

FROM CALMNESS TO HOFFMAN CONSTANTS FOR LINEAR SEMI-INFINITE INEQUALITY SYSTEMS*

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Abstract. In this paper we focus on different—global, semilocal, and local—versions of Hoffman-type inequalities expressed in a variational form. In a first stage our analysis is developed for generic multifunctions between metric spaces, and we finally deal with the feasible set mapping associated with linear semi-infinite inequality systems (finitely many variables and possibly infinitely many constraints) parameterized by their right-hand sides. The Hoffman modulus is shown to coincide with the Lipschitz upper semicontinuity modulus and the supremum of calmness moduli when confined to multifunctions with a convex graph and closed images in a reflexive Banach space, which is the case for our feasible set mapping. Moreover, for this particular multifunction a formula—involving only the system’s left-hand side—of the global Hoffman constant is derived, providing a generalization to our semi-infinite context of finite counterparts developed in the literature. In the particular case of locally polyhedral systems, the paper also provides a point-based formula for the (semilocal) Hoffman modulus in terms of the calmness moduli at certain feasible points (extreme points when the nominal feasible set contains no lines), yielding a practically tractable expression for finite systems.

Key words. Hoffman constants, Lipschitz upper semicontinuity, calmness, linear inequality systems, feasible set mapping

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1. Introduction. Concerning finite linear inequality systems parameterized by their right-hand sides, the celebrated Hoffman lemma [10] is a result of *global* nature as far as it works for *any* parameter making the system consistent and *any* point of the Euclidean space. We can also find in the literature related *semilocal* results as far as they work around a nominal (given) parameter and any point in the Euclidean space, leading to the concept of a Hoffman constant at this parameter (see, e.g., Azé and Corvellec [2] and Zălinescu [27]). In this paper we relate these global and semilocal Hoffman constants with the *local* concept of a *calmness modulus*, which involves parameters and points, both around nominal ones. Our analysis is developed in a first step in the context of generic multifunctions then subsequently moves to the particular case of the *feasible set mapping* associated with a parameterized linear semi-infinite inequality system

$$(1) \quad \sigma(b) := \{a'_t x \leq b_t, \ t \in T\},$$

where T is a compact metric space, $t \mapsto a_t \in \mathbb{R}^n$ is a fixed continuous function from T to $\mathbb{R}^n$, and $b \equiv (b_t)_{t \in T} \in C(T, \mathbb{R})$ is the parameter to be perturbed, $C(T, \mathbb{R})$ being the space of continuous functions from T to $\mathbb{R}$. We are considering column-vectors, and the prime stands for transposition, so $x'y$ denotes the usual inner product of x and y in $\mathbb{R}^n$. In this parametric context, the feasible set mapping $\mathcal{F} : C(T, \mathbb{R}) \rightrightarrows \mathbb{R}^n$

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is given by

$$(2) \quad \mathcal{F}(b) := \{x \in \mathbb{R}^n \mid a'_t x \leq b_t, \quad t \in T\}.$$

With respect to the topology, $\mathbb{R}^n$ is equipped with an arbitrary norm, $\|\cdot\|$, with *dual norm* given by $\|u\|_* = \max_{\|x\| \leq 1} |u'x|$, and the parameter space $C(T, \mathbb{R})$ is endowed with the supremum norm $\|b\|_\infty := \max_{t \in T} |b_t|$.

The particular case when T is finite is included in this framework, in which case $\mathcal{F}$ coincides with the polyhedral mapping considered in [10], and the Hoffman lemma reads as the existence of some constant $\kappa \geq 0$ such that, for all $x \in \mathbb{R}^n$ and all $b \in \text{dom } \mathcal{F}$ (the domain of $\mathcal{F}$),

$$(3) \quad d(x, \mathcal{F}(b)) \leq \kappa \max_{t \in T} [a'_t x - b_t]_+,$$

where $[\alpha]_+ := \max\{\alpha, 0\}$ is the positive part of $\alpha \in \mathbb{R}$. This result is of a global nature as far as it involves all points $x \in \mathbb{R}^n$ and all $b \in \text{dom } \mathcal{F}$. Since $\max_{t \in T} [a'_t x - b_t]_+ = d(b, \mathcal{F}^{-1}(x))$, inequality (3) can be written in a variational form, as is done in the following paragraph for a generic multifunction.

Given a multifunction $\mathcal{M} : Y \rightrightarrows X$ between metric spaces with both distances being denoted by d , we say that the (global) Hoffman property holds if there exists a constant $\kappa \geq 0$ such that

$$(4) \quad d(x, \mathcal{M}(y)) \leq \kappa d(y, \mathcal{M}^{-1}(x)) \quad \text{for all } x \in X \text{ and all } y \in \text{dom } \mathcal{M},$$

where $d(x, \Omega) := \inf\{d(x, \omega) \mid \omega \in \Omega\}$ for $x \in X$ and $\Omega \subset X$, with $\inf \emptyset := +\infty$, so that $d(x, \emptyset) = +\infty$. Since this paper is concerned with nonnegative constants, we use the convention $\sup \emptyset := 0$. Here $\text{dom } \mathcal{M}$ is the domain of $\mathcal{M}$ (recall that $y \in \text{dom } \mathcal{M} \Leftrightarrow \mathcal{M}(y) \neq \emptyset$) and $\mathcal{M}^{-1}$ denotes the inverse mapping of $\mathcal{M}$ (i.e., $y \in \mathcal{M}^{-1}(x) \Leftrightarrow x \in \mathcal{M}(y)$).

Now we write a semilocal version of (4) by fixing $y = \bar{y}$. $\mathcal{M}$ is said to be *Hoffman stable at $\bar{y} \in \text{dom } \mathcal{M}$* if there exists $\kappa \geq 0$ such that

$$(5) \quad d(x, \mathcal{M}(\bar{y})) \leq \kappa d(\bar{y}, \mathcal{M}^{-1}(x)) \quad \text{for all } x \in X.$$

When inequality (5) is only required to be satisfied in a neighborhood of $\bar{x} \in \mathcal{M}(\bar{y})$ we are dealing with the *calmness* of $\mathcal{M}$ at $(\bar{y}, \bar{x}) \in \text{gph } \mathcal{M}$, the graph of $\mathcal{M}$. Formally, the calmness of $\mathcal{M}$ at $(\bar{y}, \bar{x}) \in \text{gph } \mathcal{M}$, or equivalently the *metric subregularity* of $\mathcal{M}^{-1}$ at $(\bar{x}, \bar{y})$ (cf. [7, Theorem 3H.3 and Exercise 3H.4]), is satisfied when there exist a constant $\kappa \geq 0$ and a neighborhood U of $\bar{x}$ such that

$$(6) \quad d(x, \mathcal{M}(\bar{y})) \leq \kappa d(\bar{y}, \mathcal{M}^{-1}(x)) \quad \text{for all } x \in U.$$

The infimum of constants κ appearing in (4), (5), and (6) are called, respectively, the *global Hoffman constant* of $\mathcal{M}$, the *Hoffman modulus* of $\mathcal{M}$ at $\bar{y} \in \text{dom } \mathcal{M}$, and the *calmness modulus* of $\mathcal{M}$ at $(\bar{y}, \bar{x}) \in \text{gph } \mathcal{M}$. The three constants are denoted, respectively, by $\text{Hof } \mathcal{M}$, $\text{Hof } \mathcal{M}(\bar{y})$, and $\text{clm } \mathcal{M}(\bar{y}, \bar{x})$ and, as a consequence of the definitions, they may be written as follows:

$$(7) \quad \begin{aligned} \text{Hof } \mathcal{M} &= \sup_{(y, x) \in (\text{dom } \mathcal{M}) \times X} \frac{d(x, \mathcal{M}(y))}{d(y, \mathcal{M}^{-1}(x))}, \\ \text{Hof } \mathcal{M}(\bar{y}) &= \sup_{x \in X} \frac{d(x, \mathcal{M}(\bar{y}))}{d(\bar{y}, \mathcal{M}^{-1}(x))}, \quad \bar{y} \in \text{dom } \mathcal{M}, \\ \text{clm } \mathcal{M}(\bar{y}, \bar{x}) &= \limsup_{x \rightarrow \bar{x}} \frac{d(x, \mathcal{M}(\bar{y}))}{d(\bar{y}, \mathcal{M}^{-1}(x))}, \quad (\bar{y}, \bar{x}) \in \text{gph } \mathcal{M}, \end{aligned}$$

under the convention $\frac{0}{0} := 0$, where $\limsup$ is understood as the supremum (indeed, maximum) of all possible sequential upper limits (i.e., with (y, x) being replaced with elements of sequences $\{(y_r, x_r)\}_{r \in \mathbb{N}}$ converging to $(\bar{y}, \bar{x})$ as $r \rightarrow \infty$).

