

For any $u_0 \in H^1_0\,(\Omega)$ with $\|u_0\|_{H^1_0} \leq R$ and any $u \in \mathcal{R}$ such that $u\,(0) = u_0$, the semiflow property

$$G(t+1, u_0) \subset G(1, G(t, u_0)) \text{ and } G(t, u_0) \subset B_0, \text{ if } t \geq t_0\,(R)\,,$$

imply that

$$\|u\,(t+1)\,\|^2_{H^1_0} \leq C(1+K^2)\ \forall t \geq t_0\,(R)\,.$$

Then there exists $M > 0$ such that the closed ball B_M in $H^1_0\,(\Omega)$ centered at 0 with radius M is absorbing for G. By Lemma 3.7 the set $B_1 = \overline{G(1, B_M)}$ is an absorbing set which is compact in $H^1_0\,(\Omega)$.

□

Now, the conditions to ensure the existence of a global attractor have been established.

Theorem 3.12. *Assume the conditions of Proposition 3.11. Then the multivalued semiflow G possesses a global compact invariant attractor $\mathcal{A}$.*

Proof. From Propositions 3.9 and 3.11 we deduce that the multivalued semiflow G is upper semicontinuous with closed values and the existence of an absorbing which is compact in $H_0^1(\Omega)$. Therefore, by Theorem 0.17 the existence of the global invariant attractor and its compactness in $H_0^1(\Omega)$ follow.

□

Recalling Theorem 0.15, the global attractor can be characterized as the union of bounded complete trajectories, i.e.

$$\mathcal{A} = \{\gamma(0) : \gamma \in \mathbb{K}\} = \cup_{t\in\mathbb{R}}\{\gamma(t) : \gamma \in \mathbb{K}\}, \tag{3.3.15}$$

where $\mathbb{K}$ denotes the set of all bounded complete trajectories in $\mathcal{R}$. As $\mathcal{R}$ satisfies (K3) and (K4) by Corollary 3.8, (3.3.15) follows.

Moreover, since the set B is said to be weakly invariant if for any $x \in B$ there exists a complete trajectory γ of $\mathcal{R}$ contained in B such that $\gamma(0) = x$, characterization (3.3.15) implies that the attractor $\mathcal{A}$ is weakly invariant.

As before, we denote the set of all fixed points by $\mathfrak{R}_{\mathcal{R}}$. It can be characterized as follows.

Lemma 3.13. *Assume the conditions of Lemma 3.7. Let $\mathfrak{R}$ be the set of $z \in H^2(\Omega) \cap H_0^1(\Omega)$ such that*

$$-a(\|z\|^2_{H_0^1})\frac{d^2z}{dx^2} = \lambda f(z) + h \quad \text{in } L^2(\Omega). \tag{3.3.16}$$

Then $\mathfrak{R}_{\mathcal{R}} = \mathfrak{R}$.

Proof. If $z \in \mathfrak{R}_{\mathcal{R}}$, then $u(t) \equiv z \in \mathcal{R}$. Thus, $u(\cdot)$ satisfies (3.2.2) and

$$\frac{du}{dt} = 0 \text{ in } L^2(0,T;L^2(\Omega)),$$

so (3.3.16) is satisfied.

Let $z \in \mathfrak{R}$. Then the map $u(t) \equiv z$ satisfies (3.3.16) for any $t \geq 0$ and

$$\frac{du}{dt} = 0 \text{ in } L^2(0,T;L^2(\Omega)),$$

so (3.2.2) holds true.

$\square$

Finally, we shall obtain the characterization of the global attractor in terms of the unstable and stable sets of the stationary points.

Theorem 3.14. *Assume the conditions of Proposition 3.11. Then it holds that*

$$\mathcal{A} = M^{+}(\mathfrak{R}) = M^{-}(\mathfrak{R}),$$

where

$$M^{+}(\mathfrak{R}) = \{z : \exists \gamma(\cdot) \in \mathbb{K},\ \gamma(0) = z, \quad dist_{H_0^1}(\gamma(t), \mathfrak{R}) \to 0,\ t \to +\infty\}, \tag{3.3.17}$$

$$M^{-}(\mathfrak{R}) = \{z : \exists \gamma(\cdot) \in \mathbb{F},\ \gamma(0) = z, \quad dist_{H_0^1}(\gamma(t), \mathfrak{R}) \to 0,\ t \to -\infty\}, \tag{3.3.18}$$

and $\mathbb{F}$ denotes the set of all complete trajectories of $\mathcal{R}$ (see Definition 0.2).

Remark 3.15. In (3.3.18) it is equivalent to use $\mathbb{K}$ instead of $\mathbb{F}$ because all the solutions are bounded forward in time.

Proof. We consider the function $E : \mathcal{A} \to \mathbb{R}$

$$E(y) = \frac{1}{2} A(\|y\|^2_{H_0^1}) - \lambda \int_{\Omega} F(y(x))dx - \int_{\Omega} h(x)y(x)dx. \tag{3.3.19}$$

Note that $E(y)$ is continuous in $H_0^1(\Omega)$. Indeed, the maps

$$y \mapsto \frac{1}{2} A(\|y\|^2_{H_0^1})$$

and

$$y \mapsto \int_{\Omega} h\,(x)\,y\,(x)\,dx$$

are obviously continuous in $H_0^1(\Omega)$.
On the other hand, by the embedding $H_0^1(\Omega) \subset L^{\infty}(\Omega)$ and using Lebesgue's

theorem, the continuity of
$$y \to \int_\Omega F(y(x))dx$$
follows.

Using (3.3.2)-(3.3.3) and multiplying the equation (3.1.1) by $\frac{du}{dt}$ for any $u \in \mathcal{R}$ we can obtain the following energy equality

$$\int_s^t \|\frac{d}{dr}u(r)\|_{L^2}^2 dr + E(u(t)) = E(u(s)) \quad \text{for all } t \geq s \geq 0.$$

Hence, $E(u(t))$ is non-increasing and by $(A6)$, (3.1.8) and the boundedness of $\mathcal{A}$, it is bounded from below. Thus $E(u(t)) \to l$, as $t \to +\infty$, for some $l \in \mathbb{R}$.

Let $z \in \mathcal{A}$ and $u \in \mathbb{K}$ be such that $u(0) = z$. By contradiction, suppose the existence of $\varepsilon > 0$ and $u(t_n)$, where $t_n \to +\infty$, for which

$$dist_{H_0^1}(u(t_n), \mathfrak{R}) > \varepsilon.$$

Since $\mathcal{A}$ is compact in $H_0^1(\Omega)$, we can take a converging subsequence (relabeled the same) such that
$$u(t_n) \to y \text{ in } H_0^1(\Omega),$$

where $t_n \to \infty$. By the continuity of the function E, it follows that $E(y) = l$. We will obtain a contradiction by proving that $y \in \mathfrak{R}$. Define $v_n(\cdot) = u(\cdot + t_n)$. By Lemma 3.7, there exist $v \in \mathcal{R}$ and a subsequence satisfying $v(0) = y$ and

$$v_n(t) \to v(t) \text{ in } H_0^1(\Omega)$$

for $t \geq 0$.

Thus, from $E(v_n(t)) \to E(v(t))$ we infer that $E(v(t)) = l$. Also, $v(\cdot)$ satisfies the energy equality, so that

$$l + \int_0^t \|v_r\|_{L^2}^2 dr = E(v(t)) + \int_0^t \|v_r\|_{L^2}^2 dr = E(v(0)) = E(y) = l.$$

Therefore,
$$\frac{dv}{dt}(s) = 0 \text{ for a.a. } s$$

and then by Lemma 3.13 we have $y \in \mathfrak{R}_{\mathcal{R}} = \mathfrak{R}$. As a consequence, $\mathcal{A} \subset M^+(\mathfrak{R})$. The converse inclusion follows from (3.3.15).

As before, take arbitrary $z \in \mathcal{A}$ and $u \in \mathbb{K}$ satisfying $u(0) = z$. Since by the embedding $H_0^1(\Omega) \subset C([0,1])$ the energy function is bounded from above in $\mathcal{A}$, $E(u(t)) \to l$, as $t \to -\infty$, for some $l \in \mathbb{R}$. Suppose that there are $\varepsilon > 0$ and $u(t_n)$, where $t_n \to +\infty$, such that

$$dist_{H_0^1}(u(-t_n), \mathfrak{R}) > \varepsilon.$$

Up to a subsequence we have that $u(-t_n) \to y$ in $H_0^1(\Omega)$, $E(y) = l$. Moreover, for $v_n(\cdot) = u(\cdot - t_n)$ there are $v \in \mathcal{R}$ and a subsequence such that $v(0) = y$ and

$$v_n(t) \to v(t) \text{ in } H_0^1(\Omega)$$

for $t \geq 0$.

Therefore, $E(v_n(t)) \to E(v(t))$ gives $E(v(t)) = l$ and then by the above arguments we get a contradiction because $y \in \mathfrak{R}$. Hence, $\mathcal{A} \subset M^-(\mathfrak{R})$ and we deduce the converse inclusion from (3.3.15).

□

Finally, we are able to obtain that the global attractor is compact in the space $C^1([0,1])$. This property will be important in order to study a more precise structure of the global attractor in terms of the stationary points and their heteroclinic connections.

We define the function $w(t) = u(\alpha^{-1}(t))$, where

$$\alpha(t) = \int_0^t a(\|u(s)\|^2_{H_0^1})ds,$$

which is under the conditions of Proposition 3.11 (see [19] for more details) a strong solution to the problem

$$\begin{cases} \dfrac{\partial w}{\partial t} - \dfrac{\partial^2 w}{\partial x^2} = \dfrac{f(w)+h}{a(\|w\|^2_{H^1_0})}, \text{ in } (0,\infty)\times\Omega, \\ w = 0 \quad \text{on } (0,\infty)\times\partial\Omega, \\ w(0,x) = u_0(x) \quad \text{in } \Omega. \end{cases} \tag{3.3.20}$$

Let $V^{2r} = D(A^r)$, $r \geq 0$. We will prove that the attractor is compact in any space V^{2r} with $0 \leq r < 1$. For this aim we will need the concept of mild solution.

Definition 3.16. Let consider the auxiliary problem

$$\begin{cases} \dfrac{dv}{dt} + Av(t) = g\left(t\right), \ t > 0, \\ \qquad v\left(0\right) = u_0, \end{cases} \tag{3.3.21}$$

where $g \in L^2_{loc}\left(0,+\infty; L^2\left(\Omega\right)\right)$. The function $u \in C([0,+\infty), L^2\left(\Omega\right))$ is called a mild solution to problem (3.3.21) if

$$v\left(t\right) = e^{-At}u_0 + \int_0^t e^{-A(t-s)}g(s)ds, \ \forall t \geq 0. \tag{3.3.22}$$

Remark 3.17. In the same way as in Lemma 2 in [85] we obtain that a strong solution to problem (3.3.20) is a mild solution to problem (3.3.21) with

$$g\left(t\right) = \left(f(w\left(t\right)) + h\right)/a(\|w\left(t\right)\|^2_{H^1_0}).$$

Lemma 3.18. *Assume the conditions of Proposition 3.11. Then the global attractor $\mathcal{A}$ is compact in V^{2r} for every $0 \leq r < 1$.*

Proof. Let $z \in \mathcal{A}$ be arbitrary. Since $\mathcal{A}$ is invariant, there exist $u_0 \in \mathcal{A}$ and $u \in \mathcal{R}$ such that $z = u\left(1\right)$ and $u\left(t\right) \in \mathcal{A}$ for all $t \geq 0$. Since $w\left(t\right) = u\left(\alpha^{-1}\left(t\right)\right)$ is a mild solution of (3.3.21) with $g\left(t\right) = \left(f(w\left(t\right)) + h\right)/a(\|w\left(t\right)\|^2_{H^1_0})$, the variation of constants formula (3.3.22) gives

$$z = w(\alpha(1)) = e^{-A\alpha(1)}u_0 + \int_0^{\alpha(1)} e^{-A(\alpha(1)-s)}g(s)ds.$$

As $\mathcal{A}$ is bounded in $H_0^1(\Omega)$ (and then in $L^\infty(\Omega)$), condition (A6) and the continuity of f imply that

$$\|u_0\|_{L^2} \le C \quad \text{and} \quad \|g\|_{L^\infty(0,\alpha(1);L^2(\Omega))} \le C,$$

where $C > 0$ does not depend on z. The standard estimate $\left\|e^{-At}\right\|_{\mathcal{L}(L^2(\Omega),D(A^r))} \le M_r t^{-r} e^{-at}$, M_r, $a > 0$ [77, Theorem 37.5], implies that

$$\begin{aligned}\|A^r z\|_{L^2} &\le \left\|A^r e^{-A\alpha(1)}u_0\right\|_{L^2} + \int_0^{\alpha(1)} \left\|A^r e^{-A(\alpha(1)-s)}g(s)\right\|_{L^2} ds \\ &\le M_r e^{-a\alpha(1)}\alpha(1)^{-r} C + M_r C \int_0^{\alpha(1)} (\alpha(1)-s)^{-r} ds,\end{aligned}$$

so $\mathcal{A}$ is bounded in V^{2r} for every $0 \le r < 1$.

From the compact embedding $V^\alpha \subset V^\beta$, for $\alpha > \beta$, and the fact that $\mathcal{A}$ is closed in any V^{2r} we obtain the result.

□

Corollary 3.19. *Assume the conditions of Proposition 3.11. Then the global attractor $\mathcal{A}$ is compact in $C^1([0,1])$.*

Proof. We obtain by Lemma 37.8 in [77] the continuous embedding

$$V^{2r} \subset C^1([0,1]) \text{ if } r > \frac{3}{4}.$$

Hence, the statement follows from Lemma 3.18.

□

3.4. Fixed points

In this section we are interested in studying the fixed points of problem (3.1.1) when $h \equiv 0$, that is, the solutions of the boundary-value problem

$$\begin{cases} -a(\|u\|^2_{H^1_0})\dfrac{d^2u}{dx^2} = \lambda f(u), \; 0 < x < 1, \\ u(0) = u(1) = 0. \end{cases} \tag{3.4.1}$$

For this aim we will use the properties of the fixed points of the standard Chafee-Infante equation. In order to do that, for any $d \geq 0$ we will study the following boundary-value problem

$$\begin{cases} -a(d)\dfrac{d^2u}{dx^2} = \lambda f(u), \; 0 < x < 1, \\ u(0) = u(1) = 0, \end{cases} \tag{3.4.2}$$

as it is obvious that $u(\cdot)$ is solution to problem (3.4.1) if and only if $u(\cdot)$ is a solution to problem (3.4.2) with $d = \|u\|^2_{H^1_0}$.

3.4.1. Dependence on the parameters of the fixed points for the Chafee-Infante equation

Denoting $\widetilde{\lambda} = \dfrac{\lambda}{a(d)}$ problem (3.4.2) becomes

$$\begin{cases} -\dfrac{d^2u}{dx^2} = \widetilde{\lambda} f(u), \; 0 < x < 1, \\ u(0) = u(1) = 0. \end{cases} \tag{3.4.3}$$

Assuming conditions (A1)-(A5), it is known [18] that if $n^2\pi^2 < \widetilde{\lambda} \leq (n+1)^2\pi^2$, then this problem has exactly $2n+1$ solutions, denoted by $v_0 \equiv 0, \; v_1^{\pm}, ..., v_n^{\pm}$. The function $v_k^{\pm}$ has $k+1$ simple zeros in $[0,1]$.

We need to study the dependence of the norm of these fixed points on the parameter $\widetilde{\lambda}$. First, we will show that the H^1-norm of the fixed points of problem (3.4.3) is strictly increasing with respect to the parameter $\widetilde{\lambda}$.

Lemma 3.20. *Assume conditions (A1)-(A5). Let* $v_1 = v^+_{k,\lambda_1}$, $v_2 = v^+_{k,\lambda_2}$ *with* $k^2\pi^2 < \lambda_1 < \lambda_2$. *Then* $\|v_1\|_{H^1_0} < \|v_2\|_{H^1_0}$.

Proof. We consider the equivalent norm in $H^1_0(\Omega)$ given by $\|v'\|_{L^2}$. The fixed points are the solutions of the initial value problem

$$\begin{cases} \dfrac{d^2u}{dx^2} + \widetilde{\lambda} f(u) = 0, \\ u(0) = 0, \;\; u'(0) = v_0 \end{cases} \tag{3.4.4}$$

such that $u(1) = 0$. The solutions of (3.4.4) satisfy the relation

$$\frac{(u'(x))^2}{2} + \widetilde{\lambda} F(u(x)) = \widetilde{\lambda} E, \;\; 0 \leq x \leq 1, \tag{3.4.5}$$

for some constant $E \geq 0$.

Denote $u_{\widetilde{\lambda}} = v^+_{k,\widetilde{\lambda}}$. By Theorem 7 in [18] we have that $u_{\widetilde{\lambda}}$ is associated with a unique value $E = E^+_k(\widetilde{\lambda}) > 0$. Moreover, $E^+_k(\widetilde{\lambda})$ is a solution of one of the following equations:

$$\begin{aligned} m\tau^{\widetilde{\lambda}}_+(E) + (m-1)\tau^{\widetilde{\lambda}}_-(E) &= \frac{1}{\sqrt{2}}, \\ m\tau^{\widetilde{\lambda}}_-(E) + (m-1)\tau^{\widetilde{\lambda}}_+(E) &= \frac{1}{\sqrt{2}}, \\ m\tau^{\widetilde{\lambda}}_+(E) + m\tau^{\widetilde{\lambda}}_-(E) &= \frac{1}{\sqrt{2}}, \end{aligned} \tag{3.4.6}$$

where either $k = 2m-1$ or $k = 2m$ and

$$\tau^{\widetilde{\lambda}}_+(E) = \widetilde{\lambda}^{-1/2} \int_0^{U_+(E)} (E - F(u))^{-1/2}\, du, \tag{3.4.7}$$

$$\tau^{\widetilde{\lambda}}_-(E) = \widetilde{\lambda}^{-1/2} \int_{U_-(E)}^0 (E - F(u))^{-1/2}\, du, \tag{3.4.8}$$

being $U_+(E)$ ($U_-(E)$) the positive (negative) inverse of F at E. It is obvious that for E fixed the functions $\tau^{\widetilde{\lambda}}_+(E)$, $\tau^{\widetilde{\lambda}}_-(E)$ are strictly decreasing with respect to $\widetilde{\lambda}$.

Then from (3.4.6) we deduce that the root $E_k^+(\widetilde{\lambda})$ is strictly increasing with respect to $\widetilde{\lambda}$. Thus, If $\lambda_1 < \lambda_2$, we have

$$\sqrt{2\lambda_1(E_k^+(\lambda_1) - F(u))} < \sqrt{2\lambda_2(E_k^+(\lambda_2) - F(u))}, \tag{3.4.9}$$

for $U^-(E_k^+(\lambda_1)) \leq u \leq U^+(E_k^+(\lambda_1))$.

We will prove now that $\|u'_{\widetilde{\lambda}}\|_{L^2}$ is strictly increasing in $\widetilde{\lambda}$.
The function $u_{\widetilde{\lambda}}$ has $k+1$ simple zeros in $[0,1]$ and $u_{\widetilde{\lambda}}$ is positive in the first subinterval. Let $T_+(E_k^+(\lambda))$ be the x-time necessary to go from the initial condition $u_\lambda(0) = 0$ to the point where $u'_\lambda(T_+(E_k^+(\lambda))) = 0$. Then the length of the first subinterval is $2T_+(E_k^+(\lambda))$ [18]. By (3.4.5),

$$(u'_{\widetilde{\lambda}}(x))^2 = \sqrt{2\widetilde{\lambda}}\sqrt{E_k^+(\widetilde{\lambda}) - F(u_{\widetilde{\lambda}}(x))}\; u'_{\widetilde{\lambda}}(x),$$

so we have

$$\int_0^{T_+(E_k^+(\widetilde{\lambda}))} (u'_{\widetilde{\lambda}}(x))^2 dx = \int_0^{T_+(E_k^+(\widetilde{\lambda}))} \sqrt{2\widetilde{\lambda}}\sqrt{E_k^+(\widetilde{\lambda}) - F(u_{\widetilde{\lambda}}(x))}\; u'_{\widetilde{\lambda}}(x)dx.$$

By the change of variable $v = u_{\widetilde{\lambda}}(x)$ we obtain

$$\int_0^{T_+(E_k^+(\widetilde{\lambda}))} (u'_{\widetilde{\lambda}}(x))^2 dx = \int_0^{U^+(E_k^+(\widetilde{\lambda}))} \sqrt{2\widetilde{\lambda}}\sqrt{E_k^+(\widetilde{\lambda}) - F(v)}\; dv = g(\widetilde{\lambda}).$$

Since

$$\widetilde{\lambda} \mapsto U^+(E_k^+(\widetilde{\lambda}))$$

is strictly increasing and using (3.4.9), we conclude that the function $g(\widetilde{\lambda})$ is strictly increasing. Hence, putting $x_1(\widetilde{\lambda}) = 2T_+(E_k^+(\widetilde{\lambda}))$ we obtain that the norm of $u_{\widetilde{\lambda}}$ in the first subinterval, $\|u'_{\widetilde{\lambda}}\|_{L^2(0,x_1(\widetilde{\lambda}))}$, is strictly increasing. Arguing in the same way in the other subintervals we obtain that

$$\widetilde{\lambda} \mapsto \|u'_{\widetilde{\lambda}}\|_{L^2}$$

is strictly increasing. $\square$

Let us prove the same result but with respect to the norm $\left\|u_{\widetilde{\lambda}}\right\|_{L^p}$ with $p \geq 1$.

Lemma 3.21. *Assume conditions (A1)-(A5) and let f be odd. Let $v_1 = v^+_{k,\lambda_1}$, $v_2 = v^+_{k,\lambda_2}$ with $k^2\pi^2 < \lambda_1 < \lambda_2$. Then $\|v_1\|_{L^p} < \|v_2\|_{L^p}$ for any $p \geq 1$.*

Proof. As in the previous lemma, denote $u_{\widetilde{\lambda}} = v^+_{k,\widetilde{\lambda}}$. The function $u_{\widetilde{\lambda}}$ has $k+1$ zeros in $[0,1]$ at the points

$$0 < x_1 < x_2 < ... < x_{k-1} < 1.$$

When f is odd, by symmetry, the length of all subintervals has to be the same, so $x_j = \frac{j}{k}$ regardless the value of $\widetilde{\lambda}$.

We shall prove that in the first subinterval we have that $u_{\lambda_1}(x) < u_{\lambda_2}(x)$, for all $x \in \left(0, \frac{1}{k}\right)$. By (3.4.5) for $x \in [0, \frac{1}{2k}]$ we have

$$x = \int_0^x ds = \int_0^{u_{\widetilde{\lambda}}(x)} \frac{du}{\sqrt{2\widetilde{\lambda}\left(E_k^+\left(\widetilde{\lambda}\right) - F(u)\right)}},$$

so (3.4.9) yields

$$x = \int_0^{u_{\lambda_2}(x)} \frac{du}{\sqrt{2\lambda_2\left(E_k^+(\lambda_2) - F(u)\right)}} = \int_0^{u_{\lambda_1}(x)} \frac{du}{\sqrt{2\lambda_1\left(E_k^+(\lambda_1) - F(u)\right)}}$$

$$> \int_0^{u_{\lambda_1}(x)} \frac{du}{\sqrt{2\lambda_2\left(E_k^+(\lambda_2) - F(u)\right)}}, \quad \text{if } x \in (0, \frac{1}{2k}].$$

Thus, $u_{\lambda_1}(x) < u_{\lambda_2}(x)$, for all $x \in (0, \frac{1}{2k}]$. By symmetry we obtain that the inequality is true in $\left(0, \frac{1}{k}\right)$.

Repeating the same argument in the other subintervals we get that

$$|u_{\lambda_1}(x)| < |u_{\lambda_2}(x)| \text{ for all } x \in (0,1), \ x \neq \frac{j}{k}, \ j = 1, ...k-1.$$

This implies that $\|u_{\lambda_1}\|_{L^p} < \|u_{\lambda_2}\|_{L^p}$ for any $p \geq 1$. □

Remark 3.22. The statements in Lemmas 3.20-3.21 are also true for $v^-_{k,\widetilde{\lambda}}$, because $v^-_{k,\widetilde{\lambda}}(x) = v^+_{k,\widetilde{\lambda}}(1-x)$, so the H^1_0 and L^p norms of $v^-_{k,\widetilde{\lambda}}$ and $v^+_{k,\widetilde{\lambda}}$ are the same.

Lemma 3.23. *Assume conditions (A1)-(A5). Then* $\|v^+_{k,\widetilde{\lambda}}\|_{H^1_0}$ *and* $\|v^-_{k,\widetilde{\lambda}}\|_{H^1_0}$ *are contiunous with respect* $\widetilde{\lambda}$.

Proof. For $\lambda > (\pi k)^2$ put $\phi_{\widetilde{\lambda}} = v^{\pm}_{k,\widetilde{\lambda}}$.

Let us to show that if $\widetilde{\lambda}_n \to \widetilde{\lambda}_0 \in ((\pi k)^2, \infty)$, we must have that

$$\|\phi_{\widetilde{\lambda}_n} - \phi_{\widetilde{\lambda}_0}\|_{H^1_0} \to 0.$$

Since

$$(\phi_{\widetilde{\lambda}})_{xx}(r) + \widetilde{\lambda} f(\phi_{\widetilde{\lambda}}) = 0 \tag{3.4.10}$$

and

$$\limsup_{|u|\to+\infty} \frac{f(u)}{u} < 0$$

we have that there is a constant $M > 0$ such that

$$\int_0^1 ((\phi_{\widetilde{\lambda}})_x)^2 = \widetilde{\lambda} \int_0^1 f(\phi_{\widetilde{\lambda}})\phi_{\widetilde{\lambda}} \leq \widetilde{\lambda} M$$

Therefore, the family

$$((\pi k)^2, \infty) \ni \widetilde{\lambda} \mapsto \phi_{\widetilde{\lambda}} \in H^1_0(0,1)$$

is bounded in bounded subsets of $((\pi k)^2, \infty)$. Since $H^1_0(0,\pi) \hookrightarrow C([0,\pi])$, it is also uniformly bounded in $C([0,\pi])$ uniformly in bounded subsets of (j^2, ∞).
From the continuity of f, the same is true for

$$((\pi k)^2, \infty) \ni \widetilde{\lambda} \mapsto f \circ \phi_{\widetilde{\lambda}} \in C([0,1])$$

and, using (3.4.10), for

$$((\pi k)^2, \infty)\widetilde{\lambda} \mapsto (\phi_{\widetilde{\lambda}})_{xx} \in C([0,1]).$$

It follows from the compact embedding of $H^2(0,1)$ into $H_0^1(0,1)$ that there is a subsequence $\{\widetilde{\lambda}_{n_k}\}$ of $\{\widetilde{\lambda}_n\}_n$ such that

$$\phi_{\widetilde{\lambda}_{n_k}} \overset{k\to\infty}{\longrightarrow} w \text{ in } H_0^1(0,1).$$

Now, since

$$\int_0^1 (\phi_{\widetilde{\lambda}_{n_k}})_x v_x = \widetilde{\lambda}_n \int_0^1 f(\phi_{\widetilde{\lambda}_{n_k}})v,$$

for all $v \in H_0^1(0,1)$, passing to the limit as $k \to \infty$ we have that

$$\int_0^1 w_x v_x = \widetilde{\lambda}_0 \int_0^1 f(w)v,$$

and w is a weak solution of (3.4.10).

Hence, since w also converges in the $C^1([0,1])$ norm, we deduce that

$$w \equiv 0 \text{ or } w = \phi_{\widetilde{\lambda}_0}.$$

To see that $w \not\equiv 0$ we recall that

$$((\pi k)^2, \infty) \ni \widetilde{\lambda} \mapsto \int_0^1 ((\phi_{\widetilde{\lambda}})_x)^2$$

is an strictly increasing function of $\widetilde{\lambda}$.

This shows the continuity of the function

$$((\pi k)^2, \infty) \ni \widetilde{\lambda} \mapsto \phi_{\widetilde{\lambda}} \in H_0^1(0,1).$$

$\square$

3.4.2. Nonlocal fixed points

Although we are mainly interested in problem (3.1.1), we will study the existence of stationary points for an elliptic problem with a more general nonlocal term than in (3.4.1). Namely, let us consider the following problem:

$$\begin{cases} -a\left(l(u)\right)u_{xx} = \lambda f\left(u\right), \ 0 < x < 1, \\ u\left(0\right) = u(1) = 0, \end{cases} \tag{3.4.11}$$

where

$$l\left(u\right) = \left\|u\right\|_{H_0^1}^r$$

or

$$l\left(u\right) = \left\|u\right\|_{L^p}^r ,$$

for $p \geq 1$, $r > 0$.

Let

$$d_k = \sup\{d : \lambda > a\left(\overline{d}\right)\pi^2 k^2 \ \forall \overline{d} \leq d\}.$$

Then for any $d < d_k$ there exists the fixed point u_k^d of (3.4.2), where u_k^d is either equal to u_k^+ or u_k^-.

It is obvious that any solution of (3.4.11) is a solution of (3.4.2) with $d = l\left(u\right)$. Therefore, all the solutions to problem (3.4.11) have to be solutions u_k^d to problem (3.4.2) for a suitable d.

In the same line as for the classical Chafee-Infante equation (see [33]), we are now interested in analyzing how many equilibria there are. In this case, the nonlocal term will play an important role since it is crucial when the behaviour of the bifurcations is studied. We want to construct a sequence of bifurcations similar to the one in Theorem 1.18 where as long as the parameter $\lambda > 0$ increases, a bifurcation from 0 happens.

Theorem 3.24. *Assume conditions (A1)-(A6) and, additionally, that*

$$a\left(0\right)\pi^2 k^2 < \lambda. \tag{3.4.12}$$

Then:

- *For any* $1 \leq j \leq k$ *there exists* $d_j^* < d_k$ *such that* $u_j^{d_j^*}$ *is a fixed point of problem (3.4.11).*
- *If* $\lambda \leq a\left(0\right)\pi^2\left(k+1\right)^2$ *and* $a(0) = \min_{s\geq 0}\{a\left(s\right)\}$*, there are no fixed points for* $j > k$*.*
- *If* $N \geq k$ *is the first integer such that* $\lambda \leq \inf_{s\geq 0}\{a\left(s\right)\pi^2\left(N+1\right)^2\}$*, there are no fixed points for* $j > N$*.*
- *If* $l\left(u\right) = \|u\|_{H_0^1}^r$*,* $\lambda \leq a\left(0\right)\pi^2\left(k+1\right)^2$ *and* a *is non-decreasing, there are exactly* $2k+1$ *solutions to problem (3.4.11):* $0,\ u_{1,d_1^*}^{\pm}, \ldots, u_{k,d_k^*}^{\pm}$*.*
- *If* $l\left(u\right) = \|u\|_{L^p}^r$*,* $\lambda \leq a\left(0\right)\pi^2\left(k+1\right)^2$*,* f *is odd and* a *is non-decreasing, there are exactly* $2k+1$ *solutions to problem (3.4.11):* $0,\ u_{1,d_1^*}^{\pm}, \ldots, u_{k,d_k^*}^{\pm}$*.*

Proof. For the first statement, it is enough to prove the result for $j = k$. By condition (3.4.12) we have that $d_k \in (0, +\infty]$.

Consider first the case where d_k is finite. We need to obtain the existence of $d_k^* < d_k$ such that $l\left(u_k^{d_k^*}\right) = d_k^*$. When $d = 0$ it is clear that $l\left(u_k^0\right) > 0$. Also, we know that $l\left(u_k^{d_k}\right) = 0$. Multiplying (3.4.2) by u_k^d and using (3.1.7), (A6) and the Poincaré inequality we obtain

$$\left\|\left(u_k^d\right)'\right\|_{L^2}^2 \leq \frac{\lambda}{a\left(d\right)}\left(f\left(u_k^d\right), u_k^d\right) \leq \frac{\lambda}{m}\left(m_\varepsilon + \varepsilon\left\|u_k^d\right\|_{L^2}^2\right) \leq K_1 + \frac{1}{2}\left\|\left(u_k^d\right)'\right\|_{L^2}^2,$$

so, by using the embedding $H_0^1\left(\Omega\right) \subset L^\infty\left(\Omega\right)$, $l\left(u_k^d\right)$ is bounded in d. This implies that the function $g\left(d\right) = l\left(u_k^d\right)$ has to intersect the line $y\left(d\right) = d$ at some point d_k^*. It remains to check that $d_k^* < d_k$. For this aim we prove first that $u_k^d \underset{d\to d_k}{\to} 0$ strongly in $H_0^1\left(\Omega\right)$. Indeed, as u_k^d is bounded in $H_0^1\left(\Omega\right)$, there exist v and a sequence $\{u_k^{d_j}\}$

such that $u_k^{d_j} \to v$ in $L^2(\Omega)$. The embedding $H_0^1(\Omega) \subset C([0,1])$ and the continuity of the function $f(u)$ imply that $\{f(u_k^{d_j})\}$ is bounded in $C([0,1])$, so from

$$\left\| \left(u_k^{d_j}\right)'' \right\|_{L^2} \leq \frac{\lambda}{a(d_j)} \left\| f\left(u_k^{d_j}\right) \right\|_{L^2} \leq \frac{\lambda}{m} \left\| f\left(u_k^{d_j}\right) \right\|_{L^2} \leq C$$

we deduce that $\{u_k^{d_j}\}$ is bounded in $H^2(\Omega)$. Hence, $u_k^{d_j} \to v$ in $H_0^1(\Omega)$ and $C^1([0,1])$. Also, $f(u_k^{d_j}) \to f(v)$ in $C([0,1])$. Therefore, for any $\psi \in H_0^1(\Omega)$ we have that

$$\begin{array}{ccc} \left(\left(u_k^{d_j}\right)', \psi' \right) & = & \frac{\lambda}{a(d_j)} \left(f\left(u_k^{d_j}\right), \psi \right) \\ \downarrow & & \downarrow \\ (v', \psi') & = & \frac{\lambda}{a(d_k)} \left(f(v), \psi \right), \end{array}$$

which implies that v is a solution to problem (3.4.2) with $d = d_k$. But from $u_k^{d_j} \to v$ in $C^1([0,1])$ it follows that v cannot be a point with less than $k+1$ simple zeros in $[0,1]$ and then $\lambda / a(d_k) = k^2\pi^2$ implies that $v \equiv 0$. As the limit is the same for every converging subsequence, $u_k^d \underset{d \to d_k}{\to} 0$ strongly in $H_0^1(\Omega)$. Thus, $d_k > 0$ and $\lim_{d \to d_k} \left\| \left(u_k^d\right)' \right\|_{L^2} = 0$ imply that $d_k^* < d_k$.

Second, let $d_k = +\infty$. Then the existence of $d_k^* < +\infty$ follows by the same argument as before.

The second and third statements are a consequence of

$$\lambda \leq a(0)\pi^2 (k+1)^2 \leq a(d)\pi^2 (k+1)^2 \text{ for any } d \geq 0$$

and

$$\lambda \leq \inf_{s \geq 0} \{a(s)\} \pi^2 (N+1)^2 \leq a(d)\pi^2 (N+1)^2 \text{ for any } d \geq 0,$$

respectively, because in such a case for problem (3.4.2) the fixed points $v_j^{\pm}$, $j > k$ (respectively $j > N$), do not exist.

The last two statements are a consequence of the first two statements and of the fact that the points of intersection of the functions $g(d) = l\left(u_k^d\right)$ and $y(d) = d$ has to be unique, because if a is non-decreasing, then $g(d)$ is non-increasing by Lemmas 3.20 and 3.21. $\square$

In view of this theorem, we have exactly the same equilibria and bifurcations as in the classical Chafee-Infante equation (see [18] and [55]) when the function $a(d)$ is non-decreasing, because in this case in view of the monotone dependence between the functions $a(d)$ and $g(d)$, there is only one intersection point of the function $g(d)$ with the bisector, as it is shown in Figure 3.1. This follows from the fact that $g(d) - d$ is strictly decreasing, but there may be weaker conditions on $a(\cdot)$ that would lead $g(d) - d$ to be strictly decreasing.

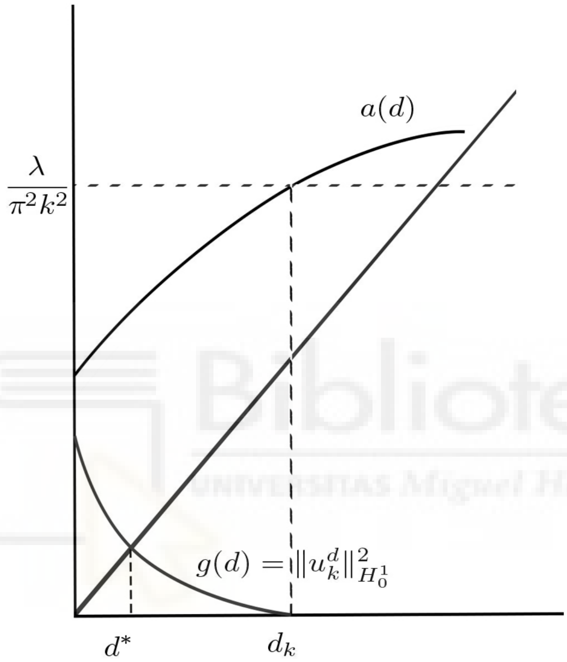


Figure 3.1: $a(d)$ non-decreasing

When the function $a(\cdot)$ is not assumed to be monotone, an interesting situation appears. More precisely, it is possible to have more than two equilibria with the same number of zeros. If $l(u) = \|u\|^2_{H_0^1}$, for the equilibria with $k+1$ zeros in $[0,1]$ this happens when the equation

$$d = \int_0^1 \left| \frac{du_k^d(x)}{dx} \right|^2 dx = g(d) \tag{3.4.13}$$

has more than one solution. For instance, if $a(0) = a(\bar{d})$ for some $0 < \bar{d} < g(0)$, then $g(0) = g(\bar{d})$. Assuming that there are $0 < d_k^1 < d_k^2 < \bar{d}$ such that $a(d_k^2) = a(d_k^1) = \frac{\lambda}{\pi^2 k^2}$, there must exist $0 < d_1^* < d_k^1 < d_k^2 < d_2^* < \bar{d}$ such that $g(d_i^*) = d_i^*$. Now, by the argument in Theorem 3.24, there must exist a $d_3^* > \bar{d}$ such that $g(d_3^*) = d_3^*$, obtaining six fixed points with $k+1$ zeros in $[0,1]$. This situation is shown in Figure 3.2, where d_1^*, d_2^* and d_3^* are solutions of (3.4.13), that is, there are three intersection points with the bisector. We notice that when $a(d) > \lambda/(\pi^2 k^2)$, the function $g(d)$ is not defined since the condition for such equilibria to exist is not satisfied, but we can make this function continuous by putting $g(d) = 0$ whenever $a(d) \geq \lambda/(\pi^2 k^2)$. This procedure establishes that, having fixed a natural number k, for any $j \in \mathbb{N}$ we may construct $a(\cdot)$ in such a way that we have $2(2j+1)$ equilibria with $k+1$ zeros in $[0,1]$.

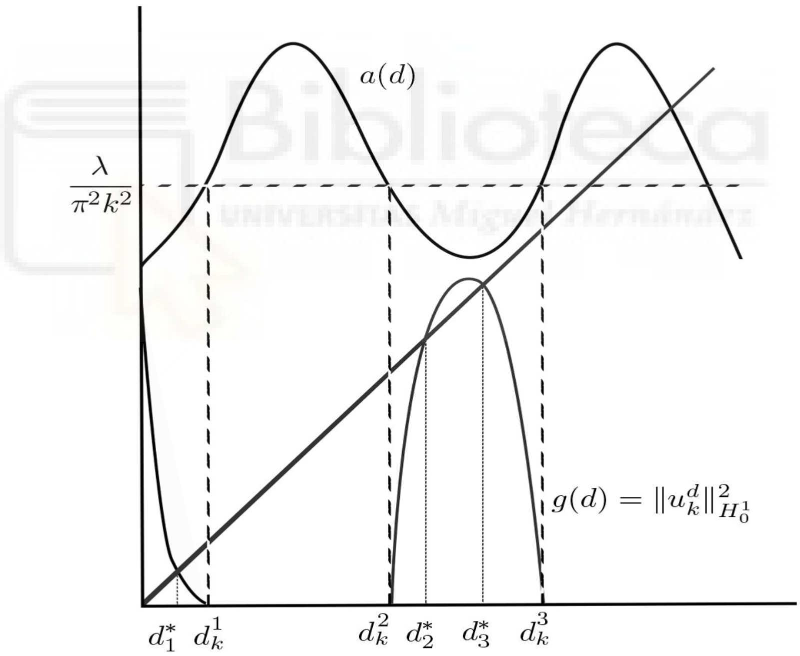


Figure 3.2: $a(d)$ whatever

At least there is always one intersection point with the bisector, but the function $g(d)$ could be even tangent to the bisector at some point or not cut it again.

3.4.3. Lap number and some forbidden connections

With Theorem 3.24 at hand we can improve the description of the global attractor given in Theorem 3.14.

Under conditions (A1)-(A6), (A8) and $h \equiv 0$, if

$$a(0)\pi^2 n^2 < \lambda \le a(0)\pi^2 (n+1)^2 \tag{3.4.14}$$

then problem (3.1.1) possesses exactly $2n+1$ fixed points: $v_0 = 0,\ u^{\pm}_{1,d_1^*}, ..., u^{\pm}_{n,d_n^*}$.

Let ϕ be a bounded complete trajectory. We know by Theorem 3.14 that

$$dist_{H_0^1}(\phi(t), \mathfrak{R}) \to 0, \text{ as } t \to \pm\infty.$$

As the number of fixed points is finite, we will prove that in fact the solution has to converge to one fixed point forwards and backwards. We recall the omega and alpha limit sets of ϕ, given by

$$\begin{aligned}\omega(\phi) &= \{y : \exists t_n \to +\infty \text{ such that } \phi(t_n) \to y\},\\ \alpha(\phi) &= \{y : \exists t_n \to -\infty \text{ such that } \phi(t_n) \to y\},\end{aligned}$$

are non-empty, compact and connected (cf. Lemma 0.18). Also, we have that $dist_{H_0^1}(\phi(t), \omega(\phi)) \underset{t\to+\infty}{\to} 0$, $dist_{H_0^1}(\phi(t), \alpha(\phi)) \underset{t\to-\infty}{\to} 0$. Since $\omega(\phi), \alpha(\phi) \subset \mathfrak{R}$ and $\mathfrak{R}$ is finite, the only possibility is that $\omega(\phi) = z_1 \in \mathfrak{R}$, $\alpha(\phi) = z_2 \in \mathfrak{R}$.

Thus, we have established the following result.

Theorem 3.25. *Let assume conditions (A1)-(A6), (A8), (3.4.14) and $h \equiv 0$. Then*

$$\mathcal{A} = \bigcup_{k=0}^{2n+1} M^+(v_k) = \bigcup_{k=0}^{2n+1} M^-(v_k),$$

where n is given in (3.4.14) and $v_0 = 0$, $v_1 = u^+_{1,d_1^*}$, $v_2 = u^-_{1,d_1^*}, \ldots$

In other words, the global attractor $\mathcal{A}$ consists of the set of stationary points $\mathfrak{R}$ (which has $2n+1$ elements) and the bounded complete trajectories that connect them (the heteroclinic connections).

Remark 3.26. As the Lyapunov function (3.3.19) is strictly decreasing along a trajectory ϕ which is not a fixed point, then there cannot exist homoclinic connections for any fixed point. This implies in particular that if $n = 0$, then $\mathcal{A} = \{0\}$.

Remark 3.27. If we use condition (A7) instead of (A8), then we cannot guarantee that the number of fixed points is finite. But if we suppose that this is the case, then the result remains valid. In this situation, there could be more than two fixed points with the same number of zeros.

Using the concept of lap number of the solutions we can discard some heteroclinic connections.

We consider the function $w(t) = u(\alpha^{-1}(t))$, which is a strong solution to problem (3.3.20). For any strong solution $u(\cdot)$ conditions (A1), (A3), (A6) and $u \in C([0,+\infty), H_0^1(\Omega))$ imply that the function

$$r(t,x) = \frac{\lambda}{a(\|w(t)\|_{H_0^1}^2)} \frac{f(w(t,x))}{w(t,x)}$$

is continuous and $w(\cdot)$ is a solution of the linear equation

$$\frac{\partial w}{\partial t} - \frac{\partial^2 w}{\partial x^2} = r(t,x)\, w. \tag{3.4.15}$$

Thus, by Theorem A.3 in the Appendix A (see also Theorem C in [2]) the number of zeros of $w(t)$ in $[0,1]$ is a nonincreasing function of t. Since $\alpha^{-1}(t)$ is an increasing function of time, the result is also true for the solution $u(\cdot)$. Making use of this property we will prove the following result.

Lemma 3.28. *Let assume conditions (A1)-(A6), $h \equiv 0$ and either (A7) or (A8). Then if $n > k$, there cannot exist a connection from the fixed point $u^{\pm}_{k,d^*_k}$ to the fixed point $u^{\pm}_{n,d^*_n}$, that is, there cannot exist a bounded complete trajectory ϕ such that*

$$\phi(t) \to u^{\pm}_{n,d^*_n} \text{ as } t \to +\infty,$$

$$\phi(t) \to u^{\pm}_{k,d^*_k} \text{ as } t \to -\infty.$$

Proof. By contradiction assume that such complete trajectory exists. Denote by $l(z)$ the number of zeros of z in $[0,1]$. Using the compactness of the attractor in $C^1([0,1])$ (see Corollary 3.19) we obtain that

$$\phi(t) \to u^{\pm}_{n,d^*_n} \text{ in } C^1([0,1]) \text{ as } t \to +\infty,$$
$$\phi(t) \to u^{\pm}_{k,d^*_k} \text{ in } C^1([0,1]) \text{ as } t \to -\infty.$$

Then, as the zeros are simple, we can choose $t_1 > 0$ large enough such that

$$l(\phi(-t_1)) = l\left(u^{\pm}_{k,d^*_k}\right) = k+1.$$

Put $u(t) = \phi(t - t_1)$, which is a strong solution of (3.1.1). Now we choose $t_2 > 0$ such that

$$l(u(t_2)) = l\left(u^{\pm}_{n,d^*_n}\right) = n+1.$$

Therefore,

$$l(u(0)) = k+1$$

and

$$l(u(t_2)) = n+1 > k+1.$$

This contradicts the fact that the number of zeros of $u(t)$ is non-increasing.

□

Lemma 3.29. *Let assume conditions (A1)-(A6), $h \equiv 0$ and either (A7) or (A8). Let $u^+_{k,d^*_k}, u^-_{k,d^*_k}$ be a pair of fixed points corresponding to the same value d^*_k. Then there cannot be an heteroclinic connection between them.*

Proof. Let k be even. The function $v(x) = u^+_{k,d^*_k}(1-x)$ is a fixed point corresponding to d^*_k as

$$-\frac{\partial^2 v}{\partial x^2}(x) = -\frac{\partial^2 u^+_{k,d^*_k}}{\partial x^2}(1-x) = \frac{\lambda}{a(d^*_k)} f\left(u^+_{k,d^*_k}(1-x)\right) = \frac{\lambda}{a(d^*_k)} f(v(x)),$$

so $u^-_{k,d^*_k}(x) = v(x) = u^+_{k,d^*_k}(1-x)$.

The equalities

$$\int_0^1 \left(\frac{\partial u^-_{k,d^*_k}}{\partial x}(x) \right)^2 dx = \int_0^1 \left(\frac{\partial u^+_{k,d^*_k}}{\partial x}(1-x) \right)^2 dx = \int_0^1 \left(\frac{\partial u^+_{k,d^*_k}}{\partial x}(y) \right)^2 dy,$$

$$\int_0^1 \int_0^{u^-_{d^*_k}(x)} f(s)\, ds dx = \int_0^1 \int_0^{u^+_{d^*_k}(1-x)} f(s)\, ds dx = \int_0^1 \int_0^{u^+_{d^*_k}(y)} f(s)\, ds dy$$

imply that $E(u^-_{k,d^*_k}) = E\left(u^+_{k,d^*_k}\right)$, where E is the Lyapunov function (3.3.19). Since this function is strictly decreasing along a trajectory ϕ which is not a fixed point, there cannot exist a heteroclinic connection between these two points.

When k is odd, we make use of the lap-number property. For a global solution $u(\cdot)$ let

$$\begin{aligned} Q^+(t) &= \{x \in (0,1) : u(t,x) > 0\}, \\ Q^-(t) &= \{x \in (0,1) : u(t,x) < 0\}. \end{aligned}$$

In the proof of Theorem A.3 in the Appendix A it is shown that if $t_1 > t_0$, then there is an injective map for the connected components of $Q^+(t_1)$ $(Q^-(t_1))$ to connected components of $Q^+(t_0)$ $(Q^-(t_0))$. Let, for example, $u(\cdot)$ be a global solution such that

$$u(t) \underset{t\to-\infty}{\to} \phi^-_{k.d^*_k},\ u(t) \underset{t\to+\infty}{\to} \phi^+_{k.d^*_k}.$$

Since we have convergence in $C^1([0,\pi])$, there are $t_0 < t_1$ such that the number of components of $Q^+(t_0)$ is equal to $(j-1)/2$ and the number of components of $Q^+(t_1)$ is equal to $(j+1)/2$. This contradicts the existence of the above injective map. Thus, such heteroclinic connection is impossible. A similar argument (but using $Q^-(t)$) is valid for the connection from $\phi^+_{k.d^*_k}$ to $\phi^-_{k.d^*_k}$.

□

Remark 3.30. In the case where condition (A7) is assumed, there could be more than two equilibria with $k+1$ zeros in $[0,1]$. In this case there could exist connections between fixed points with different values of the constant d.

3.5. Morse decomposition

In this section we study in more detail the structure of the global attractor in the case where the function f is odd. More precisely, we obtain that the m-semiflow G is dynamically gradient, which is equivalent to saying that there is a Morse decomposition of the attractor [44], and study the stability of the fixed points.

3.5.1. Aproximations

We consider now the situation when conditions (A1)-(A6), $h = 0$ and either (A7) or (A8) are satisfied and, moreover, the function f is odd.

In this section we consider the following problems:

$$\begin{cases} \dfrac{\partial u}{\partial t} - a(\|u\|^2_{H_0^1})\dfrac{\partial^2 u}{\partial x^2} = \lambda f_{\varepsilon_n}(u), \quad t > 0,\ x \in (0,1), \\ u(t,0) = 0,\ u(t,1) = 0, \\ u(0,x) = u_0(x), \end{cases} \tag{3.5.1}$$

where the function f_{ε_n} is defined below and $\varepsilon_n \to 0$, as $n \to \infty$.

Let set up the convolution. We consider the mollifier $\rho_{\varepsilon_n}(\cdot)$ in $\mathbb{R}$ with the explicit construction

$$\rho_{\varepsilon_n}(x) := \frac{\psi_{\varepsilon_n}(x)}{\int_{\mathbb{R}} \psi_{\varepsilon_n}}, \quad \text{where} \quad \psi_{\varepsilon_n}(x) := \begin{cases} \exp\left(\frac{1}{x^2-\varepsilon_n}\right), & \text{if } |x| < \varepsilon_n; \\ 0, & \text{otherwise}. \end{cases}$$

We define the function

$$f^{\varepsilon_n}(u) = \int_{\mathbb{R}} \rho_{\varepsilon_n}(s) f(u-s) ds.$$

It is well known that $f^{\varepsilon_n}(\cdot) \in C^\infty(\mathbb{R})$ and that for any compact subset $A \subset \mathbb{R}$, f^{ε_n} converges uniformly to f,

$$\|f^{\varepsilon_n} - f\|_{C^0} \to 0,$$

as $\varepsilon_n \to 0$ (see [51]).

It is clear that for $u > \varepsilon_n$ the function $f^{\varepsilon_n}(u)$ is strictly concave.

We need the approximation to fulfil (A2)-(A3). For that end, we consider the approximation except on the interval $[-\varepsilon_n, \varepsilon_n]$, for any $\varepsilon_n > 0$. There exists a polynomial of sixth degree $p(x)$ such that

$$\begin{aligned} &p(0) = 0, && p(\varepsilon_n) = f^{\varepsilon_n}(\varepsilon_n), \\ &p'(0) = 1, && p'(\varepsilon_n) = f^{\varepsilon_n\prime}(\varepsilon_n), \\ &p''(0) = 0, && p''(\varepsilon_n) = f^{\varepsilon_n\prime\prime}(\varepsilon_n), \\ &p'''(0) = -1. && \end{aligned}$$

We choose $\gamma > 0$ such that $p''(s) < 0$ for all $s \in (0, \gamma]$. We can assume that $\varepsilon_n < \gamma$ for all n.

Thus, by construction the function

$$f_{\varepsilon_n}(x) = \begin{cases} -f^{\varepsilon_n}(-x) & if \quad x < -\varepsilon_n, \\ -p(-x) & if \quad -\varepsilon_n \leq x \leq 0, \\ p(x) & if \quad 0 \leq x \leq \varepsilon_n, \\ f^{\varepsilon_n}(x) & if \quad x > \varepsilon_n \end{cases} \tag{3.5.2}$$

approximates the function f uniformly in compact sets, that is, for any $[-M, M]$ and $\delta > 0$ there exists $n_0(M, \delta) \in \mathbb{N}$ such that

$$|f(x) - f_{\varepsilon_n}(x)| < \delta, \quad \text{for all } n \geq n_0,\ x \in [-M, M]. \tag{3.5.3}$$

Also, it satisfies the following properties:

(B1) $f_{\varepsilon_n} \in C^2(\mathbb{R})$;

(B2) $f_{\varepsilon_n}(0) = 0$;

(B3) $f'_{\varepsilon_n}(0) = 1$;

(B4) f_{ε_n} is strictly concave if $u > 0$ and strictly convex if $u < 0$;

(B5) f_{ε_n} is odd.

Lemma 3.31. *Let f satisfy (A5). Then the functions f_{ε_n} satisfy condition (A5) and (3.1.7) with independent constants of ε_n.*

Proof. We assume without loss of generality that $\varepsilon_n < 1$. In order to check (3.1.2)-(3.1.3) we only need to consider u outside the interval $[-1,1]$, because the sequence $\{f_{\varepsilon_n}\}$ is uniformly bounded in any compact set of $\mathbb{R}$. Then for $u \notin [-1,1]$ the Hölder inequality and $\int_{\mathbb{R}} \rho_{\varepsilon_n}(s)\, ds = 1$ give

$$\begin{aligned}
|f_{\varepsilon_n}(u)| &= \left| \int_{\mathbb{R}} f(u-s)\, \rho_{\varepsilon_n}(s)\, ds \right| \leq \int_{\mathbb{R}} |f(u-s)|\, \rho_{\varepsilon_n}(s)\, ds \\
&\leq \int_{\mathbb{R}} \left(C_1 + C_2 |u-s|^{p-1}\right) \rho_{\varepsilon_n}(s)\, ds \\
&\leq C_1 + C_2 2^{p-2} \left(\int_{-\varepsilon_n}^{\varepsilon_n} \left(|u|^{p-1} + |s|^{p-1}\right) \rho_{\varepsilon_n}(s)\, ds \right) \\
&\leq \widetilde{C}_1 + \widetilde{C}_2\, |u|^{p-1}.
\end{aligned}$$

If f satisfies (3.1.3), then

$$\begin{aligned}
f_{\varepsilon_n}(u)\, u &= \int_{\mathbb{R}} f(u-s)\,(u-s)\, \rho_{\varepsilon_n}(s)\, ds + \int_{\mathbb{R}} f(u-s)\, s \rho_{\varepsilon_n}(s)\, ds \\
&\leq \int_{\mathbb{R}} \left(C_3 - C_4 |u-s|^p\right) \rho_{\varepsilon_n}(s)\, ds + \int_{\mathbb{R}} \left(C_1 + C_2 |u-s|^{p-1}\right) s \rho_{\varepsilon_n}(s)\, ds \\
&\leq K_1 - C_4 \int_{\mathbb{R}} \left(2^{1-p}\, |u|^p - |s|^p\right) \rho_{\varepsilon_n}(s)\, ds \\
&+ C_2 2^{p-2} \int_{\mathbb{R}} \left(|u|^{p-1} + |s|^{p-1}\right) s \rho_{\varepsilon_n}(s)\, ds \\
&\leq \widetilde{C}_3 - \widetilde{C}_4 |u|^p,
\end{aligned}$$

where we have used $|u|^p \leq 2^{p-1}\left(|s^p| + |u-s|^p\right)$ and the Young inequality.

For (3.1.7) we put in the above inequality $p = 2$, $C_3 = m_\varepsilon$, $C_4 = -\varepsilon$ and obtain

$$f_{\varepsilon_n}(u)\, u \leq \widetilde{m}_\varepsilon + \varepsilon u^2,$$

which obviously implies (3.1.4).

∎

Our next aim is to focus on the convergence of solutions of the approximations.

Theorem 3.32. *Let conditions (A1)-(A6), $h=0$ and either (A7) or (A8) be satisfied and let, moreover, the function f be odd. If $u_{\varepsilon_n,0} \to u_0$ in $H_0^1(\Omega)$ as $\varepsilon_n \to 0$, then for any sequence of solutions of (3.5.1) $u_{\varepsilon_n}(\cdot)$ with $u_{\varepsilon_n}(0) = u_{\varepsilon_n,0}$ there exists a subsequence of ε_n such that u_{ε_n} converges to some strong solution $u(\cdot)$ of (3.1.1) in the space $C([0,T], H_0^1(\Omega))$, for any $T>0$.*

Proof. Using (3.3.2) and (3.3.3) we can repeat the same lines of the proof of Theorem 3.5 and obtain the existence of a function $u(\cdot)$ and a subsequence of u_{ε_n} such that

$$u_{\varepsilon_n} \overset{*}{\rightharpoonup} u \text{ in } L^\infty(0,T;H_0^1(\Omega)),$$

$$u_{\varepsilon_n} \rightharpoonup u \text{ in } L^2(0,T;D(A)),$$

$$\frac{du_{\varepsilon_n}}{dt} \rightharpoonup \frac{du}{dt} \text{ in } L^2(0,T;L^2(\Omega)),$$

$$u_{\varepsilon_n} \to u \text{ in } C([0,T];L^2(\Omega)),$$

$$u_{\varepsilon_n} \to u \text{ in } L^2(0,T;H_0^1(\Omega)),$$

$$f_{\varepsilon_n}(u_{n_\varepsilon}) \overset{*}{\rightharpoonup} f(u) \text{ in } L^\infty(0,T;L^\infty(\Omega)),$$

$$a(\|u_{\varepsilon_n}\|^2_{H_0^1})\Delta u_{\varepsilon_n} \rightharpoonup a(\|u\|^2_{H_0^1})\Delta u \text{ in } L^2(0,T;L^2(\Omega)).$$

Also, in the same way we prove that $u(\cdot)$ is a strong solution to problem (3.1.1) such that $u(0) = u_0$.

The uniform estimate in the space $H_0^1(\Omega)$ implies also that if $t_n \to t_0$, then $u_{\varepsilon_n}(t_n) \rightharpoonup u(t_0)$ in $H_0^1(\Omega)$. We need to prove that this convergence is in fact strong, proving then the convergence in $C([0,T], H_0^1(\Omega))$ for any $T>0$.

In the same way as in the proof of Lemma 3.7 we deduce that for some $C>0$ the functions

$$Q_n(t) = A(\|u_{\varepsilon_n}(t)\|^2_{H_0^1}) - 2Ct,$$

$$Q(t) = A(\|u(t)\|^2_{H_0^1}) - 2Ct$$

are continuous and non-increasing in $[0,T]$.

Moreover,

$$Q_n(t) \to Q(t) \text{ for a.e. } t \in (0,T).$$

Let first $t_0 > 0$. Then we take $0 < t_j < t_0$ such that $t_j \to t_0$ and $Q_n(t_j) \to Q(t_j)$ for all j. Then

$$Q_n(t_n) - Q(t_0) \leq Q_n(t_j) - Q(t_0) \leq |Q_n(t_j) - Q(t_j)| + |Q(t_j) - Q(t_0)| \text{ for } t_n > t_j.$$

For any $\delta > 0$ there exist $j(\delta)$ and $N(j(\delta))$ such that $Q_n(t_n) - Q(t_0) \leq \delta$ if $n \geq N$, so $\limsup Q_n(t_n) \leq Q(t_0)$. Hence, a contradiction argument using the continuity of $A(s)$ shows that $\limsup \|u_{\varepsilon_n}(t_n)\|^2_{H_0^1} \leq \|u(t_0)\|^2_{H_0^1}$. This, together with $\liminf \|u_{\varepsilon_n}(t_n)\|^2_{H_0^1} \geq \|u(t_0)\|^2_{H_0^1}$, implies that

$$\|u_{\varepsilon_n}(t_n)\|^2_{H_0^1} \to \|u(t_0)\|^2_{H_0^1},$$

so that

$$u_{\varepsilon_n}(t_n) \to u(t_0) \text{ in } H_0^1(\Omega).$$

For the case when $t_0 = 0$ we use the same argument as in Lemma 3.7.

□

We denote by $\mathcal{A}_{\varepsilon_n}$ the global attractor for the semiflow G_{ε_n} corresponding to problem (3.5.1).

Lemma 3.33. *Assume the condition of Theorem 3.32. Then* $\cup_{n\in\mathbb{N}}\mathcal{A}_{\varepsilon_n}$ *is bounded in* $H_0^1(\Omega)$. *Hence, the set* $\overline{\cup_{n\in\mathbb{N}}\mathcal{A}_{\varepsilon_n}}$ *is compact in* $L^2(\Omega)$.

Proof. By Lemma 3.31 inequality (3.1.7) is satisfied for any n with constants which are independent of ε_n, so inequality (3.3.9) holds true with constants independent of ε_n. Thus, there a exists a common absorbing ball B_0 in $L^2(\Omega)$ (with radius $K > 0$) for problems (3.5.1). Further, by repeating the same steps as in Proposition 3.11 we obtain a common absorbing ball in $H_0^1(\Omega)$ (with radius $\widetilde{K} > 0$) as by Lemma 3.31 the constants which are involved are independent of ε_n. Thus, $\|y\|_{H_0^1} \leq \widetilde{K}$ for any $y \in \cup_{n\in\mathbb{N}}\mathcal{A}_{\varepsilon_n}$.

□

Lemma 3.34. *Assume the condition of Theorem 3.32. Then $\cup_{n\in\mathbb{N}}\mathcal{A}_{\varepsilon_n}$ is bounded in V^{2r} for any $0 \le r < 1$. Hence, $\overline{\cup_{n\in\mathbb{N}}\mathcal{A}_{\varepsilon_n}}$ is compact in V^{2r} and $C^1([0,1])$.*

Proof. Using Lemma 3.33 we obtain the boundedness of $\cup_{n\in\mathbb{N}}\mathcal{A}_{\varepsilon_n}$ in V^{2r} by repeating the same lines in Lemma 3.18. The rest of the proof follows from the compact embedding $V^\alpha \subset V^\beta$, $\alpha > \beta$, and the continuous embedding $V^{2r} \subset C^1([0,1])$ if $r > \frac{3}{4}$. □

Corollary 3.35. *Assume the condition of Theorem 3.32. Then any sequence $\xi_n \in \mathcal{A}_{\varepsilon_n}$ with $\varepsilon_n \to 0$ is relatively compact in $C^1([0,1])$.*

Lemma 3.36. *Assume the condition of Theorem 3.32. Then up to a subsequence any bounded complete trajectory u_{ε_n} of (3.5.1) converges to a bounded complete trajectory u of (3.1.1) in $C([-T,T],H_0^1(\Omega))$ for any $T > 0$. On top of that, if $y_n \in \mathcal{A}_{\varepsilon_n}$, then passing to a subsequence $y_n \to y \in \mathcal{A}$ in $H_0^1(\Omega)$. Hence,*

$$dist_{H_0^1}(\mathcal{A}_{\varepsilon_n}, \mathcal{A}) \to 0 \text{ as } n \to \infty. \tag{3.5.4}$$

Proof. Let fix $T > 0$. Up to a subsequence, by Corollary 3.35 we have

$$u_{\varepsilon_n}(-T) \to y \text{ in } H_0^1(\Omega).$$

Theorem 3.32 implies that u_{ε_n} converges in $C([-T,T],H_0^1(\Omega))$ to some solution u of (3.1.1). If we choose successive subsequences for $-2T, -3T\ldots$ and apply the standard diagonal procedure, we obtain that a subsequence u_{ε_n} converges to a complete trajectory u of (3.1.1) in $C([-T,T],H_0^1(\Omega))$ for any $T > 0$. Finally, from Lemma 3.33 this trajectory is bounded.

If $y_n \in \mathcal{A}_{\varepsilon_n}$, by Corollary 3.35 we can extract a subsequence converging to some y. If we take a sequence of bounded complete trajectories $\phi_n(\cdot)$ of (3.5.1) such that $\phi_n(0) = y_n$, then by the previous result it converges in $C([-T,T],H_0^1(\Omega))$ to some bounded complete trajectory $\phi(\cdot)$ of (3.1.1), so $y \in \mathcal{A}$.

Finally, if (3.5.4) was not true, there would exist $\delta > 0$ and a sequence $y_n \in \mathcal{A}_{\varepsilon_n}$ such that $dist_{H_0^1}(y, \mathcal{A}) > \delta$. But passing to a subsequence $y_n \to y \in \mathcal{A}$, which is a contradiction. □

Lemma 3.37. *Assume the conditions of Theorem 3.32. Let $\tau_{\pm}^{d_n,\varepsilon_n}$ be the functions (3.4.7)-(3.4.8) for problem (3.4.2) but replacing f by f_{ε_n} and d by d_n. Let $d_n, E_n \to 0$ as $n \to \infty$. Then*

$$\lim_{n\to\infty} \tau_{\pm}^{d_n,\varepsilon_n}(E_n) = \frac{\sqrt{a\,(0)}\pi}{\sqrt{2\lambda}}.$$

Proof. Let us consider

$$f_{d_n,\varepsilon_n}(u) = \frac{\lambda f_{\varepsilon_n}(u)}{a(d_n)}.$$

In view of property $(B4)$ and (3.5.3), since $f'_{\varepsilon_n}(0) = f'(0) = 1$ and $f_{\varepsilon_n}(0) = f(0) = 0$, given $\gamma \in (0,1)$ there exists $\delta > 0$ (independent of ε_n) such that

$$\begin{aligned} &(1-\gamma)u \leq f_{\varepsilon_n}(u) \leq (1+\gamma)u, \quad \text{for any } u \in (0,\delta).\\ &\frac{1}{1+\gamma} \leq \frac{u}{f_{\varepsilon_n}(u)} \leq \frac{1}{1-\gamma}, \quad \text{for any } u \in (0,\delta). \end{aligned} \tag{3.5.5}$$

The sequence $\mathcal{F}_{\varepsilon_n}(\cdot)$ converges uniformly to $\mathcal{F}(\cdot)$ in compact sets. Moreover, as $U_+(E)$ is continuous and using [70, p. 60], given $\delta > 0$, there exists $\eta > 0$ such that

$$U_+^{\varepsilon_n}(E) \leq \delta, \quad \text{for any } 0 < E \leq \eta.$$

Now, if we integrate the first inequality in (3.5.5) between 0 and u we obtain

$$\frac{1}{2}(1-\gamma)u^2 \leq \mathcal{F}_{\varepsilon_n}(u) \leq \frac{1}{2}(1+\gamma)u^2, \quad \text{for any } 0 \leq u \leq \delta.$$

Using the change of variable $E_n y^2 = \mathcal{F}_{\varepsilon_n}(u)$, we have

$$\left(\frac{1-\gamma}{2E_n}\right)^{1/2} u \leq y \leq \left(\frac{1+\gamma}{2E_n}\right)^{1/2} u, \quad \text{if } 0 < E_n \leq \eta,\ 0 \leq y \leq 1.$$

Dividing the previous expression by $\sqrt{\frac{\lambda}{a(d_n)}}\, f_{d_n,\varepsilon_n}(u)$ and using (3.5.5) we obtain

$$\left(\frac{a(d_n)(1-\gamma)}{2\lambda E_n(1+\gamma)^2}\right)^{1/2} \leq \frac{\sqrt{a\,(d_n)}y}{\sqrt{\lambda} f_{d_n,\varepsilon_n}(u)} \leq \left(\frac{a(d_n)(1+\gamma)}{2\lambda E_n(1-\gamma)^2}\right)^{1/2} \quad \text{if } 0 < E_n \leq \eta,\ 0 \leq y \leq 1.$$

Now if we multiply by $2\sqrt{E_n}(1-y^2)^{-\frac{1}{2}}$ and integrate from 0 to 1, we get

$$\pi\left(\frac{a(d_n)(1-\gamma)}{2\lambda(1+\gamma)^2}\right)^{1/2} \leq \tau_+^{\varepsilon_n}(E_n) \leq \pi\left(\frac{a(d_n)(1+\gamma)}{2\lambda(1-\gamma)^2}\right)^{1/2}, \quad \text{if } 0 < E_n \leq \eta.$$

Then the theorem follows as $a(d_n) \to a(0)$ when $n \to \infty$.

The proof for $\tau_-^{\varepsilon_n}$ is analogous.

□

Under the conditions of Theorem 3.32, if (A8) is satisfied and

$$a(0)\pi^2 k^2 < \lambda \leq a(0)\pi^2(k+1)^2, \ k \in \mathbb{Z}, \ k \geq 0, \tag{3.5.7}$$

holds, then by Theorem 3.24 problem (3.5.1) has exactly $2k+1$ fixed points (denoted by $v_0 = 0$, $v^{\pm}_{1,d_1^{\varepsilon_n}}, \ldots, v^{\pm}_{k,d_k^{\varepsilon_n}}$) and $v^{\pm}_{m,d_m^{\varepsilon_n}}$ has $m+1$ zeros in $[0,1]$ for each $1 \leq m \leq k$. The same is valid for problem (3.1.1) and we denote the $2k+1$ fixed points by $v_0 = 0$, $u^{\pm}_{1,d_1^*}, \ldots, u^{\pm}_{k,d_k^*}$.

Lemma 3.38. *Assume the conditions of Theorem 3.32, (A8) and (3.5.7). Let $m \in \mathbb{N}$, $1 \leq m \leq k$, be fixed. Then $v^+_{m,d_m^{\varepsilon_n}}$ (resp. $v^-_{m,d_m^{\varepsilon_n}}$) do not converge to 0 in $H_0^1(\Omega)$ as $\varepsilon_n \to 0$.*

Proof. Assume that $v^+_{m,d_m^{\varepsilon_n}} \to 0$ in $H_0^1(0,1)$. Then it converges to 0 in $C([0,1])$ and the equality

$$-\frac{d^2 v^+_{m,d_m^{\varepsilon_n}}}{dx^2}(x) = \frac{\lambda f_{\varepsilon_n}\left(v^+_{m,d_m^{\varepsilon_n}}(x)\right)}{a(d_m^{\varepsilon_n})}$$

implies that

$$v^+_{m,d_m^{\varepsilon_n}} \to \text{ in } C^2([0,1]).$$

In particular,

$$\frac{dv^+_{m,d_m^{\varepsilon_n}}}{dx}(0) \to 0$$

and

$$d_m^{\varepsilon_n} = \left\|v^+_{m,d_m^{\varepsilon_n}}\right\|^2_{H_0^1} \to 0.$$

The value E_n corresponding to the fixed point $v^+_{m,d^{\varepsilon_n}_m}$ is equal to

$$\frac{a\left(d^{\varepsilon_n}_m\right)}{2\lambda}\frac{dv^+_{m,d^{\varepsilon_n}_m}}{dx}(0),$$

so $E_n \to 0$. We will show that this is not possible. We know by Lemma 3.37 that

$$\lim_{n\to\infty}\tau_{\pm}^{d^{\varepsilon_n}_m,\varepsilon_n}(E_n) = \frac{\pi\sqrt{a(0)}}{\sqrt{2\lambda}}.$$

Also, since $v^+_{m,d^{\varepsilon_n}_m}$ is a fixed point with $d = d^{\varepsilon_n}_m$ one of the following conditions has to be satisfied (see (3.4.6)):

$$j\tau_{+}^{d^{\varepsilon_n}_m,\varepsilon_n}(E_n) + (j-1)\,\tau_{-}^{d^{\varepsilon_n}_m,\varepsilon_n}(E_n) = \left(\frac{1}{2}\right)^{\frac{1}{2}}, \tag{3.5.8}$$

$$j\tau_{-}^{d^{\varepsilon_n}_m,\varepsilon_n}(E_n) + (j-1)\,\tau_{+}^{d^{\varepsilon_n}_m,\varepsilon_n}(E_n) = \left(\frac{1}{2}\right)^{\frac{1}{2}},\ \text{if } m = 2j-1 \tag{3.5.9}$$

$$j\tau_{+}^{d^{\varepsilon_n}_m,\varepsilon_n}(E_n) + j\tau_{-}^{d^{\varepsilon_n}_m,\varepsilon_n}(E_n) = \left(\frac{1}{2}\right)^{\frac{1}{2}},\ \text{if } m = 2j. \tag{3.5.10}$$

Since $E_n \to 0$ and $\lambda > k^2\pi^2 a(0) \geq m^2\pi^2 a(0)$, there exists ε_{n_0} such that

$$\tau_{\pm}^{d^{\varepsilon_{n_0}}_m,\varepsilon_{n_0}}(E_{n_0}) < \frac{1}{\sqrt{2}m}.$$

Hence, neither of (3.5.8)-(3.5.10) is possible.

□

Lemma 3.39. *Assume the conditions of Theorem 3.32, (A8) and (3.5.7). Let $m \in \mathbb{N}$, $1 \leq m \leq k$, be fixed. Then $v^+_{m,d^{\varepsilon_n}_m}$ (resp. $v^-_{m,d^{\varepsilon_n}_m}$) converges to u^+_{m,d^*_m} in $H^1_0(\Omega)$ (resp. u^-_{m,d^*_m}) as $\varepsilon_n \to 0$.*

Proof. We consider $v^+_{m,d_m^{\varepsilon_n}}$. In view of Corollary 3.35, $v^+_{m,d_m^{\varepsilon_n}}$ is relatively compact in $C^1([0,1])$, so up to a subsequence

$$v^+_{m,d_m^{\varepsilon_n}} \to v \text{ in } C^1([0,1])$$

and

$$d_m^{\varepsilon_n} \to d^* = \|v\|^2_{H_0^1}.$$

The proof will be finished if we prove that $v = u^+_{m,d_m^*}$. We observe that since in such a case every subsequence would have the same limit, the whole sequence would converge to u^+_{m,d_m^*}.

In view of (3.5.3), we get that

$$f_{\varepsilon_n}(v^+_{m,d_m^{\varepsilon_n}}) \to f(v) \text{ in } C([0,1]).$$

It follows that

$$-\frac{\partial^2 v}{\partial x^2} = \frac{\lambda f(v)}{a(\|v\|^2_{H_0^1})}$$

and v is a solution of (3.4.1), so v is a fixed point of (3.1.1).

We need to prove that $v = u^+_{m,d_m^*}$. From Lemma 3.38, it follows that $v \neq 0$, and then

$$v = u^{\pm}_{j,d_j^*}, \qquad \text{for some } 1 \leq j \leq k.$$

Since $u^{\pm}_{j,d_j^*}$ has $j+1$ simple zeros, the convergence

$$v^+_{m,d_m^{\varepsilon_n}} \to u^{\pm}_{j,d_j^*} \text{ in } C^1([0,1])$$

implies that $v^+_{m,d_m^{\varepsilon_n}}$ has $j+1$ zeros for $n \geq N$. But $v^+_{m,d_m^{\varepsilon_n}}$ possesses $m+1$ zeros in $[0,1]$. Thus, $m = j$.

For the sequence $v^-_{m,d_m^{\varepsilon_n}}$ the proof is analogous.

$\square$

3.5.2. Instability

We will prove that the fixed points 0 and $u^{\pm}_{k,d^*_k}$, $k \geq 2$, are unstable under some additional assumptions on the functions f and a. For this aim we need to use the approximative problems (3.5.1).

Theorem 3.40. *Assume that the conditions (A1)-(A8), $h = 0$, (3.5.7) with $k \geq 1$ are satisfied and let, moreover, the function $f(\cdot)$ be odd and $a(\cdot)$ be globally Lipschitz continuous. Then the equilibria $v_0 = 0$ and $u^{\pm}_{j,d^*_j}$, $2 \leq j \leq k$ (if $k \geq 2$), are unstable.*

Remark 3.41. The condition that $a(\cdot)$ is globally Lipschitz continuous could be dropped, as we can replace $a(\cdot)$ in (3.5.1) by a sequence $a_{\varepsilon_n}(\cdot)$ of globally Lipschitz continuous functions.

Proof. Problem (3.5.1) generates a single-valued semigroup $\{T_{\varepsilon_n}(t); t \geq 0\}$ with a finite number of fixed points: $v_0 = 0$, $v^{\pm}_{1,d^{\varepsilon_n}_1}, ..., v^{\pm}_{k,d^{\varepsilon_n}_k}$ [32]. We know by Theorems 3.5 and 3.6 in [32] that for any $v^{+}_{j,d^{\varepsilon_n}_j}$ with $j \geq 2$ and v_0 there exists a bounded complete trajectory u^{ε_n} such that

$$u^{\varepsilon_n}(t) \to v^{+}_{j,d^{\varepsilon_n}_j} \quad \text{as } t \to -\infty, \quad \text{for } k \geq 2,$$

so v_0, $v^{+}_{j,d^{\varepsilon_n}_j}$ are unstable. The same is valid for $v^{-}_{j,d^{\varepsilon_n}_j}$. On the other hand, by Lemma 3.39 we have

$$v^{\pm}_{j,d^{\varepsilon_n}_j} \to u^{\pm}_{j,d^*_j}, \tag{3.5.11}$$

where $u^{\pm}_{j,d^*_j}$ is a fixed point of problem (3.1.1) with $j+1$ zeros in $[0,1]$. We prove the result for u^{+}_{j,d^*_j}. For u^{-}_{j,d^*_j} and v_0 the proof is the same.

By Lemma 3.36 we obtain that up to a subsequence u^{ε_n} converges to a bounded complete trajectory u of problem (3.1.1) in the space $C([-T,T], H^1_0(\Omega))$ for every $T > 0$. Thus, either $u(\cdot)$ is a fixed point v_{-1} or by Theorem 3.14 there exists a fixed point v_{-1} of problem (3.1.1) such that

$$u(t) \to v_{-1} \quad \text{as } t \to -\infty \text{ in } H^1_0(\Omega).$$

In the second case, if $v_{-1} = u^{+}_{j,d^*_j}$, the proof would be finished, so let assume the opposite.

Assume first that either $u(\cdot)$ is not a fixed point or it is a fixed point but $v_{-1} \neq u^{+}_{j,d^*_j}$. We consider $r_0 > 0$ such that the neighborhood $\mathcal{O}_{2r_0}(v_{-1})$ does not contain any other fixed point of problem (3.1.1). For any $r \leq r_0$ we can choose $t_r \to -\infty$ and n_r such that $u^{\varepsilon_n}(t_r) \in \mathcal{O}_r(v_{-1})$ for all $n \geq n_r$. On the other hand, since $u^{\varepsilon_n}(t) \to v^{+}_{j,d^{\varepsilon_n}_j}$, as $t \to -\infty$, and $v^{+}_{j,d^{\varepsilon_n}_j} \to u^{+}_{j,d^*_j} \notin B_{2r_0}(v_{-1})$, there exists $t'_r < t_r$ such that

$$u^{\varepsilon_{n_r}}(t) \in \mathcal{O}_{r_0}(v_{-1}) \text{ for } t \in (t'_r, t_r],$$

$$\|u^{\varepsilon_{n_r}}(t'_r) - v_{-1}\|_{H^1_0} = r_0.$$

Let first $t_t - t'_r \to +\infty$. We define the sequence $u_1^{\varepsilon_{n_r}}(t) = u^{\varepsilon_{n_r}}(t + t'_r)$, which passing to a subsequence converges to a bounded complete trajectory $\phi(t)$ such that $\phi(t) \in \mathcal{O}_{r_0}(v_{-1})$ for all $t \geq 0$. As there is no other fixed point in $\mathcal{O}_{2r_0}(v_{-1})$, $\phi(t) \to v_{-1}$ as $t \to +\infty$. But $\|\phi(0) - v_{-1}\| = r_0$, so $\phi(\cdot)$ is not a fixed point. Then $\phi(t) \to v_{-2}$ as $t \to -\infty$, where v_{-2} is a fixed point different from v_{-1}. Second, let $|t_t - t'_r| \leq C$. Then put $u_1^{\varepsilon_{n_r}}(t) = u^{\varepsilon_{n_r}}(t + t_r)$. Passing to a subsequence we have that

$$u_1^{\varepsilon_{n_r}}(0) \to v_{-1},$$
$$t_r - t'_r \to t_0, \text{ as } r \to 0.$$

Also, $u_1^{\varepsilon_{n_r}}(\cdot)$ converges to a bounded complete trajectory $u^1(\cdot)$ of problem (3.1.1) such that $u^1(0) = v_{-1}$. Let

$$\psi_1(t) = \begin{cases} u^1(t) & \text{if } t \leq 0, \\ v_{-1} & \text{if } t \geq 0. \end{cases}$$

We note that $\|u^1(-t_0) - v_{-1}\|_{H^1_0} = r_0$ implies that $u^1(\cdot)$ is not a fixed point. Then ψ_1 is a bounded complete trajectory of problem (3.1.1) such that $\psi_1(t) \to v_{-2} \neq v_{-1}$ as $t \to -\infty$. If $v_{-2} = u^{+}_{j,d^*_j}$, the proof is finished.

If $v_{-2} \neq u^{+}_{j,d^*_j}$, we continue constructing by the same procedure a chain of connections in which the new fixed point is always different from the previous ones, because the existence of the Lyapunov function (3.3.19) avoids the existence of a cyclic chain of connections. Since the number of fixed points is finite, at some moment we obtain a bounded complete trajectory $\phi(\cdot)$ such that

$$\phi(t) \to u^+_{j,d^*_j}, \quad \text{as } t \to -\infty,$$

proving that u^+_{j,d^*_j} is unstable.

Now let $u(\cdot) = v_{-1} = u^+_{j,d^*_j}$. Defining the neighborhood $\mathcal{O}_{2r_0}(v_{-1})$ as before, for any $r \leq r_0$ we can choose n_r such that $u^{\varepsilon_n}(0) \in \mathcal{O}_r(v_{-1})$ for all $n \geq n_r$. Also, since

$$u^{\varepsilon_n}(t) \to z^n_0, \quad \text{as } t \to +\infty,$$

where $z^n_0 \neq v^+_{j,d^{\varepsilon_n}_j}$ is a fixed point of (3.5.1), there exists $t_r > 0$ such that

$$u^{\varepsilon_{n_r}}(t) \in \mathcal{O}_{r_0}(v_{-1}) \quad \text{for } t \in [0, t_r),$$

$$\|u^{\varepsilon_{n_r}}(t_r) - v_{-1}\|_{H^1_0} = r_0.$$

The sequence $\{t_r\}$ cannot be bounded. Indeed, if $t_r \to t_0$, then

$$u^{\varepsilon_{n_r}}(t_r) \to u(t_0) = v_{-1},$$

which is a contradiction with $\|u^{\varepsilon_{n_r}}(t_0) - v_{-1}\|_{H^1_0} = r_0$. Then $t_r \to +\infty$. We define the functions $u^{\varepsilon_{n_r}}_1(t) = u^{\varepsilon_{n_r}}(t + t_r)$, which satisfy that $u^{\varepsilon_{n_r}}_1(t) \in \mathcal{O}_{r_0}(v_{-1})$ for all $t \in [-t_r, 0)$. Passing to a subsequence it converges to a bounded complete trajectory $\phi(\cdot)$ such that $\phi(t) \in \mathcal{O}_{r_0}(v_{-1})$ for all $t \leq 0$. This trajectory is not a fixed point as $\|\phi(0) - v_{-1}\|_{H^1_0} = r_0$ and

$$\phi(t) \to u^+_{j,d^*_j} \quad \text{as } t \to -\infty,$$

so u^+_{j,d^*_j} is unstable.

□

Further, we will prove that there is also a connection from 0 to the point $u^{\pm}_{k,d_k^*}$.

Theorem 3.42. *Assume the conditions of Theorem 3.40. Then there exists a bounded complete trajectory* $\phi(\cdot)$ *such that* $\phi(t) \underset{t\to-\infty}{\to} 0$, $\phi(t) \underset{t\to+\infty}{\to} u^{+}_{k,d_k^*}$ *(and the same is valid for* u^{-}_{k,d_k^*}*). Thus,* $E(0) = 0 > E(u^{\pm}_{k,d_k^*})$.

Proof. We start with the case $k = 1$. We have three fixed points: 0, $u^{+}_{1,d_1^*}, u^{-}_{1,d_1^*}$. By Theorem 3.40 there exists a bounded complete trajectory $\phi(\cdot)$ such that

$$\phi(t) \underset{t\to-\infty}{\to} 0,$$

whereas Theorem 3.14 and Remark 3.26 imply that it has to converge forward to a fixed point different from 0, that is, to either u^{+}_{1,d_1^*} or u^{-}_{1,d_1^*}. If, for example, $\phi(t) \underset{t\to+\infty}{\to} u^{+}_{1,d_1^*}$, then as the function f is odd, $\psi(t) = -\phi(t)$ is another bounded complete trajectory and $\psi(t) \underset{t\to+\infty}{\to} -u^{+}_{1,d_1^*} = u^{-}_{1,d_1^*}$.

Further we consider the problem

$$\begin{cases} \dfrac{\partial u}{\partial t} - a(\|u\|^2_{H_0^1})\dfrac{\partial^2 u}{\partial x^2} = \lambda f_k(u), \quad t>0,\ 0<x<\frac{1}{k}, \\ u(t,0) = u(t,\frac{1}{k}) = 0, \\ u(0,x) = u_0(x), \end{cases} \tag{3.5.12}$$

where $f_k(u) = \sqrt{k} f\left(u/\sqrt{k}\right)$ satisfies (A1)-(A5). In this problem, condition (3.5.7) implies that there are again three fixed points: 0, $u^{+}_{1,d_1^*,\frac{1}{k}}, u^{-}_{1,d_1^*,\frac{1}{k}}$. By the above argument there is a connection $\phi_{\frac{1}{k}}(\cdot)$ from 0 to $u^{+}_{1,d_1^*,\frac{1}{k}}$ (also to $u^{-}_{1,d_1^*,\frac{1}{k}}$). Since the function f is odd, $u^{+}_{k,d_k^*}(x)$ is equal to $\frac{1}{\sqrt{k}}u^{+}_{1,d_1^*,\frac{1}{k}}(x)$ on $[0,\frac{1}{k}]$, to $-\frac{1}{\sqrt{k}}u^{+}_{1,d_1^*,\frac{1}{k}}\left(x-\frac{1}{k}\right)$ on $[\frac{1}{k},\frac{2}{k}]$, etc. Then the function $\phi(\cdot)$ such that $\phi(t,x) = \frac{(-1)^j}{\sqrt{k}}\phi_{\frac{1}{k}}\left(t, x-\frac{j}{k}\right)$ on $[\frac{j}{k},\frac{j+1}{k}]$, $j = 0,1,...,k-1$, is a bounded complete trajectory of problem (3.1.1) which goes from 0 to u^{+}_{k,d_k^*}.

□

Remark 3.43. When $k = 1$ the structure of the global attractor is the same as in the Chafee-Infante equation.

3.5.3. Gradient structure

We will obtain that the m-semiflow G is dynamically gradient (see Definition 0.19).

Let us consider the case when the conditions of Theorem 3.40 hold. Then (3.1.1) possesses exactly $2k+1$ fixed points:

$$v_0 = 0,\ u^{\pm}_{1,d_1^*}, ..., u^{\pm}_{k,d_k^*}.$$

Also, as f is odd, $u^{+}_{j,d_j^*} = -u^{-}_{j,d_j^*}$ for any j. We define the following sets:

$$M_1 = \{u^{+}_{1,d_1^*}, u^{-}_{1,d_1^*}\}, ...,\ M_k = \{u^{+}_{k,d_k^*}, u^{-}_{k,d_k^*}\},\ M_{k+1} = \{0\}. \tag{3.5.13}$$

They are weakly invariant and using Lemma 3.29 we deduce easily that they are isolated. Then the family $\mathcal{M} = \{M_1, \ldots, M_{k+1}\}$ is a finite disjoint family of isolated weakly invariant sets.

Proposition 3.44. *Assume the conditions of Theorem 3.40. Then G is dynamically gradient with respect to the family (3.5.13) after (possibly) reordering them.*

Proof. We reorder the family (3.5.13) in such a way that if the value of the Lyapunov function E given in (3.3.19) is equal to L_i for the set $\widetilde{M}_i$, then $L_j \leq L_n$ for $j < n$. Then Theorem 25 in [44] implies that G is dynamically gradient with respect to this family.

□

We will obtain then that the fixed points $u^{+}_{1,d_1^*}, u^{-}_{1,d_1^*}$ are asymptotically stable. The compact set $M \subset \mathcal{A}$ is a local attractor for G in X if there is $\varepsilon > 0$ such that $\omega\left(O_\varepsilon(M)\right) = M$, where

$$\omega\left(B\right) = \{y : \exists t_n \to +\infty,\ y_n \in G(t_n, B) \text{ such that } y_n \to y\}$$

is the ω-limit set of B. By Lemma 14 in [44] if M is a local attractor in X, then it is stable. Thus, a local attractor is asymptotically stable.

Theorem 3.45. *Assume the conditions of Theorem 3.40. Then the stationary points $u^{+}_{1,d_1^*}, u^{-}_{1,d_1^*}$ are asymptotically stable.*

Proof. By [44, Theorem 23 and Lemma 15] $\widetilde{M}_1$ is a local attractor in X, so it is asymptotically stable. By Theorem 3.40 the sets M_j, $j \geq 2$, are unstable. Thus, $\widetilde{M}_1 = M_1$. As M_1 consists of the two elements $u^{+}_{1,d_1^*}, u^{-}_{1,d_1^*}$, which are obviously disjoint, they are asymptotically stable as well.

□

We will prove that there is a connection from 0 to any other fixed point $u^{\pm}_{j,d_j^*}$.

Theorem 3.46. *Assume the conditions of Theorem 3.40. Then there exists a bounded complete trajectory $\phi(\cdot)$ such that $\phi(t) \underset{t\to-\infty}{\to} 0$, $\phi(t) \underset{t\to+\infty}{\to} u^{+}_{j,d_j^*}$ for all $1 \leq j \leq k$ (and the same is valid for u^{-}_{j,d_j^*}).*

Proof. Let us consider problem (3.5.12) with $k = j$. The function

$$u^{+}_{1,d_j^*,\frac{1}{j}}(x) = \sqrt{j}\ u^{+}_{j,d_j^*}(x), \quad x \in [0, 1/j],$$

is the unique positive fixed point of problem (3.5.12).
Let

$$X_j^+ = \{u \in H_0^1(0,1/j)\ :\ u(x) \geq 0 \quad \forall x \in [0,1/j]\}$$

be the positive cone of $H_0^1(0,1/j)$. If we consider the restriction of the semigroup $T_j^{\varepsilon_n}(\cdot)$ of problem (3.5.1) in the interval $(0, 1/j)$ to X_j^+, denoted by $T_j^{\varepsilon_n,+}(\cdot)$, then there exists a global attractor $\mathcal{A}^{+}_{n,j}$ [31]. Since 0 and

$$v^{+}_{1,d_j^{\varepsilon_n},\frac{1}{j}} = \sqrt{j}\ v^{+}_{j,d_j^{\varepsilon_n}}\ |_{[0,\frac{1}{j}]}$$

are the unique fixed points of $T_j^{\varepsilon_n,+}$, $\mathcal{A}^{+}_{n,j}$ is connected, $v^{+}_{1,d_1^{\varepsilon_n},\frac{1}{j}}$ is stable [32] and $\mathcal{A}^{+}_{n,j}$ consists of the fixed points and their heteroclinic connections, there must exist a bounded complete trajectory $\phi_j^{\varepsilon_n}(\cdot)$ of $T_j^{\varepsilon_n,+}$ which goes from 0 to $v^{+}_{1,d_j^{\varepsilon_n},\frac{1}{j}}$.
By Lemma 3.36 up to a subsequence it converges to a bounded complete trajectory $\phi_j(\cdot)$ of problem (3.5.12) with $k = j$ such that $\phi_j(t) \geq 0$ for all $t \in \mathbb{R}$.

Since by Theorem 3.45 the fixed point $u^{+}_{1,d^*_j,\frac{1}{j}}$ is stable, the only possibility is that

$$\phi_j(t) \to 0, \qquad \text{as } t \to -\infty,$$
$$\phi_j(t) \to u^{+}_{1,d^*_j,\frac{1}{j}}, \qquad \text{as } t \to +\infty.$$

Then the function $\phi(\cdot)$ such that

$$\phi(t,x) = \frac{(-1)^i}{\sqrt{j}}\phi_j\left(t, x - \frac{i}{j}\right), \qquad \text{on } \left[\frac{i}{j}, \frac{i+1}{j}\right], \quad i = 0, 1, ..., j-1,$$

is a bounded complete trajectory of problem (3.1.1) which goes from 0 to u^{+}_{j,d^*_j}.

For u^{-}_{j,d^*_j}, noting that $u^{-}_{j,d^*_j} = -u^{+}_{j,d^*_j}$, the result follows by choosing the bounded complete trajectory $\widetilde{\phi}(t) = -\phi(t)$.

$\square$

As a consequence we obtain that the order of the family $\mathcal{M}$ has to be the one given in (3.5.13).

Theorem 3.47. *The semiflow G is dynamically gradient with respect to the family $\mathcal{M}$ in the order given in (3.5.13), that is, $\widetilde{M_i} = M_i$ for any i.*

Proof. As by Theorem 3.46 there is a connection from 0 to $u^{\pm}_{j,d^*_j}$, $1 \le j \le k$, we have proved that $\widetilde{M}_{k+1} = \{0\} = M_{k+1}$. The fact that the order of the other sets is the one given in (3.5.13) follows from Lemma 3.28.

$\square$

Conclusions and future work

This thesis begins obtaining a theoretical result about the robustness of dynamically gradient multivalued semiflows. By this way, Theorem 1.1 generalizes the previous results where the solution is unique. Subsequently, it is applied to (1.3.8), a family of problems of the Chafee-Infante type. To do this, we have analyzed the properties of the fixed points of (1.3.1), a Chafee-Infante problem with more general conditions than those usual in the literature in the reaction term, f.

The growth and dissipation conditions of the reaction term are maintained throughout the Chapter 2. In this part, we focus on the equation

$$u_t - a(\|u\|^2_{H^1_0})\Delta u = f(u) + h(t),$$

a nonlocal problem in view of the structure of the diffusion coefficient which is given by the function $a(\|u\|^2_{H^1_0})$. This leads to a number of mathematical difficulties which make the analysis of the problem particularly interesting.

In this sense, one of the problems left open consists on obtaining analogous results to those of the Chapter 2 relaxing conditions on the nonlinear term $f(u)$, without imposing upper bounds.

Actually in [3] existence of solutions and global attractor is proved for a very large class of nonlinearities that in particular covers those contemplated in this thesis. Nevertheless, the authors work with a nonlocal term $a(l(u))$ which depends on a continuous functional defined in $L^2(\Omega)$.

Therefore, it is an open and very stimulating problem that involves obtaining new results to try to adapt this theory to the conditions of the nonlocal term used in this thesis.

It should be noted that, when the study of the fixed points in Chapter 1 is carried out, the derivative of the function f is finite in 0. In addition, this condition is also imposed in Chapter 3 when the non-local case is analyzed. Therefore, there is much to know about the subject when this condition does not fulfill and we can assume

$$\lim_{u \to 0} \frac{f(u)}{u} = \infty, ,$$

i.e. the derivative of f does not exist in 0. It would be very interesting to be able to describe the properties of fixed points, analyze their connections and study their stability.

On the other hand, we have obtained another robustness result. Specifically, in [21] we consider a parametric family of reaction-diffusion equations with nonlocal viscosity depending of the continuous functional defined in $L^2(\Omega)$. In this work we obtain a robustness result of the attractors toward the corresponding minimal pullback attractor of the limiting problem. This result extends the ones obtained in [28]. Actually here all terms (reactions, external forces and nonlocal viscosity functions) may vary with the parameter. Consequently, the problem remains open for the other type of nonlocal term.

As we have seen in Chapter 3, when the function $a(\cdot)$ is not assumed to be monotone, an interesting situation appears. In fact, it is possible to have more than two equilibria with the same number of zeros, as it is shown on Figure 3.2. Under these conditions, we propose as a future work the problem about the behavior or the connections of the fixed points, even studying the structure that may exist between equilibria of the same level.

One of the questions to be addressed in this section would be to raise the equation from the point of view of stochastic dynamical systems. Indeed, in [22] we have proved the existence of weak pullback mean random attractors for a non-local stochastic reaction-diffusion equation with a nonlinear multiplicative noise. The existence and uniqueness of solutions and weak pullback mean random attractors is also established for a deterministic non-local reaction-diffusion equations with random initial data.

We apply the theory of weak mean-square random attractors developed in [87] to the following stochastic non-local reaction-diffusion problem

$$
\begin{aligned}
&du = (a(\|u\|^2_{H^1_0})\Delta u + f(u) + h(t,x))dt + \sigma(u)\, dw(t) \ \text{ in } (\tau,\infty)\times\mathcal{O},\\
&u = 0 \quad \text{on } (\tau,\infty)\times\partial\mathcal{O},\\
&u(\tau,x) = u_\tau(x) \quad \text{for } x\in\mathcal{O}.
\end{aligned}
$$

The question that remains open is to obtain analogous results for a function $a(l(u))$. The difficulty is that we have not been able to find a Lyapunov function for the solutions. This makes the structure of the attractor difficult to analyze.

Therefore, an important problem due to the theoretical implications (see e.g. Theorem 2.29) consists on proving the existence of a Lyapunov function for the nonlocal problem with $a(l(u))$. This problem could be addressed from the results in [63], although some authors suggest that such a function does not exist.

Conclusiones y trabajo futuro

Esta tesis comienza con la obtención de un resultado teórico acerca de la robustez de los semiflujos multivaluados dinámicamente gradientes. De esta manera, el Teorema 1.1 generaliza los resultados previos en los que se tiene unicidad del problema. Posteriormente, se aplica a (1.3.8), una familia de problemas de tipo Chafee-Infante. Para ello, se analiza en profundidad las propiedades de los puntos fijos de (1.3.1), un problema de tipo Chafee-Infante con unas condiciones más generales que las habituales en la literatura en el término de reacción, f.

Las condiciones de crecimiento y disipación del término de reacción se mantienen a lo largo del Capítulo 2. En esta parte, consideramos la ecuación

$$u_t - a(\|u\|^2_{H_0^1})\Delta u = f(u) + h(t),$$

un problema no local cuyo coeficiente de difusión viene dado por la función $a(\|u\|^2_{H_0^1})$. Esto conduce a una serie de dificultades matemáticas que hacen que el análisis del problema sea particularmente interesante. En este sentido, uno de los problemas que se dejan abiertos consiste en obtener resultados análogos a los del Capítulo 2 relajando las condiciones en el término no lineal $f(u)$, no imponiendo cotas superiores.

De hecho, en [3] se prueba la existencia de soluciones y de atractor global para una amplia classe de no linearidades que, en particular, incluyen las contempladas en esta tesis. No obstante, los autores manejan el funcional continuo definido en $L^2(\Omega)$, $a(l(u))$. Por tanto, es un problema abierto y muy estimulante que involucra obtener resultados nuevos para intentar adaptar esta teoría a las condiciones del término no local que se maneja en esta tesis.

Cabe destacar que, cuando se realiza el estudio de los puntos fijos en el Capítulo 1, la derivada de la función f es finita en el origen. Además, también se impone esta condición en el Capítulo 3 cuando se analiza el caso no local. Por tanto, queda mucho por conocer sobre el tema cuando esta condición no se cumple y podemos suponer

$$\lim_{u \to 0} \frac{f(u)}{u} = \infty;$$

es decir, la derivada de f no existe en 0. Sería muy interesante poder describir las propiedades de los puntos fijos, analizar sus conexiones y estudiar su estabilidad.

Por otra parte, hemos obtenido otro resultado de robustez. Concretamente, en [21] consideramos una familia paramétrica de ecuaciones de reacción-difusión con una viscosidad no local dependiendo de un funcional continuo definido en $L^2(\Omega)$. En este trabajo obtenemos un resultado de robustez de los atractores hacia el correspondiente atractor *pullback* minimal del problema límite. Este resultado extiende los obtenidos en [28]. De hecho, aquí todos los términos (reacción, fuerzas externas y funciones de viscosidad no local) pueden variar con el parámetro. En consecuencia, el problema queda abierto para el otro tipo de término no local.

Como hemos visto en el Capítulo 3, cuando la función $a(\cdot)$ no se asume monótona, aparece una interesante situación. En concreto, es posible tener más de dos puntos de equilibrio con el mismo número de ceros, como se puede ver en la Figura 3.2. Bajo estas condiciones, planteamos como trabajo futuro el problema sobre el comportamiento o las conexiones de los puntos fijos, incluso estudiar la estructura que puede haber entre puntos de un mismo nivel.

Una de las cuestiones que abordar en este apartado consistiría en plantear la ecuación desde un punto de vista de los sistemas dinámicos estocásticos. Sin embargo, en [22] hemos probado la existencia de atractores débiles aleatorio tipo *pullback* en media cuadrática para una ecuación de reacción-difusión no local con un ruido multiplicativo no lineal. La existencia y unicidad de soluciones y atractores *pullback* débiles aleatorios en media también se obtiene para una ecuación de reacción-difusión no local y determinística con una condición inicial aleatoria.

Aplicamos la teoría de los atractores *pullback* débiles aleatorios en media desarrollada en [87] al siguiente problema de reacción-difusión no local y estocástico

$$
\begin{aligned}
&du = (a(\|u\|^2_{H^1_0})\Delta u + f(u) + h(t,x))dt + \sigma\,(u)\,dw\,(t) \quad \text{en } (\tau,\infty)\times\mathcal{O},\\
&u = 0 \quad \text{ en } (\tau,\infty)\times\partial\mathcal{O},\\
&u(\tau,x) = u_\tau(x) \quad \text{ para } x\in\mathcal{O}.
\end{aligned}
$$

La cuestión que queda abierta es obtener resultados análogos para una función $a(l(u))$. La dificultad reside en que no hemos sido capaces de encontrar una función de Lyapunov para las soluciones. Esto hace que la estructura del atractor sea difícil de analizar.

Por tanto, un problema importante por las implicaciones teóricas que tiene (ver por ejemplo Teorema 2.29) consiste en probar la existencia de una función de Lyapunov para el problema no local con $a(l(u))$. Este problema podría enfrentarse a partir de los resultados de [63], aunque algunos autores sugieren que tal función no existe.

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Appendix A

In this appendix we generalize the lap number property of solutions of linear equations proved in [55] to the case when we do not have classical solutions. For this we will use a maximum principle for non-smooth functions from [56].

Let $\mathcal{O}$ be a region in $\mathbb{R}^2$ and let $(t_0, x_0) \in \mathcal{O}$ and $\rho, \sigma > 0$. We denote

$$Q_{\rho,\sigma} = \{(t,x) : t \in (t_0 - \sigma, t_0), |x - x_0| < \rho\},$$

where we assume that t_0, x_0, ρ, σ are such that $\overline{Q}_{\rho,\sigma} \subset \mathcal{O}$.

We denote by W the space of all functions from $L^2(\mathcal{O})$ such that

$$\int_{\mathcal{O}} \left(|u(t,x)|^2 + \left| \frac{\partial u}{\partial x}(t,x) \right|^2 \right) d\mu < +\infty.$$

As a particular case of Theorem 6.4 in [56] we obtain the following maximum and minimum principles.

Theorem A.1. *(Maximum principle) Let $u \in W$ be such that*

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} \leq 0 \tag{A.0.1}$$

in the sense of distributions. If

$$\sup ess_{(t,x) \in Q_{\rho\nu,\sigma_1}} u(t,x) = M,$$

for some ν, $0 < \nu < 1$, and any σ_1, where $0 < \sigma_1 < \sigma$, then $u(t,x) = M$ for a.a. $(t,x) \in Q_{\rho,\sigma}$.

Theorem A.2. *(Minimum principle) Let $u \in W$ be such that*

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} \geq 0 \tag{A.0.2}$$

in the sense of distributions. If

$$\inf ess_{(t,x)\in Q_{\rho\nu,\sigma_1}} u(t,x) = M,$$

for some ν, $0 < \nu < 1$, and any σ_1, where $0 < \sigma_1 < \sigma$, then $u\,(t,x) = M$ for a.a. $(t,x) \in Q_{\rho,\sigma}$.

We are ready to prove the lap-number property, saying that the number of zeros is a non-increasing function of time.

Theorem A.3. *Let $r\,(t,x)$ be a continuous function and $u \in C([t_0,t_1], H_0^1\,(\Omega)) \cap L^2\,(t_0,t_1; H^2\,(\Omega))$ be such that $\dfrac{du}{dt} \in L^2\,(t_0,t_1; L^2\,(\Omega))$ and satisfies the equation*

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = r(t,x)u,\ 0 < x < 1,\ t_0 < t \leq t_1. \tag{A.0.3}$$

Then the number of components of

$$\{x : 0 < x < 1,\ u\,(t,x) \neq 0\}$$

is a non-increasing function of t.

Proof. We follow similar lines as in [55, Theorem 6].

Denote

$$Q\,(t) = \{x \in (0,1) : u\,(t,x) \neq 0\}.$$

We need to show that there is an injective map from the components of $Q\,(t_1)$ to the components of $Q\,(t_0)$ if $t_1 > t_0$. If we denote by C a component of $Q\,(t_1)$ and by S_C the component of $[t_0,t_1] \times (0,1) \cap \{u\,(t,x) \neq 0)\}$ which contains C, then in order to obtain the injective map it is necessary to prove two facts:

1. $S_C \cap Q(t_0) \neq \varnothing$;

2. If C_1, C_2 are two components of $Q(t_1)$, then $S_{C_1} \cap S_{C_2} = \varnothing$.

Let us prove the first statement by contradiction, so assume that $S_C \cap Q(t_0) = \varnothing$. We can assume without loss of generality that $r(t,x) < 0$, because this property is satisfied for the function $W(t,x) = u(t,x)\, e^{-\lambda t}$ with $\lambda > 0$ large enough and the components of these two functions coincide. Consider for example that $u(t,x) > 0$ in S_C. Let $M = \max_{(t,x)\in S_C} u(t,x)$. By hypothesis and the Dirichlet boundary conditions this maximum has to be attained at a point (t', x') such that $t_0 < t' \leq t_1$, $0 < x' < 1$. Also, there has to be an $\varepsilon > 0$ such that if $(t,x) \in S_C$ and $t_0 < t \leq t_0 + \varepsilon$, then $u(t,x) < M$, as otherwise there would be a sequence $(t_n, x_n) \in S_C$, $t_n > t_0$, such that $t_n \to t_0$ and $u(t_n, x_n) = M$. By the continuity of u this would imply that $u(t_0, x_0) = M$ for some $(t_0, x_0) \in S_C$, which is a contradiction. Then we can choose t' as the first time when the maximum is attained, so $u(t,x) < M$ for all $(t,x) \in S_C$, $t_0 < t < t'$. By the continuity of u there exists a rectangle $R = [t' - \delta, t'] \times [x' - \gamma, x' + \gamma]$ such that R belongs to S_C. In order to apply Theorem A.1 we put $\mathcal{O} = R$ and

$$Q_{\gamma,\delta} = \{(t,x) : t \in (t' - \delta, t'), |x - x'| < \gamma\}.$$

We have that

$$\sup_{(t,x)\in Q_{\nu\gamma,\sigma_1}} u(t,x) = M,$$

for some $0 < \nu < 1$ and any $0 < \sigma_1 < \delta$. Since u satisfies (A.0.1), we conclude from Theorem A.1 that $u(t,x) = M$ for all $(t,x) \in Q_{\rho,\sigma}$, which is a contradiction.

For the second statement suppose the existence of two disjoints components C_1, C_2 of $Q(t_1)$ such that $S_{C_1} \cap S_{C_2} \neq \varnothing$, which implies in fact that $S_{C_1} = S_{C_2}$. In this case we can assume that $r(t,x) > 0$, being this justified by the function $W(t,x) = u(t,x)\, e^{\lambda t}$ with $\lambda > 0$ large enough. Let for example $u(t,x) > 0$ in S_{C_1} and assume that the interval C_1 has lesser values than the interval C_2. Also, it is clear that between C_1 and C_2 there must exist a point (t_1, x_0) such that $u(t_1, x_0) = 0$. On the other hand, the set $S_{C_1} \cap (t_0, t_1) \times [0,1]$ is path connected.

Thus, there exists a simple path ξ such that one end point is in $\{t_1\} \times C_1$ and the other one is in $\{t_1\} \times C_2$. Let us consider the set L of all points which are above the curve ξ and such that the function u vanishes at them. This set is non-empty because $(t_1, x_0) \in L$. Since L is compact, the function $g : L \to [t_0, t_1]$ given by $g(t,x) = t$ attains it minimum at a certain point $(t', x') \in L$ such that $t_0 < t'$. Then there exists a set $R = [t' - \delta, t') \times [x' - \gamma, x' + \gamma]$ which belongs to S_{C_1}. Let $\mathcal{O} = R$ and

$$Q_{\gamma,\delta} = \{(t,x) : t \in (t' - \delta, t'), |x - x'| < \gamma\}.$$

We have that

$$\inf_{(t,x) \in Q_{\nu\gamma,\sigma_1}} u(t,x) = 0,$$

for some $0 < \nu < 1$ and any $0 < \sigma_1 < \delta$. Since u satisfies (A.0.2), we conclude from Theorem A.2 that $u(t,x) = 0$ for all $(t,x) \in Q_{\rho,\sigma}$, which is a contradiction.

$\square$

Appendix B

In this section, we include the original manuscripts of [18], [19] and [20].

ROBUSTNESS OF DYNAMICALLY GRADIENT MULTIVALUED DYNAMICAL SYSTEMS

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To Professor Valery Melnik, in Memoriam

Abstract. In this paper we study the robustness of dynamically gradient multivalued semiflows. As an application, we describe the dynamical properties of a family of Chafee-Infante problems approximating a differential inclusion studied in [3], proving that the weak solutions of these problems generate a dynamically gradient multivalued semiflow with respect to suitable Morse sets.

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1. **Introduction.** One of the main goals of the theory of dynamical systems is to characterize the structure of global attractors. It is possible to find a wide literature about this problem for semigroups; however, it has been recently when new results in this direction for multivalued dynamical systems have been proved [3], [13], [14].

In this sense, the theory of Morse decomposition plays an important role. In fact, the existence of a Lyapunov function, the property of being a dynamically gradient semiflow and the existence of a Morse decomposition are shown to be equivalent for multivalued dynamical systems in [9].

In this work we show under suitable assumptions that a dynamically gradient multivalued semiflow is stable under perturbations, that is, the family of perturbed multivalued semiflows remains dynamically gradient.

For a fixed dynamically gradient multivalued semiflow with a global attractor we also analyze the rearrangement of a pairwise disjoint finite family of isolated weakly invariant sets, included in the attractor, in such a way that the dynamically gradient property is satisfied in the stronger sense of [16].

These results extend previous ones in the single-valued framework in [7, 1, 2] to the case where uniqueness of solution does not hold. Additionally, it is worth saying that the m-semiflows here are not supposed to be general dynamical systems as in [16], where a robustness theorem for Morse decompositions of multivalued dynamical systems is also proved under a suitable continuity assumption.

We also apply this general robustness theorem in order to show that a family of Chafee-Infante problems approximating a differential inclusion is dynamically gradient if it is close enough to the original problem.

This paper is organized as follows.

Firstly, we introduce in Section 2 basic concepts and properties related to fixed points, complete trajectories and global attractors. In this way we are able to present in Section 3 the main result about robustness of dynamically gradient multivalued semiflows. Further, in Section 4 we prove a theorem which allows us to reorder the family of weakly invariants sets, thus establishing an equivalent definition of dynamically gradient families.

Afterwards, we consider a Chafee-Infante problem in Section 5, where the equivalence of weak and strong solutions is established. Once the set of fixed points is analyzed, we consider a family of Chafee-Infante equations, approximating the differential inclusion tackled in [3]. We check that this family of Chafee-Infante equations verifies the hypotheses of the robustness theorem in order to obtain, therefore, that the multivalued semiflows generated by the solutions of the approximating problems are dynamically gradient if this family is close enough to the original one.

2. **Preliminaries.** Consider a metric space (X, d) and a family of functions $\mathcal{R} \subset \mathcal{C}(\mathbb{R}_+; X)$. Denote by $P(X)$ the class of nonempty subsets of X. Then, define the multivalued map $G : \mathbb{R}_+ \times X \to P(X)$ associated with the family $\mathcal{R}$ as follows

$$G(t, u_0) = \{u(t) : u(\cdot) \in \mathcal{R}, u(0) = u_0\}. \tag{1}$$

In this abstract setting, the multivalued map G is expected to satisfy some properties that fit in the framework of multivalued dynamical systems. The first concept is given now, although a more axiomatic construction will be provided below.

Definition 1. Let (X, d) be a metric space. A multivalued map $G : \mathbb{R}_+ \times X \to P(X)$ is a multivalued semiflow (or m-semiflow) if $G(0, x) = x$ for all $x \in X$ and

$G(t+s,x) \subset G(t,G(s,x))$ for all $t,s \geq 0$ and $x \in X$.
If the above is not only an inclusion, but an equality, it is said that the m-semiflow is strict.

Once a multivalued semiflow is defined, we recall the concepts of invariance and global attractor, with evident differences with respect to the single-valued case.

Definition 2. A map $\gamma : \mathbb{R} \to X$ is called a complete trajectory of $\mathcal{R}$ (resp. of G) if $\gamma(\cdot + h) \mid_{[0,\infty)} \in \mathcal{R}$ for all $h \in \mathbb{R}$ (resp. if $\gamma(t+s) \in G(t,\gamma(s))$ for all $s \in \mathbb{R}$ and $t \geq 0$).

A point $z \in X$ is a fixed point of $\mathcal{R}$ (resp. of G) if $\varphi(\cdot) \equiv z \in \mathcal{R}$ (resp. $z \in G(t,z)$ for all $t \geq 0$).

Definition 3. Given an m-semiflow G on a metric space (X,d) a set $B \subset X$ is said to be negatively invariant if $B \subset G(t,B)$ for all $t \geq 0$, and strictly invariant (or, simply, invariant) if the above relation is not only an inclusion but an equality.

The set B is said to be weakly invariant if for any $x \in B$ there exists a complete trajectory γ of $\mathcal{R}$ contained in B such that $\gamma(0) = x$. We observe that weak invariance implies negative invariance.

A set $\mathcal{A} \subset X$ is called a global attractor for an m-semiflow if it is negatively semi-invariant and it attracts all attainable sets through the m-semiflow starting in bounded subsets, i.e., $dist_X(G(t,B),\mathcal{A}) \to 0$ as $t \to \infty$, where $dist_X(A,B) = \sup_{a\in A} \inf_{b\in B} d(a,b)$.

Remark 1. A global attractor for an m-semiflow does not have to be unique, nor a bounded set. However, if a global attractor is bounded and closed, it is minimal among all closed sets that attract bounded sets [19]. In particular, a bounded and closed global attractor is unique.

In order to obtain a detailed characterization of the internal structure of a global attractor, we introduce an axiomatic set of properties on the set $\mathcal{R}$ (see [4] and [13]).

The set of axiomatic properties that we will deal with is the following.

(K1) For any $x \in X$ there exists at least one element $\varphi \in \mathcal{R}$ such that $\varphi(0) = x$.
(K2) $\varphi_\tau(\cdot) := \varphi(\cdot + \tau) \in \mathcal{R}$ for any $\tau \geq 0$ and $\varphi \in \mathcal{R}$ (translation property).
(K3) Let $\varphi_1, \varphi_2 \in \mathcal{R}$ be such that $\varphi_2(0) = \varphi_1(s)$ for some $s > 0$. Then, the function φ defined by
$$\varphi(t) = \begin{cases} \varphi_1(t) & 0 \leq t \leq s, \\ \varphi_2(t-s) & s \leq t, \end{cases}$$
belongs to $\mathcal{R}$ (concatenation property).
(K4) For any sequence $\{\varphi^n\} \subset \mathcal{R}$ such that $\varphi^n(0) \to x_0$ in X, there exist a subsequence $\{\varphi^{n_k}\}$ and $\varphi \in \mathcal{R}$ such that $\varphi^{n_k}(t) \to \varphi(t)$ for all $t \geq 0$.

It is immediate to observe [6, Proposition 2] or [15, Lemma 9] that $\mathcal{R}$ fulfilling (K1) and (K2) gives rise to an m-semiflow G through (1), and if besides (K3) holds, then this m-semiflow is strict. In such a case, a global bounded attractor, supposing that it exists, is strictly invariant [19, Remark 8].

Several properties concerning fixed points, complete trajectories and global attractors are summarized in the following results [13].

Lemma 1. *Let (K1)-(K2) be satisfied. Then every fixed point (resp. complete trajectory) of $\mathcal{R}$ is also a fixed point (resp. complete trajectory) of G.*

If $\mathcal{R}$ fulfills (K1)-(K4), then the fixed points of $\mathcal{R}$ and G coincide. Besides, a map $\gamma : \mathbb{R} \to X$ is a complete trajectory of $\mathcal{R}$ if and only if it is continuous and a complete trajectory of G.

The standard well-known result in the single-valued case for describing the attractor as the union of bounded complete trajectories reads in the multivalued case as follows.

Theorem 1. *Consider $\mathcal{R}$ satisfying (K1) and (K2), and either (K3) or (K4). Assume also that G possesses a compact global attractor $\mathcal{A}$. Then*

$$\mathcal{A} = \{\gamma(0) : \gamma \in \mathbb{K}\} = \cup_{t\in\mathbb{R}}\{\gamma(t) : \gamma \in \mathbb{K}\}, \tag{2}$$

where $\mathbb{K}$ denotes the set of all bounded complete trajectories in $\mathcal{R}$.

Now we recall the definitions of some important sets in the literature of dynamical systems. Let $B \subset X$ and let $\varphi \in \mathcal{R}$. We define the $\omega-$limit sets $\omega(B)$ and $\omega(\varphi)$ as follows:

$$\begin{aligned}\omega(B) =& \{y \in X : \text{ there are sequences } t_n \to \infty, y_n \in G(t_n, B) \text{ such that } y_n \to y\},\\ \omega(\varphi) =& \{y \in X : \text{ there is a sequence } t_n \to \infty \text{ such that } \varphi(t_n) \to y\}.\end{aligned}$$

If γ is a complete trajectory of $\mathcal{R}$, then the $\alpha-$limit set is defined by

$$\alpha(\gamma) = \{y \in X : \text{ there is a sequence } t_n \to -\infty \text{ such that } \gamma(t_n) \to y\}.$$

Some useful properties of these sets [4, Lemma 3.4] are summarized in the following lemma.

Lemma 2. *Assume that $(K1), (K2)$ and $(K4)$ hold. Let G be asymptotically compact, that is, every sequence $y_n \in G(t_n, B)$, where $t_n \to \infty$ and $B \subset X$ is bounded, is relatively compact. Then:*

1. *For any non-empty bounded set $B, \omega(B)$ is non-empty, compact, weakly invariant and*
$$dist_X(G(t, B), \omega(B)) \to 0, \text{ as } t \to +\infty.$$
2. *For any $\varphi \in \mathcal{R}$, $\omega(\varphi)$ is non-empty, compact, weakly invariant and*
$$dist_X(\varphi(t), \omega(\varphi)) \to 0, \text{ as } t \to +\infty.$$
3. *For any $\gamma \in \mathbb{K}$, $\alpha(\gamma)$ is non-empty, compact, weakly invariant and*
$$dist_X(\gamma(t), \alpha(\gamma)) \to 0, \text{ as } t \to -\infty.$$

In order to give a more detailed description of the internal structure of the attractor under special cases, additional concepts are required.

Definition 4. Consider a metric space (X, d) and an m-semiflow G.

1. We say that $\mathcal{S} = \{\Xi_1, \ldots, \Xi_n\}$ is a family of isolated weakly invariant sets if there exists $\delta > 0$ such that $\mathcal{O}_\delta(\Xi_i) \cap \mathcal{O}_\delta(\Xi_j) = \emptyset$ for $1 \le i < j \le n$, and each Ξ_i is the maximal weakly invariant subset in $\mathcal{O}_\delta(\Xi_i) := \{x \in X : dist_X(x, \Xi_i) < \delta\}$.
2. For an m-semiflow G on (X, d) with a global attractor $\mathcal{A}$ and a finite number of weakly invariant sets $\mathcal{S}$, a homoclinic orbit in $\mathcal{A}$ is a collection $\{\Xi_{p(1)}, \ldots, \Xi_{p(k)}\} \subset \mathcal{S}$ and a collection of complete trajectories $\{\gamma_i\}_{1\le i\le k}$ of $\mathcal{R}$ in $\mathcal{A}$ such that (putting $p(k + 1) := p(1)$)
$$\lim_{t\to-\infty} dist_X(\gamma_i(t), \Xi_{p(i)}) = 0, \lim_{t\to\infty} dist_X(\gamma_i(t), \Xi_{p(i+1)}) = 0, \ 1 \le i \le k,$$

and

$$\text{for each } i \text{ there exists } t_i \in \mathbb{R} \text{ such that } \gamma_i(t_i) \notin \Xi_{p(i)} \cup \Xi_{p(i+1)}.$$

3. We say that an m-semiflow G on (X, d) with the global attractor $\mathcal{A}$ is dynamically gradient if the following two properties hold:
(G1) there exists a finite family $\mathcal{S} = \{\Xi_1, \ldots, \Xi_n\}$ of isolated weakly invariant sets in $\mathcal{A}$ with the property that any complete trajectory γ of $\mathcal{R}$ in $\mathcal{A}$ satisfies

$$\lim_{t\to-\infty} dist_X(\gamma(t), \Xi_i) = 0, \;\; \lim_{t\to\infty} dist_X(\gamma(t), \Xi_j) = 0$$

for some $1 \le i, j \le n$;
(G2) $\mathcal{S}$ does not contain homoclinic orbits.

Remark 2. In the single-valued case, dynamically gradient semigroups have been called also gradient-like semigroups [7]. Observe that the above definitions are concerned with weakly invariant families, which need not to be unitary sets. This is to deal with the more general concept of generalized gradient-like semigroups [7], in contrast with gradient-like semigroups (when the invariant sets are unitary).

Now, we introduce the concept of unstable manifold, that will allow us to describe more precisely the structure of a global attractor of a dynamically gradient m-semiflow.

Definition 5. Let G be an m-semiflow on a metric space (X, d). The unstable manifold of a set Ξ is

$$W^u(\Xi) = \{u_0 \in X : \text{ there exists complete trajectory } \gamma \text{ of } \mathcal{R} \text{ such that } \gamma(0) = u_0 \text{ and } \lim_{t\to-\infty} dist_X(\gamma(t), \Xi) = 0\}.$$

Now the following result, relating the global attractor with unstable manifolds, is standard. The first statement is straightforward to see. The second one, supposing that the global attractor is compact, follows directly from the structure described in Theorem 1 and the definition of dynamically gradient semiflows.