Now we describe the main contributions of the paper. Clearly

$$\text{Hof } \mathcal{M} = \sup_{\bar{y} \in \text{dom } \mathcal{M}} \text{Hof } \mathcal{M}(\bar{y}),$$

and we wonder if a similar relationship between $\text{Hof } \mathcal{M}(\bar{y})$ and the supremum of all calmness moduli $\text{clm } \mathcal{M}(\bar{y}, x)$, with $x \in \mathcal{M}(\bar{y})$, works. Section 3 is devoted to this question, and Theorem 4 gives a positive answer when $\text{gph } \mathcal{M}$ is convex and $\mathcal{M}(\bar{y})$ is closed, Y being a normed space and X being a reflexive Banach space. Some examples show that the convexity assumption is not superfluous. Moreover, some intermediate constants such as the *Lipschitz upper semicontinuity modulus* are also considered.

With respect to mapping $\mathcal{F}$ our focus is on formulae only involving the system's coefficients for $\text{Hof } \mathcal{F}$ and $\text{Hof } \mathcal{F}(\bar{b})$, which are established in Theorems 5 and 6, respectively. The first one extends to the current semi-infinite framework previous results on finite linear systems (see, e.g., Burke and Tseng [4, Theorem 8], Klatte and Thiere [13, Theorem 2.7], and Peña, Vera, and Zuluaga [18, Formula (3)]); for comparative purposes, some details are gathered in section 2. Theorem 6 provides a formula for $\text{Hof } \mathcal{F}(\bar{b})$ in terms of the a_t 's, the $\bar{b}_t$'s, and some feasible points in the case when our system $\sigma(\bar{b})$ is *locally polyhedral*. Specifically, from the referred Theorem 4, we have that

$$\text{Hof } \mathcal{F}(\bar{b}) = \sup_{x \in \mathcal{F}(\bar{b})} \text{clm } \mathcal{F}(\bar{b}, x),$$

and Theorem 6 refines this expression by reducing the supremum to a smaller set (which turns out to be finite when T also is). Then, making use of the expression for $\text{clm } \mathcal{F}(\bar{b}, \bar{x})$ established in Li, Meng, and Yang [16] (recalled in Theorem 3), we derive the announced point-based formula for $\text{Hof } \mathcal{F}(\bar{b})$. Here we use the term “point-based” to emphasize the fact that the expression for $\text{Hof } \mathcal{F}(\bar{b})$ does not involve parameters different from $\bar{b}$ or points outside $\mathcal{F}(\bar{b})$. An alternative expression for $\text{Hof } \mathcal{F}(\bar{b})$ appealing to points outside $\mathcal{F}(\bar{b})$ is given in [2, Theorem 2.6] (recalled in Theorem 2). We point out the fact that Theorem 6 yields a particularly tractable procedure for computing $\text{Hof } \mathcal{F}(\bar{b})$ when T is finite.

In summary, the structure of the paper is as follows: Section 2 introduces the necessary notation and gathers some preliminary results. Section 3 analyzes the relationships among different semilocal versions of Hoffman- and Lipschitz-type properties for generic multifunctions and their moduli (Lipschitz-type properties are widely analyzed in the monographs [7, 12, 17, 22]). Section 3 also provides illustrative counterexamples. Section 4 is focused on $\text{Hof } \mathcal{F}$ and $\text{Hof } \mathcal{F}(\bar{b})$, the latter in the case of locally polyhedral systems. Before establishing the announced formula for $\text{Hof } \mathcal{F}(\bar{b})$ some technical geometrical results are proved. The paper finishes with a short section of conclusions and perspectives.

2. Preliminaries. Given $S \subset \mathbb{R}^k$, $k \in \mathbb{N}$, we denote by $\text{conv}S$, $\text{cone}S$, and $\text{span}S$ the *convex hull*, the *conical convex hull*, and the *linear hull* of S , respectively. It is assumed that $\text{cone}S$ always contains the zero-vector 0_k , in particular $\text{cone}(\emptyset) = \{0_k\}$.

Moreover, S° denotes the (negative) *polar* of S given by

$$S^\circ := \{u \in \mathbb{R}^k \mid u'x \leq 0 \text{ for all } x \in S\}$$

($S^\circ = \mathbb{R}^k$ if $S = \emptyset$). From the topological side, $\text{int}S$, $\text{cl}S$, and $\text{bd}S$ stand, respectively, for the (topological) interior, closure, and boundary of S . For a nonempty convex set $C \subset \mathbb{R}^k$, O^+C denotes its *recession cone* given by

$$O^+C := \{d \in \mathbb{R}^k \mid u + \alpha d \in C \text{ for all } u \in C \text{ and all } \alpha \geq 0\},$$

while $\text{end}C$ denotes its end set (introduced in [11]) defined as

$$\text{end}C := \{u \in \text{cl}C \mid \nexists \mu > 1 \text{ such that } \mu u \in \text{cl}C\}.$$

Moreover, $\text{extr}C$ stands for the set of extreme points of C . Recall that $x \in \text{extr}C$ if $x \in C$ and it cannot be expressed as a convex combination of two points of $C \setminus \{x\}$. In any metric space (Z, d) , the *closed* ball centered at $z \in Z$ with radius $r > 0$ is denoted by $B(z, r)$, whereas $B(S, r) := \{z \in Z \mid d(z, S) \leq r\}$, for $S \subset Z$, denotes the *r-enlargement* of S .

For comparative purposes, the next theorem gathers some results in the literature on $\text{Hof}\mathcal{F}$ when confined to finite linear systems, where $C(T, \mathbb{R}) \equiv \mathbb{R}^m$ for some $m \in \mathbb{N}$. It is adapted to our current notation and to our choice of norms. The first two expressions come from [18, formulae (3) and (4)] (see also [13, Theorem 2.7] when $\mathbb{R}^n$ is endowed with the Euclidean norm), while the third one can be derived from [4, Theorem 8], where a dual approach is followed. The last one appeals to the set

$$W_2 := \{y \in \mathbb{R}_+^m \mid \{a_t, t \in \text{supp}(y)\} \text{ lin. indep.}\},$$

where $\mathbb{R}_+^m$ is formed by the vectors of $\mathbb{R}^m$ having nonnegative coordinates and $\text{supp}(y) := \{t \in \{1, \dots, m\} \mid y_t \neq 0\}$ is the support of y ; indeed W_2 is considered a subset of the dual space of $\mathbb{R}^m$, which we are identifying with $\mathbb{R}^m$ itself.

THEOREM 1. *Consider the feasible set mapping $\mathcal{F}$ defined in (2) and assume that T is finite. We have*

$$(8) \quad \text{Hof}\mathcal{F} = \max_{\substack{J \subset T \\ 0_n \notin \text{conv}\{a_t, t \in J\}}} d_*(0_n, \text{conv}\{a_t, t \in J\})^{-1}$$

$$(9) \quad = \max_{\substack{J \subset T, \text{rank } A_J = \text{rank } A \\ \{a_t, t \in J\} \text{ lin. indep.}}} d_*(0_n, \text{conv}\{a_t, t \in J\})^{-1}$$

$$(10) \quad = \sup \{\|y\|_1 \mid y \in W_2, \|A'y\|_* = 1\},$$

where A_J and A stand for the matrices whose rows are a'_t , with $t \in J$ and $t \in T$, respectively, and d_* stands for the distance associated with the dual norm $\|\cdot\|_*$.

Proof. According to [18, formula (3)] and the subsequent comments therein, to establish (8) we only have to prove that condition $0_n \notin \text{conv}\{a_t, t \in J\}$ is equivalent to the consistency of system $\{a'_t x < 0, t \in J\}$, and this follows, for instance, from equivalence (iv) $\Leftrightarrow$ (v) in [9, Theorem 6.1]. Equality (9) comes from [18, formula (4)] with the trivial observation that instead of all linearly independent $\{a_t, t \in J\}$, with $J \subset T$, we can confine ourselves to those which are maximal with respect to the inclusion order. Indeed, the result also follows from (8), since the sufficiency of

considering those $\{a_t, t \in J\}$ which are linearly independent comes from [2, Lemma 3.1].