Lemma 3. *Consider a complete metric space (X, d) and a family $\mathcal{R} \subset \mathcal{C}(\mathbb{R}_+; X)$ satisfying (K1) and (K2). Suppose that the associated m-semiflow has a global attractor $\mathcal{A}$. Then, for any bounded set $\Xi \subset X, W^u(\Xi) \subset \bar{\mathcal{A}}$.*

Moreover, assume that $\mathcal{R}$ satisfies either (K3) or (K4), and that the global attractor $\mathcal{A}$ is compact. Suppose also that the associated m-semiflow G defined in (1) is dynamically gradient. Then

$$\mathcal{A} = \bigcup_{i=1}^{n} W^u(\Xi_i). \tag{3}$$

3. **Robustness of dynamically gradient m-semiflows.** Our first main goal is to prove that a dynamically gradient multivalued semiflow is stable under suitable perturbations, that is, a family of perturbed multivalued semiflows remains dynamically gradient if it is close enough to the original semiflow, generalizing the corresponding result in the single-valued case [7]. This is rigorously formulated in the following theorem.

Theorem 2. *Consider a complete metric space (X, d). Let η be a parameter in [0,1], $\mathcal{R}_\eta \subset \mathcal{C}(\mathbb{R}_+; X)$ fulfill (K1), (K2), (K3) and (K4), and let G_η be the corresponding m-semiflow on X having the global compact attractor $\mathcal{A}_\eta$. Assume that*

(H1) $\overline{\bigcup_{\eta\in[0,1]} \mathcal{A}_\eta}$ *is compact.*

(H2) G_0 *is a dynamically gradient m-semiflow with finitely many isolated weakly invariant sets* $\mathcal{S}^0 = \{\Xi_1^0, \ldots, \Xi_n^0\}$.

(H3) $\mathcal{A}_\eta$ *has a finite number of isolated weakly invariant sets* $\mathcal{S}_\eta = \{\Xi_1^\eta, \ldots, \Xi_n^\eta\}$, $\eta \in [0, 1]$, *which satisfy*

$$\lim_{\eta\to 0} \sup_{1\le i\le n} dist_X(\Xi_i^\eta, \Xi_i^0) = 0.$$

(H4) *Any sequence* $\{\gamma_\eta\}$ *with* $\gamma_\eta \in \mathcal{R}_\eta$ *such that* $\{\gamma_\eta(0)\}$ *converges for* $\eta \to 0^+$, *possesses a subsequence* $\{\gamma_{\eta'}\}$ *that converges uniformly in bounded intervals of* $[0, \infty)$ *to* $\gamma \in \mathcal{R}_0$.

(H5) *There exists* $\overline{\eta} > 0$ *and neighborhoods* V_i *of* Ξ_i^0 *such that* Ξ_i^η *is the maximal weakly invariant set for* G_η *in* V_i *for any* $i = 1, \ldots, n$ *and for each* $0 < \eta \le \overline{\eta}$.

Then there exists $\eta_0 > 0$ *such that for all* $\eta \le \eta_0$, $\{G_\eta\}$ *is a dynamically gradient m-semiflow. In particular, the structure of* $\mathcal{A}_\eta$ *is analogous to that given in (3).*

Proof. Observe that assumption (H5) concerning certain neighborhood V_i of Ξ_i^0 involves a hyperbolicity condition of G_0 w.r.t. each Ξ_i^0, and as far as (H3) is also assumed, there exist $\{\eta(V_i)\}_{i=1,\ldots,n}$ such that $\Xi_i^\eta \subset V_i$ for all $\eta \le \eta(V_i)$. W.l.o.g. assume that $\delta > 0$ is such that $\{x \in X : dist_X(x, \Xi_i^0) \le \delta\} \subset V_i$ for all $i = 1, \ldots, n$.

By Theorem 1, we have that $\mathcal{A}_\eta$ is composed by all the orbits of bounded complete trajectories of $\mathcal{R}_\eta$, $\mathbb{K}_\eta$.

We are going to prove by contradiction arguments that there exists $\eta_0 \in (0, 1]$ such that $\{G_\eta\}_{\eta\le\eta_0}$ is dynamically gradient.

Step 1: There exists $\eta_0 > 0$ such that for all $\eta < \eta_0$, any bounded complete trajectory ξ_η of $\mathcal{R}_\eta$ satisfies that there exist $i \in \{1, \ldots, n\}$ and t_0 such that for all $t \ge t_0$, $dist_X(\xi_\eta(t), \Xi_i^0) \le \delta$.

After proving the above claim, we consider the sets $B_\eta := \{\xi_\eta(s) : s \ge t_0\} \subset A = \{y : dist_X(y, \Xi_i^0) \le \delta\}$ and $\omega(\xi_\eta)$. It follows that $\omega(\xi_\eta) \subset A$, since $dist_X(\xi_\eta(t), \omega(\xi_\eta))$ goes to 0 as $t \to +\infty$. On the other hand, by Lemma 2 $\omega(\xi_\eta)$ is a weakly invariant set of G_η contained in V_i. By assumption (H5) we have that $\omega(\xi_\eta) \subset \Xi_i^\eta$, whence the 'forward part' of property (G1) of a dynamically gradient m-semiflow will follow immediately.

We prove this Step 1 by contradiction. Suppose it does not hold. Then, there exist a sequence $\eta_k \to 0$ (as $k \to \infty$) and bounded complete trajectories ξ_k of $\mathcal{R}_{\eta_k}$ (therefore, from $\mathcal{A}_{\eta_k}$) such that

$$\sup_{t\ge t_0} dist_X(\xi_k(t), \mathcal{S}^0) > \delta \ \forall t_0 \in \mathbb{R}. \tag{4}$$

The set $\{\xi_k(0)\} \subset \overline{\cup_{\eta\in[0,1]}\mathcal{A}_\eta}$ is relatively compact from assumption (H1). So, there exists a converging subsequence (relabeled the same) in X. From (H4), there exist a subsequence (relabeled the same, again) and $\xi_0 \in \mathcal{R}_0$, such that $\{\xi_k|_{[0,\infty)}\}$ converges to ξ_0 in bounded intervals of $[0, \infty)$. Actually, if we argue similarly not for time 0, but now for times $-1, -2, \ldots$, and use a diagonal argument, we have that $\xi_0 = \gamma_0|_{[0,\infty)}$ where $\gamma_0 \in \mathbb{K}_0$, and the convergence of (a subsequence of) $\{\xi_k\}$ toward γ_0 holds uniformly in bounded intervals $[a, b]$ of $\mathbb{R}$.

Since G_0 is dynamically gradient, there exists $i \in \{1, \ldots, n\}$ such that

$$dist_X(\gamma_0(t), \Xi_i^0) \to 0 \text{ as } t \to \infty.$$

Therefore, for all $r \in \mathbb{N}$, there exist t_r and k_r such that $dist_X(\xi_k(t_r), \Xi_i^0) < 1/r$ for all $k \geq k_r$. Indeed, this is done as follows: $dist_X(\gamma_0(s), \Xi_i^0) < 1/r$ for all $s \geq t_r$ (for some t_r, w.l.o.g. $t_r \geq r > 1/\delta$); now, combining this with the uniform convergence on $[0, t_r]$ of ξ_k toward γ_0, the existence of k_r follows.

However, from (4), there exists $t_r' > t_r$ such that $dist_X(\xi_{k_r}(t), \Xi_i^0) < \delta$ for all $t \in [t_r, t_r')$ and $dist_X(\xi_{k_r}(t_r'), \Xi_i^0) = \delta$.

Now we distinguish two cases and we will arrive to the same conclusion in both of them.

Case (1a): Suppose that $t_r' - t_r \to \infty$ as $r \to \infty$ (at least for a certain subsequence).

Since $\{\xi_{k_r}(t_r')\}$ is also relatively compact (by (H1), again), and $\xi^1_{k_r}(\cdot) = \xi_{k_r}(t_r' + \cdot)$ is a bounded complete trajectory of $\mathcal{R}_{k_r}$, from (H4) we deduce that a subsequence (relabeled the same) is converging on bounded time-intervals of $[0, \infty)$, i.e. $\gamma_1(t) := \lim_{r\to\infty} \xi_{k_r}(t + t_r')$ holds for certain $\gamma_1 \in \mathcal{R}_0$. Moreover, as before, a diagonal argument, using not t_r' above, but $t_r' - 1$, $t_r' - 2, \dots$ implies that γ_1 can be extended to the whole real line (the function will still be denoted the same; and the convergence holds in bounded time-intervals of $\mathbb{R}$), in particular, by (H1) and (H4), $\gamma_1 \in \mathbb{K}_0$.

Moreover, by its construction, we have that $dist_X(\gamma_1(t), \Xi_i^0) \leq \delta$ for all $t \leq 0$. By Lemma 2 we have that the α-limit set $\alpha(\gamma_1)$ is weakly invariant.

As long as Ξ_i^0 is the biggest weakly invariant set contained in V_i, we deduce that $dist_X(\gamma_1(\tau), \Xi_i^0) \to 0$ when $\tau \to -\infty$.

On the other hand, from (G1) and (G2) we have that $dist_X(\gamma_1(t), \Xi_j^0) \to 0$ as $t \to \infty$ for $j \neq i$.

Case (1b): Suppose that there exists $C > 0$ such that $|t_r' - t_r| \leq C$ as $r \to \infty$. (W.l.o.g. we assume that $t_r' - t_r \to t_*$.)

Recall that $dist_X(\xi_{k_r}(t_r), \Xi_i^0) < 1/r$. By [9, Lemma 19] Ξ_i^0 is closed, so, up to a subsequence $\xi_{k_r}(t_r) \to y \in \Xi_i^0$. Denote $\xi^1_{k_r}(\cdot) = \xi_{k_r}(\cdot + t_r)$. From (H4), there exist a subsequence $\{\xi^1_{k_r}\}$ and $\xi^1 \in \mathcal{R}_0$ with $\xi^1(0) = y$ such that $\xi^1_{k_r}$ converge towards ξ^1 uniformly in bounded intervals of $[0, \infty)$. In particular, $\xi^1_{k_r}(t_r' - t_r) \to \xi^1(t_*)$, so that $dist_X(\xi^1(t_*), \Xi_i^0) \geq \delta$.

Since Ξ_i^0 is weakly invariant, there exists $\gamma \in \mathbb{K}_0$ with $\gamma(0) = \xi^1(0)$ and $\gamma(t) \in \Xi_i^0$ for all $t \in \mathbb{R}$. By (K3) consider the concatenation

$$\gamma_1(t) := \begin{cases} \gamma(t), & \text{if } t \leq 0, \\ \xi^1(t), & \text{if } t \geq 0. \end{cases}$$

Then $\gamma_1 \not\equiv \xi^1$, and by (G1)-(G2) it follows that $dist_X(\gamma_1(t), \Xi_j^0) \to 0$ as $t \to \infty$ with $j \neq i$. This is exactly the same conclusion we arrived in Case (1a).

Reasoning now with the subsequence $\{\xi^1_{k_r}\}$, and proceeding as above, we obtain the existence of $\gamma_2 \in \mathbb{K}_0$ such that $dist_X(\gamma_2(t), \Xi_j^0) \to 0$ as $t \to -\infty$ and $dist_X(\gamma_2(t), \Xi_p^0) \to 0$ as $t \to \infty$, with $p \notin \{i, j\}$.

Thus, in a finite number of steps we arrive to a contradiction, since G_0 satisfies (G2). Therefore, (4) is absurd, and Step 1 is proved.

Step 2: There exists $\eta_1 > 0$ such that for all $\eta < \eta_1$, any bounded complete trajectory ξ_η of $\mathcal{R}_\eta$ satisfies that there exist $j \in \{1, \dots, n\}$ and t_1 such that $dist_X(\xi_\eta(t), \Xi_j^0) \leq \delta$ for all $t \leq t_1$.

The above claim can be proved analogously as before, and since for any bounded complete trajectory $\xi_\eta \in \mathbb{K}_\eta$, by Lemma 2, $\alpha(\xi_\eta)$ is weakly invariant for G_η, and contained in some V_j, the 'backward part' of property (G1) of a dynamically gradient m-semiflow will follow immediately. The same argument is valid for the 'forward part', and so, for all suitable small η, $\{G_\eta(t) : t \geq 0\}$ satisfies (G1).

Step 3: There exists $\eta_2 > 0$ such that $\{G_\eta\}_{\eta \leq \eta_2}$ satisfies (G2).

If not, there exist a sequence $\eta_k \to 0$, with G_{η_k} having an homoclinic structure. We may suppose that the number of elements of weakly invariant subsets connected on each homoclinic chain in $\mathcal{S}_{\eta_k}$ is the same. Moreover, by assumption (H3) each $\Xi_j^{\eta_k}$ is contained in V_j for η_k small enough and w.l.o.g. the order in the route of the homoclinics visiting the V_j sets is the same.

Therefore, for $k \geq k_0$ there exist a sequence of subsets $\Xi^{\eta_k}_{p(1)}, \dots \Xi^{\eta_k}_{p(l)}$ in $\mathcal{S}_{\eta_k}$ (with $p(l+1) = p(1)$), and a sequence of complete trajectories $\{\{\xi_i^k\}_{i=1}^l\}_k$, each collection of l elements in the corresponding attractor $\mathcal{A}_{\eta_k}$, with

$$\lim_{t\to-\infty} dist_X(\xi_i^k(t), \Xi^{\eta_k}_{p(i)}) = 0, \ \lim_{t\to\infty} dist_X(\xi_i^k(t), \Xi^{\eta_k}_{p(i+1)}) = 0, 1 \leq i \leq l.$$

If we argue now as in the proof of (G1), we may construct a homoclinic structure of G_0, getting a contradiction with the fact that the m-semiflow G_0 is dynamically gradient. □

Remark 3. The above result also applies to the particular case of a dynamically gradient m-semiflow when the weakly invariant families of the original and perturbed problems are reduced to unitary sets (Remark 2 and [7, Theorem 1.5]).

4. **An equivalent definition of dynamically gradient families.** We will give an equivalent definition of dynamically gradient families. For proving the main result in this section we will need a stronger condition than $(K4)$. Namely, we shall consider the following stronger condition:

($\overline{K}4$) For any sequence $\{\varphi^n\} \subset \mathcal{R}$ such that $\varphi^n(0) \to x_0$ in X, there exists a subsequence $\{\varphi^n\}$ and $\varphi \in \mathcal{R}$ such that φ^n converges to φ uniformly in bounded subsets of $[0, \infty)$.

Remark 4. We have seen that the property of being dynamically gradient for a disjoint family of isolated weakly invariant sets $\mathcal{S} = \{\Xi_1, \dots, \Xi_n\} \subset \mathcal{A}$ is stable under perturbations. We observe that in the paper [16] a slightly different definition was used for dynamically gradients families. Namely, instead of conditions $(G1)$-$(G2)$ it is assumed that any bounded complete trajectory $\gamma(\cdot)$ satisfies one of the following properties:

1. $\{\gamma(t) : t \in \mathbb{R}\} \subset \Xi_i$ for some i.
2. There are $i < j$ for which

$$\gamma(t) \underset{t\to\infty}{\to} \Xi_i, \ \gamma(t) \underset{t\to-\infty}{\to} \Xi_j.$$

These assumptions are clearly stronger than $(G1)$-$(G2)$ and imply that the sets Ξ_j are ordered. Our aim is to show that when $\mathcal{S}$ is a disjoint family of isolated weakly invariant sets, these conditions are equivalent. For this we will need to introduce the concept of local attractor and its repeller and study their properties.

We say that $A \subset \mathcal{A}$ is a local attractor in $\mathcal{A}$ if for some $\varepsilon > 0$ we have that $\omega(\mathcal{O}_\varepsilon(A) \cap \mathcal{A}) = A$. Let A be a local attractor in $\mathcal{A}$. Then its repeller A^* is defined by

$$A^* = \{x \in \mathcal{A} : \omega(x) \backslash A \neq \emptyset\}.$$

Some properties about local attractors and its repeller as well as the proof of the following lemmas can be found in [9].

Lemma 4. *Assume that* $(K1) - (K4)$ *hold. Then a local attractor* A *is invariant.*

Remark 5. Although in [9] the stronger assumption $(\overline{K}4)$ is assumed, the proof is valid for just $(K4)$.

Lemma 5. *Assume that* $(K1)$*-*$(K3)$*,* $(\overline{K}4)$ *hold and that a global compact attractor* $\mathcal{A}$ *exists. Then the repeller* A^* *of a local attractor* $A \subset \mathcal{A}$ *is weakly invariant and compact.*

Lemma 6. *Assume that* $(K1)$*-*$(K3)$*,* $(\overline{K}4)$ *hold and that a global compact attractor* $\mathcal{A}$ *exists. Let us consider the sequences* $x_k \in \mathcal{A}$*,* $t_k \to +\infty$ *and* $\varphi_k(\cdot) \in \mathcal{R}$ *such that* $\varphi_k(0) = x_k$*. Then from the sequence of maps* $\xi_k(\cdot) : [-t_k, +\infty) \to \mathcal{A}$ *defined by*

$$\xi_k(t) = \varphi_k(t + t_k)$$

one can extract a subsequence converging to some $\psi(\cdot) \in \mathbb{K}$ *uniformly on bounded subsets of* $\mathbb{R}$*.*

In order to prove the equivalent definition of dynamically gradient families, we have to ensure the existence of one local attractor in a family of isolated weakly invariant sets.

Lemma 7. *Assume that* $(K1)$*-*$(K3)$*,* $(\overline{K}4)$ *hold and that a global compact attractor* $\mathcal{A}$ *exists. Let* $\mathcal{S} = \{\Xi_1, \ldots, \Xi_n\} \subset \mathcal{A}$ *be a disjoint family of isolated weakly invariant sets. If* G *is dynamically gradient with respect to* $\mathcal{S}$*, then one of the sets* Ξ_j *is a local attractor in* $\mathcal{A}$*.*

Proof. Let $\delta_0 > 0$ be such that $\mathcal{O}_{\delta_0}(\Xi_i) \cap \mathcal{O}_{\delta_0}(\Xi_j) = \varnothing$ if $i \neq j$ and Ξ_j is the maximal weakly invariant set in $\mathcal{O}_{\delta_0}(\Xi_j)$ for all j. First we will prove the existence of $j \in \{1, ..., n\}$ such that for all $\delta \in (0, \delta_0)$ there exists $\delta' \in (0, \delta)$ satisfying

$$\cup_{t \geq 0} G(t, \mathcal{O}_{\delta'}(\Xi_j) \cap \mathcal{A}) \subset \mathcal{O}_{\delta}(\Xi_j). \tag{5}$$

If not, there would exist $0 < \delta < \delta_0$ and for each j sequences $t_k^j \in \mathbb{R}^+$, $x_k^j \in \mathcal{A}$, $\varphi_k^j \in \mathcal{R}$ with $\varphi_k^j(0) = x_k^j$ such that

$$\begin{aligned} d(x_k^j, \Xi_j) &< \frac{1}{k}, \\ d(\varphi_k^j(t_k^j), \Xi_j) &= \delta, \\ d(\varphi_k^j(t), \Xi_j) &< \delta \text{ for all } t \in [0, t_k^j). \end{aligned}$$

We have to consider two cases: $t_k^j \to +\infty$ or $t_k^j \leq C$.

Let $t_k^j \to +\infty$. We define the sequence

$$\psi_k^j(t) = \varphi_k^j(t + t_k^j) \text{ for } t \in [-t_k^j, \infty).$$

By Lemma 6 we obtain the existence of a complete trajectory of $\mathcal{R}$, $\psi^j(\cdot)$, such that a subsequence of ψ_k^j satisfies $\psi_k^j(t) \to \psi^j(t)$ for every $t \in \mathbb{R}$. Hence, $d(\psi^j(t), \Xi_j) \leq \delta < \delta_0$ for all $t \leq 0$. Therefore, as $\psi^j \in \mathbb{K}$, condition $(G1)$ implies that $d(\psi^j(t), \Xi_j) \to 0$ as $t \to -\infty$. On the other hand, since $d(\psi^j(0), \Xi_j) = \delta$, conditions $(G1) - (G2)$ imply that $d(\psi^j(t), \Xi_i) \to 0$ as $t \to +\infty$, where $i \neq j$.

Let now $t_k^j \leq C$. We can assume that $t_k^j \to t^j$. By $(\overline{K}4)$ we obtain the existence of $\varphi^j \in \mathcal{R}$ such that φ_k^j converges to φ^j uniformly on bounded sets of $[0, \infty)$. It is clear then that $d(\varphi^j(t^j), \Xi_j) = \delta$. As $\varphi^j(0) \in \Xi_j$ and Ξ_j is weakly invariant, there exists a complete trajectory of $\mathcal{R}$, $\psi_j^-(\cdot)$, such that $\psi_j^-(0) = \varphi^j(0)$ and $\psi_j^-(t) \in \Xi_j$ for all $t \leq 0$. Concatenating ψ_j^- and φ^j we define

$$\psi^j(t) = \begin{cases} \psi_j^-(t) \text{ if } t \leq 0, \\ \varphi^j(t) \text{ if } t \geq 0, \end{cases}$$

which is a complete trajectory by $(K3)$. Again, conditions $(G1) - (G2)$ imply that $d(\psi^j(t), \Xi_i) \to 0$ as $t \to +\infty$, where $i \neq j$.

We have obtained then a connection from Ξ_j to a different Ξ_i. Since this is true for any Ξ_j, we would obtain a homoclinic structure, which contradicts $(G2)$. Therefore, (5) holds for some j. It follows that

$$\omega(\mathcal{O}_{\delta'}(\Xi_j) \cap \mathcal{A}) \subset \overline{\mathcal{O}_\delta(\Xi_j)} \subset \mathcal{O}_{\delta_0}(\Xi_j).$$

Since $\omega(\mathcal{O}_{\delta'}(\Xi_j) \cap \mathcal{A})$ is weakly invariant, we obtain that $\omega(\mathcal{O}_{\delta'}(\Xi_j) \cap \mathcal{A}) \subset \Xi_j$. But $\Xi_j \subset G(t, \Xi_j) \subset G(t, \mathcal{O}_{\delta'}(\Xi_j) \cap \mathcal{A})$ for any $t \geq 0$ implies the converse inclusion, so that $\Xi_j = \omega(\mathcal{O}_{\delta'}(\Xi_j) \cap \mathcal{A})$. Thus, Ξ_j is a local attractor in $\mathcal{A}$. □

Now we prove the main result of this section which allows us to establish the equivalent definition of dynamically gradient families.

Theorem 3. *Assume that (K1)-(K3), $(\overline{K}4)$ hold and that a global compact attractor $\mathcal{A}$ exists. Let $\mathcal{S} = \{\Xi_1, \ldots, \Xi_n\} \subset \mathcal{A}$ be a disjoint family of isolated weakly invariant sets. Then G is dynamically gradient with respect to $\mathcal{S}$ in the sense of Definition 4 if and only if $\mathcal{S}$ can be reordered in such a way that any bounded complete trajectory $\gamma(\cdot)$ satisfies one of the following properties:*

1. *$\{\gamma(t) : t \in \mathbb{R}\} \subset \Xi_i$ for some i.*
2. *There are $i < j$ for which*

$$\gamma(t) \underset{t\to\infty}{\to} \Xi_i, \ \gamma(t) \underset{t\to-\infty}{\to} \Xi_j.$$

Proof. It is obvious that conditions 1-2 imply that G is dynamically gradient. We shall prove the converse.

By Lemma 7 one of the sets Ξ_i is a local attractor. After reordering the sets, we can say that Ξ_1 is the local attractor. Let

$$\Xi_1^* = \{x \in \mathcal{A} : \omega(x) \backslash \Xi_1 \neq \varnothing\}$$

be its repeller, which is weakly invariant by Lemma 5. Since Ξ_j are closed (cf. [9, Lemma 19]), weakly invariant and disjoint, we obtain that $\Xi_j \subset \Xi_1^*$ for $j \geq 2$.

We will consider only the dynamics inside the repeller Ξ_1^*, that is, we define the following set:

$$\mathcal{R}_1 = \{\varphi \in \mathcal{R} : \varphi(t) \in \Xi_1^* \ \forall t \geq 0\}.$$

Since Ξ_1^* is weakly invariant, $\mathcal{R}_1$ satisfies $(K1)$. Further, let $\varphi_\tau(\cdot) = \varphi(\cdot + \tau)$, where $\varphi \in \mathcal{R}_1$ and $\tau \geq 0$. Then it is clear that $\varphi_\tau(t) \in \mathcal{R}_1$ for all $t \geq 0$, and then $(K2)$ holds. If $\varphi_1(\cdot), \ \varphi_2(\cdot) \in \mathcal{R}_1$, it follows by $(K3)$ that the concatenation belongs also to $\mathcal{R}_1$. Finally, if $\varphi_n(0) \to \varphi_0$ with $\varphi_n(0) \in \Xi_1^*$ and $\varphi_n(\cdot) \in \mathcal{R}_1$, then $\varphi_0 \in \Xi_1^*$ (as Ξ_1^* is closed) and by $(\overline{K}4)$ passing to a subsequence $\varphi_n(t_n) \to \varphi(t)$, for $t_n \to t \geq 0$, where $\varphi \in \mathcal{R}$. Again, the closedness of Ξ_1^* implies that $\varphi \in \mathcal{R}_1$. Hence, $(\overline{K}4)$ also holds. We can define then the multivalued semiflow $G_1 : \mathbb{R}^+ \times \Xi_1^* \to P(\Xi_1^*)$:

$$G_1(t, x) = \{y \in \Xi_1^* : y = \varphi(t) \text{ for some } \varphi \in \mathcal{R}_1, \ \varphi(0) = x\},$$

which is strict by $(K3)$. This definition is equivalent to the following one:

$$\overline{G}_1(t,x) = G(t,x) \cap \Xi_1^* \text{ for } x \in \Xi_1^*.$$

Indeed, $G_1(t,x) \subset \overline{G}_1(t,x)$ is obvious. Conversely, let $y \in \overline{G}_1(t,x)$. Then, $y = \varphi(t)$, $\varphi(\cdot) \in \mathcal{R}$, and $y \in \Xi_1^*$. We state that $\varphi(s) \in \Xi_1^*$ for all $0 \leq s \leq t$. Assume by contradiction that $\varphi(s) \notin \Xi_1^*$ for $0 < s < t$. Therefore, $\omega(\varphi(s)) \subset \Xi_1$. But then by $(K3)$,

$$G(T,y) \subset G(T, G(t-s,\varphi(s))) \subset G(T+t-s,\varphi(s)) \to \Xi_1 \text{ as } T \to \infty,$$

which is a contradiction with $y \in \Xi_1^*$. Using again $(K3)$ one can define a function $\psi(\cdot) \in \mathcal{R}_1$ such that $\psi(0) = y$, so that $y \in G_1(t,x)$.

It is clear that G_1 possesses a global compact attractor, which is the union of all bounded complete trajectories of $\mathcal{R}_1$, and that G_1 is dynamically gradient with respect to $\{\Xi_2, \ldots, \Xi_n\}$. Then, again by Lemma 7 we can reorder the sets in such a way that Ξ_2 is a local attractor in Ξ_1^*. Let $\Xi_{2,1}^*$ be the repeller of Ξ_2 in Ξ_1^*. Then we restrict as before the dynamics to the set $\Xi_{2,1}^*$ and so on. Hence, we have reordered the sets Ξ_j in such a way that Ξ_1 is a local attractor and Ξ_j is a local attractor for the dynamics restricted to the repeller of the previous local attractor $\Xi_{j-1,j-2}^*$ for $j \geq 2$, and $\Xi_i \subset \Xi_{j-1,j-2}^*$ if $i \geq j$, where $\Xi_{1,0}^* = \Xi_1^*$.

Now, if $\gamma(\cdot)$ is a bounded complete trajectory such that

$$\gamma(t) \underset{t\to\infty}{\to} \Xi_i,\ \gamma(t) \underset{t\to-\infty}{\to} \Xi_j,$$

then we shall prove that $i \leq j$. Moreover, if $\gamma(\cdot)$ is not completely contained in some Ξ_k, then $i < j$.

If $i = 1$, then it is clear that $j \geq 1$. Also, if there exists $\gamma(t_0) \notin \Xi_1$, then $j > 1$, as Ξ_1 is a local attractor.

Let $i = 2$. Then $\gamma(t) \in \Xi_1^*$ for all $t \in \mathbb{R}$, and then $\gamma(t) \underset{t\to-\infty}{\to} \Xi_1$ is forbidden. Hence, $j \geq 2$. Again, if there exists $\gamma(t_0) \notin \Xi_2$, then the fact that Ξ_2 is a local attractor in Ξ_1^* implies that $j > 2$.

Further, note that if $i \geq 3$, then $\gamma(t) \in \Xi_1^*$ for all $t \in \mathbb{R}$. Also, by induction, it follows that $\gamma(t) \in \Xi_{k,k-1}^*$ for all $t \in \mathbb{R}$ and $2 \leq k \leq i-1$. Indeed, let $\gamma(t) \in \Xi_{k-1,k-2}^*$ for all $t \in \mathbb{R}$ with $2 \leq k \leq i-1$. Then $\gamma(t) \underset{t\to\infty}{\to} \Xi_i$ implies clearly that $\gamma(t) \in \Xi_{k,k-1}^*$ for all $t \in \mathbb{R}$. In particular, $\gamma(t) \in \Xi_{i-1,i-2}^*$ for all $t \in \mathbb{R}$. Hence, $\Xi_j \in \Xi_{i-1,i-2}^*$, so that $j \geq i$. Finally, if there exists $\gamma(t_0) \notin \Xi_i$, then $j > i$ as Ξ_i is a local attractor in $\Xi_{i-1,i-2}^*$. $\square$

To finish this section, we recall that the disjoint family of isolated weakly invariant sets $\mathcal{S} = \{\Xi_1, \ldots, \Xi_n\} \subset \mathcal{A}$ is a Morse decomposition of the global compact attractor $\mathcal{A}$ if there is a sequence of local attractors $\emptyset = A_0 \subset A_1 \subset \ldots \subset A_n = \mathcal{A}$ such that for every $k \in \{1, \ldots, n\}$ it holds

$$\Xi_k = A_k \cap A_{k-1}^*.$$

It is well known [16] that for general dynamical systems conditions 1-2 in Theorem 3 are equivalent to the fact that $\mathcal{S}$ generates a Morse decomposition. This fact can be proved also under conditions (K1)-(K3), $(\overline{K}4)$ [9].

Thus, Theorem 3 implies that under conditions $(K1)$-$(K3)$,$(\overline{K}4)$ the family $\mathcal{S}$ generates a Morse decomposition if and only if G is dynamically gradient.

5. **Application to a reaction-diffusion equation.** We will consider the Chafee-Infante problem

$$\begin{cases} \dfrac{\partial u}{\partial t} - \dfrac{\partial^2 u}{\partial x^2} = f(u),\ t > 0,\ x \in (0,1)\,, \\ u(t,0) = 0,\ u(t,1) = 0, \\ u(0,x) = u_0(x), \end{cases} \tag{6}$$

where f satisfies

(A1) $f \in C(\mathbb{R})$;
(A2) $f(0) = 0$;
(A3) $f'(0) > 0$ exists and is finite;
(A4) f is strictly concave if $u > 0$ and strictly convex if $u < 0$;
(A5) Growth condition:

$$|f(u)| \leq C_1 + C_2 |u|^{p-1},$$

where $p \geq 2, C_1, C_2 > 0$;
(A6) Dissipation condition:
(a) If $p > 2$:

$$f(u)u \leq C_3 - C_4 |u|^p, \quad C_3, C_4 > 0.$$

(b) If $p = 2$:

$$\limsup_{u \to \pm\infty} \frac{f(u)}{u} \leq 0.$$

Remark 6. Note that as a consequence of condition (A6)(b), we have that $f(u)u \leq (\lambda_1 - C_5)u^2 + C_6$, where $C_5, C_6 > 0$ and $\lambda_1 = \pi^2$ is the first eigenvalue of the operator $-\frac{\partial^2 u}{\partial x^2}$ with Dirichlet boundary conditions.

Let $\Omega = (0,1)$ and $1/p + 1/q = 1$. Denote by $(\cdot,\cdot)$ and $\|\cdot\|_{L^2}$ the scalar product and norm in $L^2(\Omega)$, by $\|\cdot\|_{H_0^1}$ the norm in $H_0^1(\Omega)$ associated to the scalar product of gradients in $L^2(\Omega)$ thanks to Poincaré's inequality. As usual, let $H^{-1}(\Omega)$ be the dual space to $H_0^1(\Omega)$. Denote by $\langle\cdot,\cdot\rangle$ pairing between the space $L^p(\Omega) \cap H_0^1(\Omega)$ and its dual $L^q(\Omega) \cap H^{-1}(\Omega)$.

Definition 6. The function $u(\cdot) \in C([0,T], L^2(\Omega))$ is called a strong solution of (6) on $[0,T]$ if:

1. $u(0) = u_0$;
2. $u(\cdot)$ is absolutely continuous on compact subsets of $(0,T)$;
3. $u(t) \in H^2(\Omega) \cap H_0^1(\Omega)$, $f(u(t)) \in L^2(\Omega)$ for a.e. $t \in (0,T)$ and

$$\frac{du(t)}{dt} - \Delta u = f(u(t)),\ \text{a.e. } t \in (0,T);$$

where the equality is understood in the sense of the space $L^2(\Omega)$.

Definition 7. The function $u(\cdot) \in C([0,T], L^2(\Omega))$ is called a weak solution of (6) on $[0,T]$ if:

1. $u \in L^\infty(0,T; L^2(\Omega))$;
2. $u \in L^2(0,T; H_0^1(\Omega)) \cap L^p(0,T; L^p(\Omega))$;
3. The equality in (6) is understood in the weak sense, i.e.

$$\frac{d}{dt}\langle u(t), v\rangle - \langle \Delta u, v\rangle = \langle f(u(t)), v\rangle,\ \forall v \in H_0^1(\Omega) \cap L^p(\Omega),$$

where the equality is understood in the sense of distributions.

Let us make some comments on the natural relation among the above two definitions. Let $u(\cdot)$ be a strong solution such that $f(u(\cdot)) \in L^2(0,T;L^2(\Omega))$. In view of [3, Proposition 2.2] we have that $u \in L^2(0,T;H_0^1(\Omega))$, so $\Delta u \in L^2(0,T;H^{-1}(\Omega))$ and then $\frac{du}{dt} \in L^2(0,T;H^{-1}(\Omega))$. Hence, by [20, Lemma 7.4] we get

$$\langle \frac{du}{dt}, v\rangle - \langle \Delta u, v\rangle = \langle f(u(t)), v\rangle,\ \forall v \in H_0^1(\Omega).$$

Using [22, p.250] we obtain

$$\frac{d}{dt}\langle u, v\rangle - \langle \Delta u, v\rangle = \langle f(u(t)), v\rangle,\ \forall v \in H_0^1(\Omega),$$

so point 3 of Definition 7 is satisfied.

Finally, if $p > 2$ by condition (A6)(a) we have

$$|u(t,x)|^p \leq \frac{C_3}{C_4} - \frac{f(u(t,x))u(t,x)}{C_4}$$

Thus, $f(u)u \in L^1((0,T)\times\Omega)$ implies that $u \in L^p((0,T)\times\Omega) = L^p(0,T;L^p(\Omega))$. Hence, $u(\cdot)$ is a weak solution as well.

In view of [8, p.283], for any $u_0 \in L^2(\Omega)$ there exists at least one weak solution. Moreover, if $f(u(\cdot)) \in L^2(0,T;L^2(\Omega))$, then putting $g(\cdot) = f(u(\cdot))$ we obtain by [5, p.189] that the problem

$$\begin{cases} \dfrac{dv}{dt} - \Delta v = g(t), \\ v(0) = u_0, \end{cases}$$

possesses a unique strong solution $v(\cdot)$. Since this problem has also a unique weak solution $\tilde{v}(\cdot)$ and the strong solution is a weak solution as well, then $v(\cdot) = \tilde{v}(\cdot) = u(\cdot)$. Hence $u(\cdot)$ is also a strong solution of problem (6).

Therefore, we have checked that the sets of weak and strong solutions satisfying $f(u(\cdot)) \in L^2(0,T;L^2(\Omega))$ coincide.

5.1. **Stationary points.** We now focus on the properties of the stationary points. To this end, we have followed the classic procedure from [11] and [12]. Moreover, we have also taken some ideas from [18].

Let $\mathcal{R} \subset C([0,\infty), L^2(\Omega))$ be the set of all weak solutions of problem (6). Properties $(K1)-(K4)$ are satisfied [cf. [13]], so that a multivalued semiflow is defined (see Section 2). It is shown in [13, Lemma 12] that v is a fixed point of $\mathcal{R}$ (equivalently, of G) if and only if $v \in H_0^1(\Omega)$ and

$$\frac{\partial^2 v}{\partial x^2} + f(v) = 0,\ \text{in } H^{-1}(\Omega). \tag{7}$$

The inclusion $H_0^1(\Omega) \subset L^\infty(\Omega)$ implies that $f(v) \in L^\infty(\Omega)$, so that $v \in H^2(\Omega) \cap H_0^1(\Omega)$. Therefore, $v(\cdot)$ is a strong solution as well.

Let consider the function $F : \mathbb{R} \to \mathbb{R}$ defined by

$$F(s) = \int_0^s f(r)\mathrm{d}r,\ s \in \mathbb{R}.$$

We define

$$a_- = inf\{s < 0 :\ \text{sgn}\ f(x) = \ \text{sgn}\ x,\ \forall x; s < x < 0\}$$

and

$$a_+ = sup\{s > 0 :\ \text{sgn}\ f(x) = \ \text{sgn}\ x,\ \forall x; 0 < x < s\}.$$

If follows from conditions (A2) and (A3) of f that $-\infty \leq a_- < 0 < a_+ \leq +\infty$. Since f is positive on $(0, a_+)$ and negative on $(a_-, 0)$, we have that F is strictly increasing on $[0, a_+)$, strictly decreasing on $(a_-, 0]$ and $F(0) = 0$. We consider $E_+, E_- \in [0, \infty]$ defined by

$$E_+ = \lim_{s \to a_+} F(s),$$
$$E_- = \lim_{s \to a_-} F(s).$$

Then, F has the inverse functions $U_+ : [0, E_+) \to [0, a_+)$, $U_- : [0, E_-) \to (a_-, 0]$.

We also define the following functions with domains $(0, E_+)$ and $(0, E_-)$, respectively, with values on $[0, \infty)$:

$$\tau_+(E) = \int_0^{U_+(E)} (E - F(u))^{-1/2} \, \mathrm{d}u, \ 0 < E < E_+,$$
$$\tau_-(E) = \int_{U_-(E)}^0 (E - F(u))^{-1/2} \, \mathrm{d}u, \ 0 < E < E_-.$$

Let us consider $v_0 \in \mathbb{R}$ and a solution u of

$$\begin{cases} \frac{\partial^2 u}{\partial x^2} + f(u) = 0, \\ u(0) = 0, u'(0) = v_0. \end{cases} \tag{8}$$

Note that the solution of the problem (8) is unique, since f is convex for $u < 0$ and concave for $u > 0$, so it is Lipschitz on compact intervals (see [27, p.4] or [10, p.8]).

If we define $E = v_0^2/2$, then:

$$\frac{(u'(x))^2}{2} + F(u(x)) = E.$$

On the other hand, the functions τ_+, τ_- evaluated in $E = v_0^2/2$ give us $\sqrt{2}$ the x-time necessary to go from the initial condition $u(0) = 0$, with initial velocity $v_0, -v_0$ respectively, to the point where $u'(T_+(E)) = 0$. Indeed, $u(x)$ satisfies $\frac{(u'(x))^2}{2} + F(u(x)) = E$, so $\frac{dx}{du} = \frac{1}{\sqrt{2}} \frac{1}{\sqrt{E - F(u)}}$. Since $u'(T_+(E)) = 0$ for $u = U_+(E)$, then

$$\sqrt{2} \int_0^{T_+(E)} 1 \, \mathrm{d}x = \int_0^{U_+(E)} \frac{1}{\sqrt{E - F(u)}} \mathrm{d}u = \tau_+(E).$$

By symmetry with respect to the u axis, the $x-$time it takes for $u(x)$ to go from $(U^+(E), 0)$ to $(0, -v_0)$ is $T_+(E)$. By this way, if $2T_+(E) = 1$, that is, $\tau^+(E) = \frac{1}{\sqrt{2}}$, then $u(\cdot)$ is a solution satisfying the boundary conditions $u(0) = u(1) = 0$. Applying a similar reasoning for $\tau^-(E)$, we obtain that u satisfies the boundary conditions if, and only if, E satisfies for some $k \in \mathbb{N}$ only one of the following conditions:

$$k\tau_+(E) + (k-1)\tau_-(E) = \frac{1}{\sqrt{2}}, \tag{9}$$

$$k\tau_-(E) + (k-1)\tau_+(E) = \frac{1}{\sqrt{2}}, \tag{10}$$

$$k\tau_+(E) + k\tau_-(E) = \frac{1}{\sqrt{2}}. \tag{11}$$

Remark 7. Note that if E satisfies (9) or (10) for a certain k, then u has $2k$ zeros and if E satisfies (11), then u has $2k + 1$ zeros. Our goal is to solve these equations for E as a function of $f'(0)$. To this end, we study the properties of $\tau_\pm$.

In order to obtain solutions of the equations (9), (10) and (11) it is necessary to make a change of variable for the functions $\tau_\pm$. Given $E \in (0, E_\pm)$, we put

$$Ey^2 = F(u),\ 0 \le y \le 1,\ 0 \le u \le U_+(E)$$

and

$$Ey^2 = F(u),\ -1 \le y \le 0,\ U_-(E) \le u \le 0.$$

Hence, $du = (2yE/f(u))dy$ and $E - F(u) = E(1-y^2)$. By this change, we obtain

$$\tau_+(E) = 2\sqrt{E}\int_0^1 (1-y^2)^{-1/2}\frac{y}{f(u)}dy, 0 < E < E_+;\ u = U_+(Ey^2), 0 \le y \le 1;$$

$$\tau_-(E) = 2\sqrt{E}\int_{-1}^0 (1-y^2)^{-1/2}\frac{y}{f(u)}dy, 0 < E < E_-; u = U_-(Ey^2), -1 \le y \le 0.$$

The next results show some properties of these functions.

Theorem 4. *The functions $\tau_\pm$ satisfy*

$$\lim_{E\to 0^+} \tau_\pm(E) = \frac{\pi}{(2f'(0))^{1/2}}.$$

Proof. Since $f'(0) > 0$ and $f(0) = 0$, given $\varepsilon \in (0,1)$, there exists $\delta > 0$ such that

$$\begin{gathered} f'(0)(1-\varepsilon)u \le f(u) \le f'(0)(1+\varepsilon)u,\quad 0 \le u \le \delta. \\ \frac{1}{f'(0)(1+\varepsilon)} \le \frac{u}{f(u)} \le \frac{1}{f'(0)(1-\varepsilon)},\quad 0 \le u \le \delta. \end{gathered} \tag{12}$$

Moreover, as $U_+(E)$ is continuous at 0, given $\delta > 0$, there exists $\eta > 0$ such that for $0 < E \le \eta$, $U_+(E) \le \delta$. Now, if we integrate (12) between 0 and u we obtain the following inequality

$$\frac{f'(0)}{2}(1-\varepsilon)u^2 \le F(u) \le \frac{f'(0)}{2}(1+\varepsilon)u^2,\ 0 \le u \le \delta.$$

Using the change of variable $Ey^2 = F(u)$, we have

$$\left(\frac{f'(0)(1-\varepsilon)}{2E}\right)^{1/2} u \le y \le \left(\frac{f'(0)(1+\varepsilon)}{2E}\right)^{1/2} u,\quad \text{for}\ \ 0 < E \le \eta,\ 0 \le y \le 1.$$

Dividing the previous expression by $f(u)$ and using (12) we obtain

$$\left(\frac{1-\varepsilon}{2Ef'(0)(1+\varepsilon)^2}\right)^{1/2} \le \frac{y}{f(u)} \le \left(\frac{1+\varepsilon}{2Ef'(0)(1-\varepsilon)^2}\right)^{1/2}, \text{for}\ \ 0 < E \le \eta,\ 0 \le y \le 1.$$

Now if we multiply by $2\sqrt{E}(1-y^2)^{-\frac{1}{2}}$ and integrate from 0 to 1, we get

$$\pi\left(\frac{1-\varepsilon}{2f'(0)(1+\varepsilon)^2}\right)^{1/2} \le \tau_+(E) \le \pi\left(\frac{1+\varepsilon}{2f'(0)(1-\varepsilon)^2}\right)^{1/2}, \text{for}\ \ 0 < E \le \eta.$$

Finally, taking $\varepsilon \to 0$, the theorem follows. The proof for τ_- is analogous. $\square$

Theorem 5. *The functions $\tau_\pm$ are strictly increasing on their domains.*

Proof. Let consider the expression of τ_+ and $0 < E_1 < E_2 < E_+$. Then,

$$\tau_+(E_2) - \tau_+(E_1) = \int_0^1 \frac{2y}{\sqrt{1-y^2}}\left[\frac{\sqrt{E_2}}{f(U^+(E_2y^2))} - \frac{\sqrt{E_1}}{f(U^+(E_1y^2))}\right]dy.$$

From [10, p.8] we have that the function f is differentiable almost everywhere in $\mathbb{R}$, so $\alpha(E) = \dfrac{\sqrt{E}}{f(U^+(Ey^2))}$ is differentiable as well. Hence,

$$\alpha'(E) = \frac{f^2(U^+(Ey^2)) - 2y^2 E f'(U^+(Ey^2))}{2\sqrt{E} f^3(U^+(Ey^2))}.$$

Recall the change of variable $F(u) = Ey^2$. Consider the numerator of α', that is, $\beta(u) = f^2(u) - 2F(u)f'(u)$. Then we obtain

$$\beta(u) = 2\int_0^u f(s)(f'(s) - f'(u))ds, \ 0 < s < u.$$

Since f is strictly concave, if $s < u$, then $f'(s) > f'(u)$ (cf. [27, p.5]). As a result, $\beta(u) > 0$.

In order to finish the proof rigorously, we have to justify the previous calculations. Indeed, from [10, p.5], we have that the function f is absolutely continuous and from [5, p.16], $f' \in L^1_{loc}$. Therefore, $\alpha' \in L^1_{loc}$ and $\alpha' > 0$ a.e., which implies that $\alpha(E)$ is strictly increasing and the proof is finished.

The claim for $\tau_-(E)$ follows analogously. □

Theorem 6. *The functions $\tau_\pm$ satisfy*

$$\lim_{E \to E^\pm} \tau_\pm(E) = \infty$$

Then, $\tau_\pm : (0, E^\pm) \to \left(\dfrac{\pi}{(2f'(0))^{1/2}}, \infty \right)$.

Proof. **Case** $a_+ < \infty$**.** Then, we have $f(a_+) = 0$ and $\bar{u}(x) = a_+$ is a constant solution to the problem $\frac{\partial^2 u}{\partial x^2} + f(u) = 0$. Let us consider $E_+ = F(a_+)$ and the solution u to this problem satisfying the conditions $u(0) = 0, u'(0) = v_0, E = \frac{1}{2}v_0^2$. As a_+ is a constant solution, by uniqueness $\tau_+(E^+) = \infty$. Therefore, given $T > 0$, there exists $\delta > 0$ such that if $E > E_+ - \delta$, then $\tau_+(E) > T$, which follows from the continuity of u with respect to its initial conditions.

Case $a_+ = \infty$. Note that if $p > 2$, then $a_+ < \infty$. Therefore, $p = 2$. In this case, $f(u) > 0$ for all $u \in (0, \infty)$. From condition (A5), there exist $\alpha, \beta > 0$ such that $f(u) \leq \alpha + \beta u$. For $u > 0$ we have

$$\frac{f(u)}{u^2} \leq \frac{\alpha}{u^2} + \frac{\beta}{u}.$$

Hence, $f(u)/u^2 \to 0$, as $u \to \infty$.

On the other hand, $\int_0^u f(s)ds \leq \int_0^u (\alpha + \beta s)\, ds$. Thus, we have $F(u) \leq \alpha u + \beta u^2/2$ and

$$0 \leq \frac{F(u)}{u^3} \leq \frac{\alpha}{u^2} + \frac{\beta}{2}\frac{1}{u}.$$

Hence, $F(u)/u^3 \to 0$, as $u \to \infty$.

We claim that $\lim_{u \to 0^+} f(u)/u^2 = \infty$. Indeed, since $f'(0)$ exists, for any $\varepsilon \in (0, f'(0))$, there exists $\delta > 0$ such that $|f'(0) - f(u)/u| < \varepsilon$, for any $|u| < \delta$. Thus, dividing by u^2, we obtain

$$\frac{u(f'(0) - \varepsilon)}{u^2} < \frac{f(u)}{u^2} < \frac{u(\varepsilon + f'(0))}{u^2}$$

and the result follows.

Since $f(u)/u^2 \to 0$, as $u \to \infty$, and $f(u)/u^2 \to \infty$, as $u \to 0^+$, for any $\varepsilon > 0$, there exists a first value $u_0 \in (0, \infty)$ where $f(u_0)/u_0^2 = \varepsilon$. Hence,

$$\frac{f(u)}{u^2} > \varepsilon,\ 0 < u < u_0.$$

From the above expression, we have $\int_0^u f(s)ds > \int_0^u \varepsilon s^2 ds$ and $\varepsilon u^3/3 < F(u)$. Then, $F(u)/u^3 > \varepsilon/3$, if $0 < u \le u_0$. Since $F(u)/u^3 \to 0$, as $u \to \infty$, we deduce that there exists a first $\overline{u} > u_0$ such that $F(\overline{u})/\overline{u}^3 = \varepsilon/3$. Hence, we have

$$\frac{F(u)}{u^3} > \frac{\varepsilon}{3},\ 0 < u < \overline{u},$$

with $F(\overline{u}) = \frac{\varepsilon}{3}\overline{u}^3$.

Now, computing τ_+ in $\overline{E} = F(\overline{u})$, we have

$$\tau_+(\overline{E}) = \int_0^{U^+(\overline{E})} \frac{1}{\sqrt{\overline{E} - F(u)}} du = \int_0^{\overline{u}} \frac{1}{\sqrt{\frac{\varepsilon}{3}\overline{u}^3 - F(u)}} du$$

$$\ge \int_0^{\overline{u}} \frac{1}{\sqrt{\frac{\varepsilon}{3}\overline{u}^3 - \frac{\varepsilon}{3}u^3}} du = \frac{\sqrt{3}}{\sqrt{\varepsilon}} \int_0^{\overline{u}} \frac{1}{\sqrt{\overline{u}^3 - u^3}} du$$

$$= \frac{\sqrt{3}}{\sqrt{\varepsilon}} \int_0^1 \frac{\overline{u}}{\sqrt{\overline{u}^3 - \overline{u}^3 t^3}} dt = \frac{\sqrt{3}}{\sqrt{\varepsilon}} \frac{\overline{u}}{\sqrt{\overline{u}^3}} \int_0^1 \left(1 - t^3\right)^{-\frac{1}{2}} dt$$

$$= \frac{\sqrt{3}}{\sqrt{\varepsilon}} \frac{\overline{u}}{\sqrt{\overline{u}^3}} \frac{1}{3} \int_0^1 s^{\frac{1}{3}-1} (1 - s)^{\frac{1}{2}-1} ds$$

$$= \frac{1}{\overline{u}^{\frac{1}{2}}} \frac{1}{\sqrt{\varepsilon}} \frac{\sqrt{3}}{3} \mathcal{B}\left(\frac{1}{2}, \frac{1}{3}\right).$$

Recall that $\varepsilon\overline{u}^3 = 3F(\overline{u})$. Then,

$$\varepsilon\overline{u} = 3\frac{F(\overline{u})}{\overline{u}^2}.$$

Taking $\varepsilon \to 0$, by construction $\overline{u} \to \infty$. Therefore, from condition (A6)(b) we have that $\lim_{u\to\infty} f(u)/u \le 0$, so the last expression tends to 0 and $\tau_+(\overline{E}) \to \infty$. □

Theorem 7. *Consider*

$$\lambda_n = n^2\pi^2.$$

Then, for each $n \ge 1$, there exist two continuous functions $E_n^{\pm} : [\lambda_n, \infty) \to [0, E_{\pm})$ with the following properties:

1. *For each integer $k \ge 1$ and for $f'(0) \in [\lambda_{2k-1}, \infty)$ the only solution of the equation (9) (resp. 10) is the value $E_{2k-1}^{+}(f'(0))$ (resp. $E_{2k-1}^{-}(f'(0))$);*
2. *For each integer $k \ge 1$ and for $f'(0) \in [\lambda_{2k}, \infty)$ the only solution of the equation (11) is the value $E_{2k}^{-}(f'(0)) = E_{2k}^{+}(f'(0)) = E_{2k}$;*
3. *For each integer $n \ge 1$, $E_n^{\pm}(f'(0)) = 0$, if $f'(0) = \lambda_n$.*

Proof. Let be $n \ge 1$. If n is odd, then $n = 2k - 1$ for $k \ge 1$. First, we prove that we can define the function

$$E_n^{\pm} : [\lambda_n, \infty) \longrightarrow [0, E_{\pm})$$

by putting $E_n^{\pm}(f'(0)) = E$, where E satisfies $k\tau_{\pm}(E) + (k-1)\tau_{\mp}(E) = 1/\sqrt{2}$.

Consider the function

$$h_{\pm}^n : (0, E_{\pm}) \longrightarrow (n\pi/\sqrt{2f'(0)}, \infty),$$

defined by $h_{\pm}^n(E) := k\tau_{\pm}(E) + (k-1)\tau_{\mp}(E)$. If $f'(0) > \lambda_n$ then, as $h_{\pm}$ is a strictly increasing function, there exists a unique $E_{2k-1}^{\pm} \in (0, E_{\pm})$ such that $h_{\pm}^n(E_{2k-1}^{\pm}) = 1/\sqrt{2}$.

Since $h_{\pm}$ has inverse, $E_{2k-1}^{\pm} = (h_{\pm}^n)^{-1}(1/\sqrt{2})$ is the solution of the expressions (9) and (10). Moreover, $E_{2k-1}^{\pm}(\lambda_n) = 0$ by construction.

Second, if n is even, then $n = 2k$ for $k \geq 1$. As before, we consider $h_{\pm}^n(E) := k\tau_{\pm}(E) + k\tau_{\mp}(E)$. Since it is an increasing function, for $f'(0) > \lambda_n$, there exists a unique $E_{2k} \in (0, E_{\pm})$ such that $h_{\pm}^n(E_{2k}) = 1/\sqrt{2}$. Analogously, we obtain the solution of the expression (11), $E_{2k}^{\pm} = (h_{\pm}^n)^{-1}(1/\sqrt{2})$, and $E_{2k-1}^{\pm}(\lambda_n) = 0$. □

Theorem 8. *For each $n \geq 0$ and $f'(0) \in [\lambda_n, \infty)$, the equation (7) has two new more solutions $v_n^{\pm}$ with the following properties:*

1. *$a_- < u_n^{\pm}(x) < a_+$ for all $x \in [0,1]$;*
2. *If $f'(0) = \lambda_n$, then $v_n^{\pm} = 0$;*
3. *For $f'(0) \in (\lambda_n, \infty)$, $v_n^{\pm}$ has $n+1$ zeros in $[0,1]$. Denoting these zeros by $x_q^{\pm}$, $q = 0, 1, \ldots, n$ with $0 = x_0^{\pm} < x_1^{\pm} < x_2^{\pm} < \ldots < x_n^{\pm} = 1$, we have $(-1)^q v_n^+(x) > 0$ for $x_q^+ < x < x_{q+1}^+, q = 0, 1, \ldots, n-1$ and $(-1)^q v_n^-(x) < 0$ for $x_q^- < x < x_{q+1}^-, q = 0, 1, \ldots, n-1$. Also, $v_n^+ = -v_n^-$, if f is odd;*

Proof. The first point follows from $F(u_n^{\pm}(x)) \leq E < E_{\pm}$.

The second point follows from the third one of Theorem 7. Indeed, for each $n \geq 1$ and $f'(0) \in [\lambda_n, \infty)$ we have the values $E_n^{\pm}(f'(0))$ by the above theorem. Also, we have a solution of the equation (7) which is denoted by $v_n^{\pm}$. If $f'(0) = \lambda_n$, then $E_n^{\pm}(\lambda_n) = 0$ and $v_0 = 0$, so $v_n^{\pm} = 0$.

The third point follows by Remark 7. If f is odd, then $-U^-(E) = U^+(E)$, $\tau_+(E) = \tau_-(E)$, so we have $v_n^+ = -v_n^-$. □

Corollary 1. *If $n^2\pi^2 < f'(0) \leq (n+1)^2\pi^2, n \in \mathbb{N}$, then there are $2n+1$ fixed points: 0, $v_1^{\pm}$, ..., $v_n^{\pm}$, where $v_j^{\pm}$ possesses $j+1$ zeros in $[0,1]$.*

5.2. **Approximations.** From now on, we shall consider the following family of Chafee-Infante equations

$$\begin{cases} \dfrac{\partial u}{\partial t} - \dfrac{\partial^2 u}{\partial x^2} = f_\varepsilon(u), \ t > 0, \ x \in (0,1), \\ u(t,0) = 0, \ u(t,1) = 0, \\ u(0,x) = u_0(x), \end{cases} \tag{13}$$

where $\varepsilon \in (0,1]$ is a small parameter and f_ε satisfies

($\widetilde{A1}$) $f_\varepsilon \in C(\mathbb{R})$ and is non-decreasing;
($\widetilde{A2}$) $f_\varepsilon(0) = 0$;
($\widetilde{A3}$) $f_\varepsilon'(0) > 0$ exists, is finite, monotone in ε and $f_\varepsilon'(0) \to \infty$, as $\varepsilon \to 0^+$;
($\widetilde{A4}$) f_ε is strictly concave if $u > 0$ and strictly convex if $u < 0$;
($\widetilde{A5}$) $-1 < f_\varepsilon(s) < 1$, for all s, and

$$|f_\varepsilon(s) - H_0(s)| < \varepsilon, \quad \text{if } |s| > \varepsilon, \tag{14}$$

where

$$H_0(u) = \begin{cases} -1, & \text{if } \; u < 0, \\ [-1,1]\,, & \text{if } \; u = 0, \\ 1, & \text{if } \; u > 0, \end{cases}$$

is the Heaviside function.

Conditions (A1)-(A6) are satisfied with $p = 2$, so problem (13) is a particular case of (6).

Our aim now is to prove that for ε sufficiently small the multivalued semiflow G_ε generated by the weak solutions of problem (13) is dynamically gradient. Problem (13) is an approximation of the following problem, governed by a differential inclusion

$$\begin{cases} \dfrac{\partial u}{\partial t} - \dfrac{\partial^2 u}{\partial x^2} \in H_0(u), \text{ on } \Omega \times (0,T), \\ u|_{\partial\Omega} = 0, \\ u(0,x) = u_0(x). \end{cases} \tag{15}$$

We say that the function $u \in C([0,T], L^2(\Omega))$ is a strong solution of (15) if

1. $u(0) = u_0$;
2. $u(\cdot)$ is absolutely continuous on $(0,T)$ and $u(t) \in H^2(\Omega) \cap H_0^1(\Omega)$ for a.e. $t \in (0,T)$;
3. There exists a function $g(\cdot)$ such that $g(t) \in L^2(\Omega)$, a.e. on $(0,T)$, $g(t,x) \in H_0(u(t,x))$, for a.e. $(t,x) \in (0,T) \times \Omega$, and
$$\frac{du}{dt} - \frac{\partial^2 u}{\partial x^2} - g(t) = 0, \text{ a.e. } t \in (0,T).$$

In this case we put $\mathcal{R}$ as the set of all strong solutions such that the map g belongs to $L^2(0,T;L^2(\Omega))$. Conditions (K1)-(K4) are satisfied (cf. [9]) and the map $G : \mathbb{R}_+ \times L^2(\Omega) \to P(L^2(\Omega))$ defined by (1) is a strict multivalued semiflow possessing a global compact attractor $\mathcal{A}_0$ (cf. [24]) in $L^2(\Omega)$, which is connected (cf. [25]). The structure of this attractor is studied in [3]. It is shown that there exists an infinite (but countable) number of fixed points

$$v_0 = 0, v_1^+, v_1^-, \ldots, v_n^+, v_n^-, \ldots,$$

and that $\mathcal{A}_0$ consists of these fixed points and all bounded complete trajectories $\psi(\cdot)$, which always connect two fixed points, that is,

$$\begin{aligned} \psi(t) &\to z_1 \text{ as } t \to \infty, \\ \psi(t) &\to z_2 \text{ as } t \to -\infty, \end{aligned} \tag{16}$$

where $z_i = 0, z_i = v_n^+$ or $z_i = v_n^-$ for some $n \geq 1$. Moreover, if ψ is not a fixed point, then either $z_2 = 0$ and $z_1 = v_n^\pm$, for some $n \geq 1$, or $z_2 = v_k^\pm$, $z_1 = v_n^\pm$ with $k > n$.

We fix some $N_0 \in \mathbb{N}$. Denote

$$Z_{N_0} = \left(\cup_{k \geq N_0} \{v_k^\pm\}\right) \cup \{v_0\}$$

and define the sets

$$\Xi_k^0 = \{v_k^+, v_k^-\}, \quad 1 \leq k \leq N_0 - 1,$$

$$\Xi_{N_0}^0 = \left\{ \begin{array}{c} y : \exists \psi \in \mathbb{K} \text{ such that (16) holds with } z_j \in Z_{N_0}, \\ j = 1,2 \text{ and } y = \psi(t) \text{ for some } t \in \mathbb{R} \end{array} \right\},$$

where $\mathbb{K}$ stands for the set of all bounded complete trajectories. We note that set $\Xi_{N_0}^0$ contains the fixed points in Z_{N_0} and all bounded complete trajectories connecting them.

Remark 8. It is known [9] that the family $\mathcal{M} = \{\Xi_1^0, \ldots, \Xi_{N_0}^0\}$ is a disjoint family of isolated weakly invariant sets and that G_0 is dynamically gradient with respect to $\mathcal{M}$ in the sense of Remark 4. Hence, G_0 is dynamically gradient with respect to $\mathcal{M}$ in the sense of Definition 4.

Now our purpose is to adapt some lemmas from [3, p.2979] to problem (13). In view of Theorems 7 and 8 and the third condition on f_ε, there exists a sequence $\overline{\varepsilon}_k \to 0$, as $k \to \infty$, such that for every $\varepsilon \in (\overline{\varepsilon}_k, \overline{\varepsilon}_{k+1}]$ and any $k \geq 1$ problem (13) has exactly $2k+1$ fixed points $\{v_0^\varepsilon = 0, \{v_{\varepsilon,j}^{+}\}_{j=1}^{k}\}$ such that for each $1 \leq n \leq k$ $v_{\varepsilon,n}^{\pm}$ has $n+1$ zeros in $[0,1]$.

Let us consider a sequence $\{\varepsilon_m\}$ converging to zero.

Lemma 8. *Let $n \in \mathbb{N}$ be fixed. Then, $v_{\varepsilon_m,n}^{+}$ (resp. $v_{\varepsilon_m,n}^{-}$) do not converge to 0 in $H_0^1(0,1)$ as $\varepsilon_m \to 0$.*

Proof. Suppose that $v_{\varepsilon_m,n}^{+} \to 0$ in $H_0^1(0,1)$. Then $v_{\varepsilon_m,n}^{+} \to 0$ in $C([0,1])$. By Remark 7, $v_{\varepsilon_m,n}^{+}$ has a unique maximum in $a \in (0, x_1^{+})$ and by the properties of τ_+ described before $a = \frac{x_1^{+}}{2}$. We may assume that x_1^{+} does not converge to 0. Let $x_0(\varepsilon_m)$ be the first point where $v_{\varepsilon_m,n}^{+}(x_0) = \varepsilon_m$ or $x_0 = a$ if such a point does not exist. We claim that $x_0(\varepsilon_m) \to 0$, as $\varepsilon_m \to 0$. It is clear that $\partial^2 v_{\varepsilon_m,n}^{+}/\partial x^2 = -f_{\varepsilon_m}(v_{\varepsilon_m,n}^{+}) < 0$ in $(0, x_1^{+})$, and then

$$\frac{v_{\varepsilon_m,n}^{+}(x_0)}{x_0} x \leq v_{\varepsilon_m,n}^{+}(x) \leq \varepsilon_m, \quad \forall x \in [0, x_0], \tag{17}$$

by concavity. Hence, integrating first on (s,a) and then on $(0,x)$ with $x \leq x_0$, we have

$$\frac{d}{dx} v_{\varepsilon_m,n}^{+}(s) = \int_s^a f_{\varepsilon_m}(v_{\varepsilon_m,n}^{+}(\tau)) d\tau, \tag{18}$$

$$v_{\varepsilon_m,n}^{+}(x) = \int_0^x \int_{x_0}^a f_{\varepsilon_m}(v_{\varepsilon_m,n}^{+}(\tau)) d\tau ds + \int_0^x \int_s^{x_0} f_{\varepsilon_m}(v_{\varepsilon_m,n}^{+}(\tau)) d\tau ds.$$

Since $f_\varepsilon(u)$ is concave, we have that $f_\varepsilon(u)/u \geq f_\varepsilon(\varepsilon)/\varepsilon$, $\forall\ 0 < u \leq \varepsilon$. Moreover, by assumption $(\widetilde{A5})$ of f_ε we get $f_\varepsilon(u) \geq \frac{1-\varepsilon}{\varepsilon} u$, for all $0 < u \leq \varepsilon$. Hence, using (17) we have

$$v_{\varepsilon_m,n}^{+}(x) \geq \int_0^x \int_s^{x_0} \frac{1-\varepsilon_m}{\varepsilon_m} v_{\varepsilon_m,n}^{+}(\tau) d\tau ds \geq \frac{1-\varepsilon_m}{\varepsilon_m} \frac{v_{\varepsilon_m,n}^{+}(x_0)}{x_0} \int_0^x \int_s^{x_0} \tau d\tau ds.$$

Thus,

$$1 \geq \frac{1-\varepsilon_m}{\varepsilon_m} \left(\frac{x x_0}{2} - \frac{x^3}{6 x_0} \right),$$

so it follows that $x_0 \to 0$, as $\varepsilon_m \to 0$.

Let $\delta_1 < 0 < \delta_2$ be such that $x_0(\varepsilon_m) \leq \delta_1 < \delta_2 \leq a(\varepsilon_m)$. Since $v_{\varepsilon_m,n}^{+}(x) \geq \varepsilon_m$ $\forall x \in [x_0, a]$, if we intregate (18) over (δ_1, x) with $\delta_1 < x \leq \delta_2$, we have

$$v_{\varepsilon_m,n}^{+}(x) - v_{\varepsilon_m,n}^{+}(\delta_1) = \int_{\delta_1}^x \int_s^a f(v_{\varepsilon_m,n}^{+}(\tau)) d\tau ds \geq (1-\varepsilon_m) \int_{\delta_1}^x \int_s^a d\tau ds,$$

which implies a contradiction if $v_{\varepsilon_m,n}^{+} \to 0$ in $C([0,1])$.

The proof is similar for $v_{\varepsilon_m,n}^{-}$. □

Lemma 9. *$v_{\varepsilon_m,k}^{+}$ (resp. $v_{\varepsilon_m,k}^{-}$) converges to v_k^{+} (resp. v_k^{-}) in $H_0^1(\Omega)$ as $m \to \infty$ for any $k \geq 1$.*

Proof. It is easy to see that $v^+_{\varepsilon_m k}$ is bounded in $H^2(\Omega)\cap H^1_0(\Omega)$, so $v^+_{\varepsilon_m k} \to v$ strongly in $H^1_0(\Omega)$ and $C([0,1])$ up to a subsequence. The proof will be finished if we prove that $v = v^+_k$. We observe that since in such a case every subsequence would have the same limit, the whole sequence would converge to v^+_k

It is clear that the functions $g_{\varepsilon_m} = f_{\varepsilon_n}(v^+_{\varepsilon_m k})$ are bounded in $L^\infty(0,1)$.

Passing to a subsequence we can then assume that g_{ε_n} converges to some g weakly in $L^2(0,1)$. It is clear that $-(\partial^2 v/\partial x^2) = g$ and v is a fixed point if we prove the inclusion $g(x) \in H_0(v(x))$ for a.e. $x \in (0,1)$. By Masur's theorem [28, p.120] there exist $z_m \in V_m = conv(\cup_{k\geq m}^\infty g_{\varepsilon_k})$ such that $z_m \to g$, as $m \to \infty$, strongly in $L^2(0,1)$. Taking a subsequence we have $z_m(x) \to g(x)$, a.e. in $(0,1)$. Since $z_m \in V_m$, we get $z_m = \sum_{i=1}^{N_m} \lambda_i g_{\varepsilon_{k_i}}$, where $\lambda_i \in [0,1], \sum_{i=1}^{N_m} \lambda_i = 1$ and $k_i \geq m$, for all i.

Now (14) implies that $|g_{\varepsilon_k}(x) - H_0(v(x))| \to 0$, as $k \to \infty$, for a.e. x. Indeed, if $v(x) = 0$, then $g_{\varepsilon_k}(x) \in [-1,1] = H_0(v(x))$. If $v(x) > 0$, then $|g_{\varepsilon_k}(x) - H_0(v(x))| = |f_{\varepsilon_k}(v_{\varepsilon_k}(x)) - 1| \to 0$, as $k \to \infty$. If $v(x) < 0$, we apply a similar argument.

Thus, for any $\delta > 0$ and a.e. x there exists $m(x,\delta)$ such that $g_{\varepsilon_k}(x) \subset [a(x) - \delta, b(x) + \delta]$, for all $k \geq m$, where $[a(x), b(x)] = H_0(v(x))$. Hence, $z_m(x) \subset [a(x) - \delta, b(x) + \delta]$, as well. Passing to the limit we obtain $g(x) \in [a(x), b(x)]$, a.e. on $(0,1)$.

To conclude the proof, we have to prove that $v = v^+_k$. By Lemma 8 $v \neq 0$. Hence, as $v^+_{\varepsilon_m k}(x) > 0$ for all $x \in (0, x^+_1(\varepsilon_m))$, $v = v^+_n$ for some $n \in \mathbb{N}$. Since v^+_n has $n+1$ zeros, the convergence $v^+_{\varepsilon_m k} \to v^+_n$ implies that $v^+_{\varepsilon_m k}$ has $n+1$ zeros for $m \geq N$. But $v^+_{\varepsilon_m k}$ possesses $k+1$ zeros. Thus, $k = n$.

For the sequence $v^-_{\varepsilon_m k}$ the proof is analogous. □

Lemma 10. *Let $\varepsilon_m \to 0$, $k_m \to \infty$ as $m \to \infty$. Then $v^+_{\varepsilon_m,k_m}$ (resp. $v^-_{\varepsilon_m,k_m}$) converges to 0 as $m \to \infty$.*

Proof. In the same way as in the proof of Lemma 9 we obtain that up to a subsequence $v^+_{\varepsilon_m,k_m} \to v$ in $H^1_0(\Omega)$ and $C([0,1])$, where v is a fixed point of problem (15). We will prove that $v = 0$ by contradiction. If not, then $v = v^\pm_n$ for some $n \in \mathbb{N}$. However, since $v^\pm_n$ has exactly $n+1$ zeros and $v^+_{\varepsilon_m,k_m} \to v$ in $C([0,1])$, we have that $v^+_{\varepsilon_m,k_m}$ has $n+1$ zeros for any $m \geq M$ with M big enough. This contradicts the fact that $v^+_{\varepsilon_m,k_m}$ possesses $k_m + 1$ zeros and $k_m \to \infty$. As the limit is 0 for every converging subsequence, the whole sequence converges to 0.

For the sequence $v^-_{\varepsilon_m k}$ the proof is analogous. □

Once we have described the preliminary properties, we are now ready to check that (13) satisfies the conditions given in Theorem 2 for certain families $\mathcal{M}_\varepsilon$. We recall that [26, Theorem 10] guarantees the existence of the global compact invariant attractors $\mathcal{A}_\varepsilon$, where each $\mathcal{A}_\varepsilon$ is the union of all bounded complete trajectories.

Let us check assumptions (H1)-(H5) of Theorem 2.

As we have seen before, condition (H2) follows from Remark 8. Therefore, we prove now condition (H1).

Multiplying the equation in (13) by u, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u\|^2_{L^2} + \|u\|^2_{H^1_0} &\leq \int_\Omega |u| dx \\ &\leq \frac{1}{2}\|u\|^2_{H^1_0} + C, \qquad (19)\end{aligned}$$

where we have used Poincaré's inequality. Denoting λ_1 the first eigenvalue of the operator $-\Delta$ in $H_0^1(\Omega)$, we have

$$\frac{d}{dt}\|u\|_{L^2}^2 \leq -\lambda_1\|u\|_{L^2}^2 + K.$$

Gronwall's lemma gives

$$\|u(t)\|_{L^2}^2 \leq e^{-\lambda_1 t}\|u(0)\|_{L^2}^2 + \frac{K}{\lambda_1}, \quad t \geq 0. \tag{20}$$

Integrating (19) over $(t, t+r)$ with $r > 0$ we have

$$\|u(t+r)\|_{L^2}^2 + \int_t^{t+r}\|u\|_{H_0^1}^2 ds \leq \|u(t)\|_{L^2}^2 + rK$$

Then by (20),

$$\int_t^{t+r}\|u\|_{H_0^1}^2 ds \leq \|u(0)\|_{L^2}^2 e^{-\lambda_1 t} + \left(\frac{1}{\lambda_1} + r\right)K. \tag{21}$$

On the other hand, multiplying (13) by $-\Delta u$ and using Young's inequality we obtain

$$\frac{d}{dt}\|u\|_{H_0^1}^2 + 2\|\Delta u\|_{L^2}^2 \leq \|f_\varepsilon(u)\|_{L^2}^2 + \|\Delta u\|_{L^2}^2 \tag{22}$$

Since $f_\varepsilon(u(\cdot)) \in L^2(0,T;L^2(\Omega)), \forall T > 0$, we obtain by [5, p.189] that

$$u \in L^\infty(\eta, T; H_0^1(\Omega)),$$

$$\frac{du}{dt} \in L^2(\eta, T; L^2(\Omega)), \quad \forall\, 0 < \eta < T.$$

This regularity guarantees that the equality

$$\frac{1}{2}\frac{d}{dt}\|u\|_{H_0^1}^2 = \langle \frac{du}{dt}, -\Delta u\rangle, \text{ for a.e. } t, \tag{23}$$

is correct [21, p.102]. Then

$$\frac{d}{dt}\|u\|_{H_0^1}^2 \leq \overline{K} + \|u\|_{H_0^1}^2.$$

We apply the uniform Gronwall lemma [22, p. 91] with $y(s) = \|u(s)\|_{H_0^1}^2$, $g(s) = 1$ and $w(s) = \overline{K}$. Also, using (21) we obtain

$$\|u(t+r)\|_{H_0^1}^2 \leq \left(\frac{\|u(0)\|_{L^2}^2 e^{-\lambda_1 t} + (\frac{1}{\lambda_1} + r)K}{r} + \overline{K}r\right)e^r \tag{24}$$

It follows from (20) that $\|y\|_{L^2} \leq \frac{K}{\lambda_1}$ for any $y \in \mathcal{A}_\varepsilon$, $0 < \varepsilon \leq 1$. Hence, $\cup_{0<\varepsilon\leq 1}\mathcal{A}_\varepsilon$ is bounded in $L^2(\Omega)$. Since $\mathcal{A}_\varepsilon \subset G_\varepsilon(t, \mathcal{A}_\varepsilon)$ for any $t \geq 0$, for any $y \in \mathcal{A}_\varepsilon$ there exists $z \in \mathcal{A}_\varepsilon$ such that $y \in G_\varepsilon(1, z)$. Then using (24) with $r = 1$ and $t = 0$ we obtain that

$$\|y\|_{H_0^1}^2 \leq \left(\|z\|_{L^2}^2 + \left(\frac{1}{\lambda_1} + 1\right)K + \overline{K}\right)e,$$

so $\cup_{0<\varepsilon\leq 1}\mathcal{A}_\varepsilon$ is bounded in $H_0^1(\Omega)$. The compact embedding $H_0^1(\Omega) \subset L^2(\Omega)$ implies that $\cup_{0<\varepsilon\leq 1}\mathcal{A}_\varepsilon$ is relatively compact in $L^2(\Omega)$. As the global attractor A_0 of the differential inclusion (15) is compact, the set $\overline{\cup_{0\leq\varepsilon\leq 1}\mathcal{A}_\varepsilon}$ is compact in $L^2(\Omega)$.

In order to establish that (13) satisfies the rest of conditions given in Theorem 2, we need to proof two previous results related to the convergence of solutions of the approximations and the connections between fixed points.

Theorem 9. *If $u_{\varepsilon_n 0} \to u_0$ in $L^2(\Omega)$ as $\varepsilon_n \to 0$, then for any sequence of solutions of (13) $u_{\varepsilon_n}(\cdot)$ with $u_{\varepsilon_n}(0) = u_{\varepsilon_n 0}$ there exists a subsequence of ε_n such that u_{ε_n} converges to some strong solution u of (15) in the space $C([0,T], L^2(\Omega))$, for any $T > 0$.*

Proof. We define $g_n(t) = f_{\varepsilon_n}(u_{\varepsilon_n}(t))$ and $u_n(t) = u_{\varepsilon_n}(t)$. From (20) we have that $\|u_n(t)\|_{L^2} \leq C_0$, for all $t \geq 0$, so that $\|g_n(t)\|_{L^2} \leq C_1$, for a.e. $t \geq 0$. Hence, there exists a subsequence such that $u_n \to u$ weakly in $L^2(0,T;L^2(\Omega))$. It follows from (22) and $\|g_n(t)\|_{L^2} \leq C_1$ that $\int_r^T \|\Delta u\|^2_{L^2} ds \leq C_1^2(T-r) + \|u_n(r)\|^2_{H_0^1}$. Using (24) we obtain that $\int_r^T \|\Delta u_n\|^2_{L^2} ds \leq C(r)$. Hence, $\frac{du_n}{dt}$ is bounded in $L^2(r,T;L^2(\Omega))$ for any $0 < r < T$, so passing to a subsequence $\frac{du_n}{dt} \to \frac{du}{dt}$ weakly in $L^2(r,T;L^2(\Omega))$.

Moreover, Ascoli-Arzelà theorem implies that for any fixed $r > 0$ we have $u_n \to u$ in $C([r,T], L^2(\Omega))$ and u is absolutely continuous on $[r,T]$.

Also, g_n converges to some $g \in L^\infty(0,T;L^2(\Omega))$ weakly star in $L^\infty(0,T;L^2(\Omega))$ and weakly in $L^2(0,T;L^2(\Omega))$. On the other hand, since $-\Delta u_n = -\frac{du_n}{dt} + g_n$, $-\Delta u_n$ converges to $l(t) = -(\frac{du}{dt}) + g$ weakly in $L^2(r,T;L^2(\Omega))$. Hence, we find at once that u satisfies

$$\frac{du}{dt} - \Delta u(t) = g(t), \text{ a.e. on } (0,T).$$

We need to prove that $u(\cdot)$ is a strong solution of (15). Now, we show that $g(t) \in H_0(u(t))$, a.e. in $(0,T)$. For this, we shall prove first that for a.e. $x \in \Omega$ and $s \in (0,T)$

$$|g_n(s,x) - H_0(u(s,x))| \to 0, \text{ as } n \to \infty.$$

Indeed, if $u(s,x) = 0$, then $g_n(s,x) = f_{\varepsilon_n}(u_n(s,x)) = 0 \in [-1,1] = H_0(u(s,x))$, for all n, so that the result is evident. If $u(s,x) < 0$, then

$$|g_n(s,x) - H_0(u(s,x))| = |f_{\varepsilon_n}(u_n(s,x)) + 1| \to 0, \text{ as } n \to \infty.$$

Finally, if $u(s,x) > 0$, then

$$|g_n(s,x) - f_0(u(s,x))| = |f_{\varepsilon_n}(u_n(s,x)) - 1| \to 0, \text{ as } n \to \infty.$$

Now, by [23, Proposition 1.1] we have that for a.e. $t \in (0,T)$

$$g(t) \in \bigcap_{n \geq 0} \overline{co} \bigcup_{k \geq n} g_k(t).$$

Then $g(t) = \lim_{n\to\infty} y_n(t)$ strongly in $L^2(\Omega)$, where

$$y_n(t) = \sum_{i=1}^{M} \lambda_i g_{k_i}(t), \sum_{i=1}^{M} \lambda_i = 1, k_i \geq n.$$

We note that for any $t \in [0,T]$ and a.e. $x \in \Omega$ we can find $n(\varepsilon, x, t)$ such that if $k \geq n$, then $|g_k(t,x) - H_0(u(t,x))| \leq \varepsilon$. Therefore,

$$|y_n(t,x) - H_0(u(t,x))| \leq \sum_{i=1}^{M} \lambda_i |g_{k_i}(t,x) - H_0(u(t,x))| \leq \varepsilon.$$

Hence, since we can assume that for a.e. $(t,x) \in (0,T) \times \Omega, y_n(t,x) \to g(t,x)$, it follows that $g(t,x) \in H_0(u(t,x))$.

It remains to check that u is continuous as $t \to 0^+$. Let $\hat{u}$ be the unique solution of

$$\begin{cases} \dfrac{du}{dt} - \Delta u = 0, \\ u|_{\partial\Omega} = 0, \\ u(0) = u_0, \end{cases}$$

and let $v_n(t) = u_n(t) - \hat{u}(t)$. Multiplying by v_n the equation

$$\frac{dv_n}{dt} - \Delta v_n = f_{\varepsilon_n}(u_n),$$

we obtain

$$\frac{1}{2}\frac{d}{dt}\|v_n\|_{L^2}^2 + \|v_n\|_{H_0^1}^2 \le (f_{\varepsilon_n}(u_n(t)), v_n) \tag{25}$$

$$\le \frac{1}{2}\|f_{\varepsilon_n}(u_n)\|_{L^2}^2 + \frac{1}{2}\|v_n\|_{L^2}^2, \tag{26}$$

so that

$$\|v_n(t)\|_{L^2}^2 \le \|v_n(0)\|_{L^2}^2 + Kt.$$

Hence, $\|u(t) - \hat{u}(t)\|_{L^2}^2 = \lim_{n\to\infty} \|v_n(t)\|_{L^2}^2 \le Kt$, for $t > 0$, and

$$\|u(t) - u_0\|_{L^2} \le \|u(t) - \hat{u}(t)\|_{L^2} + \|\hat{u}(t) - u_0\|_{L^2} < \delta,$$

as soon as $t < \varepsilon(\delta)$. Therefore, $u(\cdot)$ is a strong solution.

Finally, if $t_n \to 0$, then

$$\begin{aligned} \|u_n(t_n) - u_0\|_{L^2} &\le \|v_n(t_n)\|_{L^2} + \|\widehat{u}(t_n) - u_0\|_{L^2} \\ &\le \sqrt{\|v_n(0)\|_{L^2}^2 + Kt_n} + \|\widehat{u}(t_n) - u_0\|_{L^2} \to 0. \end{aligned}$$

Hence, $u_n \to u$ in $C([0,T], L^2(\Omega))$. By a diagonal argument we obtain that the result is true for every $T > 0$. □

As a consequence of the last theorem, condition (H4) follows.

Remark 9. Let be $u_{\varepsilon_n}(\cdot)$ a bounded complete trajectory of (13). Fix $T > 0$. Since $\bigcup_{0<\varepsilon\le\varepsilon_0} \mathcal{A}_\varepsilon$ is precompact in $L^2(\Omega)$, $u_{\varepsilon_n}(-T) \to y$ in L^2 up to a subsequence. Theorem 9 implies that u_{ε_n} converges in $C([0,T], L^2(\Omega))$ to some solution u of (15). If we choose successive subsequences for $-2T, -3T, \dots$, and apply the standard diagonal procedure, we obtain that a subsequence u_{ε_n} converges to a complete trajectory u of (15) in $C([-T,T], L^2(\Omega))$ for any $T > 0$. Since $\cup_{0<\varepsilon\le 1}\mathcal{A}_\varepsilon$ is bounded in $L^2(\Omega)$ (in fact in $H_0^1(\Omega)$), it is clear that u is a bounded complete trajectory of problem (15).

Now, we need to prove a previous lemma to obtain the convergence of solutions of the approximations in the space $C([0,T], H_0^1)$.

Lemma 11. *Any sequence $\xi_n \in A_{\varepsilon_n}$ with $\varepsilon_n \to 0$ is relatively compact in $H_0^1(\Omega)$.*

Proof. There exists a bounded complete trajectory ψ_{ε_n} of (13) with $\psi_{\varepsilon_n}(0) = \xi_n$. Denote $u_n(\cdot) = \psi_{\varepsilon_n}(-T + \cdot)$ and choose some $T > 0$. Then $\xi_n = u_n(T)$, $u_n(0) = \psi_{\varepsilon_n}(t_0 - T)$. In view of Remark 9 up to a subsequence $u_n \to u$ in $C([0,T], L^2(\Omega))$, where u is a strong solution of (15). On top of that, by (24) and the argument in the proof of Theorem 9 we obtain that for $r>0$,

$$u_n \to u \text{ weakly star in } L^\infty(r,T;H_0^1(\Omega)),$$
$$\frac{du_n}{dt} \to \frac{du}{dt} \text{ weakly in } L^2(r,T;L^2(\Omega)),$$
$$u_n \to u \text{ weakly in } L^2(r,T;H^2(\Omega)).$$

Therefore, by the Compactness Theorem [17, p.58] we have

$$u_n \to u \text{ strongly in } L^2(r,T,H_0^1(\Omega)),$$
$$u_n(t) \to u(t) \text{ in } H_0^1(\Omega) \text{ for a.a. } t\in(r,T).$$

In addition, by standard results [21, p.102] we have that $u_n,u\in C([r,T],H_0^1(\Omega))$.

Multiplying (13) by $\frac{du_n}{dt}$ and using (23), we obtain

$$\left\|\frac{du_n}{dt}\right\|_{L^2}^2+\frac{d}{dt}\|u_n\|_{H_0^1}^2 \le \|f_\varepsilon(u_n)\|_{L^2}^2.$$

Thus,

$$\|u_n(t)\|_{H_0^1}^2 \le \|u_n(s)\|_{H_0^1}^2 + C(t-s),\ C>0,\ t\ge s\ge r.$$

The same inequality is valid for the limit function $u(\cdot)$. Hence, the functions $J_n(t)=\|u_n(t)\|_{H_0^1}^2-Ct$, $J(t)=\|u(t)\|_{H_0^1}^2-Ct$, are continuous and non-increasing in $[r,T]$. Moreover, $J_n(t)\to J(t)$ for a.e. $t\in(r,T)$. Take $r<t_m<T$ such that $t_m\to T$ and $J_n(t_m)\to J(t_m)$ for all m. Then

$$J_n(T)-J(T)\le J_n(t_m)-J(T)\le |J_n(t_m)-J(t_m)|+|J(t_m)-J(T)|.$$

For any $\varepsilon>0$ there exist $m(\varepsilon)$ and $N(\varepsilon)$ such that $J_n(T)-J(T)\le\varepsilon$ if $n\ge N$. Then $\limsup J_n(T)\le J(T)$, so $\limsup\|u_n(T)\|_{H_0^1}^2\le\|u(T)\|_{H_0^1}^2$. As $u_n(T)\to u(T)$ weakly in H_0^1 implies $\liminf\|u_n(T)\|_{H_0^1}^2\ge\|u(T)\|_{H_0^1}^2$, we obtain

$$\|u_n(T)\|_{H_0^1}^2\to\|u(T)\|_{H_0^1}^2,$$

so that $u_n(T)\to u(T)$ strongly in $H_0^1(\Omega)$. Hence, the result follows. □

Corollary 2. *If $u_{\varepsilon 0}\to u_0$ in $L^2(\Omega)$, where $u_{\varepsilon 0}\in\mathcal{A}_\varepsilon$, $u_0\in\mathcal{A}_0$, then for any $T>0$ there exists a subsequence ε_n such that u_{ε_n} converges to some strong solution u of (15) in $C([0,T],H_0^1(\Omega))$.*

Proof. We know from Theorem 9 that there exists a subsequence such that u_{ε_n} converges to some strong solution u of (15) in $C([0,T],L^2(\Omega))$. Then the statement follows from the invariance of $\mathcal{A}_\varepsilon$ and Lemma 11. □

Remark 10. Let $u_{\varepsilon_n}(\cdot)$ be a bounded complete trajectory of (13). Fix $T>0$. By Lemma 11 $u_{\varepsilon_n}(-T)\to y$ in $H_0^1(\Omega)$ up to a subsequence. Corollary 2 implies then that u_{ε_n} converges in $C([0,T],H_0^1(\Omega))$ to some solution u of (15). If we choose successive subsequences for $-2T,-3T\ldots$ and apply the standard diagonal procedure we obtain that a subsequence u_{ε_n} converges to a complete trajectory u of (15) in $C([-T,T],H_0^1(\Omega))$ for any $T>0$. By Remark 9 this trajectory is bounded.

Lemma 12. *$dist_{H_0^1}(\mathcal{A}_\varepsilon,\mathcal{A}_0)\to 0$, as $\varepsilon\to 0$.*

Proof. By contradiction let there exist $\delta > 0$ and a sequence $y_{\varepsilon_n} \in \mathcal{A}_{\varepsilon_n}$ such that

$$dist_{H_0^1}(y_{\varepsilon_n}, \mathcal{A}_0) > \delta.$$

Hence, as $y_{\varepsilon_n} = u_{\varepsilon_n}(0)$, where u_{ε_n} is a bounded complete trajectory of problem (13), using Remark 10 we obtain that up to a sequence u_{ε_n} converges to a bounded complete trajectory u of the problem (15) in the spaces $C([-T,T], H_0^1(\Omega))$ for every $T > 0$. Thus, $u(t) \in \mathcal{A}_0$ for any $t \in \mathbb{R}$. We infer then that

$$y_{\varepsilon_n} = u_{\varepsilon_n}(0) \to u(0) \in \mathcal{A}_0,$$

which is a contradiction. □

We choose some $\delta > 0$ such that

$$\mathcal{O}_\delta(\Xi_i^0) \cap \mathcal{O}_\delta(\Xi_j^0) = \emptyset \text{ if } i \neq j$$

and Ξ_i^0 are maximal weakly invariant.

For problem (13) let us define the sets

$$M_i^\varepsilon = \{v_{\varepsilon,i}^+, v_{\varepsilon,i}^-\} \text{ for } 1 \leq i < N_0,$$

$$Z_{N_0}^\varepsilon = \left(\cup_{k \geq N_0} \{v_{\varepsilon,k}^\pm\}\right) \cup \{0\},$$

$$M_{N_0}^\varepsilon = \left\{ \begin{array}{c} y : \exists \psi \in \mathbb{K}^\varepsilon \text{ such that (16) holds with } z_j \in Z_{N_0}^\varepsilon, \\ j = 1,2 \text{ and } y = \psi(t) \text{ for some } t \in \mathbb{R} \end{array} \right\}$$

where $\mathbb{K}^\varepsilon$ is the set of all bounded complete trajectories of (13).

In view of Lemma 9 we have

$$dist_{H_0^1}(M_i^\varepsilon, \Xi_i^0) \to 0, \text{ as } \varepsilon \to 0, \ \ 1 \leq i < N_0$$

Lemma 13. *$dist_{H_0^1}(M_{N_0}^\varepsilon, \Xi_{N_0}^0) \to 0$, as $\varepsilon \to 0$.*

Proof. Suppose the opposite, that is, there exists $\delta > 0$ and a sequence $y_{\varepsilon_k} \in M_0^{\varepsilon_k}$ such that

$$dist_{H_0^1}(y_{\varepsilon_k}, \Xi_{N_0}^0) > \delta \text{ for all } k. \tag{27}$$

Let ξ_{ε_k} be a sequence of bounded complete trajectories of problem (13) such that $\xi_{\varepsilon_k}(0) = y_{\varepsilon_k}$ and

$$\xi_{\varepsilon_k}(t) \to z_{-1}^k \text{ as } t \to -\infty,$$

$$\xi_{\varepsilon_k}(t) \to z_0^k \text{ as } t \to \infty,$$

where $z_{-1}^k, z_0^k \in Z_{N_0}^{\varepsilon_k}$. By Lemmas 9 and 10, passing to a subsequence we have that

$$z_i^k \to z_i \in Z_{N_0}, i = -1, 0.$$

By Remark 10 we obtain that up to a subsequence ξ_{ε_k} converges to a complete trajectory ψ_0 of problem (15) in the spaces $C([-T,T], H_0^1(\Omega))$ for every $T > 0$, so $y_{\varepsilon_k} \to \psi_0(0)$ in $H_0^1(\Omega)$. Thus, either ψ_0 is equal to a fixed point $\overline{z}_0 \neq 0$ or there exist two fixed points of problem (15), denoted by $\overline{z}_{-1}, \overline{z}_0$ such that

$$E(\overline{z}_{-1}) > E(\overline{z}_0),$$

$$\psi_0(t) \to \overline{z}_{-1} \text{ as } t \to -\infty,$$

$$\psi_0(t) \to \overline{z}_0 \text{ as } t \to \infty.$$

If $\overline{z}_0 = z_0$, then $\overline{z}_{-1}, \overline{z}_0 \in Z_{N_0}$, which means that $\psi_0(0) \in \Xi_{N_0}^0$. This would imply a contradiction with (27). Therefore, we assume that $\overline{z}_0 \neq z_0$. Also, it is clear that $\overline{z}_0 = v_m^\pm \neq 0$, for some $m \in \mathbb{N}$.

Let $r_0 > 0$ be such that $\mathcal{O}_{r_0}(\overline{z}_0) \cap \mathcal{O}_{r_0}(z_0) \neq \emptyset$ and $\mathcal{O}_{2r_0}(\overline{z}_0)$ does not contain any other fixed point of problem (15). The previous convergences imply that for each $r \leq r_0$ there exist a moment of time t_r and k_r such that $\xi_{\varepsilon_k}(t_r) \in \mathcal{O}_r(\overline{z}_0)$ for all $k \geq k_r$. On the other hand, since $\xi_{\varepsilon_k}(t) \to z_0^k$, as $t \to \infty$, and $z_0^k \to z_0$, there exists $t'_r > t_r$ such that

$$\xi_{\varepsilon_{k_r}}(t) \in \mathcal{O}_{r_0}(\overline{z}_0) \text{ for all } t \in [t_r, t'_r),$$

$$\|\xi_{\varepsilon_{k_r}}(t'_r) - \overline{z}_0\|_{L^2} = r_0.$$

Let us consider two cases: 1) $t'_r - t_r \to \infty; 2)|t'_r - t_r| \leq C$. We begin with the first case. We define the sequence of bounded complete trajectories of problem (13) given by

$$\xi^1_{k_r}(t) = \xi_{\varepsilon_{k_r}}(t + t'_r).$$

By Remark 10 we can extract a subsequence of this sequence converging to a bounded complete trajectory ψ_1 of problem (15). Since $t'_r - t_r \to \infty$, we obtain that $\psi_1(t) \in \mathcal{O}_{r_0}(\overline{z}_0)$ for all $t \leq 0$. Since $\mathcal{O}_{2r_0}(\overline{z}_0)$ does not contain any other fixed point of problem (15), it follows that $\psi_1(t) \to \overline{z}_0$ as $t \to -\infty$. But $\|\psi_1(0) - \overline{z}_0\|_{L^2} = r_0$, so ψ_1 is not a fixed point. Therefore, $\psi_1(t) \to \overline{z}_1$ as $t \to \infty$, where $\overline{z}_1$ is a fixed point such that $E(\overline{z}_1) < E(\overline{z}_0)$.

In the second case we define the sequence

$$\xi^1_{k_r}(t) = \xi_{\varepsilon_{k_r}}(t + t_r).$$

Passing to a subsequence we have that

$$\xi^1_{k_r}(0) \to \overline{z}_0,$$
$$t'_r - t_r \to t'.$$

As $\xi^1_{k_r}$ converges to a solution ξ^1 of problem (15) uniformly in bounded subsets from $[0, \infty)$ such that $\xi^1(0) = \overline{z}_0$, $\xi^1_{k_r}(t'_r - t_r) \to \xi^1(t')$, so that $\|\xi^1(t') - \overline{z}_0\|_{L^2} = r_0$. We put

$$\psi_1(t) = \begin{cases} \overline{z}_0 \text{ if } t \leq 0, \\ \xi^1(t) \text{ if } t \geq 0. \end{cases}$$

Then ψ_1 is a bounded complete trajectory of problem (15) such that $\psi_1(t) \to \overline{z}_1$ as $t \to \infty$, where $\overline{z}_1$ is a fixed point satisfying $E(\overline{z}_1) < E(\overline{z}_0)$.

Now, if $\overline{z}_1 = z_0$, then we have the chain of connections

$$\psi_0(t) \to \overline{z}_{-1} \text{ as } t \to -\infty, \psi_0(t) \to \overline{z}_0 \text{ as } t \to +\infty,$$

$$\psi_1(t) \to \overline{z}_0 \text{ as } t \to -\infty, \psi_1(t) \to \overline{z}_1 \text{ as } t \to +\infty,$$

which implies that $\overline{z}_{-1}, \overline{z}_0, \overline{z}_1 \in Z_n$, an then $\psi_0(0) \in \Xi^0_n$. This would imply a contradiction with (27).

However, if $\overline{z}_1 \neq \overline{z}_0$, then we proceed in the same way and obtain a new connection from the point $\overline{z}_1$ to another fixed point with less energy. Since the number of fixed points with energy less than or equal to $E(\overline{z}_0)$ is finite, we will finally obtain a chain of connections of the form

$$\begin{aligned}
&\psi_0(t) \to \overline{z}_{-1} \text{ as } t \to -\infty,\ \psi_0(t) \to \overline{z}_0 \text{ as } t \to +\infty, \\
&\psi_1(t) \to \overline{z}_0 \text{ as } t \to -\infty,\ \psi_1(t) \to \overline{z}_1 \text{ as } t \to +\infty, \\
&\qquad \vdots \\
&\psi_n(t) \to \overline{z}_{m-1} \text{ as } t \to -\infty,\ \psi_n(t) \to \overline{z}_m = z_0 \text{ as } t \to +\infty.
\end{aligned}$$

And again, this implies a contradiction with (27). $\square$

These convergences imply the existence of ε_0 such that if $\varepsilon \leq \varepsilon_0$, then

$$M_i^\varepsilon \subset \mathcal{O}_\delta(\Xi_i^0) \text{ for any } 1 \leq i \leq N_0.$$

Further, let

$$\Xi_i^\varepsilon = \left\{ \begin{array}{c} y : \exists \psi \in \mathbb{K}^\varepsilon \text{ such that } \psi(0) = y \\ \text{and } \psi(t) \in \mathcal{O}_\delta(\Xi_i^0) \text{ for all } t \in \mathbb{R} \end{array} \right\}.$$

These sets are clearly maximal weakly invariant for G_ε in $\mathcal{O}_\delta(\Xi_i^0)$, so condition (H5) is satisfied for $V_i = \mathcal{O}_\delta(\Xi_i^0)$. As a consequence of Lemmas 9, 13, Remark 9 and the definition of δ we have

$$dist_{L^2}(\Xi_i^\varepsilon, \Xi_i^0) \to 0, \text{ as } \varepsilon \to 0, \text{ for } 1 \leq i \leq N_0.$$

Therefore, condition (H3) is satisfied.

We also get by Remark 10 and the definition of δ that

$$dist_{H_0^1}(\Xi_i^\varepsilon, \Xi_i^0) \to 0, \text{ as } \varepsilon \to 0, \text{ for } 1 \leq i \leq N_0.$$

Moreover, $\mathcal{M}^\varepsilon = \{\Xi_1^\varepsilon, \ldots, \Xi_{N_0}^\varepsilon\}$ is a disjoint family of isolated weakly invariant sets.

Applying Theorem 2 we obtain the following result.

Theorem 10. *There exists $\varepsilon_1 > 0$ such that for all $0 < \varepsilon \leq \varepsilon_1$ the multivalued semiflow G_ε is dynamically gradient with respect to the family $\mathcal{M}^\varepsilon$.*

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Existence and characterization of attractors for a nonlocal reaction-diffusion equation with an energy functional

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Dedicated to the memory of Russell Johnson

Abstract

In this paper we study a nonlocal reaction-diffusion equation in which the diffusion depends on the gradient of the solution.

Firstly, we prove the existence and uniqueness of regular and strong solutions. Secondly, we obtain the existence of global attractors in both situations under rather weak assumptions by defining a multivalued semiflow (which is a semigroup in the particular situation when uniqueness of the Cauchy problem is satisfied). Thirdly, we characterize the attractor either as the unstable manifold of the set of stationary points or as the stable one when we consider solutions only in the set of bounded complete trajectories.

Keywords: reaction-diffusion equations, nonlocal equations, global attractors, multivalued dynamical systems, structure of the attractor

AMS Subject Classification (2010): 35B40, 35B41, 35B51, 35K55, 35K57

1 Introduction

In real applications there might exist several nonlocal effects that influence the evolution of a system. For instance, usually we do not have enough information about the systems under study and its features at every point. In reality, the measurements are not made pointwise but through some local average. This is just one possible reason of introducing nonlocal terms in models. Actually, during the last decades many mathematicians have been studying nonlocal problems motivated by its various applications in physics, biology or population dynamics [13, 14, 15, 16, 17, 27].

For instance, let consider the problem of finding a function $u(t,x)$ such that

$$\begin{cases} u_t - a(\int_\Omega u(t,x)dx)\Delta u = g(t,u), \text{ in } \Omega\times(0,\infty), \\ u = 0 \quad \text{on } \partial\Omega\times(0,\infty), \\ u(0) = u_0 \quad \text{in } \Omega. \end{cases} \tag{1}$$

Here Ω is a bounded open subset in $\mathbb{R}^n$, $n \geq 1$, with smooth boundary and a is some function from $\mathbb{R}$ to $(0,+\infty)$. In such equation u could describe the density of a population subject to spreading. The diffusion coefficient a is then supposed to depend on the entire population in the domain rather than on the local density.

A wide literature with significant results about (1) have been developed during the last few decades (see for example [14, 17, 27]). However, it is possible to distinguish two basic cases of the following more general equation

$$\begin{cases} u_t - a(u)\Delta u = g(t,u), \quad t>0,\ x\in\Omega, \\ u = 0,\ \ on\ \partial\Omega\times(0,\infty), \\ u(0,x) = u_0(x) \quad x\in\Omega. \end{cases}$$

Some authors consider a depending on a linear functional $l(u)$, i.e.,

$$a(u) = a(l(u))$$

with

$$l(u) = \int_{\Omega} \Phi(x) u(x,t) dx,$$

where $\Phi(x)$ is a given function in $L^2(\Omega)$. For $g(t,u) = f(t)$ the existence and uniqueness of solutions and their asymptotic behavior are studied for example in [15, 16, 18, 32]. For $g(t,u) = f(u) + h(t)$ the existence, uniqueness and asymptotic behaviour of solutions is studied in [1, 6, 8, 9]. Moreover, the authors prove the existence of pullback attractors in $L^2(\Omega)$ and $H_0^1(\Omega)$. Extensions in this direction for equations governed by the p-laplacian operator instead of the laplacian operator Δ are given in [7, 10], whereas nonclassical diffusion equations are considered in [29].

On the other hand, it is possible to consider a function a such that $a(u) = a(\|u\|^2_{H_0^1})$. The existence and uniqueness of solutions of the following problem

$$\begin{cases} u_t - a(\|u\|^2_{H_0^1})\Delta u = f, \quad t > 0,\ x \in \Omega, \\ u = 0, \text{ on } \partial\Omega \times (0,\infty), \\ u(0,x) = u_0(x) \quad x \in \Omega. \end{cases}$$

is proved in [32, 19], where $f \in L^2(\Omega)$, $u_0 \in H_0^1(\Omega)$ and $a = a(s)$ is a continuous function such that $0 < m \leq a(s) \leq M$.

By this way, in this paper the following problem is considered

$$\begin{cases} u_t - a(\|u\|^2_{H_0^1})\Delta u = f(u) + h(t), \text{ in } \Omega \times (0,\infty), \\ u = 0 \quad \text{on } \partial\Omega \times (0,\infty), \\ u(0,x) = u_0(x) \quad \text{in } \Omega, \end{cases} \tag{2}$$

where $h \in L^2(0,T;L^2(\Omega))$, for all $T > 0$, $a : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function such that $a(s) \geq m > 0$ and f is a continuous function satisfying standard dissipative and growth conditions (see (7) below).

The aim of this paper is three-fold. First, we will prove the existence of solutions for problem (2) under different assumptions on the nonlinear function f. Second, we will obtain the existence of attractors for the semiflows generated by either regular or strong solutions in the autonomous case, that is, when h does not depend on time. Third, we establish that the global attractor can be characterized by the unstable manifold of the set of stationary points. It is important to notice that the proof of this last fact requires the existence of a Lyapunov function on the attractor, and for this aim the term $a(\|u\|^2_{H_0^1})$ is crucial. In the case when $a(u) = a(l(u))$ it is not known whether such a function exists or not.

We prove the existence of strong solutions by assuming that either the function f is continuously differentiable and $f'(s) \leq \eta$ or a more strict growth condition on f. Supposing additionaly that the function a has sublinear growth we prove the existence of regular solutions as well. Moreover, when $f'(s) \leq \eta$ and the function $s \mapsto a(s^2)s$ is non-decreasing, uniqueness is proved.

When studying the asymptotic behaviour of solutions, new challenging difficulties arise for problem (2). For this problem we consider the autonomous situation, that is, $h \in L^2(\Omega)$ does not depend on t.

If uniqueness holds, then we define classical semigroups (one for regular solutions and one for strong solutions) and prove the existence of the global attractor. Under some extra assumptions on the functions a, h we are able to obtain that the global attractor is bounded in $H^2(\Omega)$ and $L^\infty(\Omega)$.

If uniqueness is not known to be true, then we have to define a (possibly) multivalued semiflow. Then the existence of the global attractor is proved for regular solutions in the topology of the space $L^2(\Omega)$ and for strong solutions in the topology of the space $H_0^1(\Omega)$ (or $H_0^1(\Omega) \cap L^p(\Omega)$), extending in this way the known results for the local problem [21].

The structure of the global attractor is an important feature as it gives us an insight into the long-term dynamics of the solutions. In the multivalued situation it is a challenging problem that has not been completely understood yet. So far in the local case several results in this direction have been obtained for reaction-diffusion equations without uniqueness [2, 5, 21, 22].

In our nonlocal problem for both situations (for regular and strong solutions) we are able under some conditions to define a Lyapunov function on the attractor and to prove that it is characterized as the unstable set of the stationary points (denoted by $M^u(\mathfrak{R})$). Also, the attractor is equal to the stable set of the stationary points when we consider solutions only in the set of bounded complete trajectories (denoted by $M^s(\mathfrak{R})$).

2 Existence of solutions

Throughout this paper we will denote by $\|\cdot\|_X$ the norm in the Banach space X.

We consider the following nonlocal reaction-diffusion equation

$$\begin{cases} u_t - a(\|u\|^2_{H_0^1})\Delta u = f(u) + h(t), \text{ in } \Omega\times(0,\infty), \\ u = 0 \quad \text{on } \partial\Omega\times(0,\infty), \\ u(0,x) = u_0(x) \quad \text{in } \Omega, \end{cases} \tag{3}$$

where Ω is a bounded open set of $\mathbb{R}^n$ with smooth boundary $\partial\Omega$.

Let us consider the following conditions on the functions a, f, h :

$$h \in L^2(0,T;L^2(\Omega)) \ \forall T > 0, \tag{4}$$

$$a \in C(\mathbb{R}^+), \ f \in C(\mathbb{R}), \tag{5}$$

$$a(s) \geq m > 0, \tag{6}$$

$$-\kappa - \alpha_2|s|^p \leq f(s)s \leq \kappa - \alpha_1|s|^p, \tag{7}$$

where $m, \ \alpha_1, \ \alpha_2 > 0$ and $\kappa \geq 0$, $p \geq 2$. Observe that then there exists $C > 0$ such that

$$|f(s)| \leq C(1 + |s|^{p-1}) \quad \forall s \in \mathbb{R}, \tag{8}$$

and that the function $\mathcal{F}(s) := \int_0^s f(r)dr$ satisfies

$$-\widetilde{\alpha}_2|s|^p - \widetilde{\kappa} \leq \mathcal{F}(s) \leq \widetilde{\kappa} - \widetilde{\alpha}_1|s|^p \tag{9}$$

for certain positive constants $\widetilde{\alpha}_i$, $i = 1, 2$, and $\widetilde{\kappa} \geq 0$, and

$$|\mathcal{F}(s)| \leq \widetilde{C}(1 + |s|^p) \quad \forall s \in \mathbb{R}. \tag{10}$$

Conditions (4)-(7) will be always assumed throughout the paper. Sometimes, some of the following additional assumptions will also be used:

$$f \in C^1(\mathbb{R}) \text{ be such that } f'(s) \leq \eta, \ \forall s \in \mathbb{R}, \tag{11}$$

$$p \leq \frac{2n-2}{n-2}, \text{ if } n \geq 3, \tag{12}$$

$$a(s) \leq M_1 + M_2 s, \ \forall s \geq 0, \tag{13}$$

$$s \mapsto a(s^2)s \text{ is non-decreasing}, \tag{14}$$

$$a(\cdot) \in C^1\left(\mathbb{R}^+\right) \text{ and } a'(s) \geq 0, \ \forall s \geq 0, \tag{15}$$

for some constants $M_1, M_2, \eta \geq 0$.

Remark 1 $a'(s) \geq 0$ *implies that (14) holds, so condition (15) is stronger than (14). Assumption (14) is used to prove uniqueness of solutions. Assumption (15) is used to obtain the* $H^2(\Omega)$ *regularity of the global attractor.*

Definition 2 *A weak solution to (3) is a function $u(\cdot)$ such that $u \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega)) \cap L^p(0,T;L^p(\Omega))$ for any $T>0$ and satisfies the equality*

$$\frac{d}{dt}(u,v) + a(\|u(t)\|^2_{H_0^1})(\nabla u(t), \nabla v) = (f(u(t)),v) + (h(t),v) \quad \forall v \in H_0^1(\Omega) \cap L^p(\Omega), \tag{16}$$

in the sense of scalar distributions.

Here, we denote by $(\cdot,\cdot)$ the inner product in $L^2(\Omega)$ (or $\left(L^2(\Omega)\right)^d$ for $d \in \mathbb{N}$) and also the duality product between $L^p(\Omega)$ and $L^q(\Omega)$ (where q is the conjugate exponent of p, that is, $q = p/(p-1)$). The duality between $H_0^1(\Omega)$ and $H^{-1}(\Omega)$ will be denoted by $\langle \cdot,\cdot \rangle$.

We need to guarantee that the initial condition of the problem makes sense for a weak solution. This can be achieved in a standard way assuming that the function a has an upper bound, that is, there exists $M>0$ such that

$$a(s) \leq M \text{ for all } s \geq 0. \tag{17}$$

Indeed, if u is a weak solution to (3), taking into account (8) and (17) it follows that

$$u_t = a(\|u\|^2_{H_0^1})\Delta u + f(u) + h \in L^2(0,T;H^{-1}(\Omega)) + L^q(0,T;L^q(\Omega)). \tag{18}$$

Therefore, by [12, p.33] $u \in C([0,T],L^2(\Omega))$ and the initial condition makes sense when $u_0 \in L^2(\Omega)$.

For the operator $A = -\Delta$, thanks to the assumptions on the domain Ω, it is well known that $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$ [30, Proposition 6.19].

Definition 3 *A regular solution to (3) is a weak solution with the extra regularity $u \in L^\infty(\varepsilon,T;H_0^1(\Omega))$ and $u \in L^2(\varepsilon,T;D(A))$ for any $0<\varepsilon<T$.*

Remark 4 *Since $\dfrac{du}{dt} \in L^q(\varepsilon,T;L^q(\Omega))$ for any regular solution, in this case equality (16) is equivalent to the following one:*

$$\begin{aligned}&\int_\varepsilon^T \int_\Omega \frac{du(t,x)}{dt}\xi(t,x)\,dxdt - \int_\varepsilon^T a(\|u(t)\|^2_{H_0^1})\int_\Omega \Delta u \xi dxdt \\ &= \int_\varepsilon^T \int_\Omega f(u(t,x))\xi(t,x)\,dxdt + \int_\varepsilon^T \int_\Omega h(t,x)\xi(t,x)\,dxdt,\end{aligned} \tag{19}$$

for all $0<\varepsilon<T$ and $\xi \in L^p(0,T;L^p(\Omega))$.

Lemma 5 *Let $u \in L^p(\varepsilon,T;X)$, $\dfrac{du}{dt} \in L^q(\varepsilon,T;X')$ for all $0<\varepsilon<T$, where X is a reflexive and separable Banach space and X' denotes its dual space. Assume that $\beta \in C(\mathbb{R}^+)$ is such that $\beta \in W^{1,\infty}(\varepsilon,T;[\beta(\varepsilon),\beta(T)])$ and $0<\beta(\varepsilon)<\beta(T)$ for all $0<\varepsilon<T$. Then $w(\cdot) = u(\beta(\cdot)) \in L^p(\varepsilon,T;X)$, $\dfrac{dw}{dt} \in L^q(\varepsilon,T;X')$, for all $0<\varepsilon<T$, and*

$$\frac{dw}{dt}(t) = \frac{du}{dt}(\beta(t))\frac{d\beta}{dt}(t) \text{ for a.a. } t>0. \tag{20}$$

Proof. We fix arbitrary $0<\varepsilon<T$. There exists a sequence $u_n \in C^1([\beta(\varepsilon),\beta(T)],X)$ such that $u_n \to u$ in $L^p(\beta(\varepsilon),\beta(T);X)$ and $\dfrac{du_n}{dt} \to \dfrac{du}{dt}$ in $L^q(\beta(\varepsilon),\beta(T);X')$ [20, Chapter IV]. We define $w_n(t) = u_n(\beta(t))$. Following the same proof of [4, Corollary VIII.10] we obtain that $w_n(\cdot) \in W^{1,\infty}(\varepsilon,T;X)$ and

$$\frac{dw_n}{dt}(t) = \frac{du_n}{dt}(\beta(t))\frac{d\beta}{dt}(t) \text{ for a.a. } t>0.$$

It is clear that $w_n \to w$ in $L^p(\varepsilon,T;X)$ and $\dfrac{du_n}{dt}(\beta(\cdot)) \to \dfrac{du}{dt}(\beta(\cdot))$ in $L^q(\varepsilon,T;X')$. Passing to the limit we obtain that

$$\frac{dw}{dt}(\cdot) = \frac{du}{dt}(\beta(\cdot))\frac{d\beta}{dt}(\cdot)$$

in the sense of distributions $\mathcal{D}'(0,+\infty;X)$. As $\frac{du}{dt}(\beta(\cdot))\frac{d\beta}{dt}(\cdot) \in L^q(\varepsilon,T;X')$, $\frac{dw}{dt} \in L^q(\varepsilon,T;X')$ and (20) holds true. ■

We would like to avoid a being uniformly bounded by above (i.e. to relax assumption (17)). We can still prove the continuity in $L^2(\Omega)$ of u for regular solutions by assuming that a has at most linear growth.

Lemma 6 *Assume that conditions (4)-(7), (13) hold. Then any regular solution satisfies that $u \in C([0,T],L^2(\Omega))$ for all $T>0$. Moreover, $w(t) = u(\alpha^{-1}(t))$, where $\alpha(t) = \int_0^t a(\|u(s)\|^2_{H_0^1})ds$, is a regular solution to the problem*

$$\begin{cases} w_t - \Delta w = \dfrac{f(w)+h(t)}{a(\|w\|^2_{H_0^1})}, & \text{in } \Omega\times(0,\infty), \\ w = 0 \quad \text{on } \partial\Omega\times(0,\infty), \\ w(0,x) = u_0(x) \quad \text{in } \Omega. \end{cases} \tag{21}$$

Proof. Condition (13) guarantees that $a(\|u(\cdot)\|^2_{H_0^1}) \in L^1(0,T)$ if $u \in L^2(0,T;H_0^1(\Omega))$. We make the following time rescaling

$$u(t,x) = w(\alpha(t),x).$$

As $a(\|u(\cdot)\|^2_{H_0^1}) \in L^1(0,T)$, the function $t \mapsto \alpha(t)$ is continuous and $\beta(\cdot) = \alpha^{-1}(\cdot)$ satisfies the conditions of Lemma 5. It is clear that the function $w(t,x) = u(\alpha^{-1}(t),x)$ belongs to the space $L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega)) \cap L^p(0,T;L^p(\Omega))$ and also to the spaces $L^\infty(\varepsilon,T;H_0^1(\Omega))$ and $L^2(\varepsilon,T;D(A))$ for any $0<\varepsilon<T$. Moreover, $\frac{du}{dt} \in L^q(\varepsilon,T;L^q(\Omega))$ and Lemma 5 give $\frac{dw}{dt} \in L^q(\varepsilon,T;L^q(\Omega))$ and

$$\frac{dw}{dt}(t) = \frac{du}{dt}\left(\alpha^{-1}(t)\right)\frac{d}{dt}\alpha^{-1}(t) = \frac{du}{dt}\left(\alpha^{-1}(t)\right)\frac{1}{a\left(\|w(t)\|^2_{H_0^1}\right)}, \text{ for a.a. } t. \tag{22}$$

Equality (19) implies that

$$\frac{du}{dt}\left(\alpha^{-1}(t)\right) - a\left(\|u(\alpha^{-1}(t))\|^2_{H_0^1}\right)\Delta u\left(\alpha^{-1}(t)\right) = f\left(u\left(\alpha^{-1}(t)\right)\right) + h(t), \text{ for a.a. } t>0,$$

so (22) gives

$$\frac{dw}{dt}(t) - \Delta w(t) = \frac{f(w(t))}{a(\|w(t)\|^2_{H_0^1})} + \frac{h(t)}{a(\|w(t)\|^2_{H_0^1})} \text{ for a.a. } t>0.$$

Hence, w is a regular solution to problem (21). Since $0 < \frac{1}{a(s)} \leq \frac{1}{m}$, we obtain that

$$\frac{dw}{dt} \in L^2(0,T;H^{-1}(\Omega)) + L^q(0,T;L^q(\Omega)).$$

Therefore, $w \in C([0,T],L^2(\Omega))$, so that

$$u \in C([0,T],L^2(\Omega)).$$

■

Remark 7 *Under assumptions (4)-(7) any regular solution $u(\cdot)$ satisfies that $\frac{du}{dt} \in L^q(\varepsilon,T;L^q(\Omega))$ for all $0<\varepsilon<T$. Then by [12, p.33] $u \in C([\varepsilon,T],L^2(\Omega))$, $t \mapsto \|u(t)\|^2$ is absolutely continuous on $[\varepsilon,T]$ and*

$$\frac{d}{dt}\|u(t)\|^2_{L^2} = 2\left(\frac{du}{dt},u\right) \textit{ for a.a. } t>\varepsilon.$$

If the initial condition belongs to $H_0^1(\Omega) \cap L^p(\Omega)$, we can define strong solutions as well.

Definition 8 *A strong solution to (3) is a weak solution with the extra regularity* $u \in L^{\infty}(0,T; H_0^1(\Omega) \cap L^p(\Omega))$, $u \in L^2(0,T; D(A))$ *and* $\dfrac{du}{dt} \in L^2\left(0,T; L^2(\Omega)\right)$ *for any* $T > 0$.

We observe that if u is a strong solution, then $u \in C([0,T], H_0^1(\Omega))$ (see [31, p.102]). Also, $u \in L^{\infty}(0,T; L^p(\Omega))$ and $u \in C([0,T], L^2(\Omega))$ imply that $u \in C_w([0,T], L^p(\Omega))$ (see [33, p.263]). Thus, an initial condition in $H_0^1(\Omega) \cap L^p(\Omega)$ makes sense. Also, the equality $f(u) = u_t - a\left(\|u\|_{H_0^1}^2\right)\Delta u - h$ implies that $f(u) \in L^2\left(0,T; L^2(\Omega)\right)$

Also, if u is a regular solution such that $\dfrac{du}{dt} \in L^2\left(\varepsilon, T; L^2(\Omega)\right)$ for all $0 < \varepsilon < T$, then $u \in C((0,T], H_0^1(\Omega))$.

The phase space for regular solutions will be $L^2(\Omega)$, whereas for strong solutions we will use the space $H^1(\Omega) \cap L^p(\Omega)$ (or just $H_0^1(\Omega)$ when $H_0^1(\Omega) \subset L^p(\Omega)$).

The following results will be proved in Theorems 9, 10, 11, 12, 14:

- Conditions (4)-(7), (11), (13) imply the existence of at least one regular solution for any $u_0 \in L^2(\Omega)$. If, in addition, (14) holds, then it is the unique regular solution.
- Conditions (4)-(7), (11) imply the existence of at least one strong solution for any $u_0 \in H_0^1(\Omega) \cap L^p(\Omega)$. If, in addition, (14) holds, then it is the unique strong solution.
- Conditions (4)-(7), (12) imply the existence of at least one strong solution for any $u_0 \in H_0^1(\Omega)$.
- Conditions (4)-(7), (12), (17) imply the existence of at least one regular solution for any $u_0 \in L^2(\Omega)$.

To start with we prove the existence of regular solutions for initial conditions in $L^2(\Omega)$.

Theorem 9 *Assume that conditions (4)-(7), (11) and (13) hold. Then, for any* $u_0 \in L^2(\Omega)$ *there exists at least one regular solution to (3).*

Proof. We will prove the result by compactness and using Faedo-Galerkin approximations.

Consider a fixed value $T > 0$. Let $\{w_j\}_{j \geq 1}$ be a sequence of eigenfunctions of $-\Delta$ in $H_0^1(\Omega)$ with homogeneous Dirichlet boundary conditions, which forms a special basis of $L^2(\Omega)$. If Ω is a bounded regular domain, then it is well known that $\{w_j\} \subset H_0^1(\Omega) \cap L^p(\Omega)$ and that for the set $V_n = span[w_1, \ldots, w_n]$ we have that $\cup_{n \in \mathbb{N}} V_n$ is dense in $L^2(\Omega)$ and also in $H_0^1(\Omega) \cap L^p(\Omega)$ [25]. As usual, P_n will be the orthogonal projection in $L^2(\Omega)$, that is

$$z_n := P_n z = \sum_{j=1}^{n} (z, w_j) w_j,$$

and λ_j will be the eigenvalues associated to the egienfunctions w_j. For each integer $n \geq 1$, we consider the Galerkin approximations

$$u_n(t) = \sum_{j=1}^{n} \gamma_{nj}(t) w_j,$$

which satisfy the following nonlinear ODE system

$$\begin{cases} \dfrac{d}{dt}(u_n, w_i) + a(\|u_n\|_{H_0^1}^2)(\nabla u_n, \nabla w_i) = (f(u_n), w_i) + (h, w_i) \quad \forall i = 1, \ldots, n, \\ u_n(0) = P_n u_0. \end{cases} \tag{23}$$

where $P_n u_0 \to u_0$ in $L^2(\Omega)$. Since (23) can be written in the normal form with a continuous right-hand side, this Cauchy problem possesses a solution on some interval $[0, t_n)$. We claim that for any $T > 0$ such a solution can be extended to the whole interval $[0,T]$, which follows from a priori estimates in the space $L^2(\Omega)$ of the sequence $\{u_n\}$.