Formula (10) comes from [4, Theorem 8]. Let us comment that we can, alternatively, see the relationship between the second and the third expression by observing that, for any $y \in \mathbb{R}_+^m$, $y \neq 0_m$,

$$\frac{1}{\|y\|_1} A'y = \frac{1}{\|y\|_1} \sum_{i=1}^m y_i a_i \in \text{conv} \{a_t, t \in \text{supp}(y)\},$$

and that $\|A'y\|_* = 1$ is equivalent to $\|y\|_1 = \left\| \frac{1}{\|y\|_1} A'y \right\|_*^{-1}$. $\square$

Generalizations of Hoffman constants to infinite-dimensional spaces or to convex functions playing the role of the distance function can be found in [4]. Many other authors have contributed to the study of Hoffman constants and their relationship with other concepts (as Lipschitz constants). Additional references can be obtained from the reference list of the papers mentioned above as well as [2] and [27], among others. At this moment we also cite Belousov and Andronov [3], Li [15], and Robinson [20].

The following theorem provides formulae for $\text{Hof } \mathcal{F}(\bar{b})$, with $\bar{b} \in \text{dom } \mathcal{F}$, and $\text{clm } \mathcal{F}(\bar{b}, \bar{x})$, with $(\bar{b}, \bar{x}) \in \text{gph } \mathcal{F}$ through points outside $\mathcal{F}(\bar{b})$. They appeal to the supremum function $f_b : \mathbb{R}^n \rightarrow \mathbb{R}$, with $b \in C(T, \mathbb{R})$, given by

$$f_b(x) := \sup_{t \in T} (a'_t x - b_t) \quad \text{for } x \in \mathbb{R}^n,$$

which is known to be convex on $\mathbb{R}^n$. For each $x \in \mathbb{R}^n$, we consider the subset of indices

$$J_b(x) = \{t \in T \mid a'_t x - b_t = f_b(x)\}.$$

The well-known Valadier's formula works by virtue of the Ioffe–Tikhomirov theorem (see, e.g., [26, Theorem 2.4.18]), yielding

$$\partial f_b(x) = \text{conv} \{a_t, t \in J_b(x)\},$$

where $\partial f_b(x)$ stands for the usual subdifferential of convex analysis (see, e.g., [21]).

THEOREM 2. *The following statements hold:*

(i) [2, Theorem 2.6] *For any $\bar{b} \in \text{dom } \mathcal{F}$, one has*

$$\begin{aligned} \text{Hof } \mathcal{F}(\bar{b}) &= \sup_{f_{\bar{b}}(x) > 0} d_*(0_n, \partial f_{\bar{b}}(x))^{-1} \\ &= \sup_{f_{\bar{b}}(x) > 0} d_*(0_n, \text{conv} \{a_t, t \in J_{\bar{b}}(x)\})^{-1}. \end{aligned}$$

(ii) [14, Theorem 1] *For any $(\bar{b}, \bar{x}) \in \text{gph } \mathcal{F}$,*

$$\begin{aligned} \text{clm } \mathcal{F}(\bar{b}, \bar{x}) &= \limsup_{x \rightarrow \bar{x}, f_{\bar{b}}(x) > 0} d_*(0_n, \partial f_{\bar{b}}(x))^{-1} \\ &= \limsup_{x \rightarrow \bar{x}, f_{\bar{b}}(x) > 0} d_*(0_n, \text{conv} \{a_t, t \in J_{\bar{b}}(x)\})^{-1}. \end{aligned}$$

Remark 1. Observe that $\bar{b} \in \text{dom } \mathcal{F}$ and $f_{\bar{b}}(x) > 0$ mean that $\sigma(\bar{b})$ is consistent (it has some feasible solution) but $x \notin \mathcal{F}(\bar{b})$; in this case, $0_n \notin \text{conv}\{a_t, t \in J_{\bar{b}}(x)\}$, since x is not a global minimizer of the convex function $f_{\bar{b}}$. Actually, [2, Theorem 2.6] is formulated in terms of $(\text{Hof } \mathcal{F}(\bar{b}))^{-1}$, which is called there the *condition number of $f_{\bar{b}}$ at level 0*; in the terminology of [14], observe that $(\text{clm } \mathcal{F}(\bar{b}, \bar{x}))^{-1}$ is the *error bound modulus* (also known as *conditioning rate* [19]) of $f_{\bar{b}}$ at $\bar{x}$.

The following theorem is devoted to the computation of $\text{clm } \mathcal{F}(\bar{b}, \bar{x})$, $(\bar{b}, \bar{x}) \in \text{gph } \mathcal{F}$, through a point-based formula (expressed exclusively in terms of the system's coefficients and the nominal point $\bar{x}$). Now we introduce some extra notation. Given a fixed $\bar{b} \in \text{dom } \mathcal{F}$, for any $x \in \mathcal{F}(\bar{b})$, we consider (for simplicity, since there will be no ambiguity, we omit the dependence on $\bar{b}$)

$$T(x) := \{t \in T \mid a'_t x - \bar{b}_t = 0\},$$

the *subset of active indices* of system $\sigma(\bar{b})$ at x ; i.e., $T(x) = J_{\bar{b}}(x)$ if $f_{\bar{b}}(x) = 0$, while $T(x) = \emptyset$ if $f_{\bar{b}}(x) < 0$ (i.e., if x is a strict solution—*Slater point*—of the system). Let $A(x)$ be the corresponding *active cone* at x ; i.e.,

$$A(x) := \text{cone}\{a_t, t \in T(x)\}$$

(recall that $A(x) = \{0_n\}$ if $T(x) = \emptyset$). We also consider the family $\mathcal{D}(x)$ of subsets $D \subset T(x)$ such that system

$$(11) \quad \left\{ \begin{array}{ll} a'_t d = 1, & t \in D, \\ a'_t d < 1, & t \in T(x) \setminus D \end{array} \right\},$$

is consistent (in the variable $d \in \mathbb{R}^n$); i.e., $\{a_t, t \in D\}$ is contained in some hyperplane which leaves $\{0_n\} \cup \{a_t, t \in T(x) \setminus D\}$ on one of its two associated open half-spaces. With this notation, the next theorem generalizes the corresponding finite version established in [6, Theorem 4]. It appeals to the following *regularity condition* at $\bar{x}$: “There exists a neighborhood W of $\bar{x}$ such that

$$(12) \quad \mathcal{F}(\bar{b}) \cap W = (\bar{x} + A(\bar{x})^\circ) \cap W.”$$

Observe that this condition is held at all points of polyhedral sets and, for instance, at the vertex of the ice cream cone.

THEOREM 3 ([16, Corollary 2.1, Remark 2.3, and Corollary 3.2]). *Let $\bar{x} \in \mathcal{F}(\bar{b})$ such that $f_{\bar{b}}(\bar{x}) = 0$ and assume that the regularity condition (12) is held at $\bar{x}$. Then*

$$(13) \quad \text{clm } \mathcal{F}(\bar{b}, \bar{x}) = d_*(0_n, \text{end} \partial f_{\bar{b}}(\bar{x}))^{-1} = \sup_{D \in \mathcal{D}(\bar{x})} d_*(0_n, \text{conv}\{a_t, t \in D\})^{-1}.$$

Remark 2. Although condition (12) is not superfluous for establishing the first equality in (13) as [16, Example 3.3] shows (see also Example 4), the second equality does work for semi-infinite systems (1) without any additional condition. Indeed, from [16, Corollary 2.1 and Remark 2.3] we can deduce

$$(14) \quad \bigcup_{D \in \mathcal{D}(\bar{x})} \text{conv}\{a_t, t \in D\} \subset \text{end} \partial f_{\bar{b}}(\bar{x}) \subset \text{cl}\left(\bigcup_{D \in \mathcal{D}(\bar{x})} \text{conv}\{a_t, t \in D\}\right).$$

3. From calmness to Hoffman constants for a generic multifunction. The purpose of this section is to analyze the relationship among different Hoffman-

and Lipschitz-type properties, including the known Lipschitz upper semicontinuity that goes back to the classical work of Robinson [20]. At the beginning of this section $\mathcal{M} : Y \rightrightarrows X$ is a generic multifunction between metric spaces Y and X . Later we will need further structure. To start with, observe that alternatively to (5) we can write the Hoffman stability of $\mathcal{M}$ at $\bar{y} \in \text{dom } \mathcal{M}$ in terms of the existence of $\kappa \geq 0$ such that

$$d(x, \mathcal{M}(\bar{y})) \leq \kappa d(y, \bar{y}) \text{ for all } (y, x) \in \text{gph } \mathcal{M},$$

while the calmness of $\mathcal{M}$ at $(\bar{y}, \bar{x}) \in \text{gph } \mathcal{M}$, introduced in (6) in terms of the (equivalent) metric subregularity of $\mathcal{M}^{-1}$, is written as the existence of neighborhoods V of $\bar{y}$ and U of $\bar{x}$ along with a constant $\kappa \geq 0$ such that

$$d(x, \mathcal{M}(\bar{y})) \leq \kappa d(y, \bar{y}) \text{ for all } (y, x) \in (V \times U) \cap \text{gph } \mathcal{M}.$$

Moreover, the following equalities constitute well-known alternative expressions to (7) for the corresponding moduli

$$(15) \quad \begin{aligned} \text{Hof } \mathcal{M}(\bar{y}) &= \sup_{(y, x) \in \text{gph } \mathcal{M}} \frac{d(x, \mathcal{M}(\bar{y}))}{d(y, \bar{y})}, \\ \text{clm } \mathcal{M}(\bar{y}, \bar{x}) &= \limsup_{\substack{(y, x) \rightarrow (\bar{y}, \bar{x}) \\ (y, x) \in \text{gph } \mathcal{M}}} \frac{d(x, \mathcal{M}(\bar{y}))}{d(y, \bar{y})}. \end{aligned}$$

Recall that $\mathcal{M}$ is said to be *Lipschitz upper semicontinuous* at $\bar{y} \in \text{dom } \mathcal{M}$ if there exists a neighborhood V of $\bar{y}$ along with a constant $\kappa \geq 0$ such that

$$(16) \quad d(x, \mathcal{M}(\bar{y})) \leq \kappa d(y, \bar{y}) \text{ for all } (y, x) \in (V \times \mathbb{R}^n) \cap \text{gph } \mathcal{M}.$$

Here we borrow the terminology from [12] or [24], although this property, introduced in [20] as *upper Lipschitz continuity*, has been also popularized as *outer Lipschitz continuity* (see [7]). Equivalently, (16) may be written as $e(\mathcal{M}(y), \mathcal{M}(\bar{y})) \leq \kappa d(y, \bar{y})$ for all $y \in V$, where $e(A, B) := \sup_{x \in A} d(x, B)$ is the *Hausdorff excess* of A over B , with $A, B \subset X$. The associated *Lipschitz upper semicontinuity modulus*, denoted by $\text{Lipusc } \mathcal{M}(\bar{y})$, is defined as the infimum of constants κ satisfying (16) for some associated V .

In the next definition, given $\bar{y} \in \text{dom } \mathcal{M}$ and $\varepsilon > 0$, the mapping $\mathcal{M}_\varepsilon : Y \rightrightarrows X$ is defined by

$$\mathcal{M}_\varepsilon(y) := \mathcal{M}(y) \cap B(\mathcal{M}(\bar{y}), \varepsilon) \text{ for } y \in Y.$$

(For simplicity in the notation we obviate the dependence of $\mathcal{M}_\varepsilon$ on $\bar{y}$.)

DEFINITION 1. *Given $\bar{y} \in \text{dom } \mathcal{M}$, we say that $\mathcal{M}$ is uniformly calm at $\bar{y}$ if there exists a neighborhood V of $\bar{y}$ along with $\varepsilon > 0$ and $\kappa \geq 0$ such that*

$$(17) \quad d(x, \mathcal{M}(\bar{y})) \leq \kappa d(y, \bar{y}) \text{ for all } y \in V \text{ and all } x \in \mathcal{M}_\varepsilon(y),$$

or, equivalently, if $\mathcal{M}_\varepsilon$ is Lipschitz upper semicontinuous at $\bar{y}$ for some $\varepsilon > 0$.

The corresponding modulus naturally appears. Specifically, we define the *modulus of uniform calmness* of $\mathcal{M}$ at $\bar{y}$, denoted by $\text{uclm } \mathcal{M}(\bar{y})$, as the infimum of constants κ satisfying (17) for some associated V and $\varepsilon > 0$. It is straightforward to check that

$$(18) \quad \text{uclm } \mathcal{M}(\bar{y}) = \inf_{\varepsilon > 0} \text{Lipusc } \mathcal{M}_\varepsilon(\bar{y}).$$

Roughly speaking, the uniform calmness of $\mathcal{M}$ at $\bar{y}$ entails the calmness of $\mathcal{M}$ at any $(\bar{y}, x)$ for all $x \in \mathcal{M}(\bar{y})$ with the same calmness constant κ , the same neighborhood V of $\bar{y}$, and a common radius ε for all neighborhoods of points $x \in \mathcal{M}(\bar{y})$, say $U_x := B(x, \varepsilon)$. Example 1 below shows that the calmness of $\mathcal{M}$ at $(\bar{y}, x)$ for all $x \in \mathcal{M}(\bar{y})$ does not ensure the uniform calmness of $\mathcal{M}$ at $\bar{y}$.

As it occurs with the calmness property, the uniform calmness turns out to be equivalent to a certain metric regularity-type property, showing that neighborhood V in Definition 1 is redundant. The key fact is that points $x \in \mathcal{M}(y)$ which are required to satisfy (17) are those which are sufficiently close to $\mathcal{M}(\bar{y})$. This comment, which was already pointed out for polyhedral multifunctions in [20] (see the corollary after Proposition 1 therein), is formalized in the following proposition.

PROPOSITION 1. *Let $\bar{y} \in \text{dom } \mathcal{M}$. For any $\kappa > 0$, the following conditions are equivalent:*

- (i) *There exist a neighborhood V of $\bar{y}$ and $\varepsilon > 0$ such that (17) holds.*
- (ii) *There exists $\varepsilon > 0$ such that (5) holds when restricted to those $x \in B(\mathcal{M}(\bar{y}), \varepsilon)$.*

Proof. Let us establish the nontrivial implication “(i) $\Rightarrow$ (ii).” Consider V and ε as in statement (i). Take $\varepsilon_1 > 0$ such that $B(\bar{y}, \varepsilon_1) \subset V$ and define $\varepsilon_2 := \min\{\varepsilon, \kappa\varepsilon_1\} > 0$. Let us see that (ii) holds for $\varepsilon_2 > 0$. Take $x \in B(\mathcal{M}(\bar{y}), \varepsilon_2)$ and consider $y \in \mathcal{M}^{-1}(x)$. Now, we distinguish between two cases:

If $d(y, \bar{y}) \leq \varepsilon_1$, then $y \in V$ and, since we also have $x \in B(\mathcal{M}(\bar{y}), \varepsilon)$ (recall that $\varepsilon_2 \leq \varepsilon$), from (i) we conclude the aimed inequality $d(x, \mathcal{M}(\bar{y})) \leq \kappa d(y, \bar{y})$.

Otherwise, if $d(y, \bar{y}) \geq \varepsilon_1$, then $d(x, \mathcal{M}(\bar{y})) \leq \varepsilon_2 \leq \kappa\varepsilon_1 \leq \kappa d(y, \bar{y})$. $\square$

Remark 3. The statement of Proposition 1 does not hold for $\kappa = 0$. To see this, take $\mathcal{M} : \mathbb{R} \rightarrow \mathbb{R}$ (single-valued) given by $\mathcal{M}(y) := \max\{0, y - 1\}$ and let $\bar{y} = 0$. Clearly (i) holds for $V =]-1, 1[$ and $\kappa = 0$, whereas (ii) works for $\varepsilon > 0$ if and only if $\kappa \geq \varepsilon/(1 + \varepsilon)$.

COROLLARY 1. *Let $\bar{y} \in \text{dom } \mathcal{M}$. We have the following:*

- (i) *$\mathcal{M}$ is uniformly calm at $\bar{y}$ if and only if there exist $\varepsilon > 0$ and $\kappa \geq 0$ such that*

$$(19) \quad d(x, \mathcal{M}(\bar{y})) \leq \kappa d(\bar{y}, \mathcal{M}^{-1}(x)) \quad \text{for all } x \in B(\mathcal{M}(\bar{y}), \varepsilon).$$

- (ii) *The modulus of uniform calmness can be expressed as follows:*

$$\text{uclm } \mathcal{M}(\bar{y}) = \inf \{ \kappa \geq 0 \mid \exists \varepsilon > 0 \text{ such that (19) holds} \}.$$

Proof. Both (i) and (ii) come from the fact that uniform calmness at $\bar{y}$ with associated elements $V, \varepsilon > 0$, and $\kappa \geq 0$ in (17) entails the same property with $V, \varepsilon > 0$, and $\tilde{\kappa} > \kappa$. Hence the conclusions follow straightforwardly from Proposition 1. $\square$

Next we provide characterizations of $\text{Lipusc } \mathcal{M}(\bar{y})$ and $\text{uclm } \mathcal{M}(\bar{y})$ in terms of certain upper limits, which allow for a better understanding of these concepts and a clear relationship among all moduli introduced in the paper.