Multiplying by $\gamma_{ni}(t)$ and summing from $i = 1$ to n, we obtain

$$\frac{1}{2}\frac{d}{dt}\|u_n(t)\|_{L^2}^2 + a(\|u_n\|_{H_0^1}^2)\|u_n(t)\|_{H_0^1}^2 = (f(u_n(t)), u_n(t)) + (h, u_n(t)) \text{ for } a.e.\ t \in (0, t_n). \tag{24}$$

Using (7) and the Young and Poincaré inequalities we deduce that

$$(f(u_n(t)), u_n(t)) \leq \kappa|\Omega| - \alpha_1 \|u_n(t)\|_{L^p}^p,$$

$$(h(t), u_n(t)) \leq \frac{m}{2}\|u_n(t)\|_{H_0^1}^2 + \frac{1}{2\lambda_1 m}\|h(t)\|_{L^2}^2.$$

Hence, from (24) it follows that

$$\frac{1}{2}\frac{d}{dt}\|u_n(t)\|_{L^2}^2 + \frac{m}{2}\|u_n(t)\|_{H_0^1}^2 + \alpha_1\|u_n(t)\|_{L^p}^p \leq \kappa|\Omega| + \frac{1}{2\lambda_1 m}\|h(t)\|_{L^2}^2 \text{ for } a.e.\ t \in (0, t_n). \tag{25}$$

Then, integrating (25) from 0 to $t \in (0, t_n)$ we deduce

$$\begin{aligned} &\frac{1}{2}\|u_n(t)\|_{L^2}^2 + \frac{m}{2}\int_0^t \|u_n(s)\|_{H_0^1}^2 ds + \alpha_1 \int_0^t \|u_n(s)\|_{L^p}^p ds \\ &\leq \kappa|\Omega|t + \frac{1}{2\lambda_1 m}\int_0^t \|h(s)\|_{L^2}^2 ds + \frac{1}{2}\|u_n(0)\|_{L^2}^2 \leq TK_2 + K_3(T) + \frac{1}{2}\|u_n(0)\|_{L^2}^2. \end{aligned} \tag{26}$$

Therefore, the sequence $\{u_n\}$ is well defined and bounded in $L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega)) \cap L^p(0,T;L^p(\Omega))$. Also, $\{-\Delta u_n\}$ is bounded in $L^2(0,T;H^{-1}(\Omega))$.

On the other hand, by (8) it follows that

$$\int_0^T \int_\Omega |f(u(x,t))|^q dx dt \leq 2^{q-1}C^q(|\Omega|T + \int_0^T \|u(t)\|_{L^p}^p dt),$$

with $\frac{1}{p} + \frac{1}{q} = 1$. Hence, since $\{u_n\}$ is bounded in $L^p(0,T;L^p(\Omega))$, $\{f(u_n)\}$ is bounded in $L^q(0,T;L^q(\Omega))$.

On the other hand, multiplying (23) by $\lambda_i \gamma_{ni}(t)$ and summing from $i = 1$ to n, we obtain

$$\frac{1}{2}\frac{d}{dt}\|u_n\|_{H_0^1}^2 + m\|\Delta u_n\|_{L^2}^2 \leq \langle f(u_n), -\Delta u_n\rangle + (h(t), -\Delta u_n) \leq \eta\|u_n\|_{H_0^1}^2 + \frac{1}{2m}\|h(t)\|_{L^2}^2 + \frac{m}{2}\|\Delta u_n\|_{L^2}^2.$$

Integrating the previous expression between s and t, with $0 < s \leq t \leq T$, and using (11) we have

$$\frac{1}{2}\|u_n(t)\|_{H_0^1}^2 + \frac{m}{2}\int_s^t \|\Delta u_n(r)\|_{L^2}^2 dr \leq \eta \int_0^T \|u_n(r)\|_{H_0^1}^2 dr + \frac{1}{2}\|u_n(s)\|_{H_0^1}^2 + \frac{1}{2m}\int_s^t \|h(r)\|_{L^2}^2 dr. \tag{27}$$

Now, integrating in s between 0 and t, it follows that

$$t\|u_n(t)\|_{H_0^1}^2 \leq (2\eta T + 1)\int_0^T \|u_n(r)\|_{H_0^1}^2 dr + K_3(T)T.$$

Hence,

$$\|u_n(t)\|_{H_0^1}^2 \leq \frac{2\eta T + 1}{\varepsilon}\int_0^T \|u_n(r)\|_{H_0^1}^2 dr + \frac{K_3(T)T}{\varepsilon}. \tag{28}$$

for all $t \in [\varepsilon, T]$ with $\varepsilon \in (0,T)$. From the last inequality and (26) we deduce that $\{\|u_n(t)\|_{H_0^1}\}$ is uniformly bounded in $[\varepsilon, T]$ and by the continuity of the function a we get that $\{a(\|u_n(t)\|_{H_0^1}^2)\}$ is bounded in $[\varepsilon, T]$. Also, it follows that

$$\{u_n\} \text{ is bounded in } L^\infty(\varepsilon, T; H_0^1(\Omega)). \tag{29}$$

On the other hand, taking $s = \varepsilon$ and $t = T$ in (27), by (26) we obtain that

$$\{u_n\} \text{ is bounded in } L^2(\varepsilon, T; D(A)), \tag{30}$$

so $\{-\Delta u_n\}$ and $\{a(\|u_n\|_{H_0^1}^2)\Delta u_n\}$ are bounded in $L^2(\varepsilon, T; L^2(\Omega))$. Thus,

$$\left\{\frac{du_n}{dt}\right\} \text{ is bounded in } L^q(\varepsilon, T; L^q(\Omega)). \tag{31}$$

Therefore, there exists $u \in L^\infty(\varepsilon,T;H_0^1(\Omega)) \cap L^2(0,T;H_0^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega)) \cap L^2(\varepsilon,T;D(A)) \cap L^p(0,T;L^p(\Omega))$ such that $\dfrac{du}{dt} \in L^q(\varepsilon,T;L^q(\Omega))$ and a subsequence $\{u_n\}$, relabelled the same, such that

$$\begin{aligned}
u_n &\overset{*}{\rightharpoonup} u \text{ in } L^\infty(\varepsilon,T;H_0^1(\Omega)),\\
u_n &\overset{*}{\rightharpoonup} u \text{ in } L^\infty(0,T;L^2(\Omega)),\\
u_n &\rightharpoonup u \text{ in } L^2(0,T;H_0^1(\Omega)),\\
u_n &\rightharpoonup u \text{ in } L^p(0,T;L^p(\Omega)),\\
u_n &\rightharpoonup u \text{ in } L^2(\varepsilon,T;D(A)),\\
\frac{du_n}{dt} &\rightharpoonup \frac{du}{dt} \text{ in } L^q(\varepsilon,T;L^q(\Omega)),\\
f(u_n) &\rightharpoonup \chi \text{ in } L^q(0,T;L^q(\Omega)),\\
a(\|u_n\|^2_{H_0^1}) &\overset{*}{\rightharpoonup} b \text{ in } L^\infty(\varepsilon,T),
\end{aligned} \tag{32}$$

for any $0<\varepsilon<T$, where $\rightharpoonup$ means weak convergence and $\overset{*}{\rightharpoonup}$ weak star convergence.

Moreover, by (30)-(31) the Aubin-Lions Compactness Lemma gives that $u_n \to u$ in $L^2(\varepsilon,T;H_0^1(\Omega))$, so $u_n(t) \to u(t)$ in $H_0^1(\Omega)$ a.e. on (ε,T) for any $\varepsilon>0$. Consequently, there exists a subsequence $\{u_n\}$, relabelled the same, such that $u_n(t,x) \to u(t,x)$ a.e. in $\Omega\times(0,T)$. Also, we know that $P_n f(u_n) \rightharpoonup \chi$ (see [30, p.224]). Since f is continuous, it follows that $f(u_n(t,x)) \to f(u(t,x))$ a.e. in $\Omega\times(0,T)$. Therefore, in view of (32), by [26, Lemma 1.3] we have that $\chi = f(u)$.

As a consequence, by the continuity of a, we get that

$$a(\|u_n(t)\|^2_{H_0^1}) \to a(\|u(t)\|^2_{H_0^1}) \quad \text{a.e. on } (\varepsilon,T).$$

Since the sequence is bounded, by the Lebesgue theorem this convergence takes place in $L^2(\varepsilon,T)$ and $b = a(\|u\|^2_{H_0^1})$ on (ε,T). Thus,

$$a(\|u_n\|^2_{H_0^1})\Delta u_n \rightharpoonup a(\|u\|^2_{H_0^1})\Delta u, \quad \text{in } L^2(\varepsilon,T;L^2(\Omega)). \tag{33}$$

Finally, since $\{w_i\}$ is dense in $H_0^1(\Omega)\cap L^p(\Omega)$, in view of (32) and (33), we can pass to the limit in (23) and conclude that (16) holds for all $v \in H_0^1(\Omega)\cap L^p(\Omega)$.

To conclude the proof, we have to check that $u(0)=u_0$. Indeed, let be $\phi \in C^1([0,T]);H_0^1(\Omega)\cap L^p(\Omega))$, with $\phi(T)=0$, $\phi(0)\neq 0$. We consider the functions $w(t) = u(\alpha^{-1}(t))$, $w_n(t) = u_n\left(\alpha_n^{-1}(t)\right)$ (here $\alpha_n(t) = \int_0^t a(\|u_n(r)\|^2_{H_0^1} dr))$, which by Lemma 6 are regular solutions to problem (21) with initial conditions $w(0)=u_0$ and to the corresponding Galerkin approximations with initial condition $w_n(0) = u_n(0) = P_n u_0$, respectively. Since $\dfrac{dw}{dt} \in L^2(0,T;H^{-1}(\Omega)) + L^q(0,T;L^q(\Omega))$, we can multiply the equation in (21) by ϕ and integrate by parts in the t variable to obtain that

$$\int_0^T \left(-\left(w(t),\phi'(t)\right) - \langle \Delta w(t),\phi(t)\rangle\right)dt = \int_0^T \left(\frac{f(w(t))+h(t)}{a(\|w(t)\|^2_{H_0^1})},\phi(t)\right)dt + \left(w(0),\phi(0)\right), \tag{34}$$

$$\int_0^T \left(-\left(w_n(t),\phi'(t)\right) - \langle \Delta w_n(t),\phi(t)\rangle\right)dt = \int_0^T \left(\frac{P_n f(w_n(t))+P_n h(t)}{a(\|w_n(t)\|^2_{H_0^1})},\phi(t)\right)dt + \left(w_n(0),\phi(0)\right). \tag{35}$$

We can easily obtain by the previous convergences and (6) that

$$\begin{aligned}
w_n &\rightharpoonup w \text{ in } L^2\left(0,T;H_0^1(\Omega)\right),\\
\Delta w_n &\rightharpoonup \Delta w \text{ in } L^2\left(0,T;H^{-1}(\Omega)\right),\\
\frac{P_n f(w_n(t))+P_n h(t)}{a(\|w_n(t)\|^2_{H_0^1})} &\rightharpoonup \frac{f(w(t))+h(t)}{a(\|w(t)\|^2_{H_0^1})} \text{ in } L^q\left(0,T;L^q(\Omega)\right).
\end{aligned}$$

Passing to the limit in (35), taking in to account (34) and bearing in mind $w_n(0) = P_n u_0 \to u_0$ we get

$$(w(0), \phi(0)) = (u_0, \phi(0)).$$

Since $\phi(0) \in H_0^1(\Omega) \cap L^p(\Omega)$ is arbitrary, we infer that $w(0) = u(0) = u_0$.

Hence, u is a regular solution to (3) satisfying $u(0) = u_0$. ■

Second, we will prove the existence of strong solutions for initial conditions in $H_0^1(\Omega) \cap L^p(\Omega)$. In this case, we do not need to impose the upper bound (13) of the function a.

Theorem 10 *Suppose that conditions (4)-(7) and (11) are fulfilled. Then, for any $u_0 \in H_0^1(\Omega) \cap L^p(\Omega)$ there exists at least a strong solution to (3).*

Proof. We consider, as in Theorem 9, the Galerkin approximations $\{u_n\}$ and an element u for which (32) holds. Under the aforementioned conditions, we will obtain that u_n converges to a strong solution to (3). In this proof it is important to observe that $P_n u_0 \to u_0$ in the spaces $H_0^1(\Omega)$ and $L^p(\Omega)$ [30, p.199 and 220]. Thus, the sequences $\|P_n u_0\|_{H_0^1}$ and $\|P_n u_0\|_{L^p}$ are bounded.

First, we multiply the equation in (23) by $\dfrac{du_n}{dt}$ to obtain

$$\|\frac{d}{dt}u_n(t)\|_{L^2}^2 + a(\|u_n\|_{H_0^1}^2)\frac{1}{2}\frac{d}{dt}\|u_n\|_{H_0^1}^2 = \frac{d}{dt}\int_\Omega \mathcal{F}(u_n)dx + (h(t), \frac{du_n}{dt}).$$

Introducing

$$A(s) = \int_0^s a(r)dr, \tag{36}$$

we have

$$\frac{1}{2}\|\frac{d}{dt}u_n(t)\|_{L^2}^2 + \frac{d}{dt}\left[\frac{1}{2}A(\|u_n\|_{H_0^1}^2) - \int_\Omega \mathcal{F}(u_n(t))dx\right] \leq \frac{1}{2}\|h(t)\|_{L^2}^2. \tag{37}$$

Now, integrating (37) we have

$$\begin{aligned}
&\frac{1}{2}\int_0^t \|\frac{d}{ds}u_n(s)\|_{L^2}^2 ds + \frac{1}{2}A(\|u_n(t)\|_{H_0^1}^2) - \int_\Omega \mathcal{F}(u_n(t))dx \\
&\leq \frac{1}{2}A(\|u_n(0)\|_{H_0^1}^2) - \int_\Omega \mathcal{F}(u_n(0))dx + \frac{1}{2}\int_0^t \|h(s)\|_{L^2}^2 ds.
\end{aligned}$$

From (6) and (9) we get

$$\begin{aligned}
&\frac{m}{2}\|u_n(t)\|_{H_0^1}^2 + \widetilde{\alpha}_1\|u_n(t)\|_{L^p}^p + \frac{1}{2}\int_0^t \|\frac{d}{ds}u_n(s)\|_{L^2}^2 ds \\
&\leq \frac{1}{2}A(\|u_n(0)\|_{H_0^1}^2) + \widetilde{\alpha}_2\|u_n(0)\|_{L^p}^p + K(T).
\end{aligned} \tag{38}$$

Now, from (38) we obtain that

$$\left\{\frac{du_n}{dt}\right\} \text{ is bounded in } L^2(0,T;L^2(\Omega)), \tag{39}$$

so

$$\frac{du_n}{dt} \rightharpoonup \frac{du}{dt} \text{ in } L^2(0,T;L^2(\Omega)). \tag{40}$$

On the other hand, the embedding $H_0^1(\Omega) \subset\subset L^2(\Omega)$ and the Aubin-Lion Compactness Lemma imply that

$$u_n \to u \text{ in } L^2(0,T;L^2(\Omega)).$$

Hence,

$$u_n \to u \text{ for a.e. } (x,t) \in \Omega \times (0,T).$$

Moreover, thanks to

$$\|u_n(t_2) - u_n(t_1)\|_{L^2}^2 = \left\| \int_{t_1}^{t_2} \frac{d}{dt} u_n(s) ds \right\|_{L^2}^2 \leq \| \frac{d}{dt} u_n \|_{L^2(0,T;L^2(\Omega))}^2 \ |t_2 - t_1| \quad \forall t_1, t_2 \in [0, T],$$

(38), (39) and $H_0^1(\Omega) \subset\subset L^2(\Omega)$, the Ascoli-Arzelà theorem implies that $\{u_n\}$ converges strongly in the space $C([0,T]; L^2(\Omega))$ for all $T > 0$. Therefore, we obtain from (38) that $u_n(t) \rightharpoonup u(t)$ in $H_0^1(\Omega) \cap L^p(\Omega)$, for any $t \geq 0$, and

$$u_n \overset{*}{\rightharpoonup} u \text{ in } L^\infty(0, T; H_0^1(\Omega) \cap L^p(\Omega)). \tag{41}$$

Also, by the continuity of the function a, $\left\{ a(\|u_n(t)\|_{H_0^1}^2) \right\}$ is uniformly bounded in $[0, T]$.

Multiplying (23) by $\lambda_i \gamma_{ni}(t)$ and summing from $i = 1$ to n, we obtain

$$\frac{1}{2} \frac{d}{dt} \|u_n\|_{H_0^1}^2 + m\| - \Delta u_n\|_{L^2}^2 = (f(u_n), -\Delta u_n) + (h(t), -\Delta u_n) \leq \eta \|u_n\|_{H_0^1}^2 + \frac{1}{2m} \|h(t)\|_{L^2}^2 + \frac{m}{2} \| - \Delta u_n\|_{L^2}^2.$$

Integrating the previous expression between 0 and T it follows that

$$\frac{1}{2} \|u_n(T)\|_{H_0^1}^2 + \frac{m}{2} \int_0^T \|\Delta u_n(s)\|_{L^2}^2 ds \leq \eta \int_0^T \|u_n(t)\|_{H_0^1}^2 dt + \frac{1}{2} \|u_n(0)\|_{H_0^1}^2 + K(T). \tag{42}$$

Finally, taking into account (26), from (42) we deduce that

$$u_n \text{ is uniformly bounded in } L^2(0, T; D(A)),$$

so

$$u_n \rightharpoonup u \text{ in } L^2(0, T; D(A)). \tag{43}$$

Arguing as in Theorem 9 we also obtain that

$$\begin{gathered} u_n \to u \text{ in } L^2\left(0, T; H_0^1(\Omega)\right), \\ a\left(\|u_n\|_{H_0^1}^2\right) \to a\left(\|u\|_{H_0^1}^2\right) \text{ in } L^2(0, T), \\ f(u_n) \rightharpoonup f(u) \text{ in } L^q(0, T; L^q(\Omega)), \\ a\left(\|u_n\|_{H_0^1}^2\right) \Delta u_n \rightharpoonup a\left(\|u\|_{H_0^1}^2\right) \Delta u \text{ in } L^2(0, T; L^2(\Omega)). \end{gathered} \tag{44}$$

Therefore, we can pass to the limit to conclude that u is a strong solution.

It remains to show that $u(0) = u_0$. This can be done, in a similar way as in Theorem 9, by multiplying the equation in (3) by a function $\phi \in C^1([0,T]); H_0^1(\Omega) \cap L^p(\Omega))$, with $\phi(T) = 0$, $\phi(0) \neq 0$ for the Galerkin approximations u_n and the limit function u and integrating by parts. Then taking into account the above convergences and $P_n u_0 \to u_0$ in $L^2(\Omega)$ we obtain that $u(0) = u_0$. ■

We can still ensure the existence of strong solutions without using condition (11) by imposing extra assumptions on the parameter p. Indeed, if (12) is satisfied, then the embedding $H_0^1(\Omega) \subset L^{2(p-1)}(\Omega) \subset L^p(\Omega)$ and (8) imply that

$$||f(u(t))||_{L^2}^2 \leq 2C(1 + \int_\Omega |u(t,x)|^{2(p-1)} dx) \leq \widetilde{C} \left(1 + \|u(t)\|_{H_0^1}^{2(p-1)}\right), \tag{45}$$

so

$$f(u) \in L^2(0, T; L^2(\Omega)) \tag{46}$$

provided that $u \in L^\infty(0, T; H_0^1(\Omega))$. Moreover, $f(A)$ is bounded in $L^2(0, T; L^2(\Omega)$ if A is a bounded set of $L^\infty(0, T; H_0^1(\Omega))$.

Theorem 11 *Assume that (4)-(7) and (12) hold. Then for any $u_0 \in H_0^1(\Omega)$ there exists at least one strong solution to (3).*

Proof. Reasoning as in Theorem 10 and considering as well the Galerkin scheme, (32), (40) and (41) hold. We just need to check that (43) is also true and then repeat the same lines of Theorem 10.

Multiplying (23) by $\lambda_i \gamma_{ni}(t)$ and summing from $i = 1$ to n, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u_n\|^2_{H^1_0} + m\|\Delta u_n\|^2_{L^2} &= (f(u_n), -\Delta u_n) + (h(t), -\Delta u) \\ &\leq \frac{1}{2m}\|f(u_n)\|^2_{L^2} + \frac{m}{2}\| - \Delta u_n\|^2_{L^2} + \frac{1}{m}\|h(t)\|^2_{L^2} + \frac{m}{4}\|\Delta u_n\|^2_{L^2}.\end{aligned}$$

Integrating between 0 and T it follows that

$$\begin{aligned}&\frac{1}{2}\|u_n(T)\|^2_{H^1_0} + \frac{m}{4}\int_0^T \|\Delta u_n(s)\|^2_{L^2} ds \\ &\leq \frac{1}{2m}\int_0^T \|f(u_n(t))\|^2_{L^2} dt + \frac{1}{2}\|u_n(0)\|^2_{H^1_0} + \frac{1}{m}\int_0^T \|h(t)\|^2_{L^2} dt.\end{aligned} \tag{47}$$

In view of (41) and (45), we have that $f(u)$ is bounded in $L^2\left(0,T;L^2\left(\Omega\right)\right)$, so from (47) we get that $\{u_n\}$ is bounded in $L^2(0,T;D(A))$. Therefore,

$$u_n \rightharpoonup u \text{ in } L^2(0,T;D(A)), \tag{48}$$

as required. ■

Actually, in the case of regular solutions, we can get rid of the condition (11) as well by imposing the extra assumption (12) on the constant p.

Theorem 12 *Assume that (4)-(7), (12) and (17) hold. Then, for any $u_0 \in L^2(\Omega)$ there exists at least one regular solution to (3).*

Proof. Let $u_0^n \in H^1_0(\Omega)$ be a sequence such that $u_0^n \to u_0$ in $L^2(\Omega)$. By Theorem 11 there exists a strong solution $u^n(\cdot)$ of (3) with $u^n(0) = u_0^n$. Since $u^n \in L^2\left(0,T;D\left(A\right)\right)$ and $\frac{du^n}{dt} \in L^2\left(0,T;L^2\left(\Omega\right)\right)$, from [31, p.102] the equality

$$\frac{d}{dt}\|u^n\|^2_{H^1_0} = 2(-\Delta u^n, u^n_t)$$

holds true for a.a. $t > 0$.

Now, multiplying (3) by u^n and using (7) it follows that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|u^n(t)\|^2_{L^2} + m\|u^n\|^2_{H^1_0} + \alpha_1\|u^n(t)\|^p_{L^p} \\ &\leq \kappa|\Omega| + \|h(t)\|_{L^2}\|u^n(t)\|_{L^2} \leq \kappa|\Omega| + \frac{1}{2m\lambda_1}\|h(t)\|^2_{L^2} + \frac{m}{2}\|u^n(t)\|^2_{H^1_0},\end{aligned} \tag{49}$$

so

$$\|u^n(t)\|^2_{L^2} \leq \|u^n(0)\|^2_{L^2} + K_1(T). \tag{50}$$

Thus, integrating in (49) between t and $t+r$ we get

$$\begin{aligned}&\|u^n(t+r)\|^2_{L^2} + m\int_t^{t+r} \|u^n(s)\|^2_{H^1_0} ds + 2\alpha_1\int_t^{t+r} \|u^n(s)\|^p_{L^p} ds \\ &\leq 2\kappa|\Omega|r + \frac{1}{m\lambda_1}\int_t^{t+r} \|h(s)\|^2_{L^2} ds + \|u^n(t)\|^2_{L^2} \leq \|u^n(0)\|^2_{L^2} + K_2(T).\end{aligned} \tag{51}$$

Also, by (9) and (17) we deduce that

$$\begin{aligned}&\int_t^{t+r}\left(\frac{1}{2}A(\|u^n(s)\|^2_{H^1_0}) - \int_\Omega \mathcal{F}(u^n(s))dx\right)ds \\ &\leq \int_t^{t+r} \frac{M}{2}\|u^n\left(s\right)\|^2_{H^1_0} ds + \widetilde{\kappa}\left|\Omega\right|r + \widetilde{\alpha}_2\int_t^{t+r} \|u^n(s)\|^p_{L^p} ds \\ &\leq K_3(T)\left(1 + \|u^n(0)\|^2_{L^2}\right),\end{aligned} \tag{52}$$

for all $n > 0$ and $t \geq 0$.

On the other hand, multiplying (3) by u_t^n we have

$$\frac{1}{2}\|u_t^n(t)\|_{L^2}^2 + \frac{d}{dt}\left(\frac{1}{2}A(\|u^n(t)\|_{H_0^1}^2) - \int_\Omega \mathcal{F}(u^n(t))dx\right) \leq \frac{1}{2}\|h(t)\|_{L^2}^2, \tag{53}$$

where the fact that $t \mapsto \int_\Omega \mathcal{F}(u^n(t))dx$ is absolutely continuous on $[0,T]$ and

$$\frac{d}{dt}\int_\Omega \mathcal{F}(u^n(t))dx = \left(f\left(u^n\left(t\right)\right), \frac{du^n}{dt}\left(t\right)\right), \text{ for a.a. } t > 0,$$

is proved by regularization using the regularity of strong solutions and (45). By the Uniform Gronwall Lemma [34] we obtain

$$\frac{1}{2}A(\|u^n(t+r)\|_{H_0^1}^2) - \int_\Omega \mathcal{F}(u^n(t+r))dx \leq \frac{K_3(T)(1 + \|u^n(0)\|_{L^2}^2)}{r} + K_4(T), \quad \text{for all } 0 \leq t \leq t+r, \tag{54}$$

so that by (6) and (9) we obtain that

$$\|u^n(t+r)\|_{H_0^1}^2 + \|u^n\left(t+r\right)\|_{L^p}^p \leq \frac{K_5(T)(1 + \|u^n(0)\|_{L^2}^2)}{r} + K_6(T), \tag{55}$$

for all $t \geq 0$. Therefore, the sequence $u^n(\cdot)$ is bounded in $L^\infty(r,T;H_0^1(\Omega))$ for all $0 < r < T$. Consequently, $a(\|u^n\left(\cdot\right)\|_{H_0^1}^2)$ is bounded in $[r,T]$.

Integrating (53) over (r,T), from (6), (9) and (54) it follows that

$$\begin{aligned}
&\frac{1}{2}\int_r^T \|\frac{d}{dt}u^n(t)\|_{L^2}^2 dt + \frac{m}{2}\|u^n(T)\|_{H_0^1}^2 + \widetilde{\alpha}_1\|u^n(T)\|_{L^p}^p - \kappa|\Omega| \\
&\leq \frac{1}{2}\int_r^T \|\frac{d}{dt}u^n(t)\|_{L^2}^2 dt + \frac{1}{2}A(\|u^n(T)\|_{H_0^1}^2) - \int_\Omega \mathcal{F}(u^n(T))dx \\
&\leq \frac{1}{2}\int_r^T \|h(t)\|_{L^2}^2 dt + \frac{1}{2}A(\|u^n(r)\|_{H_0^1}^2) - \int_\Omega \mathcal{F}(u^n(r))dx \\
&\leq \frac{1}{2}\int_r^T \|h(t)\|_{L^2}^2 dt + \frac{K_3(T)(1 + \|u^n(0)\|_{L^2}^2)}{r} + K_4(T).
\end{aligned} \tag{56}$$

Thus $\dfrac{du^n}{dt}$ is bounded in $L^2(r,T;L^2(\Omega))$ for all $0 < r < T$.

Taking into account (45) and (55) we infer that $f\left(u^n\right)$ is bounded in $L^2\left(r,T;L^2\left(\Omega\right)\right)$. By this way, the equality $a(\|u^n\|_{H_0^1}^2)\Delta u^n = u_t^n - f(u^n) + h(t)$ implies that u^n and $a(\|u^n\|_{H_0^1}^2)\Delta u^n$ are bounded in $L^2(r,T;D(A))$ and $L^2(r,T;L^2(\Omega))$, respectively, for all $0 < r < T$.

By the compact embedding $H_0^1(\Omega) \subset L^2(\Omega)$, we can apply the Ascoli-Arzelà theorem and obtain that, up to a sequence, there exists a function u such that

$$\begin{aligned}
u^n &\overset{*}{\rightharpoonup} u \text{ in } L^\infty(r,T;H_0^1(\Omega)), \\
u^n &\to u \text{ in } C([r,T],L^2(\Omega)), \\
u^n &\rightharpoonup u \text{ in } L^2(r,T;D(A)), \\
\frac{du^n}{dt} &\rightharpoonup \frac{du}{dt} \text{ in } L^2(r,T;L^2(\Omega)),
\end{aligned} \tag{57}$$

for all $0 < r < T$.

On the other hand, from (51) we infer that u^n is bounded in $L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega)) \cap L^p(0,T;L^p(\Omega))$, for all $T > 0$. Therefore, there exists a subsequence u^n, relabelled the same, such that

$$\begin{aligned}
u^n &\overset{*}{\rightharpoonup} u \text{ in } L^\infty(0,T;L^2(\Omega)), \\
u^n &\rightharpoonup u \text{ in } L^2(0,T;H_0^1(\Omega)), \\
u^n &\rightharpoonup u \text{ in } L^p(0,T;L^p(\Omega)),
\end{aligned} \tag{58}$$

for all $T > 0$. On the other hand, arguing as in the proof of Theorem 9 we obtain that

$$\begin{gathered} f(u^n) \rightharpoonup f(u) \text{ in } L^q(0,T;L^q(\Omega)), \\ u^n \to u \text{ in } L^2(r,T;H_0^1(\Omega)), \\ a(\|u^n\|^2_{H_0^1}) \to a(\|u\|^2_{H_0^1}) \text{ in } L^2(0,T), \\ a(\|u^n(t)\|^2_{H_0^1})\Delta u^n \rightharpoonup a(\|u(t)\|^2_{H_0^1})\Delta u \quad \text{in } L^2(r,T;L^2(\Omega)). \end{gathered}$$

Passing to the limit we obtain that $u(\cdot)$ is a regular solution.

Finally, by a similar argument as in the proof of Theorem 9 we establish that $u(0) = u_0$. ■

Remark 13 *Under the conditions of Theorem 12 any regular solution $u(\cdot)$ satisfies from (45) that $f(u) \in L^2(\varepsilon, T; L^2(\Omega))$ for all $0 < \varepsilon < T$, and then $\frac{du}{dt} \in L^2(\varepsilon, T; L^2(\Omega))$ as well. Hence, $u \in C((0,T], H_0^1(\Omega))$ for all $T > 0$.*

We finish this section by giving a sufficient condition ensuring the uniqueness of solutions.

Theorem 14 *Assume the conditions of Theorem 9 and additionally that (14) is satisfied. Then there can exists at most one regular solution to the Cauchy problem (3) for $u_0 \in L^2(\Omega)$.*

If, moreover, $M_2 = 0$ in condition (13), then there can be at most one weak solution.

Under the conditions of Theorem 10 and (14), there can exists at most one strong solution to the Cauchy problem (3) for $u_0 \in H_0^1(\Omega) \cap L^p(\Omega)$.

Proof. Suppose that u and v are two regular solutions to (3) with the same initial condition $u_0 = v_0$. Then by subtraction and multiplying by $u - v$ we get by Remark 7 that

$$\frac{1}{2}\frac{d}{dt}\|u-v\|^2_{L^2} + \langle -a(\|u(t)\|^2_{H_0^1})\Delta u + a(\|v(t)\|^2_{H_0^1})\Delta v, u-v\rangle = (f(u)-f(v), u-v).$$

Let us consider

$$I = \langle -a(\|u(t)\|^2_{H_0^1})\Delta u + a(\|v(t)\|^2_{H_0^1})\Delta v, u-v\rangle.$$

After integrating by parts, we obtain

$$\begin{aligned} I &= \int_\Omega (a(\|u(t)\|^2_{H_0^1})|\nabla u|^2 - a(\|u(t)\|^2_{H_0^1})\nabla u\nabla v - a(\|v(t)\|^2_{H_0^1})\nabla u\nabla v + a(\|v(t)\|^2_{H_0^1})|\nabla v|^2)dx \\ &\geq a(\|u(t)\|^2_{H_0^1})\|u(t)\|^2_{H_0^1} - \left(a(\|u(t)\|^2_{H_0^1}) + a(\|v(t)\|^2_{H_0^1})\right)\|u(t)\|_{H_0^1}\|v(t)\|_{H_0^1} + a(\|v(t)\|^2_{H_0^1})\|v(t)\|^2_{H_0^1} \\ &= \left(a(\|u(t)\|^2_{H_0^1})\|u(t)\|_{H_0^1} - a(\|v(t)\|^2_{H_0^1})\|v(t)\|_{H_0^1}\right)\left(\|u(t)\|_{H_0^1} - \|v(t)\|_{H_0^1}\right) \geq 0, \end{aligned} \tag{59}$$

where we have used (14) in the last inequality.

Hence, from (59) and $f'(s) \leq \eta$, we infer

$$\frac{1}{2}\frac{d}{dt}\|u-v\|^2_{L^2} \leq \int_\Omega (f(u)-f(v))(u-v)dx = \int_\Omega \left(\int_v^u f'(s)ds\right)(u-v)dx \leq \eta\|u-v\|^2_{L^2}.$$

By Remark 7 it is correct to apply the Gronwall lemma over an arbitrary interval (ε, t), so

$$\|u(t)-v(t)\|^2_{L^2} \leq \|u(\varepsilon) - v(\varepsilon)\|^2_{L^2}\, e^{2\eta(t-\varepsilon)}, \quad t \geq 0.$$

Since Lemma 6 implies that $u, v \in C([0,T], L^2(\Omega))$, we pass to the limit as $\varepsilon \to 0$ to get

$$\|u(t)-v(t)\|^2_{L^2} \leq \|u(0) - v(0)\|^2_{L^2}\, e^{2\eta t}, \quad t \geq 0.$$

Hence, the uniqueness follows.

If $M_2 = 0$ in (13), then by (18) the above argument is valid for weak solutions as well.

The proof of the last statement is the same with the only difference that condition (13) is not needed.

■

3 Existence and structure of attractors

In this section we will prove the existence of global attractors for the semiflows generated by regular and strong solutions under different assumptions in the autonomous case, that is, when the function h does depend on t. We will also establish that the attractor is equal to the unstable set of the stationary points or to the stable one when we only consider solutions in the set of bounded complete trajectories.

We consider the following condition instead of (4):

$$h \in L^2(\Omega). \tag{60}$$

Throughout this section, for a metric space X with metric ρ we will denote by $dist_X(C, D)$ the Hausdorff semidistance from C to D, that is, $dist_X(C, D) = \sup_{c\in C} \inf_{d\in D} \rho(c, d)$.

It is important to observe that in the theorems of existence of solutions of the previous section we have used either assumption (11) or (12). Now, when we use condition (11) in some cases it is necessary to add a restriction on the constant p given below in (83).

We summarize the main results of this section:

- Conditions (5)-(7), (11), (17), (14) and (60) imply that the regular solutions generate a semigroup in the phase space $L^2(\Omega)$ possessing a global attractor, which is compact in $H_0^1(\Omega)$ and bounded in $L^p(\Omega)$ (Theorem 17 and Lemma 39). If, in addition, either $h \in L^\infty(\Omega)$ or $p \leq 2n/(n-2)$ for $n \geq 3$, then it is characterized by the unstable set of the stationary points (Proposition 40). Moreover, condition (15) implies that the attractor is bounded in $H^2(\Omega)$ (Proposition 19).
- Conditions (5)-(7), (17), (60) and either (12) or (11), (83) imply that the regular solutions generate a (possibly) multivalued semiflow in the phase space $L^2(\Omega)$ possessing a global attractor, which is compact in $H_0^1(\Omega)$ and $L^p(\Omega)$ and is equal to the unstable set of the stationary points (Theorems 33, 37).
- Conditions (5)-(7), (17), (60) and either (12) or (11), (83) imply that the strong solutions generate a (possibly) multivalued semiflow in the phase space $H_0^1(\Omega)$ possessing a global attractor, which is compact in $H_0^1(\Omega)$ and $L^p(\Omega)$ and is equal to the unstable set of the stationary points (Theorems 45, 48).
- Conditions (5)-(7), (11), (17), (14), (60) and (83) imply that the strong solutions generate a semigroup in the phase space $H_0^1(\Omega)$ possessing a global attractor, which is compact in $H_0^1(\Omega)$ and $L^p(\Omega)$ and is equal to the unstable set of the stationary points (Theorems 50, 53). Moreover, condition (15) implies that the attractor is bounded in $H^2(\Omega)$ (Proposition 54).
- Conditions (5)-(7), (11), (17), (14) and (60) imply that the strong solutions generate a semigroup in the phase space $H_0^1(\Omega) \cap L^p(\Omega)$ (endowed with the induced topology of $H_0^1(\Omega)$) possessing a global attractor, which is compact in $H_0^1(\Omega)$ and bounded in $L^p(\Omega)$ (Theorem 57). If, in addition, either $h \in L^\infty(\Omega)$ or $p \leq 2n/(n-2)$ for $n \geq 3$, then it is characterized by the unstable set of the stationary points (Theorem 60). Moreover, condition (15) implies that the attractor is bounded in $H^2(\Omega)$ (Proposition 61).
- In all the above situations $h \in L^\infty(\Omega)$ implies that the global attractor is bounded in $L^\infty(\Omega)$ (Theorems 18, 36, 47, 59).

3.1 Regular solutions

We split this part into three subsections.

3.1.1 The case of uniqueness

If we assume conditions (5)-(7), (11), (14), (60), then by Theorems 9 and 14 we can define the following continuous semigroup $T_r : \mathbb{R}^+ \times L^2(\Omega) \to L^2(\Omega)$:

$$T_r(t, u_0) = u(t), \tag{61}$$

where $u(\cdot)$ is the unique regular solution to (3). We denote by $\mathfrak{R}$ the set of fixed points of T_r, that is, the points z such that $T_r(t,z) = z$ for any $t \geq 0$.

We also observe that if we assume (17), then using the calculations in (52)-(55) for the Galerkin approximations of any regular solution $u(\cdot)$ one can obtain that $u \in L^{\infty}(\varepsilon, T; L^p(\Omega))$, for all $0 < \varepsilon < T$, and then $u \in C_w((0,+\infty), L^p(\Omega))$.

Our first purpose is to obtain a global attractor. We recall that the set $\mathcal{A}$ is a global compact attractor for T_r if it is compact, invariant (which means $T_r(t,\mathcal{A}) = \mathcal{A}$ for any $t \geq 0$) and it attracts any bounded set B, that is,

$$dist_{L^2}(T_r(t,B), \mathcal{A}) \to 0 \text{ as } t \to +\infty.$$

Proposition 15 *Let (5)-(7), (11), (13), (14) and (60) hold. Then the semigroup T_r has a bounded absorbing set in L^2; that is, there exists a constant K such that for any $R > 0$ there is a time $t_0 = t_0(R)$ such that*

$$\|u(t)\|_{L^2} \leq K \quad \textit{for all} \quad t \geq t_0, \tag{62}$$

where $\|u_0\|_{L^2} \leq R$, $u(t) = T_r(t,u_0)$. Moreover, there is a constant L such that

$$\int_t^{t+1} \|u(s)\|_{H_0^1}^2 ds \leq L \quad \textit{for all} \quad t \geq t_0. \tag{63}$$

Proof. Multiplying equation (3) by u and using (7) and Remark 7 we have

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|_{L^2}^2 + \frac{m}{2}\|u(t)\|_{H_0^1}^2 + \alpha_1 \|u(t)\|_{L^P}^p \leq \kappa|\Omega| + \frac{1}{2\lambda_1 m}\|h\|_{L^2}^2 = \frac{\kappa_1}{2}. \tag{64}$$

The Gronwall lemma and the inequality $\|u(t)\|_{H_0^1}^2 \geq \lambda_1 \|u(t)\|_{L^2}^2$ give

$$\|u(t)\|_{L^2}^2 \leq \|u(\varepsilon)\|_{L^2}^2 e^{-\lambda_1 m(t-\varepsilon)} + \frac{\kappa_1}{\lambda_1 m}, \text{ for any } \varepsilon > 0.$$

As $u \in C([0,T], L^2(\Omega)$ by Lemma 6, passing to the limit we have

$$\|u(t)\|_{L^2}^2 \leq \|u(0)\|_{L^2}^2 e^{-\lambda_1 m t} + \frac{\kappa_1}{\lambda_1 m}. \tag{65}$$

Hence, taking

$$t \geq t_0 \equiv \frac{1}{\lambda_1 m} \ln\left(\frac{\lambda_1 m R^2}{\kappa_1}\right)$$

we get (62) for $K = \frac{2\kappa_1}{\lambda_1 m}$. On the other hand, integrating (64) between t and $t+1$ and using (65) we obtain

$$m\int_t^{t+1} \|u(s)\|_{H_0^1}^2 ds \leq \|u(t)\|_{L^2}^2 + \kappa_1$$

and using the previous bound we get

$$\int_t^{t+1} \|u(s)\|_{H_0^1}^2 ds \leq \frac{\kappa_1}{m} + \frac{2\kappa_1}{\lambda_1 m^2}, \quad \text{for all } t \geq t_0,$$

so that (63) follows. ■

Proposition 16 *Let (5)-(7), (11), (17), (14) and (60) hold. Then there exists a bounded absorbing set in $H_0^1(\Omega) \cap L^p(\Omega)$; that is, there is a constant M such that for any $R > 0$ there is a time $t_1 = t_1(R)$ such that*

$$\|u(t)\|_{H_0^1} + \|u(t)\|_{L^p} \leq M \quad \textit{for all } t \geq t_1,$$

where $\|u_0\|_{L^2} \leq R$, $u(t) = T_r(t,u_0)$.

Proof. The following calculations are formal but can be justified by the Galerkin approximations. Arguing as in (52)-(55) we obtain the existence of a constant C such that

$$\|T_r(1,u(0))\|_{H_0^1}^2+\|T_r(1,u(0))\|_{L^p}^p\leq C(1+\|u(0)\|_{L^2}^2).$$

Hence, the semigroup property $T_r(t+1,u_0)=T_r(1,T_r(t,u_0))$ and (62) imply that

$$\|T_r(t+1,u_0)\|_{H_0^1}^2+\|T_r(t+1,u_0)\|_{L^p}^p\leq C(1+K^2)\ \forall t\geq t_0(R),$$

if $\|u_0\|_{L^2}\leq R$, which proves the statement. ■

Theorem 17 *Let (5)-(7), (11), (17), (14) and (60). Then the equation (3) has a connected global compact attractor $\mathcal{A}_r$, which is bounded in $H_0^1(\Omega)\cap L^p(\Omega)$.*

Proof. Since a bounded set in $H_0^1(\Omega)$ is relatively compact in $L^2(\Omega)$ which is a connected space, the result follows from Theorem 10.5 in [30] and Proposition 16. ■

We will also obtain the boundedness of the attractor in the spaces $L^\infty(\Omega)$ and $H^2(\Omega)$.

First, we recall that a function $\phi:\mathbb{R}\to L^2(\Omega)$ is a complete trajectory of the semigroup T_r if $\phi(t)=T_r(t-s,\phi(s))$ for any $t\geq s$. ϕ is bounded if the set $\cup_{s\in\mathbb{R}}\phi(s)$ is bounded. It is well known [24] that the global attractor is characterized by

$$\mathcal{A}_r=\{\phi(0):\phi\text{ is a bounded complete trajectory}\}. \tag{66}$$

Theorem 18 *Let (5)-(7), (11), (17), (14) and (60) hold. Then the global attractor $\mathcal{A}_r$ is bounded in $L^\infty(\Omega)$, provided that $h\in L^\infty(\Omega)$.*

Proof. We define $v_+=\max\{v,0\},\ v_-=-\max\{-v,0\}$. We multiply equation (3) by $(u-M)_+$ for some appropriate constant M and integrate over Ω to obtain

$$\frac{1}{2}\frac{d}{dt}\int_\Omega|(u-M)_+|^2dx+a(\|u(t)\|_{H_0^1}^2)\int_\Omega|\nabla(u-M)_+|^2dx=\int_\Omega(f(u(t))+h)(u-M)_+dx,$$

where we have used the equality $\frac{1}{2}\frac{d}{dt}\int_\Omega|(u-M)_+|^2dx=(u_t,(u-M)_+)$, which is proved by regularization.

Since $h\in L^\infty(\Omega)$, by (7) we deduce that

$$(f(u)+h)u\leq\widetilde{\kappa}-\widetilde{\alpha}|u|^p.$$

It follows that

$$f(u)+h\leq 0\quad\text{when}\quad u\geq(\frac{\widetilde{\kappa}}{\widetilde{\alpha}})^{1/p}=M.$$

Therefore, we have

$$(f(u)+h)(u-M)_+\leq 0.$$

Thus, by (6) and the the Poincaré inequality, we deduce that

$$\frac{d}{dt}\int_\Omega|(u-M)_+|^2dx\leq-2m\lambda_1\int_\Omega|(u-M)_+|^2dx.$$

Using the Gronwall inequality, we have

$$\int_\Omega|(u(t)-M)_+|^2dx\leq e^{-2m\lambda(t-\tau)}\int_\Omega|(u(\tau)-M)_+|^2dx.$$

For any $y\in\mathcal{A}_r$ there is by (66) a bounded complete trajectory ϕ such that $\phi(0)=y$. Then taking $t=0$ and $\tau\to-\infty$ in the last inequality, we obtain $y(x)=\phi(0,x)\leq M$, for a.a. $x\in\Omega$. The same arguments can be applied to $(u-M)_-$, which shows that

$$\|y\|_{L^\infty}\leq M,\quad\forall y\in\mathcal{A}_r.$$

■

If we assume (15), then it is possible to show that the global attractor is more regular.

Proposition 19 *Let (5)-(7), (11), (17) and (60) hold. If, additionally, (15) is satisfied, then there exists an absorbing set in $H^2(\Omega)$ and the global attractor is bounded in $H^2(\Omega)$.*

Proof. We will prove the existence of an absorbing set in $H^2(\Omega)$. The boundedness of the global attractor in this space follows then immediately. We proceed formally, but the estimates can be justified via Galerkin approximations.

Let $u(t) = T_r(t, u_0)$ with $\|u_0\|_{L^2} \leq R$. First, we differentiate the equation with respect to t

$$u_{tt} - a'(\|u\|^2_{H^1_0})\frac{d}{dt}\|u\|^2_{H^1_0}\Delta u - a(\|u\|^2_{H^1_0})\Delta u_t = f'(u)u_t.$$

Multiplying by u_t we get

$$\frac{1}{2}\frac{d}{dt}\|u_t\|^2_{L^2} + \frac{1}{2}a'(\|u\|^2_{H^1_0})(\frac{d}{dt}\|u\|^2_{H^1_0})^2 + a(\|u\|^2_{H^1_0})\|u_t\|^2_{H^1_0} = \int_\Omega f'(u)(u_t)^2 dx. \tag{67}$$

By (6), $a'(s) \geq 0$ and $f'(s) \leq \eta$ we obtain

$$\frac{1}{2}\frac{d}{dt}\|u_t\|^2_{L^2} + m\|u_t\|^2_{H^1_0} \leq \eta\|u_t\|^2_{L^2}. \tag{68}$$

Second, multiplying (3) by u_t and reordering terms, we obtain

$$\frac{d}{dt}\left(\frac{a(\|u\|^2_{H^1_0})}{2}\|u\|^2_{H^1_0} - \int_\Omega \mathcal{F}(u)dx - \int_\Omega h(x)udx\right) + \|u_t\|^2_{L^2} = \frac{a'(\|u\|^2_{H^1_0})}{2}\|u\|^2_{H^1_0}\frac{d}{dt}\|u\|^2_{H^1_0}. \tag{69}$$

Proposition 16 implies that

$$a'(\|z\|^2_{H^1_0}) \leq \gamma := sup_{|s|\leq M}a'(s^2)$$

if z belongs to the absorbing set in $H^1_0(\Omega)$. On the other hand, multiplying the equation by $-\Delta u$ and using Proposition 16, we obtain

$$\frac{d}{dt}\|u\|^2_{H^1_0} + m\|\Delta u(t)\|^2_{L^2} \leq 2\eta\|u(t)\|^2_{H^1_0} + \frac{1}{m}\|h\|^2_{L^2} \leq K_1 \quad \forall t \geq t_1(R).$$

Hence, by (69) and Proposition 16, it follows

$$\frac{d}{dt}\left(\frac{a(\|u\|^2_{H^1_0})}{2}\|u\|^2_{H^1_0} - \int_\Omega \mathcal{F}(u)dx - \int_\Omega h(x)udx\right) + \|u_t\|^2_{L^2} \leq \frac{\gamma}{2}K_1M^2, \quad \forall t \geq t_1(R). \tag{70}$$

Multiplying both sides of the inequality $f'(s) \leq \eta$ by s and integrating between 0 and s, we obtain

$$sf(s) \leq \mathcal{F}(s) + \frac{s^2}{2}\eta, \quad \forall s \in \mathbb{R}. \tag{71}$$

Moreover, integrating $f'(s) \leq \eta$ twice between 0 and s, we infer

$$\mathcal{F}(s) \leq \frac{\eta}{2}s^2 + Cs, \quad \forall s \in \mathbb{R}. \tag{72}$$

Now, we multiply (3) by u and integrate between t and $t+1$ to obtain

$$\frac{1}{2}\|u(t+1)\|^2_{L^2} + \int_t^{t+1}\left(a(\|u\|^2_{H^1_0})\|u(s)\|^2_{H^1_0} - \int_\Omega f(u)udx - \int_\Omega h(x)udx\right)ds = \frac{1}{2}\|u(t)\|^2_{L^2}. \tag{73}$$

From (71), (73) and Proposition 15 it follows

$$\int_t^{t+1}\left(\frac{a(\|u\|^2_{H^1_0})}{2}\|u\|^2_{H^1_0} - \int_\Omega \mathcal{F}(u)dx - \int_\Omega h(x)udx\right)ds \leq \frac{1}{2}\|u(t)\|^2_{L^2} + \frac{\eta}{2}\int_t^{t+1}\|u\|^2_{L^2}ds \leq \widetilde{L} \quad \forall t \geq t_0$$

The last inequality allows us to apply the Uniform Gronwall Lemma to (70) in order to obtain

$$\frac{a(\|u\|_{H_0^1}^2)}{2}\|u\|_{H_0^1}^2 - \int_\Omega \mathcal{F}(u)dx - \int_\Omega h(x)udx \leq \widetilde{L} + \frac{\gamma}{2}K_1M^2 \quad \forall t \geq t_1 + 1. \tag{74}$$

Using (6) and (72) we get

$$\frac{a(\|u\|_{H_0^1}^2)}{2}\|u\|_{H_0^1}^2 - \int_\Omega \mathcal{F}(u)dx - \int_\Omega h(x)udx \geq -\frac{\eta}{2}\|u\|_{L^2}^2 - \widetilde{C}\|u\|_{L^2}. \tag{75}$$

Now, integrating (70) from t to $t+1$, using (74), (75), by Proposition 15 we have

$$\int_t^{t+1} \|u_s\|_{L^2}^2 ds \leq \widetilde{L} + \gamma K_1 M^2 + \frac{\eta}{2}K^2 + \widetilde{C}K = \rho_1, \quad \forall t \geq t_1 + 1. \tag{76}$$

Hence, the last equation allow us to apply to (68) the Uniform Gronwall Lemma [34] to obtain

$$\|\frac{du}{dt}(t)\|_{L^2}^2 \leq \rho_2, \quad \forall t \geq t_1 + 2. \tag{77}$$

Finally, we multiply (3) by $-\Delta u$ and use (6) to obtain

$$\frac{m}{2}\|\Delta u\|_{L^2}^2 \leq \eta\|u\|_{H_0^1}^2 + \frac{1}{m}\|h\|_{L^2}^2 + \frac{1}{m}\|u_t\|_{L^2}^2.$$

Thus, by Proposition 16 and (77), we deduce that

$$\|u(t)\|_{H^2}^2 \leq \rho_3 \quad \forall t \geq t_1 + 2.$$

■

3.1.2 Abstract theory of attractors for multivalued semiflows

Prior to studying the case of non-uniqueness, we recall some well-known results concerning the structure of attractors for multivalued semiflows.

Consider a metric space (X, d) and a family of functions $\mathcal{R} \subset \mathcal{C}(\mathbb{R}_+; X)$. Denote by $P(X)$ the class of nonempty subsets of X. Then we define the multivalued map $G : \mathbb{R}_+ \times X \to P(X)$ associated with the family $\mathcal{R}$ as follows

$$G(t, u_0) = \{u(t) : u(\cdot) \in \mathcal{R}, u(0) = u_0\}. \tag{78}$$

In this abstract setting, the multivalued map G is expected to satisfy some properties that fit in the framework of multivalued dynamical systems. The first concept is given now.

Definition 20 *A multivalued map $G : \mathbb{R}_+ \times X \to P(X)$ is a multivalued semiflow (or m-semiflow) if $G(0, x) = x$ for all $x \in X$ and $G(t + s, x) \subset G(t, G(s, x))$ for all $t, s \geq 0$ and $x \in X$.*
If the above is not only an inclusion, but an equality, it is said that the m-semiflow is strict.

Once a multivalued semiflow is defined, we recall the following concepts.

Definition 21 *A map $\gamma : \mathbb{R} \to X$ is called a complete trajectory of $\mathcal{R}$ (resp. of G) if $\gamma(\cdot + h)\mid_{[0,\infty)} \in \mathcal{R}$ for all $h \in \mathbb{R}$ (resp. if $\gamma(t + s) \in G(t, \gamma(s))$ for all $s \in \mathbb{R}$ and $t \geq 0$).*

A point $z \in X$ is a fixed point of $\mathcal{R}$ if $\varphi(\cdot) \equiv z \in \mathcal{R}$. The set of all fixed points will be denoted by $\mathfrak{R}_\mathcal{R}$.

A point $z \in X$ is a stationary point of G if $z \in G(t, z)$ for all $t \geq 0$.

Definition 22 *Given an m-semiflow G a set $B \subset X$ is said to be negatively (positively) invariant if $B \subset G(t, B)$ $(G(t, B) \subset B)$ for all $t \geq 0$, and strictly invariant (or, simply, invariant) if it is both negatively and positively invariant.*

The set B is said to be weakly invariant if for any $x \in B$ there exists a complete trajectory γ of $\mathcal{R}$ contained in B such that $\gamma(0) = x$. We observe that weak invariance implies negative invariance.

Definition 23 *A set $\mathcal{A} \subset X$ is called a global attractor for the m-semiflow G if it is negatively invariant and it attracts all bounded subsets, i.e., $dist_X(G(t,B),\mathcal{A}) \to 0$ as $t \to +\infty$.*

Remark 24 *When $\mathcal{A}$ is compact, it is the minimal closed attracting set [28, Remark 5].*

In order to obtain a detailed characterization of the internal structure of a global attractor, we introduce an axiomatic set of properties on the set $\mathcal{R}$.

(K1) For any $x \in X$ there exists at least one element $\varphi \in \mathcal{R}$ such that $\varphi(0) = x$.

(K2) $\varphi_\tau(\cdot) := \varphi(\cdot + \tau) \in \mathcal{R}$ for any $\tau \geq 0$ and $\varphi \in \mathcal{R}$ (translation property).

(K3) Let $\varphi_1, \varphi_2 \in \mathcal{R}$ be such that $\varphi_2(0) = \varphi_1(s)$ for some $s > 0$. Then, the function φ defined by

$$\varphi(t) = \begin{cases} \varphi_1(t) & 0 \leq t \leq s, \\ \varphi_2(t-s) & s \leq t, \end{cases}$$

belongs to $\mathcal{R}$ (concatenation property).

(K4) For any sequence $\{\varphi^n\} \subset \mathcal{R}$ such that $\varphi^n(0) \to x_0$ in X, there exist a subsequence $\{\varphi^{n_k}\}$ and $\varphi \in \mathcal{R}$ such that $\varphi^{n_k}(t) \to \varphi(t)$ for all $t \geq 0$.

Remark 25 *If in assumption (K1), for every $x \in X$, there exists a unique $\varphi \in \mathcal{R}$ such that $\varphi(0) = x$, then the set $\{\varphi \in \mathcal{R} : \varphi(0) = x\}$ consists of a single trajectory φ, and the equality $G(t,x) = \varphi(t)$ defines a classical semigroup $G : \mathbb{R}^+ \times X \to X$.*

It is immediate to observe [11, Proposition 2] or [23, Lemma 9] that $\mathcal{R}$ fulfilling (K1) and (K2) gives rise to an m-semiflow G through (78), and if besides (K3) holds, then this m-semiflow is strict. In such a case, a global bounded attractor, supposing that it exists, is strictly invariant [28, Remark 8].

Several properties concerning fixed points, complete trajectories and global attractors are summarized in the following results [21].

Lemma 26 *Let (K1)-(K2) be satisfied. Then every fixed point (resp. complete trajectory) of $\mathcal{R}$ is also a fixed point (resp. complete trajectory) of G.*

If $\mathcal{R}$ fulfills (K1)-(K4), then the fixed points of $\mathcal{R}$ and G coincide. Besides, a map $\gamma : \mathbb{R} \to X$ is a complete trajectory of $\mathcal{R}$ if and only if it is continuous and a complete trajectory of G.

The standard well-known result in the single-valued case for describing the attractor as the union of bounded complete trajectories (see [24]) reads in the multivalued case as follows.

Theorem 27 *Consider $\mathcal{R}$ satisfying (K1) and (K2) and either (K3) or (K4). Assume that G possesses a compact global attractor $\mathcal{A}$. Then*

$$\mathcal{A} = \{\gamma(0) : \gamma \in \mathbb{K}\} = \cup_{t \in \mathbb{R}} \{\gamma(t) : \gamma \in \mathbb{K}\}, \tag{79}$$

where $\mathbb{K}$ denotes the set of all bounded complete trajectories in $\mathcal{R}$. Hence, $\mathcal{A}$ is weakly invariant.

We finish this section by stating a general result about the existence of attractors. We recall that the map $t \mapsto G(t,x)$ is upper semicontinuous if for any $x \in X$ and any neighborhood $O(G(t,x))$ in X there exists $\delta > 0$ such that if $d(y,x) < \delta$, then $G(t,y) \subset O$.

Theorem 28 *[28, Theorem 4 and Remark 8] Let the map $t \mapsto G(t,x)$ be upper semicontinuous with closed values. If there exists a compact attracting set K, that is,*

$$dist_X(G(t,B),K) \to 0, \text{ as } t \to +\infty,$$

for any bounded set B, then G possesses a global compact attractor $\mathcal{A}$, which is the minimal closed attracting set. If, moreover, G is strict, then $\mathcal{A}$ is invariant.

We observe that, although in the papers [28], [21] the space X is assumed to be complete, the results are true in a non-complete space.

3.1.3 The case of non-uniqueness

If we do not assume the additional assumptions on the function $a(\cdot)$ of Section 3.1.1 ensuring uniqueness of the Cauchy problem, we have to define a multivalued semiflow.

We have two possibilities: either to consider the conditions of Theorem 9 with an extra growth assumption or to use the conditions of Theorem 12.

If we assume conditions (5)-(7), (12), (17) and (60), then by Theorem 12 for any $u_0 \in L^2(\Omega)$ there exists at least one regular solution and (45) implies that $f(u) \in L^2(\varepsilon, T; L^2(\Omega))$ for any regular solution, so $\frac{du}{dt} \in L^2(\varepsilon, T; L^2(\Omega))$ as well. In this case, as $H_0^1(\Omega) \subset L^p(\Omega)$, we have that $u \in C((0, +\infty), H_0^1(\Omega)) \subset C((0, +\infty), L^p(\Omega))$.

If we assume conditions (5)-(7), (11), (13) and (60) as well, then we known by Theorem 9 that for any $u_0 \in L^2(\Omega)$ there exists at least one regular solution.

In order to obtain the necessary estimates leading to the existence of a global attractor, we need to ensure that

$$\frac{du}{dt} \in L^2(\varepsilon, T; L^2(\Omega)), \text{ for all } 0 < \varepsilon < T, \tag{80}$$

holds, as by [31, p.102] we obtain that

$$\frac{d}{dt}\|u\|_{H_0^1}^2 = 2(-\Delta u, u_t) \text{ for a.a. } t. \tag{81}$$

and $u \in C((0, +\infty), H_0^1(\Omega))$.

We note that the set of regular solutions of that kind is non-empty if we assume (17), as using inequalities (52)-(56) in the proof of Theorem 9 we prove that the regular solution satisfies (80).