PROPOSITION 2. *Let $\mathcal{M} : Y \rightrightarrows X$ be a multifunction between metric spaces, and let $\bar{y} \in \text{dom } \mathcal{M}$. Then*

- (i) $\text{Lipusc } \mathcal{M}(\bar{y}) = \limsup_{y \rightarrow \bar{y}} \left(\sup_{x \in \mathcal{M}(y)} \frac{d(x, \mathcal{M}(\bar{y}))}{d(y, \bar{y})} \right);$
- (ii) $\text{uclm } \mathcal{M}(\bar{y}) = \limsup_{d(x, \mathcal{M}(\bar{y})) \rightarrow 0} \frac{d(x, \mathcal{M}(\bar{y}))}{d(\bar{y}, \mathcal{M}^{-1}(x))}.$

Proof. (i) For the sake of simplicity, let us denote by s the right-hand side of (i) and

$$(20) \quad K := \{\kappa \geq 0 \mid \exists V \text{ neighborhood of } \bar{y} \text{ verifying (16)}\}.$$

We start by establishing inequality “ $\leq$ ”. Since $\text{Lipusc } \mathcal{M}(\bar{y}) = \inf K$, we can write $\text{Lipusc } \mathcal{M}(\bar{y}) = \lim_{r \rightarrow \infty} \kappa_r$ for some $\{\kappa_r\} \subset K$. For each r take a neighborhood V_r associated with κ_r according to (20) and define

$$\bar{\kappa}_r := \sup_{y \in V_r \cap B(\bar{y}, 1/r)} \left(\sup_{x \in \mathcal{M}(y)} \frac{d(x, \mathcal{M}(\bar{y}))}{d(y, \bar{y})} \right) \leq \kappa_r.$$

By definition $\bar{\kappa}_r \in K$, having $V_r \cap B(\bar{y}, 1/r)$ as an associated neighborhood, so that we have $\text{Lipusc } \mathcal{M}(\bar{y}) = \lim_{r \rightarrow \infty} \bar{\kappa}_r$.

Finally, for each r , consider any $y_r \in V_r \cap B(\bar{y}, 1/r)$ such that

$$\bar{\kappa}_r - \frac{1}{r} \leq \sup_{x \in \mathcal{M}(y_r)} \frac{d(x, \mathcal{M}(\bar{y}))}{d(y_r, \bar{y})} \leq \bar{\kappa}_r.$$

Obviously, $\{y_r\}_{r \in \mathbb{N}}$ converges to $\bar{y}$, and then

$$\text{Lipusc } \mathcal{M}(\bar{y}) = \lim_{r \rightarrow \infty} \sup_{x \in \mathcal{M}(y_r)} \frac{d(x, \mathcal{M}(\bar{y}))}{d(y_r, \bar{y})} \leq s.$$

In order to prove “ $\geq$ ” in (i), we may assume the nontrivial case $s > 0$ and write

$$s = \lim_{r \rightarrow \infty} \sup_{x \in \mathcal{M}(\tilde{y}_r)} \frac{d(x, \mathcal{M}(\bar{y}))}{d(\tilde{y}_r, \bar{y})}$$

for some $\{\tilde{y}_r\}_{r \in \mathbb{N}}$ converging to $\bar{y}$. It is clear that we may replace $\{\tilde{y}_r\}_{r \in \mathbb{N}}$ with a suitable subsequence (denoted as the whole sequence for simplicity) such that $\tilde{y}_r \in V_r$, and then

$$s \leq \lim_{r \rightarrow \infty} \kappa_r = \text{Lipusc } \mathcal{M}(\bar{y}).$$

(ii) The procedure is analogous to the previous one by considering

$$\hat{K} = \{\kappa \geq 0 \mid \exists \varepsilon > 0 \text{ such that (19) holds}\}.$$

□

As a direct consequence of the expressions in (15) for $\text{clm } \mathcal{M}(\bar{y}, \bar{x})$ and $\text{Hof } \mathcal{M}(\bar{y})$, together with (18) and the previous proposition, we conclude the following corollary. Observe that the smaller $\varepsilon > 0$, the smaller $\text{Lipusc } \mathcal{M}_\varepsilon(\bar{y})$, and $\text{Lipusc } \mathcal{M}(\bar{y})$ corresponds to $\varepsilon = +\infty$.

COROLLARY 2. *Let $\bar{y} \in \text{dom } \mathcal{M}$. We have*

$$(21) \quad \sup_{x \in \mathcal{M}(\bar{y})} \text{clm } \mathcal{M}(\bar{y}, x) \leq \text{uclm } \mathcal{M}(\bar{y}) \leq \text{Lipusc } \mathcal{M}(\bar{y}) \leq \text{Hof } \mathcal{M}(\bar{y}).$$

Remark 4. The previous corollary yields (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv), where

- (i) $\mathcal{M}$ is Hoffman stable at $\bar{y}$;
- (ii) $\mathcal{M}$ is Lipschitz upper semicontinuous at $\bar{y}$;
- (iii) $\mathcal{M}$ is uniformly calm at $\bar{y}$;
- (iv) $\mathcal{M}$ is calm at every $(\bar{y}, x) \in \text{gph } \mathcal{M}$.

The next three examples show that all converse implications in the previous remark may fail for a suitable multifunction.

Example 1. Let $\mathcal{M} : \mathbb{R} \rightrightarrows \mathbb{R}$ be given by $\mathcal{M}(y) = \{h_r(y), r \in \mathbb{N}\}$, where

$$h_r(y) = \begin{cases} r + y & \text{if } y \leq \frac{1}{r}, \\ r + \frac{1}{r} + r(y - \frac{1}{r}) & \text{if } y > \frac{1}{r}. \end{cases}$$

For $\bar{y} = 0$, it is easy to check that $\text{clm } \mathcal{M}(\bar{y}, x) = 1$ for all $x \in \mathcal{M}(\bar{y})$. Hence, $\sup_{x \in \mathcal{M}(\bar{y})} \text{clm } \mathcal{M}(\bar{y}, x) = 1$. Nevertheless, it is impossible to find $\varepsilon > 0$ that meets the conditions for uniform calmness; i.e., $\text{uclm } \mathcal{M}(\bar{y}) = +\infty$. More specifically, take $\varepsilon_r := r^{-1} + r^{-1/2}$ for all $r \in \mathbb{N}$, $r \geq 8$ (to ensure $\varepsilon_r < 1/2$) and consider $y_r := r^{-1} + r^{-3/2}$ and $x_r := h_r(y_r) = r + r^{-1} + r^{-1/2} \in \mathcal{M}_{\varepsilon_r}(y_r)$. Then

$$\frac{d(x_r, \mathcal{M}(0))}{d(y_r, 0)} = \frac{r^{-1} + r^{-1/2}}{r^{-1} + r^{-3/2}} \rightarrow +\infty \text{ as } r \rightarrow +\infty.$$

Example 2. Consider $\mathcal{M} : \mathbb{R} \rightarrow \mathbb{R}$ (single-valued) given by $\mathcal{M}(y) = 0$ if $y \leq 0$ and $\mathcal{M}(y) = 1$ if $y > 0$. It is clear that $\mathcal{M}$ is uniformly calm at $\bar{y} = 0$ (take $\varepsilon = 1/2$) but not Lipschitz upper semicontinuous by just considering $y_r = 1/r$ for $r \in \mathbb{N}$.

Example 3. Let $\mathcal{M} : \mathbb{R} \rightrightarrows \mathbb{R}$ be given by

$$\mathcal{M}(y) = [0, 1] \text{ if } y < 0, \quad \mathcal{M}(y) = [0, +\infty[\text{ if } y \geq 0.$$

It is clear that $\mathcal{M}$ is Lipschitz upper semicontinuous, with zero modulus, at any $y \in \mathbb{R}$. Nevertheless, it is not Hoffman stable at any $\bar{y} < 0$.

The next theorem establishes that all inequalities in (21) become equalities under the convexity of $\text{gph } \mathcal{M}$ together with the closedness of $\mathcal{M}(\bar{y})$, provided that Y is a normed space and X is a reflexive Banach space. As an obvious consequence, all properties in Remark 4 become equivalent in such a case. First, we include two lemmas.

LEMMA 1. *Let X be a normed space and $\emptyset \neq C \subset X$ be a closed set. Take any $x \in X$ and assume that there exists a best approximation, $\bar{x}$, of x in C . Then $\bar{x}$ is a best approximation of $x_\lambda := (1 - \lambda)\bar{x} + \lambda x$ in C for all $\lambda \in [0, 1]$.*

Proof. Reasoning by contradiction, suppose that for some $\lambda \in [0, 1]$ there exists $\hat{x} \in C$ such that $\|\hat{x} - x_\lambda\| < \|\bar{x} - x_\lambda\|$. Then

$$\begin{aligned} \|\hat{x} - x\| &\leq \|\hat{x} - x_\lambda\| + \|x_\lambda - x\| < \|\bar{x} - x_\lambda\| + \|x_\lambda - x\| \\ &= \lambda \|\bar{x} - x\| + (1 - \lambda) \|\bar{x} - x\| = \|\bar{x} - x\|, \end{aligned}$$

which contradicts the fact that $\bar{x}$ is a best approximation of x in C . $\square$

In the next result X is assumed to be a reflexive Banach space in order to ensure the existence of best approximations on nonempty closed convex sets; see, e.g., [26, Theorem 3.8.1].

LEMMA 2. *Let $\mathcal{M} : Y \rightrightarrows X$ be a multifunction between a normed space Y and a reflexive Banach space X , and assume that $\text{gph } \mathcal{M}$ is a nonempty convex set. Let $\bar{y} \in \text{dom } \mathcal{M}$ and suppose that $\mathcal{M}(\bar{y})$ is closed. Consider any $(y, x) \in \text{gph } \mathcal{M}$ and let $\bar{x}$ be a best approximation of x in $\mathcal{M}(\bar{y})$. Then*

$$\frac{d(x, \mathcal{M}(\bar{y}))}{d(y, \bar{y})} \leq \text{clm } \mathcal{M}(\bar{y}, \bar{x}).$$

Proof. By the convexity assumption, for each $\lambda \in [0, 1]$,

$$(y_\lambda, x_\lambda) := (1 - \lambda)(\bar{y}, \bar{x}) + \lambda(y, x) \in \text{gph } \mathcal{M}.$$

According to Lemma 1, $\bar{x}$ is also a best approximation of x_λ in $\mathcal{M}(\bar{y})$ for each $\lambda \in [0, 1]$. Therefore,

$$\frac{d(x, \mathcal{M}(\bar{y}))}{d(y, \bar{y})} = \frac{\|x - \bar{x}\|}{\|y - \bar{y}\|} = \frac{\|x_\lambda - \bar{x}\|}{\|y_\lambda - \bar{y}\|} = \frac{d(x_\lambda, \mathcal{M}(\bar{y}))}{d(y_\lambda, \bar{y})} \quad \text{for all } \lambda \in]0, 1].$$

Since, letting $\lambda \rightarrow 0$, we have $(y_\lambda, x_\lambda) \rightarrow (\bar{y}, \bar{x})$, by the definition of the calmness modulus (recall (15)) we conclude that

$$\text{clm } \mathcal{M}(\bar{y}, \bar{x}) \geq \limsup_{\lambda \rightarrow 0} \frac{d(x_\lambda, \mathcal{M}(\bar{y}))}{d(y_\lambda, \bar{y})} = \frac{d(x, \mathcal{M}(\bar{y}))}{d(y, \bar{y})}. \quad \square$$

THEOREM 4. Let $\mathcal{M} : Y \rightrightarrows X$, with Y being a normed space and X being a reflexive Banach space, and assume that $\text{gph } \mathcal{M}$ is a nonempty convex set. Let $\bar{y} \in \text{dom } \mathcal{M}$ with $\mathcal{M}(\bar{y})$ closed. Then one has

$$\sup_{x \in \mathcal{M}(\bar{y})} \text{clm } \mathcal{M}(\bar{y}, x) = \text{uclm } \mathcal{M}(\bar{y}) = \text{Lipusc } \mathcal{M}(\bar{y}) = \text{Hof } \mathcal{M}(\bar{y}).$$

Proof. We only have to prove $\text{Hof } \mathcal{M}(\bar{y}) \leq \sup_{x \in \mathcal{M}(\bar{y})} \text{clm } \mathcal{M}(\bar{y}, x)$, according to (21).

Take any $(\tilde{y}, \tilde{x}) \in \text{gph } \mathcal{M}$ and let $\bar{x}$ be a best approximation of $\tilde{x}$ in $\mathcal{M}(\bar{y})$. Lemma 2 ensures that

$$\frac{d(\tilde{x}, \mathcal{M}(\bar{y}))}{d(\tilde{y}, \bar{y})} \leq \text{clm } \mathcal{M}(\bar{y}, \bar{x}) \leq \sup_{x \in \mathcal{M}(\bar{y})} \text{clm } \mathcal{M}(\bar{y}, x).$$

Then, recalling (15), we conclude that

$$\text{Hof } \mathcal{M}(\bar{y}) = \sup_{(\tilde{y}, \tilde{x}) \in \text{gph } \mathcal{M}} \frac{d(\tilde{x}, \mathcal{M}(\bar{y}))}{d(\tilde{y}, \bar{y})} \leq \sup_{x \in \mathcal{M}(\bar{y})} \text{clm } \mathcal{M}(\bar{y}, x). \quad \square$$

We finish this section by observing that the global Hoffman constant for the whole graph can be larger than the Hoffman modulus for a specific $\bar{y}$. Just consider $\mathcal{M} : \mathbb{R} \rightrightarrows \mathbb{R}$ given by

$$\mathcal{M}(y) =]-\infty, y] \quad \text{if } y < 0, \quad \mathcal{M}(y) =]-\infty, 0] \quad \text{if } y \geq 0.$$

Then clearly $\text{Hof } \mathcal{M}(\bar{y}) = 1$ if $\bar{y} < 0$ and $\text{Hof } \mathcal{M}(\bar{y}) = 0$ if $\bar{y} \geq 0$, so that $\text{Hof } \mathcal{M} = 1$.

4. Hoffman and calmness moduli for linear semi-infinite inequality systems. This section aims to obtain expressions for $\text{Hof } \mathcal{F}$ and $\text{Hof } \mathcal{F}(\bar{b})$, $\bar{b} \in \text{dom } \mathcal{F}$, in terms of the system's data. These expressions are established in Theorems 5 and 6, respectively. The first result generalizes Theorem 1 to the current semi-infinite framework, while the second provides an alternative expression to Theorem 2(i), via points inside $\mathcal{F}(\bar{b})$, for locally polyhedral systems. In the case of finite linear systems Theorem 6 is particularly useful as far as it establishes an implementable procedure for computing $\text{Hof } \mathcal{F}(\bar{b})$.

THEOREM 5. Consider $\mathcal{F} : C(T, \mathbb{R}) \rightrightarrows \mathbb{R}^n$ defined in (2). We have

$$\text{Hof } \mathcal{F} = \sup_{\substack{J \subset T \text{ compact} \\ 0_n \notin \text{conv}\{a_t, t \in J\}}} d_*(0_n, \text{conv}\{a_t, t \in J\})^{-1}.$$

Proof. It is clear that $\text{Hof } \mathcal{F} = \sup_{b \in \text{dom } \mathcal{F}} \text{Hof } \mathcal{F}(b)$, and applying Theorem 2 we have

$$(22) \quad \text{Hof } \mathcal{F} = \sup_{b \in \text{dom } \mathcal{F}} \sup_{x \notin \mathcal{F}(b)} d_*(0_n, \text{conv}\{a_t, t \in J_b(x)\})^{-1}.$$

Hence, inequality “ $\leq$ ” comes from (22) taking into account that $b \in \text{dom } \mathcal{F}$ and $x \notin \mathcal{F}(b)$ imply $0_n \notin \text{conv}\{a_t, t \in J_b(x)\}$ (recall Remark 1). Also take into account that each $J_b(x)$ is compact since it is closed in T as far as $J_b(x)$ is the preimage of $\{f_b(x)\}$ by the continuous function $t \mapsto a'_t x - b_t$.