We also observe that we can force all the regular solutions to satisfy $\frac{du}{dt} \in L^2(\varepsilon, T; L^2(\Omega))$ with an additional assumption on the constant p, which is weaker than (12). This is achieved by obtaining that $f(u) \in L^2(\varepsilon, T; L^2(\Omega))$, which can be done by using an interpolation inequality. Indeed, for $u \in L^\infty(\varepsilon, T; H_0^1(\Omega)) \cap L^2(\varepsilon, T; D(A))$ we have the interpolation inequality

$$\|u\|_{L^{2(\gamma+1)}(\varepsilon,T;L^{2(\gamma+1)}(\Omega))}^{2(\gamma+1)} \leq \|u\|_{L^\infty(\varepsilon,T;L^{p_1}(\Omega))}^{2\gamma} \|u\|_{L^2(\varepsilon,T;L^{p_2}(\Omega))}^2, \tag{82}$$

where $\gamma = \frac{4}{n-2}$, $p_1 = \frac{2n}{n-2}$, $p_2 = \frac{2n}{n-4}$, provided that $n > 4$; $\gamma < 2$, $p_1 = 4$, $p_2 = \frac{4}{2-\gamma}$ if $n = 4$; $\gamma = 3$, $p_1 = 6$, $p_2 = +\infty$ if $n = 3$; and $\gamma \geq 0$ is arbitrary for $n = 1, 2$. We have used the embeddings $H_0^1(\Omega) \subset L^{p_1}(\Omega)$, $H^2(\Omega) \subset L^{p_2}(\Omega)$ and [35, Lemma II.4.1, p. 72]. Thus, (8) implies that $f(u) \in L^2(\varepsilon, T; L^2(\Omega))$ if

$$p \leq \gamma + 2 \tag{83}$$

and also that

$$\|f(u)\|_{L^2(\varepsilon,T;L^2(\Omega))}^2 = \int_\varepsilon^T \int_\Omega |f(u(x,t))|^2 dxdt \leq C_1 + C_2 \int_\varepsilon^T \int_\Omega |u(x,t)|^{2(\gamma+1)} dxdt. \tag{84}$$

Condition (83) also implies $H_0^1(\Omega) \subset L^p(\Omega)$, so $u \in C((0, +\infty), L^p(\Omega))$.

Another necessary property to obtain estimates is the fact that $t \mapsto \int_\Omega \mathcal{F}(u(t))dx$ is absolutely continuous on $[\varepsilon, T]$ for all $0 < \varepsilon < T$ and

$$\frac{d}{dt}\int_\Omega \mathcal{F}(u(t))dx = \left(f(u(t)), \frac{du}{dt}(t)\right), \text{ for a.a. } t > 0. \tag{85}$$

This can be proved by regularization in both situations by using the regularity of regular solutions and either (45) or (84).

Therefore, under either the conditions of Theorem 9 with the extra assumption (83) or the conditions of Theorem 12 we define the set

$$\mathcal{R} = K_r^+ := \{u(\cdot) : u \text{ is a regular solution of (3)}\}.$$

We define the (possibly multivalued) map $G_r : \mathbb{R}^+ \times L^2(\Omega) \to P(L^2(\Omega))$ by

$$G_r(t, u_0) = \{u(t) : u \in K_r^+ \text{ and } u(0) = u_0\}.$$

With respect to the axiomatic properties $(K1) - (K4)$ given above, we observe that obviously $(K1)$ is true, and $(K2)$ can be proved easily using equality (19). Therefore, G_r is a multivalued semiflow by the results of the previous section. In this case we are not able to prove $(K3)$, so G_r could be non-strict. Further we will prove that $(K4)$ holds true.

Lemma 29 *Let us assume (5)-(7), (17) and (60). Additionally, assume one of the following assumptions:*

1. *(11) and (83) hold;*
2. *(12) is true.*

Given a sequence $\{u^n\} \subset K_r^+$ such that $u^n(0) \to u_0$ weakly in $L^2(\Omega)$, there exists a subsequence of $\{u^n\}$ (relabeled the same) and $u \in K_r^+$, satisfying $u(0) = u_0$, such that

$$u^n(t) \to u(t) \text{ strongly in } H_0^1(\Omega) \quad \forall t > 0.$$

Proof. We take an arbitrary $T > 0$. Arguing as in the proof of Theorem 9 we obtain the existence of a subsequence of u^n such that

$$\begin{aligned} \{u^n\} &\text{ is bounded in } L^\infty(0,T;L^2(\Omega)), \\ \{u^n\} &\text{ is bounded in } L^p(0,T;L^p(\Omega)), \\ \{f(u^n)\} &\text{ is bounded in } L^q(0,T;L^q(\Omega)). \end{aligned} \tag{86}$$

The only difference is that we obtain inequality (26) in an arbitrary interval $[\varepsilon, T]$ and then pass to the limit as $\varepsilon \to 0$ (see the proof of Proposition 15).

Since $\dfrac{du^n}{dt} \in L^2(\varepsilon, T; L^2(\Omega))$, for any $\varepsilon > 0$, we have that $u \in C((0,T], H_0^1(\Omega))$ and we know that (81), (85) are true. Therefore, arguing as in the proofs of Theorems 9 and 12 and using (84) and (45) there exists $u \in L^\infty(\varepsilon,T;L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega))$ and a subsequence $\{u^n\}$, relabelled the same, such that

$$\begin{aligned} u_n &\overset{*}{\rightharpoonup} u \text{ in } L^\infty(0,T;L^2(\Omega)) \\ u_n &\overset{*}{\rightharpoonup} u \text{ in } L^\infty(\varepsilon,T;H_0^1(\Omega)) \\ u_n &\rightharpoonup u \text{ in } L^2(0,T;H_0^1(\Omega)) \\ u_n &\rightharpoonup u \text{ in } L^p(0,T;L^p(\Omega)) \\ u_n &\rightharpoonup u \text{ in } L^2(\varepsilon,T;D(A)), \\ \frac{du_n}{dt} &\rightharpoonup \frac{du}{dt} \text{ in } L^2(\varepsilon,T;L^2(\Omega)) \\ f(u_n) &\rightharpoonup f(u) \text{ in } L^q(0,T;L^q(\Omega)), \\ f(u_n) &\rightharpoonup f(u) \text{ in } L^2(\varepsilon,T;L^2(\Omega)), \\ a(\|u_n\|^2_{H_0^1})\Delta u_n &\rightharpoonup a(\|u\|^2_{H_0^1})\Delta u \text{ in } L^2(\varepsilon,T;L^2(\Omega)). \end{aligned} \tag{87}$$

In view of (87), the Aubin-Lions Compactness Lemma gives

$$u_n \to u \text{ in } L^2(\varepsilon,T;H_0^1(\Omega)). \tag{88}$$

Since the sequence $\{u^n\}$ is equicontinuous in $L^2(\Omega)$ on $[\varepsilon, T]$ and bounded in $C([\varepsilon,T], H_0^1(\Omega))$, by the compact embedding $H_0^1(\Omega) \subset L^2(\Omega)$ and the Ascoli-Arzelà theorem, a subsequence fulfills

$$u^n \to u \text{ in } C([\varepsilon,T], L^2(\Omega)),$$

$$u^n(t) \rightharpoonup u(t) \text{ in } H_0^1(\Omega) \quad \forall t \in [\varepsilon, T].$$

By a similar argument as in the proof of Theorem 9 we establish that $u \in K_r^+$, $u(0) = u_0$.

Finally, we shall prove that $u^n(t) \to u(t)$ in $H_0^1(\Omega)$ for all $t \in [\varepsilon, T]$.

Multiplying (3) by u_t^n and using (36), (81), and (85) we obtain

$$\frac{1}{2}\left\|\frac{du^n}{dt}\right\|_{L^2}^2 + \frac{d}{dt}\left(\frac{1}{2}A(\|u^n(t)\|_{H_0^1}^2 - \int_\Omega \mathcal{F}(u^n(t))dx\right) \leq \frac{1}{2}\|h\|_{L^2}^2 = D.$$

Thus,

$$\frac{1}{2}A(\|u^n(t)\|_{H_0^1}^2) - \int_\Omega \mathcal{F}(u^n(t))dx \leq \frac{1}{2}A(\|u^n(s)\|_{H_0^1}^2) - \int_\Omega \mathcal{F}(u^n(s))dx + D(t-s),\ t \geq s \geq \varepsilon > 0.$$

The same inequality is valid for the limit function $u(\cdot)$. We observe that the map $y \longmapsto \int_\Omega \mathcal{F}(y(x))dx$ is continuous in the topology of $H_0^1(\Omega)$, which follows easily from $H_0^1(\Omega) \subset L^p(\Omega)$ and (10) using Lebesgue's theorem. Hence, the functions $J_n(t) = \frac{1}{2}A(\|u^n(t)\|_{H_0^1}^2) - \int_\Omega \mathcal{F}(u^n(t))dx - Dt$, $J(t) = \frac{1}{2}A(\|u(t)\|_{H_0^1}^2) - \int_\Omega \mathcal{F}(u(t))dx - Dt$ are continuous and non-increasing in $[\varepsilon, T]$. Moreover, from (88) we deduce that $J_n(t) \to J(t)$ for a.e. $t \in (\varepsilon, T)$. Take $\varepsilon < t_m < T$ such that $t_m \to T$ and $J_n(t_m) \to J(t_m)$ for all m. Then

$$J_n(T) - J(T) \leq J_n(t_m) - J(T) \leq |J_n(t_m) - J(t_m)| + |J(t_m) - J(T)|.$$

For any $\delta > 0$ there exist $m(\delta)$ and $N(m(\delta))$ such that $J^n(T) - J(T) \leq \delta$ if $n \geq N$. Then $\limsup J_n(T) \leq J(T)$, so $\limsup \|u^n(T)\|_{H_0^1}^2 \leq \|u(T)\|_{H_0^1}^2$ (see the explanation below). As $u^n(T) \to u(T)$ weakly in $H_0^1(\Omega)$ implies $\liminf \|u^n(T)\|_{H_0^1}^2 \geq \|u(T)\|_{H_0^1}^2$, we obtain

$$\|u^n(T)\|_{H_0^1}^2 \to \|u(T)\|_{H_0^1}^2,$$

so that $u^n(T) \to u(T)$ strongly in $H_0^1(\Omega)$.

In order to finish the proof rigorously, we have to justify that $\limsup J_n(T) \leq J(T)$ implies the inequality $\limsup \|u^n(T)\|_{H_0^1}^2 \leq \|u(T)\|_{H_0^1}^2$. First, we observe that by (10) we have

$$\left|\int_\Omega \mathcal{F}(u_n(T,x))dx\right| \leq C\int_\Omega \left(1 + |u_n(T,x)|^p\right)dx,$$

so the boundedness of $u_n(T)$ in $L^p(\Omega)$ implies that $-\int_\Omega \mathcal{F}(u_n(T,x))dx < \infty$. Also, (9) gives $-\mathcal{F}(u_n(T,x)) \geq -\widetilde{\kappa}$, so by Fatou's lemma we obtain

$$\begin{aligned}\liminf\ \left(-\int_\Omega \mathcal{F}(u_n(T,x))dx\right) &\geq \int_\Omega \liminf\left(-\mathcal{F}(u_n(T,x))\right)dx \\ &= -\int_\Omega \mathcal{F}(u(T,x))dx,\end{aligned}$$

where we have used that $\mathcal{F}(u_n(T,x) \to \mathcal{F}(u(T,x))$ for a.a. $x \in \Omega$. By contradiction let us assume that $\limsup \|u_n(T)\|_{H_0^1} > \|u(T)\|$. Then using the continuity of the function $A(s)$ we have

$$\begin{aligned}&\limsup\left(\frac{1}{2}A\left(\|u^n(T)\|_{H_0^1}^2\right) - \int_\Omega \mathcal{F}(u_n(T,x))dx\right) \\ &\geq \limsup\frac{1}{2}\int_0^{\|u^n(T)\|_{H_0^1}^2} a(s)\,ds + \liminf\left(-\int_\Omega \mathcal{F}(u_n(T,x))dx\right) \\ &\geq \frac{1}{2}\int_0^{\limsup\|u^n(T)\|_{H_0^1}^2} a(s)\,ds - \int_\Omega \mathcal{F}(u_n(T,x))dx \\ &> \frac{1}{2}\int_0^{\|u(T)\|_{H_0^1}^2} a(s)\,ds - \int_\Omega \mathcal{F}(u_n(T,x))dx,\end{aligned}$$

which is a contradiction with $\limsup J_n(T) \leq J(T)$. ∎

Corollary 30 *Assume the conditions of Lemma 29. Then the set K_r^+ satisfies condition $(K4)$.*

Proposition 31 *Assume the conditions of Lemma 29. The multivalued semiflow G_r is upper semicontinuous for all $t \geq 0$, that is, for any neighborhood $O(G_r(t,u_0))$ in $L^2(\Omega)$ there exists $\delta > 0$ such that if $\|u_0 - v_0\| < \delta$, then $G_r(t,v_0) \subset O$. Also, it has compact values.*

Proof. We argue by contradiction. Assume that there exists $t \geq 0, u_0 \in L^2(\Omega)$, a neighbourhood $O(G_r(t,u_0))$ and a sequence $\{y_n\}$ which fulfills that each $y_n \in G_r(t,u_0^n)$, where u_0^n converges strongly to u_0 in $L^2(\Omega)$, and $y_n \notin O(G_r(t,u_n))$ for all $n \in \mathbb{N}$. Since $y_n \in G_r(t,u_0^n)$ for all n, there exists $u^n \in K_r^+$, $u^n(0) = u_0^n$, such that $y_n = u^n(t)$. Now, since $\{u_0^n\}$ is a convergent sequence of initial data, making use of Lemma 29 there exists a subsequence of $\{u^n\}$ which converges to a function $u \in K_r^+$. Hence, $y_n \to y \in G_r(t,u_0)$. This is a contradiction because $y_n \notin O(G_r(t,u_0))$ for any $n \in \mathbb{N}$. ∎

Proposition 32 *Assume the conditions of Lemma 29. Then there exists an absorbing set B_1 for G_r, which is compact in $H_0^1(\Omega)$ and $L^p(\Omega)$.*

Proof. Reasoning as in Proposition 15, we obtain an absorbing set B_0 in $L^2(\Omega)$.

Let $K > 0$ be such that $\|y\| \leq K$ for all $y \in B_0$. Since $\dfrac{du}{dt} \in L\left(\varepsilon, T; L^2(\Omega)\right)$ and (85) holds, we are allowed to multiply (3) by u_t, use (81) and argue as in (52)-(55) to obtain the existence of a constant C such that

$$\|u(1)\|_{H_0^1}^2 + \|u(1)\|_{L^p}^p \leq C(1 + \|u(0)\|_{L^2}^2), \tag{89}$$

for any regular solution $u(\cdot)$ with initial condition $u(0)$.

For any $u_0 \in L^2(\Omega)$ with $\|u_0\|_{L^2} \leq R$ and any $u \in K_r^+$ such that $u(0) = u_0$, the semiflow property $G_r(t+1,u_0) \subset G_r(1, G_r(t,u_0))$ and $G_r(t,u_0) \subset B_0$, if $t \geq t_0(R)$, imply that

$$\|u(t+1)\|_{H_0^1}^2 + \|u(t+1)\|_{L^p}^p \leq C(1+K^2)\ \forall t \geq t_0(R).$$

Then there exists $M > 0$ such that the closed ball B_M in $H_0^1(\Omega)$ centered at 0 with radius M is absorbing for G_r.

By Lemma 29 the set $B_1 = \overline{G_r(1,B_M)}$ is an absorbing set which is compact in $H_0^1(\Omega)$. The embedding $H_0^1(\Omega) \subset L^p(\Omega)$ implies that it is compact in $L^p(\Omega)$ as well. ∎

Theorem 33 *Assume the conditions of Lemma 29. Then the multivalued semiflow G_r possesses a global compact attractor $\mathcal{A}_r$. Moreover, for any set B bounded in $L^2(\Omega)$ we have*

$$dist_{H_0^1}(G_r(t,B), \mathcal{A}_r) \to 0 \quad \text{as } t \to \infty. \tag{90}$$

Also $\mathcal{A}_r$ is compact in $H_0^1(\Omega)$ and $L^p(\Omega)$.

Proof. From Propositions 31 and 32 we deduce that the multivalued semiflow G_r is upper semicontinuous with closed values and the existence of an absorbing which is compact in $H_0^1(\Omega)$ and $L^p(\Omega)$. Therefore, by Theorem 28 the existence of the global attractor and its compactness in $H_0^1(\Omega)$ and $L^p(\Omega)$ follow.

The proof of (90) is analogous to that in Theorem 29 in [21]. ∎

The set of all complete trajectories of K_r^+ (see Definition 21) will be denoted by $\mathbb{F}_r$. Moreover, we write $\mathbb{K}_r$ as the set of all complete trajectories which are bounded in $L^2(\Omega)$, and $\mathbb{K}_r^1$ as the ones bounded in $H_0^1(\Omega)$.

Lemma 34 *Assume the conditions of Lemma 29. Then the sets defined above coincide, that is, $\mathbb{K}_r = \mathbb{K}_r^1$.*

Proof. Let $\gamma(\cdot) \in \mathbb{K}_r$. Then there is C such that $\|\gamma(t)\|_{L^2} \leq C$ for any $t \in \mathbb{R}$. Let $u_\tau(\cdot) = \gamma(\cdot + \tau)$ for any τ, which is a regular solution. Since $\dfrac{du}{dt} \in L^2(\varepsilon, T; L^2(\Omega))$, for any $\varepsilon > 0$, the equality (81) holds true. Also, (85) is satisfied. Therefore, we can multiply the equation in (3) by u_t and apply again similar arguments as in Theorem 12 to deduce that

$$\|u(t+r)\|_{H_0^1}^2 \leq \frac{K_1(T)(1 + \|u(0)\|_{L^2}^2)}{r} + K_2(T) \text{ for any } 0 < r < T. \tag{91}$$

Denote $B_\gamma = \cup_{t\in\mathbb{R}}\gamma(t)$. Therefore,

$$B_\gamma \subset G_r(1, B_\gamma)$$

and (91) implies that B_γ is bounded in $H_0^1(\Omega)$, so $\gamma(\cdot) \in \mathbb{K}_r^1$.
The other inclusion is obvious. ■

In view of Corollary 30 and Theorem 27, the global attractor is characterized in terms of bounded complete trajectories:

$$\begin{aligned} \mathcal{A}_r &= \{\gamma(0) : \gamma(\cdot) \in \mathbb{K}_r\} = \{\gamma(0) : \gamma(\cdot) \in \mathbb{K}_r^1\} \\ &= \bigcup_{t\in\mathbb{R}}\{\gamma(t) : \gamma(\cdot) \in \mathbb{K}_r\} = \bigcup_{t\in\mathbb{R}}\{\gamma(t) : \gamma(\cdot) \in \mathbb{K}_r^1\}. \end{aligned} \tag{92}$$

The set $\mathfrak{R}_{K_r^+}$ was defined in the previous section as the set of fixed points of K_r^+, which means that $z \in \mathfrak{R}_{K_r^+}$ if the function $u(\cdot)$ defined by $u(t) = z$, for all $t \geq 0$, belongs to K_r^+. This set can be characterized as follows.

Lemma 35 *Assume the conditions of Lemma 29. Let $\mathfrak{R}$ be the set of $z \in H^2(\Omega) \cap H_0^1(\Omega)$ such that*

$$-a(\|z\|^2_{H_0^1})\Delta z = f(z) + h \text{ in } L^2(\Omega). \tag{93}$$

Then $\mathfrak{R}_{K_r^+} = \mathfrak{R}$.

Proof. If $z \in \mathfrak{R}_{K_r^+}$, then $u(t) \equiv z \in K_r^+$. Thus, $u(\cdot)$ satisfies (19) and $\dfrac{du}{dt} = 0$ in $L^2\left(0, T; L^2(\Omega)\right)$, so (93) is satisfied.

Let $z \in \mathfrak{R}$. Then the map $u(t) \equiv z$ satisfies (93) for any $t \geq 0$ and $\dfrac{du}{dt} = 0$ in $L^2\left(0, T; L^2(\Omega)\right)$, so (19) holds true. ■

The following result is proved exactly as Theorem 18.

Theorem 36 *Assume the conditions of Lemma 29. Then the global attractor $\mathcal{A}$ is bounded in $L^\infty(\Omega)$, provided that $h \in L^\infty(\Omega)$.*

We are now ready to obtain the characterization of the global attractor.

Theorem 37 *Assume the conditions of Lemma 29. Then it holds that*

$$\mathcal{A}_r = M_r^u(\mathfrak{R}) = M_r^s(\mathfrak{R}),$$

where

$$M_r^s(\mathfrak{R}) = \{z : \exists\gamma(\cdot) \in \mathbb{K}_r,\ \gamma(0) = z,\ \ dist_{\ L^2(\Omega)}(\gamma(t), \mathfrak{R}) \to 0,\ t \to +\infty\}, \tag{94}$$

$$M_r^u(\mathfrak{R}) = \{z : \exists\gamma(\cdot) \in \mathbb{F}_r,\ \gamma(0) = z,\ \ dist_{\ L^2(\Omega)}(\gamma(t), \mathfrak{R}) \to 0,\ t \to -\infty\}. \tag{95}$$

Remark 38 *In the definition of $M_r^u(\mathfrak{R})$ we can replace $\mathbb{F}_r$ by $\mathbb{K}_r$. Also, as the global attractor $\mathcal{A}$ is compact in $H_0^1(\Omega)$, in the definitions of $M_r^s(\mathfrak{R})$ and $M_r^u(\mathfrak{R})$, it is equivalent to write $H_0^1(\Omega)$ instead of $L^2(\Omega)$.*

Proof. We consider the function $E : \mathcal{A}_r \to \mathbb{R}$

$$E(y) = \frac{1}{2}A(\|y\|^2_{H_0^1}) - \int_\Omega \mathcal{F}(y(x))dx - \int_\Omega h(x)\, y(x)\, dx, \tag{96}$$

where $A(r) = \int_0^r a(s)ds$. We observe that $E(y)$ is continuous in $H_0^1(\Omega)$. Indeed, the maps $y \mapsto \frac{1}{2}A(\|y\|^2_{H_0^1})$, $y \mapsto \int_\Omega h(x)\, y(x)\, dx$ are obviously continuous in $H_0^1(\Omega)$. On the other hand, both conditions (12) and (83) imply that $H_0^1(\Omega) \subset L^p(\Omega)$, so making use of the Lebesgue theorem the continuity of $y \mapsto \int_\Omega \mathcal{F}(y(x))dx$ follows as well.

Since $\frac{du}{dt} \in L^2\left(\varepsilon, T; L^2(\Omega)\right)$ and (85) holds for any $u \in K_r^+$ and $0 < \varepsilon < T$, we obtain the energy equality

$$\int_s^t \|\frac{d}{dr}u(r)\|_{L^2}^2 dr + E(u(t)) = E(u(s)) \quad \text{for all } t \geq s > 0. \tag{97}$$

Hence, $E(u(t))$ is non-increasing and, by (6) and (9), bounded from below. Thus, $E(u(t)) \to l$, as $t \to +\infty$, for some $l \in \mathbb{R}$.

Let $x \in \mathcal{A}_r$ and $\gamma(0) = x$, where $\gamma \in \mathbb{K}_r$. We reason by contradiction, so let suppose that there exists $\varepsilon > 0$ and a sequence $\gamma(t_n)$, $t_n \to +\infty$, such that

$$dist_{\ L^2(\Omega)}(\gamma(t_n), \mathfrak{R}) > \varepsilon.$$

In view of Theorem 33, $\mathcal{A}_r$ is compact in $H_0^1(\Omega)$, so we can take a converging subsequence (relabeled the same) such that $\gamma(t_n) \to y$ in $H_0^1(\Omega)$, where $t_n \to +\infty$. Since the function $E : H_0^1(\Omega) \to \mathbb{R}$ is continuous, it follows that $E(y) = l$. We obtain a contradiction by proving that $y \in \mathfrak{R}$. In view of Lemma 29, there exists $v \in K_r^+$ and a subsequence $v_n(\cdot) = \gamma(\cdot + t_n)$ such that $v(0) = y$ and $v_n(t) \to v(t) = z$ in $H_0^1(\Omega)$ for $t > 0$. Thus, $E(v_n(t)) \to E(z)$ implies that $E(z) = l$. Also, $v(\cdot)$ satisfies the energy equality for all $0 \leq s \leq t$, so that

$$l + \int_0^t \|v_r\|_{L^2}^2 dr = E(z) + \int_0^t \|v_r\|_{L^2}^2 dr = E(v(0)) = E(y) = l.$$

Therefore, $\frac{dv}{dt}(t) = 0$ for a.a. t, and then by Lemma 35 we have $y \in \mathfrak{R}_{K_r^+} = \mathfrak{R}$. As a consequence, $\mathcal{A}_r \subset M_r^s(\mathfrak{R})$. The converse inclusion follows from (92).

For the second equality we observe that for any $\gamma \in \mathbb{F}_r$ the energy equality (97) is satisfied for all $-\infty < s \leq t$. Let $x \in \mathcal{A}_r$ and let $\gamma \in \mathbb{K}_r = \mathbb{K}_r^1$ (cf. Lemma 34) be such that $\gamma(0) = x$. Since the second term of the energy function is bounded from above by (9), $E(\gamma(t)) \to l$, as $t \to -\infty$, for some $l \in \mathbb{R}$. We reason as before, so let suppose that there exists $\varepsilon > 0$ and a sequence $\gamma(-t_n)$, $t_n \to \infty$, such that

$$dist_{\ L^2(\Omega)}(\gamma(-t_n), \mathfrak{R}) > \varepsilon,$$

and we have that $\gamma(-t_n) \to y$ in $H_0^1(\Omega)$, $E(y) = l$. Moreover, for a fixed $t > 0$, there exists $v \in K_r^+$ and a subsequence of $v_n(\cdot) = \gamma(\cdot - t_n)$ (relabeled the same) such that $v(0) = y$ and $v_n(t) \to v(t) = z$ in $H_0^1(\Omega)$. Therefore, $E(v_n(t)) \to E(z)$ implies that $E(z) = l$ and reasoning as before we get a contradiction since it follows that $y \in \mathfrak{R}$. Hence, $\mathcal{A}_r \subset M_r^u(\mathfrak{R})$ and the converse inclusion follows from (92). ■

We can improve the regularity of the global attractor of the semigroup T_r of Section 3.1.1 and obtain its characterization

Lemma 39 *Let the conditions of Theorem 17 hold. Then the global attractor $\mathcal{A}_r$ of the semigroup T_r is compact in $H_0^1(\Omega)$, bounded in $L^p(\Omega)$ and the convergence takes place in the topology of $H_0^1(\Omega)$, that is,*

$$dist_{H_0^1(\Omega)}(T_r(t, B), \mathcal{A}) \to 0, \ \text{as } t \to +\infty,$$

for any set B bounded in $L^2(\Omega)$.

Proof. The estimates of Lemma 29 can be justified for T_r via Galerkin approximations, so in this case we do not need to impose assumption (83) in order to use (85). Thus, the proof follows the same lines as in Proposition 32 and Theorem 33. ■

Proposition 40 *Let the conditions of Theorem 17 hold. Also, assume one of the following conditions:*

1. $h \in L^\infty(\Omega)$;
2. $p \leq \frac{2n}{n-2}$ *if* $n \geq 3$.

Then the global attractor $\mathcal{A}_r$ can be characterized as follows:

$$\mathcal{A}_r = M_r^u(\mathfrak{R}) = M_r^s(\mathfrak{R}),$$

where $M_r^s(\mathfrak{R})$, $M_r^u(\mathfrak{R})$ are defined in (94)-(95).

Proof. We recall that a function $E : \mathcal{A} \to \mathbb{R}$ is a Lyapunov functional if E is continuous (with respect to the topology of $H_0^1(\Omega)$), for any $u_0 \in \mathcal{A}$ the map $t \mapsto E(T_r(t, u_0))$ is non-increasing and $E(T_r(\tau, u_0)) = E(u_0)$, for some $\tau > 0$, implies that $u(\cdot)$ is a fixed point. We estate that the function E given in (96) is a Lyapunov functional for the semigroup T_r.

We prove that $E(y)$ is continuous. First, the maps $y \mapsto \frac{1}{2}A(\|y\|^2_{H_0^1})$, $y \mapsto \int_\Omega h(x)\, y(x)\, dx$ are obviously continuous in $H_0^1(\Omega)$. Second, if $h \in L^\infty(\Omega)$, taking into account that $\mathcal{A}$ is bounded in $L^\infty(\Omega)$ by Theorem 18, it follows that

$$\left| \int_\Omega \mathcal{F}(y_1) - \mathcal{F}(y_2) dx \right| = \left| \int_\Omega \int_{y_2(x)}^{y_1(x)} f(s) ds dx \right| \leq \int_\Omega C_1 |y_1(x) - y_2(x)| dx \leq C_2 \|y_1 - y_2\|_{L^2},$$

so $y \mapsto \int_\Omega \mathcal{F}(y(x))dx$ is continuous as well. In the case of the second condition, this result follows from the embedding $H_0^1(\Omega) \subset L^p(\Omega)$ and the Lebesgue theorem.

Multiplying the equation in (3) by u_t we obtain the energy inequality

$$\int_s^t \|\frac{d}{dr}u(r)\|^2_{L^2} dr + E(u(t)) \leq E(u(s)), \quad \text{for all } t \geq s,$$

if $u(\cdot)$ is a bounded complete trajectory of T_r. This calculation is rigorous when $h \in L^\infty(\Omega)$ as the boundedness of the solutions in $L^\infty(\mathbb{R}; L^\infty(\Omega))$ implies by regularization that (85) is true. Under the second condition, the calculations are formal but can be justified via Galerkin approximations. Hence, $E(u(t))$ is non-increasing as a function of t. Also, if $E(u(\tau)) = E(u_0)$, then $\|\frac{du}{dt}(t)\|^2_{L^2} = 0$ for a.a. $0 < t < \tau$, so u must be a fixed point.

The result follows then from [3, p.160]. ■

3.2 Strong solutions

We split this part into two cases.

3.2.1 Attractor in the phase space $H_0^1(\Omega)$

If we assume conditions (5)-(7), (60) and that either p satisfies (12) or that (11) is satisfied, then we know by Theorems 10 and 11 that for any $u_0 \in H_0^1(\Omega) \cap L^p(\Omega)$ there exists at least one strong solution $u(\cdot)$.

In the first case, $H_0^1(\Omega) \subset L^p(\Omega)$ implies that $H_0^1(\Omega) \cap L^p(\Omega) = H_0^1(\Omega)$. This is also true in the second case if we assume additionally that (83) holds true. Under such assumptions we define then the set

$$\mathcal{R} = K_s^+ := \{u(\cdot) : u \text{ is a strong solution of (3) with } u(0) \in H_0^1(\Omega)\}.$$

We define the (possibly multivalued) map $G_s : \mathbb{R}^+ \times H_0^1(\Omega) \to P(H_0^1(\Omega))$ by

$$G_s(t, u_0) = \{u(t) : u \in K_s^+ \text{ and } u(0) = u_0\}.$$

With respect to the axiomatic properties $(K1) - (K4)$ given above, property $(K1)$ is obviously true, and $(K2) - (K3)$ can be proved easily using equality (19). Therefore, G_s is a strict multivalued semiflow by the results of Section 3.1.2.

We shall obtain a similar result as in Lemma 29.

Lemma 41 *Let assume conditions (5)-(7), (60). Additionally, assume one of the following assumptions:*

1. *(11) and (83) hold;*

2. (12) is true.

Given a sequence $\{u^n\} \subset K_s^+$ such that $u^n(0) \to u_0$ weakly in $H_0^1(\Omega)$, there exists a subsequence of $\{u^n\}$ (relabeled the same) and $u \in K_s^+$, satisfying $u(0) = u_0$, such that

$$u^n(t) \to u(t) \quad \text{in } H_0^1(\Omega),\ \forall t > 0.$$

Proof. Since $\dfrac{du^n}{dt} \in L^2\left(0,T;L^2\left(\Omega\right)\right)$ and (85) hold, we can use (81) and multiplying (3) by u_t and integrating between s and t we obtain

$$\int_s^t \|\frac{d}{dr}\|u(r)\|_{L^2}^2 dr + E(u(t)) = E(u(s)) \quad \text{ for all } t \geq s \geq 0,$$

where E was defined in (96). Therefore, by (6) and (9) we have that

$$\int_0^t \|\frac{d}{dr}u(r)\|_{L^2}^2 dr + \frac{m}{4}\|u(t)\|_{H_0^1}^2 + \widetilde{\alpha}_1\|u(t)\|_{L^p}^p \leq \frac{1}{2}A(\|u(0)\|_{H_0^1}^2) + \widetilde{\alpha}_2\|u(0)\|_{L^p}^p + K_1\left\|u\left(0\right)\right\|_{L^2}^2 + K_2 \quad (98)$$

holds for all $t > 0$.

In the first case, multiplying by $-\Delta u$, integrating over $(0,T)$ and using (98) it follows that

$$\frac{1}{2}\|u(T)\|_{H_0^1}^2 + \frac{m}{2}\int_0^T \|\Delta u(s)\|_{L^2}^2 ds \leq \eta\int_0^T \|u(s)\|_{H_0^1}^2 ds + \frac{1}{2}\|u(0)\|_{H_0^1}^2 + K_3 \leq K_4\left(T\right), \quad (99)$$

for all $T > 0$. In the second case, combining (98) with (45) the boundedness of $f\left(u^n\right)$ in $L^2\left(0,T;L^2\left(\Omega\right)\right)$ follows for any $T > 0$. Hence, the equality

$$a\left(\|u\|_{H_0^1}^2\right)\Delta u = \frac{du^n}{dt} - f\left(u^n\right) - h$$

and (6) imply that u^n is bounded in $L^2\left(0,T;D(A)\right)$.

Thus, the sequence $\{u^n\}$ is bounded in $L^\infty(0,T;H_0^1(\Omega)) \cap L^2(0,T;D(A))$ and $\dfrac{du^n}{dt}$, $f\left(u^n\right)$ are bounded in $L^2(0,T;L^2(\Omega))$, for all $T > 0$. Therefore, there is u such that

$$u^n \stackrel{*}{\rightharpoonup} u \text{ in } L^\infty(0,T;H_0^1(\Omega)),$$

$$u^n \rightharpoonup u \text{ in } L^2(0,T;D(A)),$$

$$u_t^n \rightharpoonup u_t \text{ in } L^2(0,T;L^2(\Omega)),.$$

Arguing in a similar way as in Theorem 9 we have

$$u_n \to u \text{ in } L^2(0,T;H_0^1(\Omega)),$$

$$u_n(t,x) \to u(t,x) \text{ a.e. on } (0,T)\times\Omega,$$

$$f\left(u^n\right) \rightharpoonup f\left(u\right) \text{ in } L^2(0,T;L^2(\Omega)),$$

$$a(\|u_n\|_{H_0^1}^2)\Delta u_n \rightharpoonup a(\|u\|_{H_0^1}^2)\Delta u \text{ in } L^2(0,T;L^2(\Omega)).$$

Hence, we can pass to the limit and obtain that $u \in K_s^+$. Following the same lines of Theorem 10 we check that $u\left(0\right) = u_0$.

Moreover, arguing as in Lemma 29 we obtain

$$u^n(t) \to u(t) \text{ in } H_0^1(\Omega) \text{ for all } t > 0.$$

■

Corollary 42 *Assume the conditions of Lemma 41. Then the set K_s^+ satisfies condition $(K4)$.*

Using Lemma 41 and reasoning as before the following result holds.

Proposition 43 *Assume the conditions of Lemma 41. Then the map $G_s(t,\cdot)$ is upper semicontinuous for all $t \geq 0$ with compact values.*

Proposition 44 *Assume the conditions of Lemma 41 and (17). Then there exists an absorbing set B_1 for G_s, which is compact in $H_0^1(\Omega)$ and $L^p(\Omega)$.*

Proof. The proof follows the same lines of that in Proposition 32 but using Lemma 41. ■

From these results and Theorem 28 we obtain the existence of the global attractor.

Theorem 45 *Assume the conditions of Lemma 41 and (17). Then the multivalued semiflow G_s possesses a global compact invariant attractor $\mathcal{A}_s$, which is compact in $L^p(\Omega)$.*

Lemma 46 *Assume the conditions of Lemma 41 and (17). Then $\mathcal{A}_s = \mathcal{A}_r$, where $\mathcal{A}_r$ is the global attractor in Theorem 33.*

Proof. Since $G_s(t,u_0) \subset G_r(t,u_0)$ for all $u_0 \in H_0^1(\Omega)$, it is clear that $\mathcal{A}_r$ is a compact attracting set. Hence, the minimality of the global attractor gives $\mathcal{A}_s \subset \mathcal{A}_r$.

Let $z \in \mathcal{A}_r$. Since $z = \gamma(0)$, where $\gamma \in \mathbb{K}_r^1$, and $\gamma\mid_{[s,+\infty)}$ is a strong solution of (3) for any $s \in \mathbb{R}$, we get that $z \in G_s(t_n, \gamma(-t_n))$ for $t_n \to +\infty$. Hence,

$$dist\,(z, \mathcal{A}_s) \leq dist\,(G_s(t_n, \gamma(-t_n)), \mathcal{A}_s) \to 0 \text{ as } n \to \infty,$$

so $z \in \mathcal{A}_s$. ■

The set of all complete trajectories of K_s^+ (see Definition 21) will be denoted by $\mathbb{F}_s$. Let $\mathbb{K}_s$ be the set of all complete trajectories which are bounded in $H_0^1(\Omega)$.

In view of Theorem 27, the global attractor is characterized in terms of bounded complete trajectories:

$$\mathcal{A}_s = \{\gamma(0) : \gamma(\cdot) \in \mathbb{K}_s\} = \bigcup_{t\in\mathbb{R}} \{\gamma(t) : \gamma(\cdot) \in \mathbb{K}_s\}. \tag{100}$$

In the same way as in Lemma 35 we obtain that $\mathfrak{R}_{K_s^+} = \mathfrak{R}$.

Reasoning as in Theorem 18 we obtain the following result.

Theorem 47 *Assume the conditions of Lemma 41 and (17). Then the global attractor $\mathcal{A}_s$ is bounded in $L^\infty(\Omega)$, provided that $h \in L^\infty(\Omega)$.*

Following the same procedure of Theorem 37 we can prove an analogous characterization of the global attractor.

Theorem 48 *Assume the conditions of Lemma 41 and (17).Then it holds that*

$$\mathcal{A}_s = M_s^u(\mathfrak{R}) = M_s^s(\mathfrak{R}),$$

where

$$M_s^s(\mathfrak{R}) = \{z : \exists\gamma(\cdot) \in \mathbb{K}_s,\ \gamma(0) = z,\ \ dist\ _{H_0^1(\Omega}(\gamma(t), \mathfrak{R}) \to 0,\ t \to +\infty\}, \tag{101}$$

$$M_s^u(\mathfrak{R}) = \{z : \exists\gamma(\cdot) \in \mathbb{F}_s,\ \gamma(0) = z,\ \ dist_{H_0^1(\Omega}(\gamma(t), \mathfrak{R}) \to 0,\ t \to -\infty\}. \tag{102}$$

Remark 49 *In the definition of $M_s^u(\mathfrak{R})$ we can replace $\mathbb{F}_r$ by $\mathbb{K}_r$.*

Let us consider now the particular situation when G_s is single-valued semigroup. Under the conditions (5)-(7), (11), (60), (83), if we assume additionally that (14) is satisfied, then by Theorem 14 for any $u_0 \in H_0^1(\Omega)$ there exists a unique strong solution $u(\cdot)$. Then we can define the following semigroup $T_s : \mathbb{R}^+ \times H_0^1(\Omega) \to H_0^1(\Omega)$:

$$T_s(t, u_0) = u(t),$$

where $u(\cdot)$ is the unique strong solution to (3). We recall also that $u \in C([0,T], H_0^1(\Omega))$ for any $T > 0$. Also, by Lemma 41 if $u_0^n \to u_0$ weakly in $H_0^1(\Omega)$, then $T_s(t, u_0^n) \to T(t, u_0)$ in $H_0^1(\Omega)$ for all $t > 0$.

Since $T_s = G_s$, by Theorems 45, 47, 48 and Lemma 46 we obtain the following results.

Theorem 50 *Assume the conditions (5)-(7), (11), (17), (60), (83) and (14). Then the semigroup T_s possesses a global invariant attractor $\mathcal{A}_s$, which is compact in $H_0^1(\Omega)$ and $L^p(\Omega)$.*

Lemma 51 *Under the conditions of Theorem 50, $\mathcal{A}_s = \mathcal{A}_r$, where $\mathcal{A}_r$ is the attractor of Theorem 17.*

Theorem 52 *Assume the conditions of Theorem 50. Then the global attractor $\mathcal{A}_s$ is bounded in $L^\infty(\Omega)$ provided that $h \in L^\infty(\Omega)$.*

As before, we denote by $\mathfrak{R}$ the set of fixed points of T_s. Also, the global attractor is the union of all bounded complete trajectories

$$\mathcal{A}_s = \{\phi(0) : \phi \text{ is a bounded complete trajectory of } T_s\}.$$

Theorem 53 *Assume the conditions of Theorem 50. Then the global attractor $\mathcal{A}_s$ can be characterized as follows*

$$\mathcal{A}_s = M_s^u(\mathfrak{R}) = M_s^s(\mathfrak{R}),$$

where the sets $M_s^u(\mathfrak{R})$, $M_s^s(\mathfrak{R})$ are defined in (101)-(102).

In this case we can obtain additionally that the attractor is bounded in $H^2(\Omega)$.

Proposition 54 *Assume the conditions of Theorem 50 and also that (15) holds true. Then $\mathcal{A}_s$ is bounded in $H^2(\Omega)$.*

Proof. The proof follows the same lines as in Proposition 19, so we omit it. ■

3.2.2 Attractor in the phase space $H_0^1(\Omega) \cap L^p(\Omega)$

We consider the metric space $X = H_0^1(\Omega) \cap L^p(\Omega)$ endowed with the induced topology of the space $H_0^1(\Omega)$.

If we assume conditions (5)-(7), (11), (14) and (60), then by Theorems 10 and 14 for any $u_0 \in H_0^1(\Omega) \cap L^p(\Omega)$ there exists a unique strong solution $u(\cdot)$. Then we can define the following semigroup $T_s : \mathbb{R}^+ \times X \to X$:

$$T_s(t, u_0) = u(t),$$

where $u(\cdot)$ is the unique strong solution to (3). We recall also that $u \in C([0,T], H_0^1(\Omega)) \cap C_w([0,T], L^p(\Omega))$ for any $T > 0$.

Lemma 55 *Assume conditions (5)-(7), (11), (14) and (60). If $u_0^n \to u_0$ weakly in $H_0^1(\Omega) \cap L^p(\Omega)$, then $T_s(t, u_0^n) \to T_s(t, u_0)$ strongly in $H_0^1(\Omega)$ and weakly in $L^p(\Omega)$ for any $t > 0$.*

Proof. Repeating the same proof of Lemma 41 we obtain that $T_s(t, u_0^n) \to T_s(t, u_0)$ strongly in $H_0^1(\Omega)$ for all $t > 0$. We observe that in this case the estimates are justified via Galerkin approximations, so we do not need condition (83) in order to provide property (85).

Finally, by the Ascoli-Arzelà theorem we deduce

$$u^n \to u \text{ in } C([0,T], L^2(\Omega))$$

and combining this with (98) we infer that

$$u^n(t) \rightharpoonup u(t) \text{ in } L^p(\Omega) \ \forall t \geq 0.$$

■

Proposition 56 *Assume the conditions of Lemma 55 and (17). Then there exists an absorbing set B_1 for T_s, which is compact in $H_0^1(\Omega)$ and bounded $L^p(\Omega)$.*

Proof. Following the same lines of that in Proposition 32 (and justifying the estimates via Galerkin approximations), we obtain that there exists $M > 0$ such that the closed ball B_M in $H_0^1(\Omega) \cap L^p(\Omega)$ centered at 0 with radius M is absorbing for T_s. By Lemma 55 the set $B_1 = \overline{T_s(1, B_M)}$ is an absorbing set which is compact in $H_0^1(\Omega)$ and bounded in $L^p(\Omega)$. ■

Theorem 57 *We assume the conditions of Lemma 55 and (17). Then the semigroup T_s possesses a global attractor $\mathcal{A}_s$, which is compact in X and bounded in $L^p(\Omega)$.*

Proof. We cannot apply directly the general theory of attractors for semigroup because we do not know whether the semigroup T_s is continuous with respect to the initial datum in X.

We state that

$$\mathcal{A}_s = \omega(B_1) = \{y : \exists t_n \to +\infty,\ y_n \in T_s(t_n, B_1) \text{ such that } y_n \to y \text{ in } X\}$$

is a global compact attractor. The fact that set $\omega(B_1)$ is non-empty, compact and the minimal closed set attracting B_1 can be proved in a standard way (see for example Theorem 10.5 in [30]). Since B_1 is absorbing, $\omega(B_1)$ attracts any bounded set B. As $\omega(B_1) \subset B_1$, $\mathcal{A}_s$ is bounded in $L^p(\Omega)$.

We need to prove that it is invariant.

First, we prove that it is negatively invariant. Let $y \in \mathcal{A}_s$ and $t > 0$ be arbitrary. We take a sequence $y_n \in T_s(t_n, B_1)$ such that $y_n \to y$, $t_n \to +\infty$. Since $T_s(t_n, B_1) = T_s(t, T_s(t_n - t, B_1))$, there are $x_n \in T_s(t_n - t, B_1)$ such that $y_n = T_s(t, x_n)$. As for n large $T_s(t_n - t, B_1) \subset B_1$, the sequence $\{x_n\}$ is bounded in $L^p(\Omega)$ and relatively compact in $H_0^1(\Omega)$. Hence, up to a subsequence $x_n \to x \in \mathcal{A}_s$ weakly in $L^p(\Omega)$ and strongly in $H_0^1(\Omega)$. We deduce by Lemma 55 that $T_s(t, x_n) \to T_s(t, x)$ weakly in $L^p(\Omega)$ and strongly in $H_0^1(\Omega)$. Thus, $y = T_s(t, x) \subset T_s(t, \mathcal{A}_s)$.

Second, we prove that it is positively invariant. As $\mathcal{A}_s = T_s(\tau, \mathcal{A}_s)$ for any $\tau \geq 0$, this follows from

$$dist_X(T_s(t, \mathcal{A}_s), \mathcal{A}_s) = dist_X(T_s(t, T_s(\tau, \mathcal{A}_s)), \mathcal{A}_s) = dist_X(T_s(t + \tau, \mathcal{A}_s), \mathcal{A}_s) \underset{\tau \to +\infty}{\to} 0.$$

■

Lemma 58 *Under the conditions of Theorem 57, $\mathcal{A}_s = \mathcal{A}_r$, where $\mathcal{A}_r$ is the attractor of Theorem 17.*

Proof. Since $T_r(t, u_0) = T_s(t, u_0)$ for any $u_0 \in X$, we have

$$dist_{L^2}(\mathcal{A}_s, \mathcal{A}_r) = dist_{L^2}(T_s(t, \mathcal{A}_s), \mathcal{A}_r) = dist_{L^2}(T_r(t, \mathcal{A}_s), \mathcal{A}_r) \underset{t \to +\infty}{\to} 0,$$

so $\mathcal{A}_s \subset \mathcal{A}_r$. In the same way,

$$dist_X(\mathcal{A}_r, \mathcal{A}_s) = dist_X(T_r(t, \mathcal{A}_r), \mathcal{A}_s) = dist_X(T_s(t, \mathcal{A}_r), \mathcal{A}_s) \underset{t \to +\infty}{\to} 0,$$

and then $\mathcal{A}_r \subset \mathcal{A}_s$. ■

The following two theorems are proved in the same way as Theorem 18 and Proposition 40

Theorem 59 *Assume the conditions of Theorem 57. Then the global attractor $\mathcal{A}_s$ is bounded in $L^\infty(\Omega)$ provided that $h \in L^\infty(\Omega)$.*

As before, we denote by $\mathfrak{R}$ the set of fixed points of T_s. Also, the global attractor is the union of all bounded complete trajectories

$$\mathcal{A}_s = \{\phi(0) : \phi \text{ is a bounded complete trajectory of } T_s\}.$$

Theorem 60 *We assume the conditions of Theorem 57 and one of the following assumptions:*

1. $h \in L^\infty(\Omega)$;
2. $p \leq \frac{2n}{n-2}$ *if* $n \geq 3$.

Then the global attractor $\mathcal{A}_s$ can be characterized as follows

$$\mathcal{A}_s = M_s^u(\mathfrak{R}) = M_s^s(\mathfrak{R}),$$

where the sets $M_s^u(\mathfrak{R})$, $M_s^s(\mathfrak{R})$ are defined in (101)-(102).

We obtain additionally that the attractor is bounded in $H^2(\Omega)$.

Proposition 61 *Assume the conditions of Theorem 57 and also that (15) is satisfied. Then $\mathcal{A}_s$ is bounded in $H^2(\Omega)$.*

Proof. The proof follows the same lines as in Proposition 19, so we omit it. ■

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About the structure of attractors for a nonlocal Chafee-Infante problem

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Abstract

In this paper, we study the structure of the global attractor for the multivalued semiflow generated by a nonlocal reaction-diffusion equation in which we cannot guarantee uniqueness of the Cauchy problem.

First, we analyse the existence and properties of stationary points, showing that the problem undergoes the same cascade of bifurcations as in the Chafee-Infante equation. Second, we study the stability of the fixed points and establish that the semiflow is dynamically gradient. We prove that the attractor consists of the stationary points and their heteroclinic connections and analyse some of the possible connections.

Keywords: reaction-diffusion equations, nonlocal equations, global attractors, multivalued dynamical systems, structure of the attractor, stability, Morse decomposition

AMS Subject Classification (2010): 35B40, 35B41, 35B51, 35K55, 35K57

1 Introduction

Ordinary and partial differential equations play a key role in modeling for all sciences: Engineering, Physics, Chemistry, Biology, Medicine, Economy and many others. The right understanding of the behavior of solutions (in particular, well-posedness versus blow-up) means not only to predict the future of trajectories but also to establish strategies for control (i.e. optimization). Concerning PDE and Economy, it is interesting to cite the nice survey [9] and the references therein on many different problems dealing with effects as aggregation and repulsion, optimal control, mean-field games, and so on as applications.

Parabolic PDE models reflect the diffusion phenomena due to *local touching of molecules* and dissipation of energy, and when different internal and external factors come into play, it links naturally to some reaction-diffusion models, as the growth versus capacity of the environment in Biology or the endogenous growth versus the neoclassical theories in economy. In particular, capital accumulation distribution in space and time following spatial extensions of the continuous Ramsey model [35] by Brito [6, 7, 8] and others later uses the semilinear parabolic PDE

$$\partial_t u - \alpha \Delta u = f(u) - c.$$

This spatiality introduces important issues about the steady states distribution as well as the dynamic evolution, convergence, local interaction among local agents, and so on.

Not for the sake of generality but for real modeling purposes, in the last two decades the increment of nonlocal PDE models that attempt to capture in a more accurate way the real spreading of the problem (density of population, capital accumulation, consumption or prices and innovation indexes, and so on) has been very important. Firstly we might comment about extensions by using some nonlocal operators acting in the right-hand side of the PDE and/or the boundary conditions as integral operators, leading to integro-differential equations. Among others we can cite [4] for a system coupling capital and pollution stock model, a population dynamic model in [28]

$$\partial_t u - \alpha \Delta u = u \left(f(u) - \alpha \int_{\mathbb{R}^N} g(x-y) u(y,t) dy \right),$$

the elliptic (stationary) counterpart in population/physics models as the Fischer-KPP [2], or a logistic model [27]. Secondly, we wish to point out that the nonlocal extensions have also been performed on the diffusion operators as well. The literature about fractional laplacian is vast nowadays. However, let us concentrate in an intermediate step. Coming originally from modeling of bacteria population in Biology, the introduction of a nonlocal viscosity in front of the laplacian has become an interesting problem for different applications and for its mathematical study, as for example occurs in the equation

$$u_t - a(\int_\Omega g(y) u(t,y) dy) \Delta u = f(t).$$

In this way, the spreading (or aggregating/concentrating) effects are given by the increasing (resp. non-increasing) function a as a viscosity nonlocal coefficient. One should cite Prof. Chipot and his collaborators [20, 21, 22, 23, 24, 25, 39] among others for a detailed analysis including existence, uniqueness, steady states and convergence of evolutionary solutions to equilibria.

When the reaction term f depends on the unknown u

$$u_t - a(\Phi_\Omega(u(t)) \Delta u = f(t,u) \tag{1}$$

(here the functional Φ_Ω may represent a general nonlocal functional acting over the whole domain Ω, for instance $\|u(t)\|^2_{H^1_0}$ or $\int_\Omega g(y)u(t,y)dy$) equilibria are difficult to analyse. Opposite to ordinary differential equations, the analysis of existence of stationary states for the above problem is much more involved. Also, comparing with reaction-diffusion equations with local diffusion, another difficulty is that in general a Lyapunov functional is not known to exist in most cases.

The dynamical analysis of problem (1) and in particular the existence of global attractors have been established till now in several papers (cf. [3, 11, 12, 14, 15]). Other differential operators as the p-laplacian coupled with nonlocal viscosity has also been considered (cf. [13, 15, 16]). However, in general little is known about the internal structure of the attractor, which is very important as it gives us a deep insight into the long-term dynamics of the problem. When we manage to obtain a Lyapunov functional some insights can be obtained.

If we consider the non-local equation

$$\frac{\partial u}{\partial t} - a(\|u\|^2_{H^1_0}) \frac{\partial^2 u}{\partial x^2} = \lambda f(u) \tag{2}$$

with Dirichlet boundary conditions, then it is possible to define a suitable Lypaunov functional. In [11] it is shown that regular and strong solutions generate (possibly) multivalued semiflows having a global attractor which is described by the unstable set of the stationary points. Although this is already a good piece of information, our goal is to describe the structure of the attractor as accurately as possible. For this aim we need to study the particular situation where the domain is one-dimensional and the function f is of the type of the standard Chafee-Infante problem, for which the dynamics inside the attractor has been completely understood [29].

The first step when studying the structure of the attractor consists in analysing the stationary points. In the case where the function f is odd and equation (2) generates a continuous semigroup the existence of fixed points of the type given in the Chafee-Infante problem was established in [18]. Moreover, if a is non-decreasing, then they coincide with the ones in the Chafee-Infante problem and, moreover, in [19] the stability and hyperbolicity of the fixed points is studied. In this paper we extend these results for a more general function f (not necessarily odd and for which we do not known whether the Cauchy problem has a unique solution or not), showing that equation (2) undergoes the same cascade of bifurcations as the Chafee-Infante equation. Moreover, when we allow the function a to decrease, though the problem possesses at least the same fixed point as in the Chafee-Infante problem, we show that more equilibria can appear. For a non-decreasing function a and an odd function f we prove also that even when uniqueness fails the stability of the fixed points is the same as for the corresponding ones in the Chafee-Infante problem. Finally, we are able to prove that in this last case the semiflow is dynamically gradient with respect to the disjoint family of isolated weakly invariant sets generated by the equilibria, which is ordered by the number of zeros of the fixed points. More precisely, the attractor consists of the set of equilibria and their heteroclinic connections and a connection from a fixed point to another is allowed only if the number of zeros of the first one is greater.

In Section 3 we study the existence of strong solutions of the Cauchy problem in the space H_0^1. In Section 4 we prove that strong solutions generate a multivalued semiflow in H_0^1 having a global attractor which is equal to the unstable set of the stationary points. In Section 5 we study the existence and properties of equilibria. In Section 6 we analyse the stability of the fixed points and establish that the semiflow is dynamically gradient.

2 Setting of the problem

Let us consider the following problem

$$\begin{cases} \dfrac{\partial u}{\partial t} - a(\|u\|^2_{H_0^1})\dfrac{\partial^2 u}{\partial x^2} = \lambda f(u) + h(t), \quad t>0, x\in\Omega, \\ u(t,0)=u(t,1)=0, \\ u(0,x)=u_0(x), \end{cases} \tag{3}$$

where $\Omega=(0,1)$ and $\lambda>0$. Throughout the paper we will use the following conditions (but not all of them at the same time):

(A1) $f\in C(\mathbb{R})$.

(A2) $f(0)=0$.

(A3) $f'(0)$ exists and $f'(0)=1$.

(A4) f is strictly concave if $u>0$ and strictly convex if $u<0$.

(A5) Growth and dissipation conditions: for $p\geq 2,\ C_i>0,\ i=1,..,4$, we have

$$|f(u)|\leq C_1+C_2|u|^{p-1}, \tag{4}$$

$$f(u)u\leq C_3-C_4|u|^p, \text{ if } p>2, \tag{5}$$

$$\limsup_{u\to\pm\infty}\frac{f(u)}{u}\leq 0, \text{ if } p=2. \tag{6}$$

(A6) The function $a\in C(\mathbb{R}^+)$ satisfies:

$$a(s)\geq m>0.$$

(A7) The function $a\in C(\mathbb{R}^+)$ satisfies:

$$a(s)\leq M_1,\quad \forall s\geq 0,$$

where $M_1>0$.

(A8) The function $a\in C(\mathbb{R}^+)$ is non-decreasing.

(A9) $h\in L^2_{loc}\left(0,+\infty;L^2\left(\Omega\right)\right).$

(A10) h does not depend on time and $h \in L^2(\Omega)$.

We define the function $\mathcal{F}(u) = \int_0^u f(s)ds$. We observe that from (4) we have

$$|\mathcal{F}(s)| \leq \widetilde{C}(1 + |s|^p) \quad \forall s \in \mathbb{R}, \tag{7}$$

whereas (5) implies

$$\mathcal{F}(s) \leq \widetilde{\kappa} - \widetilde{\alpha}_1 |s|^p. \tag{8}$$

Also, from condition (6) it follows that for all $\varepsilon > 0$, there exists a constant $M > 0$ such that $\frac{f(u)}{u} \leq \varepsilon$, for all $|u| \geq M$. Hence, there exists $m_\varepsilon > 0$ such that

$$f(u)u \leq m_\varepsilon + \varepsilon u^2, \quad \forall u \in \mathbb{R}. \tag{9}$$

In addition, it follows that

$$\mathcal{F}(u) \leq \varepsilon u^2 + C_\varepsilon, \tag{10}$$

where $C_\varepsilon > 0$. These two inequaities are also true under condition (5).

The main aim of this paper consists in describing in as much detail as possible the internal structure of the global attractor in a similar way as for the classical Chafee-Infante equation.

Some of these conditions will be used all the time, whereas other ones will be used only in certain results. In particular, the function h will be considered as a time-dependent function satisfying (A9) only for establishing the existence of solution for problem (3). However, since we will study the asymptotic behaviour of solutions in the autonomous situation, for the second part concerning the existence and properties of global attractors the function h will be time-independent, so assumption (A10) will be used instead. Finally, in order to study the structure of the global attractors in terms of the stationary points and their possible heteroclinic connections we will assume that $h \equiv 0$.

Throughout the paper, $\|\cdot\|_X$ will denote the norm in the Banach space X.

3 Existence of solutions

In this section we will establish the existence of strong solutions for problem (3) with initial condition in the phase space $H_0^1(\Omega)$. Although we will follow the same lines of a similar result given in [11], we would like to point out that in the present case, as we are working in a one-dimensional problem, the assumptions on the function f are much weaker. In particular, we do not need to impose a growth assumption of any kind.

Definition 1 *For $u_0 \in L^2(\Omega)$, a weak solution to (3) is an element $u \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega))$, for any $T > 0$, such that*

$$\frac{d}{dt}(u,v) + a(\|u\|^2_{H_0^1})(\nabla u, \nabla v) = \lambda(f(u),v) + (h(t),v) \quad \forall v \in H_0^1(\Omega), \tag{11}$$

where the equation is understood in the sense of distributions.

As usual, let $A : D(A) \to H,\ D(A) = H^2(\Omega) \cap H_0^1(\Omega)$, be the operator $A = -\dfrac{d^2}{dx^2}$ with Dirichlet boundary conditions. This operator is the generator of a C_0-semigroup $T(t) = e^{-At}$.

Definition 2 *For $u_0 \in H_0^1(\Omega)$, a strong solution to (3) is a weak solution with the extra regularity $u \in L^\infty(0,T;H_0^1(\Omega))$, $u \in L^2(0,T;D(A))$ and $\dfrac{du}{dt} \in L^2(0,T;L^2(\Omega))$ for any $T > 0$.*

Remark 3 *We observe that if u is a strong solution, then $u \in C([0,T];H_0^1(\Omega))$ (see [37, p.102]). By this way, the initial condition makes sense.*

Remark 4 *Since* $\frac{du}{dt} \in L^2\left(0,T;L^2\left(\Omega\right)\right)$ *for any strong solution, in this case equality (11) is equivalent to the following one:*

$$\int_0^T \int_\Omega \frac{du\left(t,x\right)}{dt}\xi\left(t,x\right)dxdt - \int_0^T a(\|u(t)\|^2_{H_0^1}) \int_\Omega \frac{\partial^2 u}{\partial x^2}\xi dxdt \tag{12}$$
$$= \int_0^T \int_\Omega \lambda f\left(u\left(t,x\right)\right)\xi\left(t,x\right)dxdt + \int_0^T \int_\Omega h\left(t,x\right)\xi\left(t,x\right)dxdt,$$

for all $\xi \in L^2\left(0,T;L^2\left(\Omega\right)\right)$.

Theorem 5 *Assume conditions (A1), (A6) and (A9). Assume also the existence of constants* $\beta,\gamma > 0$ *such that*

$$f\left(u\right)u \leq \gamma + \beta u^2 \text{ for all } u \in \mathbb{R}. \tag{13}$$

Then, for any $u_0 \in H_0^1(\Omega)$ *problem (3) has at least one strong solution.*

Remark 6 *Assumption (13) is weaker than the dissipative property (9) as the constant* ε *is arbitrarily small. Due to the fact that we are working in a one-dimensional domain, no growth condition of the type given in (A5) is necessary in order to prove existence of solutions. Also, (13) implies that*

$$F\left(u\right) \leq \widetilde{\gamma} + \widetilde{\beta}u^2 \tag{14}$$

for some constants $\widetilde{\gamma}, \widetilde{\beta} > 0$.

Proof. Consider a fixed value $T > 0$. In order to use the Faedo-Galerkin method let $\{w_j\}_{j\geq 1}$ be the sequence of eigenfunctions of $-\Delta$ in $H_0^1(\Omega)$ with homogeneous Dirichlet boundary conditions, which forms a special basis of $L^2(\Omega)$. Since Ω is a bounded regular domain, it is known that $\{w_j\} \subset H_0^1(\Omega)$ and that $\cup_{n\in\mathbb{N}} V_n$ is dense in the spaces $L^2(\Omega)$ and $H_0^1(\Omega)$, where $V_n = span[w_1,\ldots,w_n]$. As usual, P_n will be the orthogonal projection in $L^2\left(\Omega\right)$, that is

$$z_n := P_n z = \sum_{j=1}^n (z, w_j)w_j,$$

and λ_j will be the eigenvalues associated to the eigenfunctions w_j. For each integer $n \geq 1$, we consider the Galerkin approximations

$$u_n(t) = \sum_{j=1}^n \gamma_{nj}(t)w_j,$$

which are given by the following nonlinear ODE system

$$\begin{cases} \frac{d}{dt}(u_n, w_i) + a(\|u_n\|^2_{H_0^1})(\nabla u_n, \nabla w_i) = \lambda(f(u_n), w_i) + (h, w_i) \quad \forall i = 1,\ldots,n, \\ u_n(0) = P_n u_0. \end{cases} \tag{15}$$

We observe that $P_n u_0 \to u_0$ in $H_0^1(\Omega)$. This Cauchy problem possesses a solution on some interval $[0, t_n)$ and by the estimates in the space $L^2(\Omega)$ of the sequence $\{u_n\}$ given below for any $T > 0$ such a solution can be extended to the whole interval $[0, T]$.

Firstly, multiplying the equation in (15) by $\gamma_{ni}(t)$ and summing from $i = 1$ to n, we obtain

$$\frac{1}{2}\frac{d}{dt}\|u_n(t)\|^2_{L^2} + a(\|u_n\|^2_{H_0^1})\|u_n(t)\|^2_{H_0^1} = \lambda(f(u_n(t), u_n(t)) + (h(t), u_n(t)) \quad \text{for a.e. } t \in (0, t_n). \tag{16}$$

Using the Young and Poincaré inequalities we deduce that

$$(h(t), u_n(t)) \leq \frac{m}{2}\|u_n(t)\|^2_{H_0^1} + \frac{1}{2\lambda_1 m}\|h(t)\|^2_{L^2},$$

where m is the constant from (A6). Hence, from (A6), (13) and (16) it follows that

$$\frac{1}{2}\frac{d}{dt}\|u_n(t)\|_{L^2}^2 + \frac{m}{2}\|u_n(t)\|_{H_0^1}^2 \leq \lambda\gamma|\Omega| + \beta\lambda\|u_n(t)\|_{L^2}^2 + \frac{1}{2\lambda_1 m}\|h(t)\|_{L^2}^2.$$

We infer that

$$\begin{aligned} \|u_n(t)\|_{L^2}^2 &\leq \|u_n(0)\|_{L^2}^2 e^{2\beta\lambda t} + \int_0^t e^{2\beta\lambda(t-s)}\left(2\lambda\gamma|\Omega| + \frac{1}{\lambda_1 m}\|h(s)\|_{L^2}^2\right)ds \\ &\leq \|u_n(0)\|_{L^2}^2 e^{2\beta\lambda T} + K_1(T). \end{aligned} \tag{17}$$

Therefore, the solution exists on any given interval $[0,T]$ and

$$\{u_n\} \text{ is bounded in } L^\infty(0,T;L^2(\Omega)). \tag{18}$$

Now, we multiply the equation (3) by $\frac{du_n}{dt}$ to obtain

$$\|\frac{du_n}{dt}(t)\|_{L^2}^2 + a(\|u_n\|_{H_0^1}^2)\frac{1}{2}\frac{d}{dt}\|u_n\|_{H_0^1}^2 = \frac{d}{dt}\int_\Omega \lambda\mathcal{F}(u_n)dx + (h(t),\frac{du_n}{dt}).$$

Introducing

$$A(s) = \int_0^s a(r)dr \tag{19}$$

we have

$$\frac{1}{2}\|\frac{du_n}{dt}(t)\|_{L^2}^2 + \frac{d}{dt}\left(\frac{1}{2}A(\|u_n\|_{H_0^1}^2) - \int_\Omega \lambda\mathcal{F}(u_n)dx\right) \leq \frac{1}{2}\|h(t)\|_{L^2}^2.$$

Integrating the previous expression between 0 and t we get

$$\begin{aligned} &\frac{1}{2}A(\|u_n(t)\|_{H_0^1}^2) + \lambda\int_\Omega \mathcal{F}(u_n(0))dx + \frac{1}{2}\int_0^t \|\frac{d}{ds}u_n(s)\|_{L^2}^2 ds \\ &\leq \frac{1}{2}A(\|u_n(0)\|_{H_0^1}^2) + \lambda\int_\Omega \mathcal{F}(u_n(t))dx + \frac{1}{2}\int_0^t \|h(s)\|_{L^2}^2 ds. \end{aligned} \tag{20}$$

By (A6), (14) and (17) it follows that

$$\begin{aligned} &\frac{m}{2}\|u_n(t)\|_{H_0^1}^2 + \lambda\int_\Omega \mathcal{F}(u_n(0))dx + \frac{1}{2}\int_0^t \|\frac{d}{ds}u_n(s)\|_{L^2}^2 ds \\ &\leq \frac{1}{2}A(\|u_n(0)\|_{H_0^1}^2) + \lambda\widetilde{\beta}\|u_n(t)\|_{L^2}^2 + \lambda\widetilde{\gamma}|\Omega| + K_2(T) \\ &\leq \frac{1}{2}A(\|u_n(0)\|_{H_0^1}^2) + \lambda\widetilde{\beta}e^{2\beta\lambda T}\|u_n(0)\|_{L^2}^2 + K_3(T). \end{aligned} \tag{21}$$

Since $dim(\Omega)=1$, $H_0^1(\Omega)\subset L^\infty(\Omega)$, so $u_n(0)$ is bounded in $L^\infty(\Omega)$. Thus, as f maps bounded sets of $\mathbb{R}$ into bounded ones, $\mathcal{F}(u_n(0))$ is bounded in $L^\infty(\Omega)$ as well. Therefore, we deduce that

$$\{u_n\} \text{ is bounded in } L^\infty(0,T;H_0^1(\Omega))$$

and

$$\frac{du_n}{dt} \text{ is bounded in } L^2(0,T;L^2(\Omega)). \tag{22}$$

Using again the embedding $H_0^1(\Omega)\subset L^\infty(\Omega)$ we obtain that u_n is bounded in the space $L^\infty(0,T;L^\infty(\Omega))$. Thus,

$$f(u_n) \text{ is bounded in } L^\infty(0,T;L^\infty(\Omega)). \tag{23}$$

Also, we deduce that $\|u_n(t)\|_{H_0^1}^2$ is uniformly bounded in $[0,T]$ and then by the continuity of the function $a(\cdot)$ we get that the sequence $a\left(\|u_n(t)\|_{H_0^1}^2\right)$ is also uniformly bounded in $[0,T]$.

Finally, multiplying (15) by $\lambda_j \gamma_{ni}(t)$ and summing from $i = 1$ to n we obtain

$$\frac{1}{2}\frac{d}{dt}\|u_n\|^2_{H^1_0} + m\|\Delta u_n\|^2_{L^2} \leq \lambda(f(u_n), -\Delta u_n) + (h(t), -\Delta u).$$

By (23) and applying the Young inequality, we get

$$\frac{1}{2}\frac{d}{dt}\|u_n\|^2_{H^1_0} + m\|\Delta u_n\|^2_{L^2} \leq \frac{\lambda^2}{m}\|f(u_n)\|^2_{L^2} + \frac{m}{4}\|\Delta u_n\|^2_{L^2} + \frac{1}{m}\|h(t)\|^2_{L^2} + \frac{m}{4}\|\Delta u\|^2_{L^2}.$$

Integrating the previous expression between 0 and t, it follows that

$$\|u_n(t)\|^2_{H^1_0} + m\int_0^t \|\Delta u_n(s)\|^2_{L^2} ds \leq \|u_n(0)\|^2_{H^1_0} + \frac{2\lambda^2}{m}\int_0^t \|f(u_n(s))\|^2_{L^2} ds + \frac{2}{m}\int_0^t \|h(s)\|^2_{L^2} ds.$$

Taking into account (23), the last inequality implies that

$$u_n \text{ is bounded in } L^2(0,T;D(A)), \tag{24}$$

so $\{-\Delta u_n\}$ and $\{a(\|u_n\|^2_{H^1_0})\Delta u_n\}$ are bounded in $L^2(0,T;L^2(\Omega))$.

As a consequence, there exists $u \in L^\infty(0,T;H^1_0(\Omega))$ and a subsequence u_n (relabeled the same) such that

$$\begin{aligned} u_n &\overset{*}{\rightharpoonup} u \text{ in } L^\infty(0,T;H^1_0(\Omega)),\\ u_n &\rightharpoonup u \text{ in } L^2(0,T;D(A)),\\ f(u_n) &\overset{*}{\rightharpoonup} \chi \text{ in } L^\infty(0,T;L^\infty(\Omega)),\\ a(\|u_n\|^2_{H^1_0}) &\overset{*}{\rightharpoonup} b \text{ in } L^\infty(0,T), \end{aligned} \tag{25}$$

where $\rightharpoonup$ ($\overset{*}{\rightharpoonup}$) stands for the weak (weak star) convergence. By (22) and (24) the Aubin-Lions Compactness Lemma gives that $u_n \to u$ in $L^2(0,T;H^1_0(\Omega))$, so $u_n(t) \to u(t)$ in $H^1_0(\Omega)$ a.e. on $(0,T)$. Consequently, there exists a subsequence u_n, relabelled the same, such that $u_n(t,x) \to u(t,x)$ a.e. in $\Omega \times (0,T)$.

Moreover, thanks to the inequality

$$\|u_n(t_2) - u_n(t_1)\|^2_{L^2} = \left\|\int_{t_1}^{t_2} \frac{d}{dt}u_n(s)ds\right\|^2_{L^2} \leq \|\frac{d}{dt}u_n\|^2_{L^2(0,T;L^2(\Omega))}\ |t_2 - t_1| \quad \forall t_1, t_2 \in [0,T],$$

(21), (22) and $H^1_0(\Omega) \subset\subset L^2(\Omega)$, the Ascoli-Arzelà theorem implies that $\{u_n\}$ converges strongly in $C([0,T];L^2(\Omega))$ for all $T > 0$. Therefore, we obtain from (21) that $u_n(t) \rightharpoonup u(t)$ in $H^1_0(\Omega)$, for any $t \geq 0$.

Also, by (25) we have that $P_n f(u_n)) \rightharpoonup \chi$ in $L^q(0,T;L^q(\Omega))$ for any $q \geq 1$ (see [36, p.224]). Since f is continuous, it follows that $f(u_n(t,x)) \to f(u(t,x))$ a.e. in $\Omega \times (0,T)$. Therefore, in view of (25), by [33, Lemma 1.3] we have that $\chi = f(u)$.

As a consequence, by the continuity of a we get that

$$a(\|u_n(t)\|^2_{H^1_0}) \to a(\|u(t)\|^2_{H^1_0}) \quad \text{a.e. on } (0,T).$$

Since the sequence is uniformly bounded, by Lebesgue's theorem this convergence takes place in $L^2(0,T)$, so $b = a(\|u\|^2_{H^1_0})$. Thus,

$$a(\|u_n\|^2_{H^1_0})\Delta u_n \rightharpoonup a(\|u\|^2_{H^1_0})\Delta u, \quad \text{in} \quad L^2(0,T;L^2(\Omega)).$$

Therefore, we can pass to the limit to conclude that u is a strong solution.

It remains to show that $u(0) = u_0$ which makes sense since $u \in C([0,T];H^1_0(\Omega))$ (see Remark 4). Indeed, let be $\phi \in C^1([0,T];H^1_0(\Omega))$ with $\phi(T) = 0$, $\phi(0) \neq 0$. We multiply the equation in (3) and (15) by ϕ and integrate by parts in the t variable to obtain that

$$\begin{aligned} &\int_0^T \Big(-(u(t),\phi'(t)) - a(\|u(t)\|^2_{H^1_0})(\Delta u(t),\phi(t))\Big)\,dt \\ &= \int_0^T (\lambda f(u(t)) + h(t),\phi(t))dt + (u(0),\phi(0)), \end{aligned} \tag{26}$$

$$
\begin{aligned}
&\int_0^T \left(-(u_n(t),\phi'(t)) - a(\|u_n(t)\|^2_{H_0^1})(\Delta u_n(t),\phi(t))\right) dt \qquad (27)\\
&= \int_0^T (\lambda f(u_n(t)) + h(t),\phi(t))dt + (u_n(0),\phi(0)).
\end{aligned}
$$

In view of the previous convergences, we can pass to the limit in (27). Taking into account (26) and bearing in mind $u_n(0) = P_n u_0 \to u_0$, since $\phi(0) \in H_0^1(\Omega)$ is arbitrary, we infer that $u(0) = u_0$. ■

4 Existence and structure of attractors

In this section, we will prove the existence of a global attractor for the semiflow generated by strong solutions in the autonomous case. Thus, the function h will be an independent of time function satisfying (A10) instead of (A9). Also, we will establish that the attractor is equal to the unstable set of the stationary points (see the definition in (45)).

Throughout this section, for a metric space X with metric d we will denote by $dist_X(C,D)$ the Hausdorff semidistance from C to D, that is,

$$dist_X(C,D) = \sup_{c\in C}\inf_{d\in D}\rho(c,d).$$

Let us consider the phase space $X = H_0^1(\Omega)$ and the sets

$$K(u_0) = \{u(\cdot) : u \text{ is a strong solution of (3) such that } u(0) = u_0\},$$

$$\mathcal{R} = \cup_{u_0\in X} K(u_0).$$

Denote by $P(X)$ the class of nonempty subsets of X. We define the (possibly multivalued) map $G : \mathbb{R}^+ \times X \to P(X)$ by

$$G(t,u_0) = \{u(t) : u \in \mathcal{R} \text{ and } u(0) = u_0\}. \qquad (28)$$

In order to study the map G let us consider the following axiomatic properties of the set $\mathcal{R}$:

(K1) For every $x \in X$ there is $\phi \in \mathcal{R}$ satisfying $\phi(0) = x$.

(K2) $\phi_\tau(\cdot) := \phi(\cdot + \tau) \in \mathcal{R}$ for every $\tau \geq 0$ and $\phi \in \mathcal{R}$ (translation property).

(K3) Let $\phi_1, \phi_2 \in \mathcal{R}$ be such that $\phi_2(0) = \phi_1(s)$ for some $s > 0$. Then, the function ϕ defined by

$$\phi(t) = \begin{cases} \phi_1(t) & 0 \leq t \leq s, \\ \phi_2(t-s) & s \leq t, \end{cases}$$

belongs to $\mathcal{R}$ (concatenation property).

(K4) For every sequence $\{\phi^n\} \subset \mathcal{R}$ satisfying $\phi^n(0) \to x_0$ in X, there is a subsequence $\{\phi^{n_k}\}$ and $\phi \in \mathcal{R}$ such that $\phi^{n_k}(t) \to \phi(t)$ for every $t \geq 0$.

Assuming conditions (A1), (A6), (A10) and (13) property (K1) follows from Theorem 5, whereas (K2)-(K3) can be proved easily using equality (12). By [17, Proposition 2] or [32, Lemma 9] we know that $\mathcal{R}$ fulfilling (K1) and (K2) gives rise to a multivalued semiflow G through (28) (m-semiflow for short), which means that:

- $G(0,x) = x$ for all $x \in X$;
- $G(t+s,x) \subset G(t,G(s,x))$ for all $t,s \geq 0$ and $x \in X$.

Moreover, (K3) implies that the m-semiflow is strict, that is, $G(t+s,x) = G(t,G(s,x))$ for all $t,s \geq 0$ and $x \in X$.

We will show first that the m-semiflow G possesses a bounded absorbing set in the space $L^2(\Omega)$ and that property (K4) is satisfied.

Lemma 7 *Assume conditions (A1), (A6), (A10) and (13). Given* $\{u^n\} \subset \mathcal{R}$, $u^n(0) \to u_0$ *weakly in* $H_0^1(\Omega)$, *there exists a subsequence of* $\{u^n\}$ *(relabeled the same) and* $u \in K(u_0)$ *such that*

$$u^n(t) \to u(t) \quad \text{in } H_0^1(\Omega),\ \forall t > 0.$$

Also, if $u^n(0) \to u_0$ *strongly in* $H_0^1(\Omega)$, *then for* $t_n \to 0$ *we get* $u^n(t_n) \to u_0$ *strongly in* $H_0^1(\Omega)$.

Proof. Since $\dfrac{du^n}{dt} \in L^2(0,T;L^2(\Omega))$ and $u^n \in L^2(0,T;H_0^1(\Omega))$, we have by [37, pg. 102] that

$$\frac{d}{dt}\|u^n\|^2_{H_0^1} = 2(-\Delta u^n, u_t^n) \ \text{ for a.a. } t \tag{29}$$

and $u^n \in C([0,T];H_0^1(\Omega))$. Also, as $f(u^n) \in L^\infty(0,T;L^\infty(\Omega))$, by regularization one can show that $(F(u^n(t)),1)$ is an absolutely continuous function on $[0,T]$ and

$$\frac{d}{dt}(F(u^n(t)),1) = (f(u^n(t)), \frac{du^n}{dt}) \ \text{ for a.a. } t > 0. \tag{30}$$

By a similar argument as in Theorem 5, there is a subsequence of u^n such that

$$\begin{aligned} u^n &\text{ is bounded in } L^\infty(0,T;L^\infty(\Omega)),\\ u^n &\text{ is bounded in } L^\infty(0,T;H_0^1(\Omega)),\\ f(u^n) &\text{ is bounded in } L^\infty(0,T;L^\infty(\Omega)),\\ u^n &\text{ is bounded in } L^2(0,T;D(A)). \end{aligned} \tag{31}$$

Therefore, arguing as in the proof of Theorem 5, there exists $u \in K(u_0)$ and a subsequence u^n, relabelled the same, such that

$$\begin{aligned} u^n &\overset{*}{\rightharpoonup} u \text{ in } L^\infty(0,T;H_0^1(\Omega))\\ u_n &\rightharpoonup u \text{ in } L^2(0,T;D(A))\\ f(u^n) &\overset{*}{\rightharpoonup} f(u) \text{ in } L^\infty(0,T;L^\infty(\Omega))\\ \frac{du^n}{dt} &\rightharpoonup \frac{du}{dt} \text{ in } L^2(0,T;L^2(\Omega))\\ a(\|u^n\|^2_{H_0^1})\Delta u^n &\rightharpoonup a(\|u\|^2_{H_0^1})\Delta u \text{ in } L^2(0,T;L^2(\Omega)),\\ u^n &\to u \text{ in } L^2(0,T;H_0^1(\Omega)),\\ u^n &\to u \text{ in } C([0,T],L^2(\Omega)),\\ u^n(t) &\rightharpoonup u(t) \text{ in } H_0^1(\Omega) \quad \forall t \in (0,T]. \end{aligned} \tag{32}$$

We also need to prove that $u^n(t) \to u(t)$ in $H_0^1(\Omega)$ for all $t \in (0,T]$. For this end, we multiply (3) by u_t^n and using (A10), (29) and (31) we have

$$\frac{1}{2}\left\|\frac{du^n}{dt}\right\|^2_{L^2} + \frac{d}{dt}\left(\frac{1}{2}A(\|u^n(t)\|^2_{H_0^1})\right) \le C.$$

Thus, we obtain

$$A(\|u^n(t)\|^2_{H_0^1}) \le A(\|u^n(s)\|^2_{H_0^1}) + 2C(t-s), \quad t \ge s \ge 0.$$

Since this inequality is also true for $u(\cdot)$, the functions $Q_n(t) = A(\|u^n(t)\|^2_{H_0^1}) - 2Ct$, $Q(t) = A(\|u(t)\|^2_{H_0^1}) - 2Ct$ are continuous and non-increasing in $[0,T]$. Moreover, from (32) we deduce that

$$Q_n(t) \to Q(t) \quad \text{ for a.e. } t \in (0,T).$$

Take $0 < t \le T$ and $0 < t_j < t$ such that $t_j \to t$ and $Q_n(t_j) \to Q(t_j)$ for all j. Then

$$Q_n(t) - Q(t) \le Q_n(t_j) - Q(t) \le |Q_n(t_j) - Q(t_j)| + |Q(t_j) - Q(t)|.$$

For any $\delta > 0$ there exist $j(\delta)$ and $N(j(\delta))$ such that $Q_n(t) - Q(t) \leq \delta$ if $n \geq N$. Then $\limsup Q_n(t) \leq Q(t)$, so $\limsup \|u^n(t)\|^2_{H^1_0} \leq \|u(t)\|^2_{H^1_0}$, which follows by contradiction using the continuity of the function $A(s)$. As $u^n(t) \to u(t)$ weakly in $H^1_0(\Omega)$ implies that $\liminf \|u^n(t)\|^2_{H^1_0} \geq \|u(t)\|^2_{H^1_0}$, we obtain

$$\|u^n(t)\|^2_{H^1_0} \to \|u(t)\|^2_{H^1_0},$$

so that $u^n(t) \to u(t)$ strongly in $H^1_0(\Omega)$.

Finally, if $u^n(0) \to u_0$ strongly in $H^1_0(\Omega)$ and we take $t_n \to 0$, then

$$Q_n(t_n) - Q(0) \leq Q_n(0) - Q(0) = A(\|u^n(0)\|^2_{H^1_0}) - A(\|u_0\|^2_{H^1_0}) \to 0,$$

so $\limsup Q_n(t_n) \leq Q(0)$. Repeating the above argument, we infer that $u^n(t_n) \to u_0$ strongly in $H^1_0(\Omega)$. ■

Corollary 8 *Assume the conditions of Lemma 7. Then the set $\mathcal{R}$ satisfies condition $(K4)$.*

The map $t \mapsto G(t,x)$ is said to be upper semicontinuous if for every $x \in X$ and for an arbitrary neighborhood $O(G(t,x))$ in X there is $\delta > 0$ such that as soon as $d(y,x) < \delta$, we have $G(t,y) \subset O$.

Proposition 9 *Assume the conditions of Lemma 7. The multivalued semiflow G is upper semicontinuous for all $t \geq 0$. Also, it has compact values.*

Proof. By contradiction let us assume that there exist $t \geq 0$, $u_0 \in H^1_0(\Omega)$, a neighbourhood $O(G(t,u_0))$ and sequences $\{y_n\}$, $\{u^n_0\}$ such that $y_n \in G(t,u^n_0)$, u^n_0 converges strongly to u_0 in $H^1_0(\Omega)$ and $y_n \notin O(G(t,u_n))$ for all $n \in \mathbb{N}$. Thus, there exist $u^n \in K(u^n_0)$ such that $y_n = u^n(t)$. From Lemma 7 there exists a subsequence of y_n which converges to some $y \in G(t,u_0)$. This contradicts $y_n \notin O(G(t,u_0))$ for any $n \in \mathbb{N}$. ■

In order to prove the existence of an absorbing set in the space $L^2(\Omega)$ we need to use the stronger condition (A5) instead of (13).

Proposition 10 *Assume that conditions (A1), (A5), (A6) and (A10) hold. Then the m-semiflow G has a bounded absorbing set in $L^2(\Omega)$, that is, there exists a constant $K > 0$ such that for any $R > 0$ there is a time $t_0 = t_0(R)$ such that*

$$\|y\|_{L^2} \leq K \quad \text{for all} \quad t \geq t_0,\ y \in G(t,u_0), \tag{33}$$

where $\|u_0\|_{L^2} \leq R$. Moreover, there is $L > 0$ such that

$$\int_t^{t+1} \|u(s)\|^2_{H^1_0} ds \leq L \quad \text{for all} \quad t \geq t_0,\ u \in K(u_0). \tag{34}$$

Proof. Multiplying equation (3) by u and using (A6) and (9) we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u(t)\|^2_{L^2} + m\|u(t)\|^2_{H^1_0} &\leq (f(u),u) + (h,u) \\ &\leq m_\varepsilon|\Omega| + \varepsilon\|u(t)\|^2_{L^2} + \frac{1}{2\lambda_1 m}\|h\|^2_{L^2} + \frac{\lambda_1 m}{2}\|u\|^2_{L^2}.\end{aligned} \tag{35}$$

Using the Poincaré inequality it follows that

$$\frac{d}{dt}\|u\|^2_{L^2} \leq 2m_\varepsilon|\Omega| + 2(\varepsilon - \frac{m}{2}\lambda_1)\|u(t)\|^2_{L^2} + \frac{1}{\lambda_1 m}\|h\|^2_{L^2} = -\delta\|u(t)\|^2_{L^2} + \kappa,$$

where $\delta = m\lambda_1 - 2\varepsilon$, $\kappa = 2m_\varepsilon|\Omega| + \frac{1}{\lambda_1 m}\|h\|^2_{L^2}$. We take $\varepsilon > 0$ small enough, so $\delta > 0$. Then Gronwall's lemma gives

$$\|u(t)\|^2_{L^2} \leq \|u(0)\|^2_{L^2} e^{-\delta t} + \frac{\kappa}{\delta}. \tag{36}$$

Hence, taking

$$t \geq t_0 = \frac{1}{\delta} \ln \left(\frac{\delta R^2}{\kappa} \right)$$

we get (33) for $K = \sqrt{\frac{2\kappa}{\delta}}$.

On the other hand, using again the Poincaré inequality from (35) we get

$$\frac{d}{dt}\|u(t)\|_{L^2}^2 + \left(\frac{m\lambda_1 - 2\varepsilon}{\lambda_1} \right) \|u(t)\|_{H_0^1}^2 \leq \kappa$$

and integrating from t to $t+1$ we obtain

$$\left(\frac{m\lambda_1 - 2\varepsilon}{\lambda_1} \right) \int_t^{t+1} \|u(s)\|_{H_0^1}^2 ds \leq \|u(t)\|_{L^2}^2 + \kappa.$$

Therefore, applying (33), (34) follows. ■

Further, in order to obtain an absorbing set in $H_0^1(\Omega)$ we need to assume additionally that either the function $a(\cdot)$ is bounded above or that it is non-decreasing.

Proposition 11 *Assume the conditions in Proposition 10 and that either (A7) or (A8) holds true. Then there exists an absorbing set B_1 for G, which is compact in $H_0^1(\Omega)$.*

Proof. In view of Proposition 10 we have an absorbing set B_0 in $L^2(\Omega)$. Let $K > 0$ be such that $\|y\| \leq K$ for all $y \in B_0$.

Multiplying (3) by u and using (9) and (36) we get

$$\begin{aligned}\frac{d}{dt}\|u(t)\|_{L^2}^2 + a\left(\|u(t)\|_{H_0^1}^2\right)\|u(t)\|_{H_0^1}^2 &\leq 2m_\varepsilon|\Omega| + 2\varepsilon\|u(t)\|_{L^2}^2 + \frac{1}{\lambda_1 m}\|h\|_{L^2}^2 \\ &\leq K_1 + K_2\|u(0)\|_{L^2}^2.\end{aligned}$$

Thus, integrating between t and $t+r$, $0 < r \leq 1$, we deduce by using (36) again that

$$\begin{aligned}&\|u(t+r)\|_{L^2}^2 + \int_t^{t+r} a\left(\|u(s)\|_{H_0^1}^2\right)\|u(s)\|_{H_0^1}^2 ds \\ &\leq K_1 + K_2\|u(0)\|_{L^2}^2 + \|u(t)\|_{L^2}^2 \leq K_3\|u(0)\|_{L^2}^2 + K_4.\end{aligned} \tag{37}$$

Also, if $p > 2$ in (A5), we multiply again by (3) by u and use (5) and (A6) to obtain

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|_{L^2}^2 + \frac{m}{2}\|u(t)\|_{H_0^1}^2 + C_4\|u(t)\|_{L^p}^p \leq C_3 + \frac{1}{2\lambda_1 m}\|h\|_{L^2}^2.$$

Integrating over $(t, t+r)$ we have

$$\|u(t+r)\|_{L^2}^2 + 2C_4\int_t^{t+r}\|u(s)\|_{L^p}^p ds \leq K_5 + \|u(t)\|_{L^2}^2 \leq K_6 + \|u(0)\|_{L^2}^2. \tag{38}$$

If we assume (A7), by (37) and (A6) we have that

$$\int_t^{t+r} A(\|u(s)\|_{H_0^1}^2)ds \leq \int_t^{t+r} M_1\|u(s)\|_{H_0^1}^2 ds \leq K_7(1 + \|u(0)\|_{L^2}^2). \tag{39}$$

If we assume (A8), by (37) we obtain

$$\begin{aligned}\int_t^{t+r} A(\|u(s)\|_{H_0^1}^2)ds &= \int_t^{t+r}\int_0^{\|u(s)\|_{H_0^1}^2} a(r)\,drds \\ &\leq \int_t^{t+r} a\left(\|u(s)\|_{H_0^1}^2\right)\|u(s)\|_{H_0^1}^2 ds \leq K_3\|u(0)\|_{L^2}^2 + K_4.\end{aligned} \tag{40}$$

On the other hand, by (7) we get

$$-\int_{\Omega} F\left(u\left(t\right)\right) dx \geq -\widetilde{C} \int_{\Omega} (1 + |u\left(t\right)|^{p}) dx. \tag{41}$$

Using (29) and (30) we can argue as in Theorem 5 to obtain

$$\frac{1}{2}\|u_t\|_{L^2}^2 + \frac{d}{dt}\left(\frac{1}{2}A(\|u(t)\|_{H_0^1}^2 - \int_{\Omega} \lambda \mathcal{F}(u_n) dx\right) \leq \frac{1}{2}\|h\|_{L^2}^2.$$

Since (38)-(41) imply that

$$\int_t^{t+r} \left(\frac{1}{2}A(\|u(s)\|_{H_0^1}^2 - \int_{\Omega} \lambda \mathcal{F}(u\left(s\right)) dx\right) ds \leq K_8 + K_9\|u(0)\|_{L^2}^2,$$

we can apply the Uniform Gronwall Lemma to get

$$\frac{1}{2}A(\|u(t+r)\|_{H_0^1}^2) - \int_{\Omega} \lambda \mathcal{F}(u\left(t+r\right)) dx \leq \frac{K_8 + K_9\|u(0)\|_{L^2}^2}{r} + K_{10}, \quad \text{for all } t \geq 0,$$

so by condition $(A6)$, (10) and (36) it follows that

$$\|u(t+1)\|_{H_0^1}^2 \leq K_{11} + K_{12}\|u(0)\|_{L^2}^2,$$

for all $t \geq 0$. In particular,

$$\|u(1)\|_{H_0^1}^2 \leq K_{11} + K_{12}\|u(0)\|_{L^2}^2,$$

for any strong solution $u(\cdot)$ with initial condition $u(0)$.

For any $u_0 \in H_0^1(\Omega)$ with $\|u_0\|_{H_0^1} \leq R$ and any $u \in \mathcal{R}$ such that $u(0) = u_0$, the semiflow property $G(t+1, u_0) \subset G(1, G(t, u_0))$ and $G(t, u_0) \subset B_0$, if $t \geq t_0(R)$, imply that

$$\|u\left(t+1\right)\|_{H_0^1}^2 \leq C(1 + K^2)\ \forall t \geq t_0(R).$$

Then there exists $M > 0$ such that the closed ball B_M in $H_0^1(\Omega)$ centered at 0 with radius M is absorbing for G.

By Lemma 7 the set $B_1 = \overline{G(1, B_M)}$ is an absorbing set which is compact in $H_0^1(\Omega)$. ■

Given an m-semiflow G, a set $B \subset X$ is said to be negatively (positively) invariant if $B \subset G(t, B)$ ($G(t, B) \subset B$) for all $t \geq 0$, and strictly invariant (or, simply, invariant) if it is both negatively and positively invariant.

We recall that a set $\mathcal{A} \subset X$ is called a global attractor for the m-semiflow G if it is negatively invariant and attracts all bounded subsets, i.e., $dist_X(G(t, B), \mathcal{A}) \to 0$ as $t \to +\infty$. When $\mathcal{A}$ is compact, it is the minimal closed attracting set [34, Remark 5].

Theorem 12 *Assume the conditions of Proposition 11. Then the multivalued semiflow G possesses a global compact invariant attractor $\mathcal{A}$.*

Proof. From Propositions 9 and 11 we deduce that the multivalued semiflow G is upper semicontinuous with closed values and the existence of an absorbing which is compact in $H_0^1(\Omega)$. Therefore, by [34, Theorem 4 and Remark 8] the existence of the global invariant attractor and its compactness in $H_0^1(\Omega)$ follow. ■

We recall some concepts which are necessary to study the structure of the global attractor.

Definition 13 *A map $\phi : \mathbb{R} \to X$ is a complete trajectory of $\mathcal{R}$ if $\phi(\cdot + s)\mid_{[0,\infty)} \in \mathcal{R}$ for all $s \in \mathbb{R}$. It is a complete trajectory of G if $\phi(t+s) \in G(t, \phi(s))$ for every $s \in \mathbb{R}$, $t \geq 0$.*

An element $z \in X$ is a fixed point of $\mathcal{R}$ if $\varphi(\cdot) \equiv z \in \mathcal{R}$. We denote the set of all fixed points by $\mathfrak{R}_{\mathcal{R}}$.

An element $z \in X$ is a fixed point of G if $z \in G(t, z)$ for every $t \geq 0$.

Several properties concerning fixed points, complete trajectories and global attractors are summarized in the following results [31].

Lemma 14 *Let (K1)-(K2) hold. Then each fixed point (complete trajectory) of $\mathcal{R}$ is also a fixed point (complete trajectory) of G.*

Let (K1)-(K4) hold. Then the fixed points of $\mathcal{R}$ and G are the same. In addition, a map $\phi : \mathbb{R} \to X$ is a complete trajectory of $\mathcal{R}$ if and only if it is continuous and a complete trajectory of G.

The standard well-known result in the single-valued case for describing the attractor as the union of bounded complete trajectories reads in the multivalued case as follows.

Theorem 15 *Suppose that (K1)-(K2) are satisfied and that either (K3) or (K4) holds true. The semiflow G is assumed to have a compact global attractor $\mathcal{A}$. Then*

$$\mathcal{A} = \{\gamma(0) : \gamma \in \mathbb{K}\} = \cup_{t\in\mathbb{R}}\{\gamma(t) : \gamma \in \mathbb{K}\}, \tag{42}$$

where $\mathbb{K}$ stands for the set of all bounded complete trajectories in $\mathcal{R}$.

In view of Theorem 15, as $\mathcal{R}$ satisfies (K3) and (K4) (by Corollary 8), the global attractor is characterized in terms of bounded complete trajectories, so (42) follows.

The set B is said to be weakly invariant if for any $x \in B$ there exists a complete trajectory γ of $\mathcal{R}$ contained in B such that $\gamma(0) = x$. Characterization (42) implies that the attractor $\mathcal{A}$ is weakly invariant.

The set of fixed points $\mathfrak{R}_{\mathcal{R}}$ is characterized as follows.

Lemma 16 *Assume the conditions of Lemma 7. Let $\mathfrak{R}$ be the set of $z \in H^2(\Omega) \cap H_0^1(\Omega)$ such that*

$$-a(\|z\|^2_{H_0^1})\frac{d^2z}{dx^2} = \lambda f(z) + h \quad \text{in } L^2(\Omega). \tag{43}$$

Then $\mathfrak{R}_{\mathcal{R}} = \mathfrak{R}$.

Proof. If $z \in \mathfrak{R}_{\mathcal{R}}$, then $u(t) \equiv z \in \mathcal{R}$. Thus, $u(\cdot)$ satisfies (12) and $\frac{du}{dt} = 0$ in $L^2(0,T;L^2(\Omega))$, so (43) is satisfied. Let $z \in \mathfrak{R}$. Then the map $u(t) \equiv z$ satisfies (43) for any $t \geq 0$ and $\frac{du}{dt} = 0$ in $L^2(0,T;L^2(\Omega))$, so (12) holds true. ■

Finally, we shall obtain the characterization of the global attractor in terms of the unstable and stable sets of the stationary points.

Theorem 17 *Assume the conditions of Proposition 11. Then it holds that*

$$\mathcal{A} = M^+(\mathfrak{R}) = M^-(\mathfrak{R}),$$

where

$$M^+(\mathfrak{R}) = \{z : \exists\gamma(\cdot) \in \mathbb{K},\ \gamma(0) = z, \quad dist_{\,H_0^1}(\gamma(t), \mathfrak{R}) \to 0,\ t \to +\infty\}, \tag{44}$$

$$M^-(\mathfrak{R}) = \{z : \exists\gamma(\cdot) \in \mathbb{F},\ \gamma(0) = z, \quad dist_{\,H_0^1}(\gamma(t), \mathfrak{R}) \to 0,\ t \to -\infty\}, \tag{45}$$

and $\mathbb{F}$ denotes the set of all complete trajectories of $\mathcal{R}$ (see Definition 13).

Remark 18 *In (45) it is equivalent to use $\mathbb{K}$ instead of $\mathbb{F}$ because all the solutions are bounded forward in time.*

Proof. We consider the function $E : \mathcal{A} \to \mathbb{R}$

$$E(y) = \frac{1}{2}A(\|y\|^2_{H_0^1}) - \lambda\int_\Omega F(y(x))dx - \int_\Omega h(x)y(x)dx. \tag{46}$$

Note that $E(y)$ is continuous in $H_0^1(\Omega)$. Indeed, the maps $y \mapsto \frac{1}{2}A(\|y\|^2_{H_0^1})$ and $y \mapsto \int_\Omega h\,(x)\,y\,(x)\,dx$ are obviously continuous in $H_0^1(\Omega)$. On the other hand, by the embedding $H_0^1(\Omega) \subset L^\infty(\Omega)$ and using Lebesgue's theorem, the continuity of $y \to \int_\Omega F(y(x))dx$ follows.

Using (29)-(30) and multiplying the equation (3) by $\dfrac{du}{dt}$ for any $u \in \mathcal{R}$ we can obtain the following energy equality

$$\int_s^t \|\frac{d}{dr}u(r)\|_{L^2}^2 dr + E(u(t)) = E(u(s)) \quad \text{ for all } t \geq s \geq 0.$$

Hence, $E(u(t))$ is non-increasing and by $(A6)$, (10) and the boundedness of $\mathcal{A}$, it is bounded from below. Thus $E(u(t)) \to l$, as $t \to +\infty$, for some $l \in \mathbb{R}$.

Let $z \in \mathcal{A}$ and $u \in \mathbb{K}$ be such that $u(0) = z$. By contradiction, suppose the existence of $\varepsilon > 0$ and $u(t_n)$, where $t_n \to +\infty$, for which $dist_{H_0^1}(u(t_n), \mathfrak{R}) > \varepsilon$. Since $\mathcal{A}$ is compact in $H_0^1(\Omega)$, we can take a converging subsequence (relabeled the same) such that $u(t_n) \to y$ in $H_0^1(\Omega)$, where $t_n \to \infty$. By the continuity of the function E, it follows that $E(y) = l$. We will obtain a contradiction by proving that $y \in \mathfrak{R}$. Define $v_n(\cdot) = u(\cdot + t_n)$. By Lemma 7, there exist $v \in \mathcal{R}$ and a subsequence satisfying $v(0) = y$ and $v_n(t) \to v(t)$ in $H_0^1(\Omega)$ for $t \geq 0$. Thus, from $E(v_n(t)) \to E(v(t))$ we infer that $E(v(t)) = l$. Also, $v(\cdot)$ satisfies the energy equality, so that

$$l + \int_0^t \|v_r\|_{L^2}^2 dr = E(v(t)) + \int_0^t \|v_r\|_{L^2}^2 dr = E(v(0)) = E(y) = l.$$

Therefore, $\dfrac{dv}{dt}(s) = 0$ for a.a. s, and then by Lemma 16 we have $y \in \mathfrak{R}_{\mathcal{R}} = \mathfrak{R}$. As a consequence, $\mathcal{A} \subset M^+(\mathfrak{R})$. The converse inclusion follows from (42).

As before, take arbitrary $z \in \mathcal{A}$ and $u \in \mathbb{K}$ satisfying $u(0) = z$. Since by the embedding $H_0^1(\Omega) \subset C([0,1])$ the energy function is bounded from above in $\mathcal{A}$, $E(u(t)) \to l$, as $t \to -\infty$, for some $l \in \mathbb{R}$. Suppose that there are $\varepsilon > 0$ and $u(t_n)$, where $t_n \to +\infty$, such that $dist_{H_0^1}(u(-t_n), \mathfrak{R}) > \varepsilon$. Up to a subsequence we have that $u(-t_n) \to y$ in $H_0^1(\Omega)$, $E(y) = l$. Moreover, for $v_n(\cdot) = u(\cdot - t_n)$ there are $v \in \mathcal{R}$ and a subsequence such that $v(0) = y$ and $v_n(t) \to v(t)$ in $H_0^1(\Omega)$ for $t \geq 0$. Therefore, $E(v_n(t)) \to E(v(t))$ gives $E(v(t)) = l$ and then by the above arguments we get a contradiction because $y \in \mathfrak{R}$. Hence, $\mathcal{A} \subset M^-(\mathfrak{R})$ and we deduce the converse inclusion from (42). ■

Finally, we are able to obtain that the global attractor is compact in the space $C^1([0,1])$. This property will be important in order to study a more precise structure of the global attractor in terms of the stationary points and their heteroclinic connections.

We define the function $w(t) = u\left(\alpha^{-1}(t)\right)$, where $\alpha(t) = \int_0^t a(\|u(s)\|_{H_0^1}^2)ds$, which is under the conditions of Proposition 11 (see [11] for more details) a strong solution to the problem

$$\begin{cases} \dfrac{\partial w}{\partial t} - \dfrac{\partial^2 w}{\partial x^2} = \dfrac{f(w)+h}{a(\|w\|_{H_0^1}^2)}, \text{ in } (0,\infty) \times \Omega, \\ w = 0 \quad \text{on } (0,\infty) \times \partial\Omega, \\ w(0,x) = u_0(x) \quad \text{in } \Omega. \end{cases} \tag{47}$$

Let $V^{2r} = D(A^r)$, $r \geq 0$. We will prove first that the attractor is compact in any space V^{2r} with $0 \leq r < 1$. For this aim we will need the concept of mild solution. We consider the auxiliary problem

$$\begin{cases} \dfrac{dv}{dt} + Av(t) = g(t), \ t > 0, \\ \qquad v(0) = u_0, \end{cases} \tag{48}$$

where $g \in L^2_{loc}\left(0, +\infty; L^2(\Omega)\right)$. The function $u \in C([0,+\infty), L^2(\Omega))$ is called a mild solution to problem (48) if

$$v(t) = e^{-At}u_0 + \int_0^t e^{-A(t-s)}g(s)ds, \ \forall t \geq 0. \tag{49}$$

In the same way as in Lemma 2 in [40] we obtain that a strong solution to problem (47) is a mild solution to problem (48) with $g(t) = (f(w(t)) + h)/a(\|w(t)\|_{H_0^1}^2)$.

Lemma 19 *Assume the conditions of Proposition 11. Then the global attractor $\mathcal{A}$ is compact in V^{2r} for every $0 \leq r < 1$.*

Proof. Let $z \in \mathcal{A}$ be arbitrary. Since $\mathcal{A}$ is invariant, there exist $u_0 \in \mathcal{A}$ and $u \in \mathcal{R}$ such that $z = u(1)$ and $u(t) \in \mathcal{A}$ for all $t \geq 0$. Since $w(t) = u(\alpha^{-1}(t))$ is a mild solution of (48) with $g(t) = (f(w(t)) + h)/a(\|w(t)\|^2_{H_0^1})$, the variation of constants formula (49) gives

$$z = w(\alpha(1)) = e^{-A\alpha(1)}u_0 + \int_0^{\alpha(1)} e^{-A(\alpha(1)-s)}g(s)ds.$$

As $\mathcal{A}$ is bounded in $H_0^1(\Omega)$ (and then in $L^\infty(\Omega)$), condition (A6) and the continuity of f imply that

$$\|u_0\|_{L^2} \leq C,\ \|g\|_{L^\infty(0,\alpha(1);L^2(\Omega))} \leq C,$$

where $C > 0$ does not depend on z. The standard estimate $\|e^{-At}\|_{\mathcal{L}(L^2(\Omega),D(A^r))} \leq M_r t^{-r} e^{-at},\ M_r, a > 0$ [37, Theorem 37.5], implies that

$$\begin{aligned}
\|A^r z\|_{L^2} &\leq \left\|A^r e^{-A\alpha(1)}u_0\right\|_{L^2} + \int_0^{\alpha(1)} \left\|A^r e^{-A(\alpha(1)-s)}g(s)\right\|_{L^2} ds \\
&\leq M_r e^{-a\alpha(1)}\alpha(1)^{-r} C + M_r C \int_0^{\alpha(1)} (\alpha(1) - s)^{-r} ds,
\end{aligned}$$

so $\mathcal{A}$ is bounded in V^{2r} for every $0 \leq r < 1$.

From the compact embedding $V^\alpha \subset V^\beta$, for $\alpha > \beta$, and the fact that $\mathcal{A}$ is closed in any V^{2r} we obtain the result. ■

Corollary 20 *Assume the conditions of Proposition 11. Then the global attractor $\mathcal{A}$ is compact in $C^1([0,1])$.*

Proof. We obtain by Lemma 37.8 in [37] the continuous embedding

$$V^{2r} \subset C^1([0,1]) \text{ if } r > \frac{3}{4}.$$

Hence, the statement follows from Lemma 19. ■

5 Fixed points

In this section we are interested in studying the fixed points of problem (3) when $h \equiv 0$, that is, the solutions of the boundary-value problem

$$\begin{cases} -a(\|u\|^2_{H_0^1})\dfrac{d^2u}{dx^2} = \lambda f(u),\ 0 < x < 1, \\ u(0) = u(1) = 0. \end{cases} \tag{50}$$

For this aim we will use the properties of the fixed points of the standard Chafee-Infante equation. In order to do that, for any $d \geq 0$ we will study the following boundary-value problem

$$\begin{cases} -a(d)\dfrac{d^2u}{dx^2} = \lambda f(u),\ 0 < x < 1, \\ u(0) = u(1) = 0, \end{cases} \tag{51}$$

as it is obvious that $u(\cdot)$ is solution to problem (50) if and only if $u(\cdot)$ is a solution to problem (51) with $d = \|u\|^2_{H_0^1}$.

5.1 Dependence on the parameters of the fixed points for the Chafee-Infante equation

Denoting $\widetilde{\lambda} = \dfrac{\lambda}{a(d)}$ problem (51) becomes

$$\begin{cases} -\dfrac{d^2u}{dx^2} = \widetilde{\lambda} f(u),\ 0 < x < 1, \\ u(0) = u(1) = 0. \end{cases} \tag{52}$$

Assuming conditions (A1)-(A5), it is known [10] that if $n^2\pi^2 < \widetilde{\lambda} \leq (n+1)^2 \pi^2$, then this problem has exactly $2n+1$ solutions, denoted by $v_0 \equiv 0,\ v_1^{\pm}, ..., v_n^{\pm}$. The function $v_k^{\pm}$ has $k+1$ simple zeros in $[0,1]$.

We need to study the dependence of the norm of these fixed points on the parameter $\widetilde{\lambda}$. First, we will show that the H^1-norm of the fixed points of problem (52) is strictly increasing with respect to the parameter $\widetilde{\lambda}$.

Lemma 21 *Assume conditions (A1)-(A5). Let* $v_1 = v_{k,\lambda_1}^+,\ v_2 = v_{k,\lambda_2}^+$ *with* $k^2\pi^2 < \lambda_1 < \lambda_2$. *Then* $\|v_1\|_{H_0^1} < \|v_2\|_{H_0^1}$.

Proof. We consider the equivalent norm in $H_0^1(\Omega)$ given by $\|v'\|_{L^2}$. The fixed points are the solutions of the initial value problem

$$\begin{cases} \dfrac{d^2u}{dx^2} + \widetilde{\lambda} f(u) = 0, \\ u(0) = 0,\ u'(0) = v_0 \end{cases} \tag{53}$$

such that $u(1) = 0$. The solutions of (53) satisfy the relation

$$\frac{(u'(x))^2}{2} + \widetilde{\lambda} F(u(x)) = \widetilde{\lambda} E,\ 0 \leq x \leq 1, \tag{54}$$

for some constant $E \geq 0$. Denote $u_{\widetilde{\lambda}} = v_{k,\widetilde{\lambda}}^+$. By Theorem 7 in [10] we have that $u_{\widetilde{\lambda}}$ is associated with a unique value $E = E_k^+(\widetilde{\lambda}) > 0$. Moreover, $E_k^+(\widetilde{\lambda})$ is a solution of one of the following equations:

$$\begin{aligned} m\tau_+^{\widetilde{\lambda}}(E) + (m-1)\tau_-^{\widetilde{\lambda}}(E) &= \frac{1}{\sqrt{2}}, \\ m\tau_-^{\widetilde{\lambda}}(E) + (m-1)\tau_+^{\widetilde{\lambda}}(E) &= \frac{1}{\sqrt{2}}, \\ m\tau_+^{\widetilde{\lambda}}(E) + m\tau_-^{\widetilde{\lambda}}(E) &= \frac{1}{\sqrt{2}}, \end{aligned} \tag{55}$$

where either $k = 2m-1$ or $k = 2m$ and

$$\tau_+^{\widetilde{\lambda}}(E) = \widetilde{\lambda}^{-1/2} \int_0^{U_+(E)} (E - F(u))^{-1/2}\, du, \tag{56}$$

$$\tau_-^{\widetilde{\lambda}}(E) = \widetilde{\lambda}^{-1/2} \int_{U_-(E)}^0 (E - F(u))^{-1/2}\, du, \tag{57}$$

being $U_+(E)$ ($U_-(E)$) the positive (negative) inverse of F at E. It is obvious that for E fixed the functions $\tau_+^{\widetilde{\lambda}}(E)$, $\tau_-^{\widetilde{\lambda}}(E)$ are strictly decreasing with respect to $\widetilde{\lambda}$. Then from (55) we deduce that the root $E_k^+(\widetilde{\lambda})$ is strictly increasing with respect to $\widetilde{\lambda}$. Thus, If $\lambda_1 < \lambda_2$, we have

$$\sqrt{2\lambda_1(E_k^+(\lambda_1) - F(u))} < \sqrt{2\lambda_2(E_k^+(\lambda_2) - F(u))},\quad U^-(E_k^+(\lambda_1)) \leq u \leq U^+(E_k^+(\lambda_1)). \tag{58}$$

We will prove now that $\|u'_{\widetilde{\lambda}}\|_{L^2}$ is strictly increasing in $\widetilde{\lambda}$.

The function $u_{\widetilde{\lambda}}$ has $k+1$ simple zeros in $[0,1]$ and $u_{\widetilde{\lambda}}$ is positive in the first subinterval. Let $T_+(E_k^+(\lambda))$ be the x-time necessary to go from the initial condition $u_\lambda(0)=0$ to the point where $u'_\lambda(T_+(E_k^+(\lambda)))=0$. Then the length of the first subinterval is $2T_+(E_k^+(\lambda))$ [10]. By (54),

$$(u'_{\widetilde{\lambda}}(x))^2 = \sqrt{2\widetilde{\lambda}}\sqrt{E_k^+(\widetilde{\lambda}) - F(u_{\widetilde{\lambda}}(x))}\,u'_{\widetilde{\lambda}}(x),$$

so we have

$$\int_0^{T_+(E_k^+(\widetilde{\lambda}))} (u'_{\widetilde{\lambda}}(x))^2 dx = \int_0^{T_+(E_k^+(\widetilde{\lambda}))} \sqrt{2\widetilde{\lambda}}\sqrt{E_k^+(\widetilde{\lambda}) - F(u_{\widetilde{\lambda}}(x))}\,u'_{\widetilde{\lambda}}(x)dx.$$

By the change of variable $v = u_{\widetilde{\lambda}}(x)$ we obtain

$$\int_0^{T_+(E_k^+(\widetilde{\lambda}))} (u'_{\widetilde{\lambda}}(x))^2 dx = \int_0^{U^+(E_k^+(\widetilde{\lambda}))} \sqrt{2\widetilde{\lambda}}\sqrt{E_k^+(\widetilde{\lambda}) - F(v)}dv = g(\widetilde{\lambda}).$$

Since $\widetilde{\lambda} \mapsto U^+(E_k^+(\widetilde{\lambda}))$ is strictly increasing and using (58), we conclude that the function $g(\widetilde{\lambda})$ is strictly increasing. Hence, putting $x_1(\widetilde{\lambda}) = 2T_+(E_k^+(\widetilde{\lambda}))$ we obtain that the norm of $u_{\widetilde{\lambda}}$ in the first subinterval, $\left\|u'_{\widetilde{\lambda}}\right\|_{L^2(0,x_1(\widetilde{\lambda}))}$, is strictly increasing. Arguing in the same way in the other subintervals we obtain that $\widetilde{\lambda} \mapsto \|u'_{\widetilde{\lambda}}\|_{L^2}$ is strictly increasing. ■

Let us prove the same result but with respect to the norm $\left\|u_{\widetilde{\lambda}}\right\|_{L^p}$ with $p \geq 1$.

Lemma 22 *Assume conditions (A1)-(A5) and let f be odd. Let $v_1 = v^+_{k,\lambda_1}$, $v_2 = v^+_{k,\lambda_2}$ with $k^2\pi^2 < \lambda_1 < \lambda_2$. Then $\|v_1\|_{L^p} < \|v_2\|_{L^p}$ for any $p \geq 1$.*

Proof. As in the previous lemma, denote $u_{\widetilde{\lambda}} = v^+_{k,\widetilde{\lambda}}$. The function $u_{\widetilde{\lambda}}$ has $k+1$ zeros in $[0,1]$ at the points $0 < x_1 < x_2 < ... < x_{k-1} < 1$. When f is odd, by symmetry, the length of all subintervals has to be the same, so $x_j = \frac{j}{k}$ regardless the value of $\widetilde{\lambda}$.

We shall prove that in the first subinterval we have that $u_{\lambda_1}(x) < u_{\lambda_2}(x)$, for all $x \in \left(0, \frac{1}{k}\right)$. By (54) for $x \in [0, \frac{1}{2k}]$ we have

$$x = \int_0^x ds = \int_0^{u_{\widetilde{\lambda}}(x)} \frac{du}{\sqrt{2\widetilde{\lambda}\left(E_k^+\left(\widetilde{\lambda}\right) - F(u)\right)}},$$

so (58) yields

$$\begin{aligned} x &= \int_0^{u_{\lambda_2}(x)} \frac{du}{\sqrt{2\lambda_2\left(E_k^+(\lambda_2) - F(u)\right)}} = \int_0^{u_{\lambda_1}(x)} \frac{du}{\sqrt{2\lambda_1\left(E_k^+(\lambda_1) - F(u)\right)}} \\ &> \int_0^{u_{\lambda_1}(x)} \frac{du}{\sqrt{2\lambda_2\left(E_k^+(\lambda_2) - F(u)\right)}}, \text{ if } x \in (0, \frac{1}{2k}]. \end{aligned}$$

Thus, $u_{\lambda_1}(x) < u_{\lambda_2}(x)$, for all $x \in (0, \frac{1}{2k}]$. By symmetry we obtain that the inequality is true in $\left(0, \frac{1}{k}\right)$.

Repeating the same argument in the other subintervals we get that

$$|u_{\lambda_1}(x)| < |u_{\lambda_2}(x)| \text{ for all } x \in (0,1),\ x \neq \frac{j}{k},\ j = 1, ...k-1.$$

This implies that $\|u_{\lambda_1}\|_{L^p} < \|u_{\lambda_2}\|_{L^p}$ for any $p \geq 1$. ■

Remark 23 *The statements in Lemmas 21-22 are also true for $v^-_{k,\widetilde{\lambda}}$, because $v^-_{k,\widetilde{\lambda}}(x) = v^+_{k,\widetilde{\lambda}}(1-x)$, so the H_0^1 and L^p norms of $v^-_{k,\widetilde{\lambda}}$ and $v^+_{k,\widetilde{\lambda}}$ are the same.*

5.2 Nonlocal fixed points

Although in this paper we are mainly interested in problem (3), we will study the existence of stationary points for an elliptic problem with a more general nonlocal term than in (50). Namely, let us consider the following problem:

$$\begin{cases} -a\left(l(u)\right)u_{xx} = \lambda f\left(u\right),\ 0 < x < 1, \\ u\left(0\right) = u(1) = 0, \end{cases} \tag{59}$$

where

$$l\left(u\right) = \left\Vert u\right\Vert_{H_0^1}^r \text{ or } \left\Vert u\right\Vert_{L^p}^r,\ p \geq 1,\, r > 0.$$

Let

$$d_k = \sup\{d : \lambda > a\left(\overline{d}\right)\pi^2 k^2\ \forall \overline{d} \leq d\}.$$

Then for any $d < d_k$ there exists the fixed point u_k^d of (51), where u_k^d is either equal to u_k^+ or u_k^-.

It is obvious that any solution of (59) is a solution of (51) with $d = l\left(u\right)$. Therefore, all the solutions to problem (59) have to be solutions u_k^d to problem (51) for a suitable d.

Theorem 24 *Assume conditions (A1)-(A6) and, additionally, that*

$$a\left(0\right)\pi^2 k^2 < \lambda. \tag{60}$$

Then:

- *For any $1 \leq j \leq k$ there exists $d_j^* < d_k$ such that $u_j^{d_j^*}$ is a fixed point of problem (59).*
- *If $\lambda \leq a\left(0\right)\pi^2\left(k+1\right)^2$ and $a(0) = \min_{s\geq 0}\{a\left(s\right)\}$, there are no fixed points for $j > k$.*
- *If $N \geq k$ is the first integer such that $\lambda \leq \inf_{s\geq 0}\{a\left(s\right)\pi^2\left(N+1\right)^2\}$, there are no fixed points for $j > N$.*
- *If $l\left(u\right) = \left\Vert u\right\Vert_{H_0^1}^r$, $\lambda \leq a\left(0\right)\pi^2\left(k+1\right)^2$ and a is non-decreasing, there are exactly $2k+1$ solutions to problem (59): $0,\ u_{1,d_1^*}^{\pm}, ..., u_{k,d_k^*}^{\pm}$.*
- *If $l\left(u\right) = \left\Vert u\right\Vert_{L^p}^r$, $\lambda \leq a\left(0\right)\pi^2\left(k+1\right)^2$, f is odd and a is non-decreasing, there are exactly $2k+1$ solutions to problem (59): $0,\ u_{1,d_1^*}^{\pm}, ..., u_{k,d_k^*}^{\pm}$.*

Proof. For the first statement, it is enough to prove the result for $j = k$. By condition (60) we have that $d_k \in (0, +\infty]$.

Consider first the case where d_k is finite. We need to obtain the existence of $d_k^* < d_k$ such that $l\left(u_k^{d_k^*}\right) = d_k^*$. When $d = 0$ it is clear that $l\left(u_k^0\right) > 0$. Also, we know that $l\left(u_k^{d_k}\right) = 0$. Multiplying (51) by u_k^d and using (9), (A6) and the Poincaré inequality we obtain

$$\left\Vert \left(u_k^d\right)'\right\Vert_{L^2}^2 \leq \frac{\lambda}{a\left(d\right)}\left(f\left(u_k^d\right), u_k^d\right) \leq \frac{\lambda}{m}\left(m_\varepsilon + \varepsilon\left\Vert u_k^d\right\Vert_{L^2}^2\right) \leq K_1 + \frac{1}{2}\left\Vert \left(u_k^d\right)'\right\Vert_{L^2}^2,$$

so, by using the embedding $H_0^1\left(\Omega\right) \subset L^\infty\left(\Omega\right)$, $l\left(u_k^d\right)$ is bounded in d. This implies that the function $g\left(d\right) = l\left(u_k^d\right)$ has to intersect the line $y\left(d\right) = d$ at some point d_k^*. It remains to check that $d_k^* < d_k$. For this aim we prove first that $u_k^d \underset{d\to d_k}{\to} 0$ strongly in $H_0^1\left(\Omega\right)$. Indeed, as u_k^d is bounded in $H_0^1\left(\Omega\right)$, there exist v and a sequence $\{u_k^{d_j}\}$ such that $u_k^{d_j} \to v$ in $L^2\left(\Omega\right)$. The embedding $H_0^1\left(\Omega\right) \subset C\left([0,1]\right)$ and the continuity of the function $f\left(u\right)$ imply that $\{f(u_k^{d_j})\}$ is bounded in $C([0,1])$, so from

$$\left\Vert \left(u_k^{d_j}\right)''\right\Vert_{L^2} \leq \frac{\lambda}{a\left(d_j\right)}\left\Vert f\left(u_k^{d_j}\right)\right\Vert_{L^2} \leq \frac{\lambda}{m}\left\Vert f\left(u_k^{d_j}\right)\right\Vert_{L^2} \leq C$$

we deduce that $\{u_k^{d_j}\}$ is bounded in $H^2(\Omega)$. Hence, $u_k^{d_j} \to v$ in $H_0^1(\Omega)$ and $C^1([0,1])$. Also, $f(u_k^{d_j}) \to f(v)$ in $C([0,1])$. Therefore, for any $\psi \in H_0^1(\Omega)$ we have that

$$\begin{array}{ccc} \left(\left(u_k^{d_j}\right)', \psi'\right) & = & \frac{\lambda}{a(d_j)}\left(f\left(u_k^{d_j}\right), \psi\right) \\ \downarrow & & \downarrow \\ (v', \psi') & = & \frac{\lambda}{a(d_k)}\left(f(v), \psi\right), \end{array}$$

which implies that v is a solution to problem (51) with $d = d_k$. But from $u_k^{d_j} \to v$ in $C^1([0,1])$ it follows that v cannot be a point with less than $k+1$ simple zeros in $[0,1]$ and then $\lambda / a(d_k) = k^2\pi^2$ implies that $v \equiv 0$. As the limit is the same for every converging subsequence, $u_k^d \underset{d \to d_k}{\to} 0$ strongly in $H_0^1(\Omega)$. Thus, $d_k > 0$ and $\lim_{d \to d_k} \left\|\left(u_k^d\right)'\right\|_{L^2} = 0$ imply that $d_k^* < d_k$.