Let us prove the converse inequality “ $\geq$ ”. Observe that for $J = \emptyset$ we have $d_*(0_n, \text{conv}\{a_t, t \in J\})^{-1} = d_*(0_n, \emptyset)^{-1} = 0$. Fix a nonempty compact set $\hat{J} \subset T$ such that $0_n \notin \text{conv}\{a_t, t \in \hat{J}\}$ and let us define $\hat{b} \in C(T, \mathbb{R})$ such that

$$\hat{J} = J_{\hat{b}}(\hat{x}) \text{ for some } \hat{x} \notin \mathcal{F}(\hat{b}), \hat{b} \in \text{dom } \mathcal{F}.$$

First, by separation, since $0_n \notin \text{conv}\{a_t, t \in \hat{J}\}$, there exists $0_n \neq \hat{x} \in \mathbb{R}^n$, such that

$$a'_t \hat{x} \geq \hat{x}' \hat{x} \text{ for all } t \in \hat{J},$$

where $\hat{x}$ is the best approximation of 0_n in the compact set $\text{conv}\{a_t, t \in \hat{J}\}$ with respect to the Euclidean norm in $\mathbb{R}^n$. Define

$$\hat{b}_t := \max\{a'_t \hat{x}, \frac{1}{2} \hat{x}' \hat{x}\} - \varphi(t) \frac{1}{2} \hat{x}' \hat{x}, \quad t \in T,$$

where

$$\varphi(t) = 1 - d(t, \hat{J}) \text{ for all } t \in T.$$

Observe that $\hat{b} \in \text{dom } \mathcal{F}$ since $\hat{b}_t \geq \frac{1}{2} (1 - \varphi(t)) \hat{x}' \hat{x} \geq 0$ for all $t \in T$ and, for instance, $0_n \in \mathcal{F}(\hat{b})$. On the other hand, $\hat{x} \notin \mathcal{F}(\hat{b})$ since

$$a'_t \hat{x} - \hat{b}_t = a'_t \hat{x} - (a'_t \hat{x} - \varphi(t) \frac{1}{2} \hat{x}' \hat{x}) = \frac{1}{2} \hat{x}' \hat{x} > 0 \text{ if } t \in \hat{J}.$$

Finally, observe that

$$a'_t \hat{x} - \hat{b}_t \leq a'_t \hat{x} - a'_t \hat{x} + \varphi(t) \frac{1}{2} \hat{x}' \hat{x} < \frac{1}{2} \hat{x}' \hat{x} \text{ whenever } t \in T \setminus \hat{J}.$$

So

$$\hat{J} = \left\{ t \in T \mid a'_t \hat{x} - \hat{b}_t = f_{\hat{b}}(\hat{x}) \right\},$$

in other words, $\hat{J} = J_{\hat{b}}(\hat{x})$, which finishes the proof. $\square$

Remark 5. Theorem 5 is the only result in this paper which uses the fact that T is assumed to be a compact metric space. The rest of the results work for T being a compact Hausdorff space, which is the framework of the so-called *continuous systems* in [9].

The rest of this section is focused on $\text{Hof } \mathcal{F}(\bar{b})$, provided that $\bar{b} \in \text{dom } \mathcal{F}$. To start with, as a consequence of Theorem 4, we always have

$$(23) \quad \text{Hof } \mathcal{F}(\bar{b}) = \sup_{x \in \mathcal{F}(\bar{b})} \text{clm } \mathcal{F}(\bar{b}, x) = \sup_{x \in \text{bd } \mathcal{F}(\bar{b})} \text{clm } \mathcal{F}(\bar{b}, x), \quad \bar{b} \in \text{dom } \mathcal{F},$$

where the last equality comes from the fact that $\text{clm } \mathcal{F}(\bar{b}, x) = 0$ when $x \in \text{int } \mathcal{F}(\bar{b})$ (the trivial case $\text{bd } \mathcal{F}(\bar{b}) = \emptyset$, equivalently $\mathcal{F}(\bar{b}) = \mathbb{R}^n$, is included; recall $\sup \emptyset := 0$). From now on we are devoted to refining (23) by replacing $\text{bd } \mathcal{F}(\bar{b})$ with a smaller subset. The concluding result is Theorem 6. First, we establish some technical results.

PROPOSITION 3. *Let $x^1, x^2 \in \text{bd } \mathcal{F}(\bar{b})$ such that $T(x^1) \subset T(x^2)$. Then*

- (i) $\text{end } \partial f_{\bar{b}}(x^1) \subset \text{end } \partial f_{\bar{b}}(x^2)$;
- (ii) *if the regularity condition (12) is held at x^i , $i = 1, 2$, then*

$$\text{clm } \mathcal{F}(\bar{b}, x^1) \leq \text{clm } \mathcal{F}(\bar{b}, x^2).$$

Proof. (i) First, $x^i \in \text{bd } \mathcal{F}(\bar{b})$ implies $f_{\bar{b}}(x^i) = 0$, and so $T(x^i) \neq \emptyset$, $i = 1, 2$, by the compactness of T together with the continuity of $t \mapsto \begin{pmatrix} a_t \\ b_t \end{pmatrix}$. Recall that $\partial f_{\bar{b}}(x^i) = \text{conv} \{a_i, i \in T(x^i)\}$, $i = 1, 2$, and hence $\partial f_{\bar{b}}(x^1) \subset \partial f_{\bar{b}}(x^2)$.

Assume, arguing by contradiction, that there exists $a \in \text{end } \partial f_{\bar{b}}(x^1) \setminus \text{end } \partial f_{\bar{b}}(x^2)$. Since, by compactness, $\text{end } \partial f_{\bar{b}}(x^1) \subset \partial f_{\bar{b}}(x^1) \subset \partial f_{\bar{b}}(x^2)$, we have $a \in \partial f_{\bar{b}}(x^2) \setminus \text{end } \partial f_{\bar{b}}(x^2)$. Then we have $\lambda a \in \partial f_{\bar{b}}(x^2)$ for some $\lambda > 1$ and can write

$$(24) \quad \lambda a = \sum_{t \in T(x^1)} \lambda_t a_t + \sum_{t \in T(x^2) \setminus T(x^1)} \lambda_t a_t$$

for some $\{\lambda_t\}_{t \in T(x^2)} \subset \mathbb{R}_+$ such that $\{\lambda_t \mid \lambda_t \neq 0, t \in T(x^2)\}$ is a finite set.

On the other hand, consider $d := x^1 - x^2$ and observe that

$$\begin{cases} a'_t d = 0, & t \in T(x^1), \\ a'_t d = a'_t x^1 - a'_t x^2 < \bar{b}_t - \bar{b}_t = 0, & t \in T(x^2) \setminus T(x^1). \end{cases}$$

Then, multiplying (with the inner product) both members of (24) by d , we deduce

$$0 = \lambda a' d = \sum_{t \in T(x^2) \setminus T(x^1)} \lambda_t a'_t d,$$

which yields $\lambda_t = 0$ for all $t \in T(x^2) \setminus T(x^1)$. So, we attain the contradiction $\lambda a = \sum_{t \in T(x^1)} \lambda_t a_t \in \partial f_{\bar{b}}(x^1)$.

Statement (ii) follows straightforwardly from Theorem 3. $\square$

The following example shows that the regularity condition assumed in statement (ii) of the previous proposition is not superfluous. The example comes from modifying Example 1 in [6] (revisited in [16, Example 3.3]).

Example 4. Let us consider the system, in $\mathbb{R}^2$ endowed with the Euclidean norm, given by

$$\sigma(\bar{b}) := \left\{ \begin{array}{ll} t(\cos t)x_1 + t(\sin t)x_2 \leq t, & t \in [0, \pi], \\ x_1 \leq 1, & t = 4, \\ -x_1 - x_2 \leq 1, & t = 5 \end{array} \right\};$$

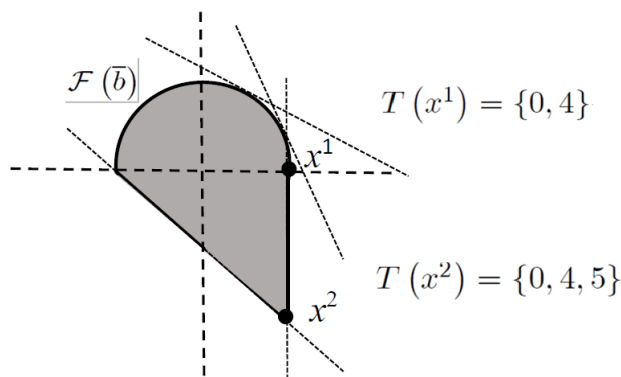


FIG. 1. Illustration of Example 4.

i.e., $T := [0, \pi] \cup \{4, 5\}$, $a_t := t(\cos t, \sin t)'$ for $t \in [0, \pi]$, $a_4 := (1, 0)'$, and $a_5 := (-1, -1)'$; $\bar{b} \in C([0, \pi] \cup \{4, 5\}, \mathbb{R})$ is given by $\bar{b}_t = t$, $t \in [0, \pi]$, $\bar{b}_4 = 1$, and $\bar{b}_5 = 1$. Consider the feasible points $x^1 = (1, 0)'$ and $x^2 = (1, -2)'$. The feasible set of $\sigma(\bar{b})$ is represented in Figure 1.

As proved in [6, Example 1], we have that

$$\text{clm } \mathcal{F}(\bar{b}, x^1) = +\infty.$$

Alternatively, we can apply Theorem 2(ii) with sequence $x^r = (1 + \frac{1}{r}) (\frac{\cos \frac{1}{r}}{\sin \frac{1}{r}})$. It is clear that the regularity condition (12) is not satisfied at x^1 . Indeed $(0, 1)' \in A(x^1)^\circ = \text{cone}\{(1, 0)'\}^\circ = \mathbb{R}_- \times \mathbb{R}$, but $x^1 + \varepsilon(0, 1)' \notin \mathcal{F}(\bar{b})$ for any $\varepsilon > 0$. Moreover,

$$\partial f_{\bar{b}}(x^1) = \text{conv}\{(0, 0)', (1, 0)'\}$$

and $\text{end} \partial f_{\bar{b}}(x^1) = \{(1, 0)'\}$. Hence, $\text{clm } \mathcal{F}(\bar{b}, x^1) \neq d_*(0_2, \text{end} \partial f_{\bar{b}}(x^1))^{-1}$.