Second, let $d_k = +\infty$. Then the existence of $d_k^* < +\infty$ follows by the same argument as before.

The second and third statements are a consequence of

$$\lambda \le a(0)\,\pi^2 (k+1)^2 \le a(d)\,\pi^2 (k+1)^2 \text{ for any } d \ge 0$$

and

$$\lambda \le \inf_{s \ge 0}\{a(s)\}\pi^2 (N+1)^2 \le a(d)\,\pi^2 (N+1)^2 \text{ for any } d \ge 0,$$

respectively, because in such a case for problem (51) the fixed points $v_j^{\pm}$, $j > k$ (respectively $j > N$), do not exist.

The last two statements are a consequence of the first two statements and of the fact that the points of intersection of the functions $g(d) = l\left(u_k^d\right)$ and $y(d) = d$ has to be unique, because if a is non-decreasing, then $g(d)$ is non-increasing by Lemmas 21 and 22. ■

In view of this theorem, we have exactly the same equilibria and bifurcations as in the classical Chafee-Infante equation (see [29], [10]) when the function $a(d)$ is non-decreasing, because in this case in view of the monotone dependence between the functions $a(d)$ and $g(d)$, there is only one intersection point of the function $g(d)$ with the bisector, as it is shown in Figure 1. This follows from the fact that $g(d) - d$ is strictly decreasing, but there may be weaker conditions on $a(\cdot)$ that would lead $g(d) - d$ to be strictly decreasing.

When the function $a(\cdot)$ is not assumed to be monotone, an interesting situation appears. More precisely, it is possible to have more than two equilibria with the same number of zeros. If $l(u) = \|u\|_{H_0^1}^2$, for the equilibria with $k+1$ zeros in $[0,1]$ this happens when the equation

$$d = \int_0^1 \left|\frac{du_k^d(x)}{dx}\right|^2 dx = g(d) \tag{61}$$

has more than one solution. For instance, if $a(0) = a(\bar{d})$ for some $0 < \bar{d} < g(0)$, then $g(0) = g(\bar{d})$. Assuming that there are $0 < d_k^1 < d_k^2 < \bar{d}$ such that $a(d_k^2) = a(d_k^1) = \frac{\lambda}{\pi^2 k^2}$, there must exist $0 < d_1^* < d_k^1 < d_k^2 < d_2^* < \bar{d}$ such that $g(d_i^*) = d_i^*$. Now, by the argument in Theorem 24, there must exist a $d_3^* > \bar{d}$ such that $g(d_3^*) = d_3^*$, obtaining six fixed points with $k+1$ zeros in $[0,1]$. This situation is shown in Figure 2, where d_1^*, d_2^* and d_3^* are solutions of (61), that is, there are three intersection points with the bisector. We notice that when $a(d) > \lambda/(\pi^2 k^2)$, the function $g(d)$ is not defined since the condition for such equilibria to exist is not satisfied, but we can make this function continuous by putting $g(d) = 0$ whenever $a(d) \ge \lambda/(\pi^2 k^2)$. This procedure establishes that, having fixed a natural number k, for any $j \in \mathbb{N}$ we may construct $a(\cdot)$ in such a way that we have $2(2j+1)$ equilibria with $k+1$ zeros in $[0,1]$.

At least there is always one intersection point with the bisector, but the function $g(d)$ could be even tangent to the bisector at some point or not cut it again.

Figure 1: $a(d)$ non-decreasing

Figure 2: $a(d)$ whatever

5.3 Lap number and some forbidden connections

With Theorem 24 at hand we can improve the description of the global attractor given in Theorem 17.

Under conditions (A1)-(A6), (A8) and $h \equiv 0$, if

$$a(0)\,\pi^2 n^2 < \lambda \leq a(0)\,\pi^2 (n+1)^2 \tag{62}$$

then problem (3) possesses exactly $2n+1$ fixed points: $v_0 = 0,\ u^{\pm}_{1,d_1^*}, ..., u^{\pm}_{n,d_n^*}$.

Let ϕ be a bounded complete trajectory. We know by Theorem 17 that

$$dist_{H_0^1}(\phi(t), \mathfrak{R}) \to 0, \text{ as } t \to \pm\infty.$$

As the number of fixed points is finite, we will prove that in fact the solution has to converge to one fixed point forwards and backwards. We recall the omega and alpha limit sets of ϕ, given by

$$\omega(\phi) = \{y : \exists t_n \to +\infty \text{ such that } \phi(t_n) \to y\},$$
$$\alpha(\phi) = \{y : \exists t_n \to -\infty \text{ such that } \phi(t_n) \to y\},$$

are non-empty, compact and connected [5, Lemma 3.4 and Proposition 4.1]. Also, $dist_{H_0^1}(\phi(t), \omega(\phi)) \underset{t\to+\infty}{\to} 0$, $dist_{H_0^1}(\phi(t), \alpha(\phi)) \underset{t\to-\infty}{\to} 0$. Since $\omega(\phi), \alpha(\phi) \subset \mathfrak{R}$ and $\mathfrak{R}$ is finite, the only possibility is that $\omega(\phi) = z_1 \in \mathfrak{R}$, $\alpha(\phi) = z_2 \in \mathfrak{R}$.

Thus, we have established the following result.

Theorem 25 *Let assume conditions (A1)-(A6), (A8), (62) and $h \equiv 0$. Then*

$$\mathcal{A} = \cup_{k=0}^{2n+1} M^+(v_k) = \cup_{k=0}^{2n+1} M^-(v_k),$$

where n is given in (62) and $v_0 = 0$, $v_1 = u^+_{1,d_1^*}$, $v_2 = u^-_{1,d_1^*}, ...$

In other words, the global attractor $\mathcal{A}$ consists of the set of stationary points $\mathfrak{R}$ (which has $2n+1$ elements) and the bounded complete trajectories that connect them (the heteroclinic connections).

Remark 26 *As the Lyapunov function (46) is strictly decreasing along a trajectory ϕ which is not a fixed point, then there cannot exist homoclinic connections for any fixed point. This implies in particular that if $n = 0$, then $\mathcal{A} = \{0\}$.*

Remark 27 *If we use condition (A7) instead of (A8), then we cannot guarantee that the number of fixed points is finite. But if we suppose that this is the case, then the result remains valid. In this situation, there could be more than two fixed points with the same number of zeros.*

Lemma 28 *Let assume conditions (A1)-(A6), $h \equiv 0$ and either (A7) or (A8). Let $u^+_{k,d_k^*}, u^-_{k,d_k^*}$ be a pair of fixed points corresponding to the same value d_k^*. Then there cannot be an heteroclinic connection between them.*

Proof. The function $v(x) = u^+_{k,d_k^*}(1-x)$ is a fixed point corresponding to d_k^* as

$$-\frac{\partial^2 v}{\partial x^2}(x) = -\frac{\partial^2 u^+_{k,d_k^*}}{\partial x^2}(1-x) = \frac{\lambda}{a(d_k^*)} f\left(u^+_{k,d_k^*}(1-x)\right) = \frac{\lambda}{a(d_k^*)} f(v(x)),$$

so $u^-_{k,d_k^*}(x) = v(x) = u^+_{k,d_k^*}(1-x)$. The equalities

$$\int_0^1 \left(\frac{\partial u^-_{k,d_k^*}}{\partial x}(x)\right)^2 dx = \int_0^1 \left(\frac{\partial u^+_{k,d_k^*}}{\partial x}(1-x)\right)^2 dx = \int_0^1 \left(\frac{\partial u^+_{k,d_k^*}}{\partial x}(y)\right)^2 dy,$$

$$\int_0^1 \int_0^{u^-_{d_k^*}(x)} f(s)\, ds dx = \int_0^1 \int_0^{u^+_{d_k^*}(1-x)} f(s)\, ds dx = \int_0^1 \int_0^{u^+_{d_k^*}(y)} f(s)\, ds dy$$

imply that $E(u^-_{k,d_k^*}) = E\left(u^+_{k,d_k^*}\right)$, where E is the Lyapunov function (46). Since this function is strictly decreasing along a trajectory ϕ which is not a fixed point, there cannot exist a heteroclinic connection between these two points. ■

Remark 29 *In the case where condition (A7) is assumed, there could be more than two equilibria with $k+1$ zeros in $[0,1]$. In this case there could exist connections between fixed points with different values of the constant d.*

Using the concept of lap number of the solutions we can discard some more heteroclinic connections.

We consider the function $w(t) = u(\alpha^{-1}(t))$, which is a strong solution to problem (47). For any strong solution $u(\cdot)$ conditions (A1), (A3), (A6) and $u \in C([0,+\infty), H_0^1(\Omega))$ imply that the function

$$r(t,x) = \frac{\lambda}{a(\|w(t)\|^2_{H_0^1})} \frac{f(w(t,x))}{w(t,x)}$$

is continuous and $w(\cdot)$ is a solution of the linear equation

$$\frac{\partial w}{\partial t} - \frac{\partial^2 w}{\partial x^2} = r(t,x)\, w. \tag{63}$$

Thus, by Theorem 51 in the Appendix (see also Theorem C in [1]) the number of zeros of $w(t)$ in $[0,1]$ is a nonincreasing function of t. Since $\alpha^{-1}(t)$ is an increasing function of time, the result is also true for the solution $u(\cdot)$. Making use of this property we will prove the following result.

Lemma 30 *Let assume conditions (A1)-(A6), $h \equiv 0$ and either (A7) or (A8). Then if $n > k$, there cannot exist a connection from the fixed point $u^{\pm}_{k,d_k^*}$ to the fixed point $u^{\pm}_{n,d_n^*}$, that is, there cannot exist a bounded complete trajectory ϕ such that*

$$\phi(t) \to u^{\pm}_{n,d_n^*} \text{ as } t \to +\infty,\ \phi(t) \to u^{\pm}_{k,d_k^*} \text{ as } t \to -\infty.$$

Proof. By contradiction assume that such complete trajectory exists. Denote by $l(z)$ the number of zeros of z in $[0,1]$. Using the compactness of the attractor in $C^1([0,1])$ (see Corollary 20) we obtain that

$$\phi(t) \to u^{\pm}_{n,d_n^*} \text{ in } C^1([0,1]) \text{ as } t \to +\infty,$$
$$\phi(t) \to u^{\pm}_{k,d_k^*} \text{ in } C^1([0,1]) \text{ as } t \to -\infty.$$

Then, as the zeros are simple, we can choose $t_1 > 0$ large enough such that $l(\phi(-t_1)) = l\left(u^{\pm}_{k,d_k^*}\right) = k+1$. Put $u(t) = \phi(t - t_1)$, which is a strong solution of (3). Now we choose $t_2 > 0$ such that $l(u(t_2)) = l\left(u^{\pm}_{n,d_n^*}\right) = n+1$. Then $l(u(0)) = k+1$ and $l(u(t_2)) = n+1 > k+1$. This contradicts the fact that the number of zeros of $u(t)$ is non-increasing. ∎

6 Morse decomposition

In this section we study in more detail the structure of the global attactor in the case where the function f is odd. More precisely, we obtain that the m-semiflow G is dynamically gradient, which is equivalent to saying that there is a Morse decomposition of the attractor [26], and study the stability of the fixed points.

6.1 Aproximations

We consider now the situation when conditions (A1)-(A6), $h = 0$ and either (A7) or (A8) are satisfied and, moreover, the function f is odd.

In this section we consider the following problems:

$$\begin{cases} \dfrac{\partial u}{\partial t} - a(\|u\|^2_{H_0^1}) \dfrac{\partial^2 u}{\partial x^2} = \lambda f_{\varepsilon_n}(u), \quad t > 0,\ x \in (0,1), \\ u(t,0) = 0,\ u(t,1) = 0, \\ u(0,x) = u_0(x), \end{cases} \tag{64}$$

where the function f_{ε_n} is defined below and $\varepsilon_n \to 0$, as $n \to \infty$.

Let $\rho_{\varepsilon_n}(\cdot)$ be a mollifier in $\mathbb{R}$. We define the function $f^{\varepsilon_n}(u) = \int_{\mathbb{R}} \rho_{\varepsilon_n}(s) f(u-s) ds$. It is well known that $f^{\varepsilon_n}(\cdot) \in C^{\infty}(\mathbb{R})$ and that for any compact subset $A \subset \mathbb{R}$ we have $f^{\varepsilon_n} \to f$ uniformly on A. It is clear that for $u > \varepsilon_n$ the function $f^{\varepsilon_n}(u)$ is strictly concave.

We need the approximation to fulfil (A2)-(A3). For that end, we consider the approximation except on the interval $[-\varepsilon_n, \varepsilon_n]$, for any $\varepsilon_n > 0$. There exists a polynomial of sixth degree $p(x)$ such that

$$\begin{array}{ll} p(0) = 0, & p(\varepsilon_n) = h(\varepsilon_n), \\ p'(0) = 1, & p'(\varepsilon_n) = h'(\varepsilon_n), \\ p''(0) = 0, & p''(\varepsilon_n) = h''(\varepsilon_n), \\ p'''(0) = -1. & \end{array}$$

We choose $\gamma > 0$ such that $p''(s) < 0$ for all $s \in (0, \gamma]$. We can assume that $\varepsilon_n < \gamma$ for all n.

Thus, by construction the function

$$f_{\varepsilon_n}(x) = \begin{cases} -f^{\varepsilon_n}(-x) & if \quad x < -\varepsilon_n, \\ -p(-x) & if \quad -\varepsilon_n \le x \le 0, \\ p(x) & if \quad 0 \le x \le \varepsilon_n, \\ f^{\varepsilon_n}(x) & if \quad x > \varepsilon_n \end{cases} \tag{65}$$

approximates the function f uniformly in compact sets, that is, for any $[-M, M]$ and $\delta > 0$ there exists $n_0(M, \delta) \in \mathbb{N}$ such that

$$|f(x) - f_{\varepsilon_n}(x)| < \delta, \quad \text{for all } n \ge n_0,\ x \in [-M, M]. \tag{66}$$

Also, it satisfies the following properties:

(B1) $f_{\varepsilon_n} \in C^2(\mathbb{R})$;

(B2) $f_{\varepsilon_n}(0) = 0$;

(B3) $f'_{\varepsilon_n}(0) = 1$;

(B4) f_{ε_n} is strictly concave if $u > 0$ and strictly convex if $u < 0$;

(B5) f_{ε_n} is odd.

Lemma 31 *Let f satisfy (A5). Then the functions f_{ε_n} satisfy condition (A5) and (9) with independent constants of ε_n.*

Proof. We assume without loss of generality that $\varepsilon_n < 1$. In order to check (4)-(5) we only need to consider u outside the interval $[-1, 1]$, because the sequence $\{f_{\varepsilon_n}\}$ is uniformly bounded in any compact set of $\mathbb{R}$. Then for $u \notin [-1, 1]$ the Hölder inequality and $\int_{\mathbb{R}} \rho_{\varepsilon_n}(s)\, ds = 1$ give

$$\begin{aligned} |f_{\varepsilon_n}(u)| &= \left| \int_{\mathbb{R}} f(u-s) \rho_{\varepsilon_n}(s)\, ds \right| \le \int_{\mathbb{R}} |f(u-s)| \rho_{\varepsilon_n}(s)\, ds \\ &\le \int_{\mathbb{R}} \left(C_1 + C_2 |u-s|^{p-1} \right) \rho_{\varepsilon_n}(s)\, ds \\ &\le C_1 + C_2 2^{p-2} \left(\int_{-\varepsilon_n}^{\varepsilon_n} \left(|u|^{p-1} + |s|^{p-1} \right) \rho_{\varepsilon_n}(s)\, ds \right) \\ &\le \widetilde{C}_1 + \widetilde{C}_2 |u|^{p-1}. \end{aligned}$$

If f satisfies (5), then

$$\begin{aligned}
f_{\varepsilon_n}(u)\,u &= \int_{\mathbb{R}} f(u-s)(u-s)\,\rho_{\varepsilon_n}(s)\,ds + \int_{\mathbb{R}} f(u-s)\,s\rho_{\varepsilon_n}(s)\,ds \\
&\leq \int_{\mathbb{R}} (C_3 - C_4|u-s|^p)\,\rho_{\varepsilon_n}(s)\,ds + \int_{\mathbb{R}} \left(C_1 + C_2|u-s|^{p-1}\right) s\rho_{\varepsilon_n}(s)\,ds \\
&\leq K_1 - C_4 \int_{\mathbb{R}} \left(2^{1-p}|u|^p - |s|^p\right)\rho_{\varepsilon_n}(s)\,ds \\
&+ C_2 2^{p-2} \int_{\mathbb{R}} \left(|u|^{p-1} + |s|^{p-1}\right) s\rho_{\varepsilon_n}(s)\,ds \\
&\leq \widetilde{C}_3 - \widetilde{C}_4|u|^p,
\end{aligned}$$

where we have used $|u|^p \leq 2^{p-1}\left(|s^p| + |u-s|^p\right)$ and the Young inequality.

For (9) we put in the above inequality $p=2,\ C_3 = m_\varepsilon,\ C_4 = -\varepsilon$ and obtain

$$f_{\varepsilon_n}(u)\,u \leq \widetilde{m}_\varepsilon + \varepsilon u^2,$$

which obviously implies (6). ■

Our next aim is to focus on the convergence of solutions of the approximations.

Theorem 32 *Let conditions (A1)-(A6), $h=0$ and either (A7) or (A8) be satisfied and let, moreover, the function f be odd. If $u_{\varepsilon_n,0} \to u_0$ in $H_0^1(\Omega)$ as $\varepsilon_n \to 0$, then for any sequence of solutions of (64) $u_{\varepsilon_n}(\cdot)$ with $u_{\varepsilon_n}(0) = u_{\varepsilon_n,0}$ there exists a subsequence of ε_n such that u_{ε_n} converges to some strong solution $u(\cdot)$ of (3) in the space $C([0,T],H_0^1(\Omega))$, for any $T>0$.*

Proof. Using (29) and (30) we can repeat the same lines of the proof of Theorem 5 and obtain the existence of a function $u(\cdot)$ and a subsequence of u_{ε_n} such that

$$\begin{aligned}
u_{\varepsilon_n} &\overset{*}{\rightharpoonup} u \text{ in } L^\infty(0,T;H_0^1(\Omega)),\\
u_{\varepsilon_n} &\rightharpoonup u \text{ in } L^2(0,T;D(A)),\\
\frac{du_{\varepsilon_n}}{dt} &\rightharpoonup \frac{du}{dt} \text{ in } L^2(0,T;L^2(\Omega)),\\
u_{\varepsilon_n} &\to u \text{ in } C([0,T];L^2(\Omega)),\\
u_{\varepsilon_n} &\to u \text{ in } L^2(0,T;H_0^1(\Omega)),\\
f_{\varepsilon_n}(u_{n_\varepsilon}) &\overset{*}{\rightharpoonup} f(u) \text{ in } L^\infty(0,T;L^\infty(\Omega)),\\
a(\|u_{\varepsilon_n}\|^2_{H_0^1})\Delta u_{\varepsilon_n} &\rightharpoonup a(\|u\|^2_{H_0^1})\Delta u \text{ in } L^2(0,T;L^2(\Omega)).
\end{aligned}$$

Also, in the same way we prove that $u(\cdot)$ is a strong solution to problem (3) such that $u(0)=u_0$.

The uniform estimate in the space $H_0^1(\Omega)$ implies also that if $t_n \to t_0$, then $u_{\varepsilon_n}(t_n) \rightharpoonup u(t_0)$ in $H_0^1(\Omega)$. We need to prove that this convergence is in fact strong, proving then the convergence in $C([0,T],H_0^1(\Omega))$ for any $T>0$.

In the same way as in the proof of Lemma 7 we deduce that for some $C>0$ the functions $Q_n(t) = A(\|u_{\varepsilon_n}(t)\|^2_{H_0^1}) - 2Ct$, $Q(t) = A(\|u(t)\|^2_{H_0^1}) - 2Ct$ are continuous and non-increasing in $[0,T]$. Moreover, $Q_n(t) \to Q(t)$ for a.e. $t \in (0,T)$. Let first $t_0 > \dot{0}$. Then we take $0 < t_j < t_0$ such that $t_j \to t_0$ and $Q_n(t_j) \to Q(t_j)$ for all j. Then

$$Q_n(t_n) - Q(t_0) \leq Q_n(t_j) - Q(t_0) \leq |Q_n(t_j) - Q(t_j)| + |Q(t_j) - Q(t_0)| \text{ for } t_n > t_j.$$

For any $\delta > 0$ there exist $j(\delta)$ and $N(j(\delta))$ such that $Q_n(t_n) - Q(t_0) \leq \delta$ if $n \geq N$, so $\limsup Q_n(t_n) \leq Q(t_0)$. Hence, a contradiction argument using the continuity of $A(s)$ shows that $\limsup \|u_{\varepsilon_n}(t_n)\|^2_{H_0^1} \leq \|u(t_0)\|^2_{H_0^1}$. This, together with $\liminf \|u_{\varepsilon_n}(t_n)\|^2_{H_0^1} \geq \|u(t_0)\|^2_{H_0^1}$, implies that $\|u_{\varepsilon_n}(t_n)\|^2_{H_0^1} \to \|u(t_0)\|^2_{H_0^1}$, so that $u_{\varepsilon_n}(t_n) \to u(t_0)$ strongly in $H_0^1(\Omega)$. For the case when $t_0 = 0$ we use the same argument as in Lemma 7. ■

We denote by $\mathcal{A}_{\varepsilon_n}$ the global attractor for the semiflow G_{ε_n} corresponding to problem (64).

Lemma 33 *Assume the condition of Theorem 32. Then* $\cup_{n\in\mathbb{N}}\mathcal{A}_{\varepsilon_n}$ *is bounded in* $H_0^1(\Omega)$. *Hence, the set* $\overline{\cup_{n\in\mathbb{N}}\mathcal{A}_{\varepsilon_n}}$ *is compact in* $L^2(\Omega)$.

Proof. By Lemma 31 inequality (9) is satisfied for any n with constants which are independent of ε_n, so inequality (36) holds true with constants independent of ε_n. Thus, there a exists a common absorbing ball B_0 in $L^2(\Omega)$ (with radius $K>0$) for problems (64). Further, by repeating the same steps as in Proposition 11 we obtain a common absorbing ball in $H_0^1(\Omega)$ (with radius $\widetilde{K}>0$) as by Lemma 31 the constants which are involved are independent of ε_n. Thus, $\|y\|_{H_0^1}\leq\widetilde{K}$ for any $y\in\cup_{n\in\mathbb{N}}\mathcal{A}_{\varepsilon_n}$. ■

Lemma 34 *Assume the condition of Theorem 32. Then* $\cup_{n\in\mathbb{N}}\mathcal{A}_{\varepsilon_n}$ *is bounded in* V^{2r} *for any* $0\leq r<1$. *Hence,* $\overline{\cup_{n\in\mathbb{N}}\mathcal{A}_{\varepsilon_n}}$ *is compact in* V^{2r} *and* $C^1([0,1])$.

Proof. Using Lemma 33 we obtain the boundedness of $\cup_{n\in\mathbb{N}}\mathcal{A}_{\varepsilon_n}$ in V^{2r} by repeating the same lines in Lemma 19. The rest of the proof follows from the compact embedding $V^{\alpha}\subset V^{\beta}$, $\alpha>\beta$, and the continuous embedding $V^{2r}\subset C^1([0,1])$ if $r>\frac{3}{4}$. ■

Corollary 35 *Assume the condition of Theorem 32. Then any sequence* $\xi_n\in\mathcal{A}_{\varepsilon_n}$ *with* $\varepsilon_n\to 0$ *is relatively compact in* $C^1([0,1])$.

Lemma 36 *Assume the condition of Theorem 32. Then up to a subsequence any bounded complete trajectory* u_{ε_n} *of (64) converges to a bounded complete trajectory* u *of (3) in* $C([-T,T],H_0^1(\Omega))$ *for any* $T>0$. *On top of that, if* $y_n\in\mathcal{A}_{\varepsilon_n}$, *then passing to a subsequence* $y_n\to y\in\mathcal{A}$ *in* $H_0^1(\Omega)$. *Hence,*

$$dist_{H_0^1}(\mathcal{A}_{\varepsilon_n},\mathcal{A})\to 0 \text{ as } n\to\infty. \tag{67}$$

Proof. Let fix $T>0$. By Corollary 35 $u_{\varepsilon_n}(-T)\to y$ in $H_0^1(\Omega)$ up to a subsequence. Theorem 32 implies that u_{ε_n} converges in $C([-T,T],H_0^1(\Omega))$ to some solution u of (3). If we choose successive subsequences for $-2T,-3T\ldots$ and apply the standard diagonal procedure, we obtain that a subsequence u_{ε_n} converges to a complete trajectory u of (3) in $C([-T,T],H_0^1(\Omega))$ for any $T>0$. Finally, from Lemma 33 this trajectory is bounded.

If $y_n\in\mathcal{A}_{\varepsilon_n}$, by Corollary 35 we can extract a subsequence converging to some y. If we take a sequence of bounded complete trajectories $\phi_n(\cdot)$ of (64) such that $\phi_n(0)=y_n$, then by the previous result it converges in $C([-T,T],H_0^1(\Omega))$ to some bounded complete trajectory $\phi(\cdot)$ of (3), so $y\in\mathcal{A}$.

Finally, if (67) was not true, there would exist $\delta>0$ and a sequence $y_n\in\mathcal{A}_{\varepsilon_n}$ such that $dist_{H_0^1}(y,\mathcal{A})>\delta$. But passing to a subsequence $y_n\to y\in\mathcal{A}$, which is a contradiction. ■

Lemma 37 *Assume the conditions of Theorem 32. Let* $\tau_{\pm}^{d_n,\varepsilon_n}$ *be the functions (56)-(57) for problem (51) but replacing* f *by* f_{ε_n} *and* d *by* d_n. *Let* $d_n,E_n\to 0$ *as* $n\to\infty$. *Then*

$$\lim_{n\to\infty}\tau_{\pm}^{d_n,\varepsilon_n}(E_n)=\frac{\sqrt{a(0)}\pi}{\sqrt{2\lambda}}.$$

Proof. Let us consider $f_{d_n,\varepsilon_n}(u)=\frac{\lambda f_{\varepsilon_n}(u)}{a(d_n)}$. In view of property $(B4)$ and (66), since $f'_{\varepsilon_n}(0)=f'(0)=1$ and $f_{\varepsilon_n}(0)=f(0)=0$, given $\gamma\in(0,1)$ there exists $\delta>0$ (independent of ε_n) such that

$$\begin{array}{c}(1-\gamma)u\leq f_{\varepsilon_n}(u)\leq(1+\gamma)u, \quad \text{for any } u\in(0,\delta).\\ \frac{1}{1+\gamma}\leq\frac{u}{f_{\varepsilon_n}(u)}\leq\frac{1}{1-\gamma}, \quad \text{for any } u\in(0,\delta).\end{array} \tag{68}$$

The sequence $\mathcal{F}_{\varepsilon_n}(\cdot)$ converges uniformly to $\mathcal{F}(\cdot)$ in compact sets. Moreover, as $U_+(E)$ is continuous and using [38, p. 60], given $\delta>0$, there exists $\eta>0$ such that $U_+^{\varepsilon_n}(E)\leq\delta$ for any $0<E\leq\eta$. Now, if we integrate the first inequality in (68) between 0 and u we obtain

$$\frac{1}{2}(1-\gamma)u^2\leq\mathcal{F}_{\varepsilon_n}(u)\leq\frac{1}{2}(1+\gamma)u^2, \quad \text{for any } 0\leq u\leq\delta.$$

Using the change of variable $E_n y^2=\mathcal{F}_{\varepsilon_n}(u)$, we have

$$\left(\frac{1-\gamma}{2E_n}\right)^{1/2}u\leq y\leq\left(\frac{1+\gamma}{2E_n}\right)^{1/2}u, \quad \text{if } 0<E_n\leq\eta,\ 0\leq y\leq 1.$$

Dividing the previous expression by $\sqrt{\frac{\lambda}{a(d_n)}}f_{d_n,\varepsilon_n}(u)$ and using (68) we obtain

$$\left(\tfrac{a(d_n)(1-\gamma)}{2\lambda E_n(1+\gamma)^2}\right)^{1/2} \leq \frac{\sqrt{a(d_n)}y}{\sqrt{\lambda}f_{d_n,\varepsilon_n}(u)} \leq \left(\tfrac{a(d_n)(1+\gamma)}{2\lambda E_n(1-\gamma)^2}\right)^{1/2}, \text{if } 0 < E_n \leq \eta,\ 0 \leq y \leq 1.$$

Now if we multiply by $2\sqrt{E_n}(1-y^2)^{-\frac{1}{2}}$ and integrate from 0 to 1, we get

$$\pi\left(\tfrac{a(d_n)(1-\gamma)}{2\lambda(1+\gamma)^2}\right)^{1/2} \leq \tau_+^{\varepsilon_n}(E_n) \leq \pi\left(\tfrac{a(d_n)(1+\gamma)}{2\lambda(1-\gamma)^2}\right)^{1/2}, \quad \text{if } 0 < E_n \leq \eta.$$

Then the theorem follows as $a(d_n) \to a(0)$ when $n \to \infty$. The proof for $\tau_-^{\varepsilon_n}$ is analogous. ■

Under the conditions of Theorem 32, if (A8) is satisfied and

$$a(0)\pi^2k^2 < \lambda \leq a(0)\pi^2(k+1)^2,\ k \in \mathbb{Z},\ k \geq 0, \tag{69}$$

holds, then by Theorem 24 problem (64) has exactly $2k+1$ fixed points (denoted by $v_0 = 0,\ v^{\pm}_{1,d_1^{\varepsilon_n}}, ..., v^{\pm}_{k,d_k^{\varepsilon_n}}$) and $v^{\pm}_{m,d_m^{\varepsilon_n}}$ has $m+1$ zeros in $[0,1]$ for each $1 \leq m \leq k$. The same is valid for problem (3) and we denote the $2k+1$ fixed points by $v_0 = 0,\ u^{\pm}_{1,d_1^*}, ..., u^{\pm}_{k,d_k^*}$.

Lemma 38 *Assume the conditions of Theorem 32, (A8) and (69). Let $m \in \mathbb{N}$, $1 \leq m \leq k$, be fixed. Then $v^+_{m,d_m^{\varepsilon_n}}$ (resp. $v^-_{m,d_m^{\varepsilon_n}}$) do not converge to 0 in $H_0^1(\Omega)$ as $\varepsilon_n \to 0$.*

Proof. Assume that $v^+_{m,d_m^{\varepsilon_n}} \to 0$ in $H_0^1(0,1)$. Then it converges to 0 in $C([0,1])$ and the equality

$$-\frac{d^2v^+_{m,d_m^{\varepsilon_n}}}{dx^2}(x) = \frac{\lambda f_{\varepsilon_n}\left(v^+_{m,d_m^{\varepsilon_n}}(x)\right)}{a(d_m^{\varepsilon_n})}$$

implies that $v^+_{m,d_m^{\varepsilon_n}} \to 0$ in $C^2([0,1])$. In particular, $\dfrac{dv^+_{m,d_m^{\varepsilon_n}}}{dx}(0) \to 0$ and $d_m^{\varepsilon_n} = \left\|v^+_{m,d_m^{\varepsilon_n}}\right\|^2_{H_0^1} \to 0$. The value E_n corresponding to the fixed point $v^+_{m,d_m^{\varepsilon_n}}$ is equal to $\dfrac{a(d_m^{\varepsilon_n})}{2\lambda}\dfrac{dv^+_{m,d_m^{\varepsilon_n}}}{dx}(0)$, so $E_n \to 0$. We will show that this is not possible. We know by Lemma 37 that

$$\lim_{n\to\infty} \tau_{\pm}^{d_m^{\varepsilon_n},\varepsilon_n}(E_n) = \frac{\pi\sqrt{a(0)}}{\sqrt{2\lambda}}.$$

Also, since $v^+_{m,d_m^{\varepsilon_n}}$ is a fixed point with $d = d_m^{\varepsilon_n}$ one of the following conditions has to be satisfied (see (55)):

$$j\tau_+^{d_m^{\varepsilon_n},\varepsilon_n}(E_n) + (j-1)\tau_-^{d_m^{\varepsilon_n},\varepsilon_n}(E_n) = \left(\frac{1}{2}\right)^{\frac{1}{2}}, \tag{70}$$

$$j\tau_-^{d_m^{\varepsilon_n},\varepsilon_n}(E_n) + (j-1)\tau_+^{d_m^{\varepsilon_n},\varepsilon_n}(E_n) = \left(\frac{1}{2}\right)^{\frac{1}{2}},\ \text{if } m = 2j-1 \tag{71}$$

$$j\tau_+^{d_m^{\varepsilon_n},\varepsilon_n}(E_n) + j\tau_-^{d_m^{\varepsilon_n},\varepsilon_n}(E_n) = \left(\frac{1}{2}\right)^{\frac{1}{2}},\ \text{if } m = 2j. \tag{72}$$

Since $E_n \to 0$ and $\lambda > k^2\pi^2a(0) \geq m^2\pi^2a(0)$, there exists ε_{n_0} such that

$$\tau_{\pm}^{d_m^{\varepsilon_{n_0}},\varepsilon_{n_0}}(E_{n_0}) < \frac{1}{\sqrt{2}m}.$$

Hence, neither of (70)-(72) is possible. ■

Lemma 39 *Assume the conditions of Theorem 32, (A8) and (69). Let $m \in \mathbb{N}$, $1 \leq m \leq k$, be fixed. Then $v^+_{m,d_m^{\varepsilon_n}}$ (resp. $v^-_{m,d_m^{\varepsilon_n}}$) converges to u^+_{m,d_m^*} in $H_0^1(\Omega)$ (resp. u^-_{m,d_m^*}) as $\varepsilon_n \to 0$.*

Proof. We consider $v^+_{m,d_m^{\varepsilon_n}}$. In view of Corollary 35, $v^+_{m,d_m^{\varepsilon_n}}$ is relatively compact in $C^1([0,1])$, so up to a subsequence $v^+_{m,d_m^{\varepsilon_n}} \to v$ strongly in $C^1([0,1])$ and $d_m^{\varepsilon_n} \to d^* = \|v\|^2_{H_0^1}$. The proof will be finished if we prove that $v = u^+_{m,d_m^*}$. We observe that since in such a case every subsequence would have the same limit, the whole sequence would converge to u^+_{m,d_m^*}.

In view of (66) $f_{\varepsilon_n}(v^+_{m,d_m^{\varepsilon_n}})$ converges to $f(v)$ in $C([0,1])$. It follows that

$$-\frac{\partial^2 v}{\partial x^2} = \frac{\lambda f(v)}{a(\|v\|^2_{H_0^1})}$$

and v is a solution of (50), so v is a fixed point of (3). We need to prove that $v = u^+_{m,d_m^*}$. By Lemma 38 $v \neq 0$, and then $v = u^{\pm}_{j,d_j^*}$ for some $1 \leq j \leq k$. Since $u^{\pm}_{j,d_j^*}$ has $j+1$ simple zeros, the convergence $v^+_{m,d_m^{\varepsilon_n}} \to u^{\pm}_{j,d_j^*}$ in $C^1([0,1])$ implies that $v^+_{m,d_m^{\varepsilon_n}}$ has $j+1$ zeros for $n \geq N$. But $v^+_{m,d_m^{\varepsilon_n}}$ possesses $m+1$ zeros in $[0,1]$. Thus, $m = j$.

For the sequence $v^-_{m,d_m^{\varepsilon_n}}$ the proof is analogous. ■

6.2 Instability

We will prove that the fixed points 0 and $u^{\pm}_{k,d_k^*}$, $k \geq 2$, are unstable under some additional assumptions on the functions f and a. For this aim we need to use the approximative problems (64).

Theorem 40 *Assume that the conditions (A1)-(A8), $h = 0$, (69) with $k \geq 1$ are satisfied and let, moreover, the function $f(\cdot)$ be odd and $a(\cdot)$ be globally Lipschitz continuous. Then the equilibria $v_0 = 0$ and $u^{\pm}_{j,d_j^*}$, $2 \leq j \leq k$ (if $k \geq 2$), are unstable.*

Remark 41 *The condition that $a(\cdot)$ is globally Lipschitz continuous could be dropped, as we can replace $a(\cdot)$ in (64) by a sequence $a_{\varepsilon_n}(\cdot)$ of globally Lipschitz continuous functions.*

Proof. Problem (64) generates a single-valued semigroup $\{T_{\varepsilon_n}(t); t \geq 0\}$ with a finite number of fixed points: $v_0 = 0$, $v^{\pm}_{1,d_1^{\varepsilon_n}}, ..., v^{\pm}_{k,d_k^{\varepsilon_n}}$ [19]. We know by Theorems 3.5 and 3.6 in [19] that for any $v^+_{j,d_j^{\varepsilon_n}}$ with $j \geq 2$ and v_0 there exists a bounded complete trajectory u^{ε_n} such that

$$u^{\varepsilon_n}(t) \to v^+_{j,d_j^{\varepsilon_n}} \quad \text{as } t \to -\infty, \quad \text{for } k \geq 2,$$

so v_0, $v^+_{j,d_j^{\varepsilon_n}}$ are unstable. The same is valid for $v^-_{j,d_j^{\varepsilon_n}}$. On the other hand, by Lemma 39 we have

$$v^{\pm}_{j,d_j^{\varepsilon_n}} \to u^{\pm}_{j,d_j^*}, \tag{73}$$

where $u^{\pm}_{j,d_j^*}$ is a fixed point of problem (3) with $j+1$ zeros in $[0,1]$. We prove the result for u^+_{j,d_j^*}. For u^-_{j,d_j^*} and v_0 the proof is the same.

By Lemma 36 we obtain that up to a subsequence u^{ε_n} converges to a bounded complete trajectory u of problem (3) in the space $C([-T,T], H_0^1(\Omega))$ for every $T > 0$. Thus, either $u(\cdot)$ is a fixed point v_{-1} or by Theorem 17 there exists a fixed point v_{-1} of problem (3) such that

$$u(t) \to v_{-1} \quad \text{as } t \to -\infty \text{ in } H_0^1(\Omega).$$

In the second case, if $v_{-1} = u^+_{j,d_j^*}$, the proof would be finished, so let assume the opposite.

Assume first that either $u(\cdot)$ is not a fixed point or it is a fixed point but $v_{-1} \neq u^+_{j,d_j^*}$. We consider $r_0 > 0$ such that the neighborhood $\mathcal{O}_{2r_0}(v_{-1})$ does not contain any other fixed point of problem (3). For any $r \le r_0$ we can choose $t_r \to -\infty$ and n_r such that $u^{\varepsilon_n}(t_r) \in \mathcal{O}_r(v_{-1})$ for all $n \ge n_r$. On the other hand, since $u^{\varepsilon_n}(t) \to v^+_{j,d_j^{\varepsilon_n}}$, as $t \to -\infty$, and $v^+_{j,d_j^{\varepsilon_n}} \to u^+_{j,d_j^*} \notin B_{2r_0}(v_{-1})$, there exists $t'_r < t_r$ such that

$$u^{\varepsilon_{n_r}}(t) \in \mathcal{O}_{r_0}(v_{-1}) \text{ for } t \in (t'_r, t_r],$$

$$\|u^{\varepsilon_{n_r}}(t'_r) - v_{-1}\|_{H_0^1} = r_0.$$

Let first $t_t - t'_r \to +\infty$. We define the sequence $u_1^{\varepsilon_{n_r}}(t) = u^{\varepsilon_{n_r}}(t + t'_r)$, which passing to a subsequence converges to a bounded complete trajectory $\phi(t)$ such that $\phi(t) \in \mathcal{O}_{r_0}(v_{-1})$ for all $t \ge 0$. As there is no other fixed point in $\mathcal{O}_{2r_0}(v_{-1})$, $\phi(t) \to v_{-1}$ as $t \to +\infty$. But $\|\phi(0) - v_{-1}\| = r_0$, so $\phi(\cdot)$ is not a fixed point. Then $\phi(t) \to v_{-2}$ as $t \to -\infty$, where v_{-2} is a fixed point different from v_{-1}. Second, let $|t_t - t'_r| \le C$. Then put $u_1^{\varepsilon_{n_r}}(t) = u^{\varepsilon_{n_r}}(t + t_r)$. Passing to a subsequence we have that

$$\begin{aligned} &u_1^{\varepsilon_{n_r}}(0) \to v_{-1}, \\ &t_r - t'_r \to t_0, \text{ as } r \to 0. \end{aligned}$$

Also, $u_1^{\varepsilon_{n_r}}(\cdot)$ converges to a bounded complete trajectory $u^1(\cdot)$ of problem (3) such that $u^1(0) = v_{-1}$. Let

$$\psi_1(t) = \begin{cases} u^1(t) \text{ if } t \le 0, \\ v_{-1} \text{ if } t \ge 0. \end{cases}$$

We note that $\left\|u^1(-t_0) - v_{-1}\right\|_{H_0^1} = r_0$ implies that $u^1(\cdot)$ is not a fixed point. Then ψ_1 is a bounded complete trajectory of problem (3) such that $\psi_1(t) \to v_{-2} \ne v_{-1}$ as $t \to -\infty$. If $v_{-2} = u^+_{j,d_j^*}$, the proof is finished.

If $v_{-2} \ne u^+_{j,d_j^*}$, we continue constructing by the same procedure a chain of connections in which the new fixed point is always different from the previous ones, because the existence of the Lyapunov function (46) avoids the existence of a cyclic chain of connections. Since the number of fixed points is finite, at some moment we obtain a bounded complete trajectory $\phi(\cdot)$ such that $\phi(t) \to u^+_{j,d_j^*}$ as $t \to -\infty$, proving that u^+_{j,d_j^*} is unstable.

Now let $u(\cdot) = v_{-1} = u^+_{j,d_j^*}$. Defining the neighborhood $\mathcal{O}_{2r_0}(v_{-1})$ as before, for any $r \le r_0$ we can choose n_r such that $u^{\varepsilon_n}(0) \in \mathcal{O}_r(v_{-1})$ for all $n \ge n_r$. Also, since $u^{\varepsilon_n}(t) \to z_0^n$, as $t \to +\infty$, where $z_0^n \ne v^+_{j,d_j^{\varepsilon_n}}$ is a fixed point of (64), there exists $t_r > 0$ such that

$$u^{\varepsilon_{n_r}}(t) \in \mathcal{O}_{r_0}(v_{-1}) \text{ for } t \in [0, t_r),$$

$$\|u^{\varepsilon_{n_r}}(t_r) - v_{-1}\|_{H_0^1} = r_0.$$

The sequence $\{t_r\}$ cannot be bounded. Indeed, if $t_r \to t_0$, then $u^{\varepsilon_{n_r}}(t_r) \to u(t_0) = v_{-1}$, which is a contradiction with $\|u^{\varepsilon_{n_r}}(t_0) - v_{-1}\|_{H_0^1} = r_0$. Then $t_r \to +\infty$. We define the functions $u_1^{\varepsilon_{n_r}}(t) = u^{\varepsilon_{n_r}}(t + t_r)$, which satisfy that $u_1^{\varepsilon_{n_r}}(t) \in \mathcal{O}_{r_0}(v_{-1})$ for all $t \in [-t_r, 0)$. Passing to a subsequence it converges to a bounded complete trajectory $\phi(\cdot)$ such that $\phi(t) \in \mathcal{O}_{r_0}(v_{-1})$ for all $t \le 0$. This trajectory is not a fixed point as $\|\phi(0) - v_{-1}\|_{H_0^1} = r_0$ and $\phi(t) \to u^+_{j,d_j^*}$ as $t \to -\infty$, so u^+_{j,d_j^*} is unstable. ■

Further, we will prove that there is also a connection from 0 to the point $u^\pm_{k,d_k^*}$.

Theorem 42 *Assume the conditions of Theorem 40. Then there exists a bounded complete trajectory* $\phi(\cdot)$ *such that* $\phi(t) \underset{t \to -\infty}{\to} 0$, $\phi(t) \underset{t \to +\infty}{\to} u^+_{k,d_k^*}$ *(and the same is valid for* u^-_{k,d_k^*}*). Thus,* $E(0) = 0 > E(u^\pm_{k,d_k^*})$.

Proof. We start with the case where $k = 1$. We have three fixed points: 0, $u^+_{1,d_1^*}, u^-_{1,d_1^*}$. By Theorem 40 there exists a bounded complete trajectory $\phi(\cdot)$ such that $\phi(t) \underset{t \to -\infty}{\to} 0$, whereas Theorem 17 and Remark 26 imply that it has to converge forward to a fixed point different from 0, that is, to either

u^+_{1,d_1^*} or u^-_{1,d_1^*}. If, for example, $\phi(t) \underset{t\to+\infty}{\to} u^+_{1,d_1^*}$, then as the function f is odd, $\psi(t) = -\phi(t)$ is another bounded complete trajectory and $\psi(t) \underset{t\to+\infty}{\to} -u^+_{1,d_1^*} = u^-_{1,d_1^*}$.

Further we consider the problem

$$\begin{cases} \dfrac{\partial u}{\partial t} - a(\|u\|^2_{H_0^1})\dfrac{\partial^2 u}{\partial x^2} = \lambda f_k(u), \quad t>0,\ 0<x<\frac{1}{k}, \\ u(t,0) = u(t,\frac{1}{k}) = 0, \\ u(0,x) = u_0(x), \end{cases} \tag{74}$$

where $f_k(u) = \sqrt{k} f\left(u/\sqrt{k}\right)$ satisfies (A1)-(A5). In this problem, condition (69) implies that there are again three fixed points: 0, $u^+_{1,d_1^*,\frac{1}{k}}, u^-_{1,d_1^*,\frac{1}{k}}$. By the above argument there is a connection $\phi_{\frac{1}{k}}(\cdot)$ from 0 to $u^+_{1,d_1^*,\frac{1}{k}}$ (also to $u^-_{1,d_1^*,\frac{1}{k}}$). Since the function f is odd, $u^+_{k,d_k^*}(x)$ is equal to $\frac{1}{\sqrt{k}}u^+_{1,d_1^*,\frac{1}{k}}(x)$ on $[0,\frac{1}{k}]$, to $-\frac{1}{\sqrt{k}}u^+_{1,d_1^*,\frac{1}{k}}\left(x-\frac{1}{k}\right)$ on $[\frac{1}{k},\frac{2}{k}]$, etc. Then the function $\phi(\cdot)$ such that $\phi(t,x) = \frac{(-1)^j}{\sqrt{k}}\phi_{\frac{1}{k}}\left(t, x-\frac{j}{k}\right)$ on $[\frac{j}{k},\frac{j+1}{k}]$, $j = 0,1,...,k-1$, is a bounded complete trajectory of problem (3) which goes from 0 to u^+_{k,d_k^*}. ■

Remark 43 *When $k = 1$ the structure of the global attractor is the same as in the Chafee-Infante equation.*

6.3 Gradient structure

We will obtain that the m-semiflow G is dynamically gradient. Let us recall this concept.

A weakly invariant set M of X is isolated if there is a neighborhood O of M such that M is the maximal weakly invariant subset on $\mathcal{O}$. If M belongs to the global attractor $\mathcal{A}$, then it is compact [26, Lemma 19]. In this case, it is equivalent to use a δ-neighborhood $\mathcal{O}_\delta(M) = \{y \in X : dist(y, M) < \delta\}$.

Suppose that there is a finite disjoint family of isolated weakly invariant sets $\mathcal{M} = \{M_1, \ldots, M_m\}$ in $\mathcal{A}$, that is, for every $j \in \{1, \ldots, n\}$ there is $\epsilon_j > 0$ such that $M_j \subset \mathcal{A}$ is the maximal weakly invariant set on $\mathcal{O}_{\epsilon_j}(M_j)$, and suppose that there exists $\delta > 0$ such that $\mathcal{O}_\delta(M_i) \cap \mathcal{O}_\delta(M_j) = \emptyset$, if $i \neq j$.

Definition 44 *We say the m-semiflow $G\colon \mathbb{R}^+ \times X \to P(X)$ is dynamically gradient with respect to the disjoint family of isolated weakly invariant sets $\mathcal{M} = \{M_1, \ldots, M_m\}$ if for every complete and bounded trajectory ψ of $\mathcal{R}$ we have that either $\psi(\mathbb{R}) \subset M_j$, for some $j \in \{1, \ldots, m\}$, or $\alpha(\psi) \subset M_i$ and $\omega(\psi) \subset M_j$ with $1 \leq j < i \leq m$.*

Let us consider the case when the conditions of Theorem 40 hold. Then (3) possesses exactly $2k+1$ fixed points: $v_0 = 0$, $u^\pm_{1,d_1^*}, ..., u^\pm_{k,d_k^*}$. Also, as f is odd, $u^+_{j,d_j^*} = -u^-_{j,d_j^*}$ for any j. We define the following sets:

$$M_1 = \{u^+_{1,d_1^*}, u^-_{1,d_1^*}\}, ..., \ M_k = \{u^+_{k,d_k^*}, u^-_{k,d_k^*}\}, \ M_{k+1} = \{0\}. \tag{75}$$

They are weakly invariant and using Lemma 28 we deduce easily that they are isolated. Then the family $\mathcal{M} = \{M_1, \ldots, M_{k+1}\}$ is a finite disjoint family of isolated weakly invariant sets.

Proposition 45 *Assume the conditions of Theorem 40. Then G is dynamically gradient with respect to the family (75) after (possibly) reordering them.*

Proof. We reorder the family (75) in such a way that if the value of the Lyapunov function E given in (46) is equal to L_i for the set $\widetilde{M}_i$, then $L_j \leq L_n$ for $j < n$. Then Theorem 25 in [26] implies that G is dynamically gradient with respect to this family. ■

We will obtain then that the fixed points $u^+_{1,d_1^*}, u^-_{1,d_1^*}$ are asymptotically stable. The compact set $M \subset \mathcal{A}$ is a local attractor for G in X if there is $\varepsilon > 0$ such that $\omega(O_\varepsilon(M)) = M$, where

$$\omega(B) = \{y : \exists t_n \to +\infty,\ y_n \in G(t_n, B) \text{ such that } y_n \to y\}$$

is the ω-limit set of B. By Lemma 14 in [26] if M is a local attractor in X, then it is stable. Thus, a local attractor is asymptotically stable.

Theorem 46 *Assume the conditions of Theorem 40. Then the stationary points $u^{+}_{1,d_1^*}, u^{-}_{1,d_1^*}$ are asymptotically stable.*

Proof. By [26, Theorem 23 and Lemma 15] $\widetilde{M}_1$ is a local attractor in X, so it is asymptotically stable. By Theorem 40 the sets M_j, $j \geq 2$, are unstable. Thus, $\widetilde{M}_1 = M_1$. As M_1 consists of the two elements $u^{+}_{1,d_1^*}, u^{-}_{1,d_1^*}$, which are obviously disjoint, they are asymptotically stable as well. ■

We will prove that there is a connection from 0 to any other fixed point $u^{\pm}_{j,d_j^*}$.

Theorem 47 *Assume the conditions of Theorem 40. Then there exists a bounded complete trajectory $\phi(\cdot)$ such that $\phi(t) \underset{t\to-\infty}{\to} 0$, $\phi(t) \underset{t\to+\infty}{\to} u^{+}_{j,d_j^*}$ for all $1 \leq j \leq k$ (and the same is valid for u^{-}_{j,d_j^*}).*

Proof. Let us consider problem (74) with $k = j$. The function $u^{+}_{1,d_j^*,\frac{1}{j}}(x) = \sqrt{j}u^{+}_{j,d_j^*}(x)$, $x \in [0, \frac{1}{j}]$, is the unique positive fixed point of problem (74). Let $X_j^+ = \{u \in H_0^1\left(0, \frac{1}{j}\right) : u(x) \geq 0 \ \forall x \in [0, \frac{1}{j}]\}$ be the positive cone of $H_0^1\left(0, \frac{1}{j}\right)$. If we consider the restriction of the semigroup $T_j^{\varepsilon_n}(\cdot)$ of problem (64) in the interval $\left(0, \frac{1}{j}\right)$ to X_j^+, denoted by $T_j^{\varepsilon_n,+}(\cdot)$, then there exists a global attractor $\mathcal{A}^{+}_{n,j}$ [18]. Since 0 and $v^{+}_{1,d_j^{\varepsilon_n},\frac{1}{j}} = \sqrt{j}v^{+}_{j,d_j^{\varepsilon_n}} \mid_{[0,\frac{1}{j}]}$ are the unique fixed points of $T_j^{\varepsilon_n,+}$, $\mathcal{A}^{+}_{n,j}$ is connected, $v^{+}_{1,d_1^{\varepsilon_n},\frac{1}{j}}$ is stable [19] and $\mathcal{A}^{+}_{n,j}$ consists of the fixed points and their heteroclinic connections, there must exist a bounded complete trajectory $\phi_j^{\varepsilon_n}(\cdot)$ of $T_j^{\varepsilon_n,+}$ which goes from 0 to $v^{+}_{1,d_j^{\varepsilon_n},\frac{1}{j}}$. By Lemma 36 up to a subsequence it converges to a bounded complete trajectory $\phi_j(\cdot)$ of problem (74) with $k = j$ such that $\phi_j(t) \geq 0$ for all $t \in \mathbb{R}$. Since by Theorem 46 the fixed point $u^{+}_{1,d_j^*,\frac{1}{j}}$ is stable, the only possibility is that $\phi_j(t) \to 0$, as $t \to -\infty$, $\phi_j(t) \to u^{+}_{1,d_j^*,\frac{1}{j}}$, as $t \to +\infty$. Then the function $\phi(\cdot)$ such that $\phi(t,x) = \frac{(-1)^i}{\sqrt{j}}\phi_j\left(t, x - \frac{i}{j}\right)$ on $[\frac{i}{j}, \frac{i+1}{j}]$, $i = 0, 1, ..., j-1$, is a bounded complete trajectory of problem (3) which goes from 0 to u^{+}_{j,d_j^*}.

For u^{-}_{j,d_j^*}, noting that $u^{-}_{j,d_j^*} = -u^{+}_{j,d_j^*}$, the result follows by choosing the bounded complete trajectory $\widetilde{\phi}(t) = -\phi(t)$. ■

As a consequence we obtain that the order of the family $\mathcal{M}$ has to be the one given in (75).

Theorem 48 *The semiflow G is dynamically gradient with respect to the family $\mathcal{M}$ in the order given in (75), that is, $\widetilde{M}_i = M_i$ for any i.*

Proof. As by Theorem 47 there is a connection from 0 to $u^{\pm}_{j,d_j^*}$, $1 \leq j \leq k$, we have proved that $\widetilde{M}_{k+1} = \{0\} = M_{k+1}$. The fact that the order of the other sets is the one given in (75) follows from Lemma 30. ■

7 Appendix

In this appendix we generalize the lap number property of solutions of linear equations proved in [29] to the case when we do not have classical solutions. For this we will use a maximum principle for non-smooth functions from [30].

Let $\mathcal{O}$ be a region in $\mathbb{R}^2$ and let $(t_0, x_0) \in \mathcal{O}$ and $\rho, \sigma > 0$. We denote

$$Q_{\rho,\sigma} = \{(t,x) : t \in (t_0 - \sigma, t_0), |x - x_0| < \rho\},$$

where we assume that t_0, x_0, ρ, σ are such that $\overline{Q}_{\rho,\sigma} \subset \mathcal{O}$.

We denote by W the space of all functions from $L^2(\mathcal{O})$ such that

$$\int_{\mathcal{O}} \left(|u(t,x)|^2 + \left| \frac{\partial u}{\partial x}(t,x) \right|^2 \right) d\mu < +\infty.$$

As a particular case of Theorem 6.4 in [30] we obtain the following maximum and minimum principles.

Theorem 49 *(Maximum principle) Let $u \in W$ be such that*

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} \leq 0 \tag{76}$$

in the sense of distributions. If

$$\sup ess_{(t,x)\in Q_{\rho\nu,\sigma_1}} u(t,x) = M,$$

for some ν, $0 < \nu < 1$, and any σ_1, where $0 < \sigma_1 < \sigma$, then $u(t,x) = M$ for a.a. $(t,x) \in Q_{\rho,\sigma}$.

Theorem 50 *(Minimum principle) Let $u \in W$ be such that*

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} \geq 0 \tag{77}$$

in the sense of distributions. If

$$\inf ess_{(t,x)\in Q_{\rho\nu,\sigma_1}} u(t,x) = M,$$

for some ν, $0 < \nu < 1$, and any σ_1, where $0 < \sigma_1 < \sigma$, then $u(t,x) = M$ for a.a. $(t,x) \in Q_{\rho,\sigma}$.

We are ready to prove the lap-number property, saying that the number of zeros is a non-increasing function of time.

Theorem 51 *Let $r(t,x)$ be a continuous function and $u \in C([t_0,t_1], H_0^1(\Omega)) \cap L^2\left(t_0,t_1; H^2(\Omega)\right)$ be such that $\frac{du}{dt} \in L^2\left(t_0,t_1; L^2(\Omega)\right)$ and satisfies the equation*

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = r(t,x)u,\ 0 < x < 1,\ t_0 < t \leq t_1. \tag{78}$$

Then the number of components of

$$\{x : 0 < x < 1,\ u(t,x) \neq 0\}$$

is a non-increasing function of t.

Proof. We follow similar lines as in [29, Theorem 6].

Denote $Q(t) = \{x \in (0,1) : u(t,x) \neq 0\}$. We need to show that there is an injective map from the components of $Q(t_1)$ to the components of $Q(t_0)$ if $t_1 > t_0$. If we denote by C a component of $Q(t_1)$ and by S_C the component of $[t_0,t_1] \times (0,1) \cap \{u(t,x) \neq 0)\}$ which contains C, then in order to obtain the injective map it is necessary to prove two facts:

1. $S_C \cap Q(t_0) \neq \varnothing$;
2. If C_1, C_2 are two components of $Q(t_1)$, then $S_{C_1} \cap S_{C_2} = \varnothing$.

Let us prove the first statement by contradiction, so assume that $S_C \cap Q(t_0) = \varnothing$. We can assume without loss of generality that $r(t,x) < 0$, because this property is satisfied for the function $W(t,x) = u(t,x)e^{-\lambda t}$ with $\lambda > 0$ large enough and the components of these two functions coincide. Consider for example that $u(t,x) > 0$ in S_C. Let $M = \max_{(t,x)\in S_C} u(t,x)$. By hypothesis and the Dirichlet boundary conditions this maximum has to be attained at a point (t',x') such that $t_0 < t' \leq t_1$, $0 < x' < 1$. Also, there has to be an $\varepsilon > 0$ such that if $(t,x) \in S_C$ and $t_0 < t \leq t_0 + \varepsilon$, then $u(t,x) < M$, as otherwise there would be a sequence $(t_n,x_n) \in S_C$, $t_n > t_0$, such that $t_n \to t_0$ and $u(t_n,x_n) = M$. By the continuity of u this would imply that $u(t_0,x_0) = M$ for some $(t_0,x_0) \in S_C$, which is a contradiction. Then we can choose t' as the first time when the maximum is attained, so $u(t,x) < M$ for all $(t,x) \in S_C$, $t_0 < t < t'$. By the continuity of u there exists a rectangle $R = [t'-\delta, t'] \times [x'-\gamma, x'+\gamma]$ such that R belongs to S_C. In order to apply Theorem 49 we put $\mathcal{O} = R$ and

$$Q_{\gamma,\delta} = \{(t,x) : t \in (t'-\delta, t'), |x - x'| < \gamma\}.$$

We have that

$$\sup_{(t,x)\in Q_{\nu\gamma,\sigma_1}} u(t,x) = M,$$

for some $0 < \nu < 1$ and any $0 < \sigma_1 < \delta$. Since u satisfies (76), we conclude from Theorem 49 that $u(t,x) = M$ for all $(t,x) \in Q_{\rho,\sigma}$, which is a contradiction.

For the second statement suppose the existence of two disjoints components C_1, C_2 of $Q(t_1)$ such that $S_{C_1} \cap S_{C_2} \neq \varnothing$, which implies in fact that $S_{C_1} = S_{C_2}$. In this case we can assume that $r(t,x) > 0$, being this justified by the function $W(t,x) = u(t,x)e^{\lambda t}$ with $\lambda > 0$ large enough. Let for example $u(t,x) > 0$ in S_{C_1} and assume that the interval C_1 has lesser values than the interval C_2. Also, it is clear that between C_1 and C_2 there must exist a point (t_1, x_0) such that $u(t_1, x_0) = 0$. On the other hand, the set $S_{C_1} \cap (t_0, t_1) \times [0,1]$ is path connected. Thus, there exists a simple path ξ such that one end point is in $\{t_1\} \times C_1$ and the other one is in $\{t_1\} \times C_2$. Let us consider the set L of all points which are above the curve ξ and such that the function u vanishes at them. This set is non-empty because $(t_1, x_0) \in L$. Since L is compact, the function $g : L \to [t_0, t_1]$ given by $g(t,x) = t$ attains it minimum at a certain point $(t', x') \in L$ such that $t_0 < t'$. Then there exists a set $R = [t' - \delta, t') \times [x' - \gamma, x' + \gamma]$ which belongs to S_{C_1}. Let $\mathcal{O} = R$ and

$$Q_{\gamma,\delta} = \{(t,x) : t \in (t' - \delta, t'), |x - x'| < \gamma\}.$$

We have that

$$\inf_{(t,x)\in Q_{\nu\gamma,\sigma_1}} u(t,x) = 0,$$

for some $0 < \nu < 1$ and any $0 < \sigma_1 < \delta$. Since u satisfies (77), we conclude from Theorem 50 that $u(t,x) = 0$ for all $(t,x) \in Q_{\rho,\sigma}$, which is a contradiction. ■

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