With respect to point x^2 , one easily sees that condition (12) is satisfied, where $A(x^2)^\circ = \{u \in \mathbb{R}^2 \mid -u_1 - u_2 \leq 0, u_1 \leq 0\}$. In this case, $\partial f_{\bar{b}}(x^2) = \text{conv}\{(0, 0)', (1, 0)', (-1, -1)'\}$. Hence, from Theorem 3 we have

$$\begin{aligned} \text{clm } \mathcal{F}(\bar{b}, x^2) &= d_*(0_2, \text{end} \partial f_{\bar{b}}(x^2))^{-1} \\ &= d_*(0_2, \text{conv}\{(1, 0)', (-1, -1)'\})^{-1} = \sqrt{5}. \end{aligned}$$

PROPOSITION 4. Let C be a nonempty closed convex subset of $\mathbb{R}^n$ different from a singleton with $\text{extr } C \neq \emptyset$ and let $x^0 \in C \setminus \text{extr } C$. Then there exist $y^0 \in \text{extr } C$, $z^0 \in C$, and $\mu \in]0, 1[$ such that $x^0 = (1 - \mu)y^0 + \mu z^0$.

Proof. The assumption $\text{extr } C \neq \emptyset$ is equivalent to the fact that C contains no lines (i.e., its lineality space is $\{0_n\}$). According to [21, Corollary 14.6.1], this is also equivalent to $\text{int}(O^+C)^\circ \neq \emptyset$, recalling that O^+C is the recession cone of C . Pick $0_n \neq u \in \text{int}(O^+C)^\circ$ and consider

$$K := C \cap \{x \in \mathbb{R}^n \mid u'x \geq u'x^0 - 1\}.$$

Let us see that K is bounded, i.e., $O^+K = \{0_n\}$ (see [21, Theorem 8.4]). Reasoning by contradiction, assume the existence of $0_n \neq v \in O^+K$. Then $x^0 + \lambda v \in K$ and,

accordingly, $u'(x^0 + \lambda v) \geq u'x^0 - 1$ for all $\lambda > 0$. Letting $\lambda \rightarrow +\infty$ we obtain $u'v \geq 0$. On the other hand, $v \in O^+C$ and, for $\alpha > 0$ small enough, we have $u + \alpha v \in (O^+C)^\circ$, yielding the contradiction $0 \geq (u + \alpha v)'v \geq \alpha v'v$.

Once we know that K is a nonempty convex compact set, by applying the Minkowski–Carathéodory theorem (see, e.g., [23, Theorem 8.11]), we have $K = \text{conv}(\text{extr } K)$ and can write

$$(25) \quad x^0 = \sum_{i=1}^k \lambda_i x^i$$

with $\{x^1, \dots, x^k\} \subset \text{extr } K$ being affinely independent, $\sum_{i=1}^k \lambda_i = 1$, and $\lambda_i > 0$ for all $i = 1, \dots, k$. Clearly it is not restrictive to assume $u'x^1 \geq u'x^0$, which easily entails $x^1 \in \text{extr } C$. More specifically, if x^1 were a midpoint of distinct points in C , we could replace these points with others in the same segment verifying $u'x \geq u'x^0 - 1$, and hence these points would be in K , contradicting $x^1 \in \text{extr } K$.

On the other hand, by applying [9, Theorem A.7], (25) entails that x^0 is in the relative interior of $\text{conv}\{x^1, \dots, x^k\}$ (i.e., the interior relative to the affine hull of these points), and then $z^0 := x^1 + \beta(x^0 - x^1) \in \text{conv}\{x^1, \dots, x^k\} \subset C$ for a small enough $\beta > 1$. Finally, let us write

$$x^0 = \left(1 - \frac{1}{\beta}\right)x^1 + \frac{1}{\beta}z^0,$$

which provides the desired result with $y^0 = x^1$ and $\mu = \frac{1}{\beta}$. $\square$

The following theorem appeals to locally polyhedral (LOP, in brief) systems. Recall that given $\bar{b} \in \text{dom } \mathcal{F}$, $\sigma(\bar{b})$ is a LOP system if and only if

$$(26) \quad D(\mathcal{F}(\bar{b}), \bar{x}) = A(\bar{x})^\circ \quad \text{for all } \bar{x} \in \mathcal{F}(\bar{b}),$$

where $D(\mathcal{F}(\bar{b}), \bar{x})$ denotes the *cone of feasible directions* of $\mathcal{F}(\bar{b})$ at $\bar{x}$; i.e., $d \in D(\mathcal{F}(\bar{b}), \bar{x})$ if there exists $\varepsilon > 0$ such that $\bar{x} + \alpha d \in \mathcal{F}(\bar{b})$ for all $\alpha \in [0, \varepsilon]$. See [1] for a comprehensive analysis of LOP systems (see also [9]). At this moment we recall a characterization of LOP systems in terms of the regularity condition (26) which can be derived from Corollary 3.3 in [16].

LEMMA 3 (see [16, Corollary 3.3]). *Let $\bar{b} \in \text{dom } \mathcal{F}$. The following conditions are equivalent:*

- (i) $D(\mathcal{F}(\bar{b}), \bar{x}) = A(\bar{x})^\circ$ for all $\bar{x} \in \mathcal{F}(\bar{b})$.
- (ii) The regularity condition (12) is held at any $\bar{x} \in \mathcal{F}(\bar{b})$.

From now on we consider the set

$$(27) \quad \mathcal{E}(\bar{b}) := \text{extr}(\mathcal{F}(\bar{b}) \cap \text{span}\{a_t, t \in T\}), \quad \text{with } \bar{b} \in \text{dom } \mathcal{F}.$$

Observe that $\mathcal{E}(\bar{b})$ is always a nonempty and finite set when T is finite; moreover,

$$\mathcal{E}(\bar{b}) = \text{extr } \mathcal{F}(\bar{b}) \Leftrightarrow \text{extr } \mathcal{F}(\bar{b}) \neq \emptyset;$$

in fact, $\text{extr } \mathcal{F}(\bar{b}) \neq \emptyset$ if and only if $\mathcal{F}(\bar{b})$ does not contain any line, which is equivalent to the fact that $\text{span}\{a_t, t \in T\} = \mathbb{R}^n$. This construction is inspired by that of [15, p. 142], and is used in [8] to compute the calmness modulus of the optimal value function of finite linear optimization problems.

THEOREM 6. Let $\bar{b} \in \text{dom } \mathcal{F}$ and assume that $\sigma(\bar{b})$ is a LOP system. Then

$$\text{Hof } \mathcal{F}(\bar{b}) = \sup_{x \in \mathcal{E}(\bar{b})} \text{clm } \mathcal{F}(\bar{b}, x) = \sup_{x \in \mathcal{E}(\bar{b})} \sup_{D \in \mathcal{D}(x)} d_*(0_n, \text{conv}\{a_t, t \in D\})^{-1}.$$

Proof. To start with, we recall (23):

$$\text{Hof } \mathcal{F}(\bar{b}) = \sup_{x \in \text{bd } \mathcal{F}(\bar{b})} \text{clm } \mathcal{F}(\bar{b}, x).$$

Since $\mathcal{E}(\bar{b}) \subset \text{bd } \mathcal{F}(\bar{b})$, the inequality $\text{Hof } \mathcal{F}(\bar{b}) \geq \sup_{x \in \mathcal{E}(\bar{b})} \text{clm } \mathcal{F}(\bar{b}, x)$ follows trivially.

Let us see that $\text{Hof } \mathcal{F}(\bar{b}) \leq \sup_{x \in \mathcal{E}(\bar{b})} \text{clm } \mathcal{F}(\bar{b}, x)$. Specifically, let us prove that for every $x \in \text{bd } \mathcal{F}(\bar{b})$ there exists $\tilde{x} \in \mathcal{E}(\bar{b})$ such that $\text{clm } \mathcal{F}(\bar{b}, x) \leq \text{clm } \mathcal{F}(\bar{b}, \tilde{x})$.

Fix arbitrarily $x \in \text{bd } \mathcal{F}(\bar{b})$ and write $x = y + z$, where $y \in \text{span}\{a_t, t \in T\}$ and $z \in \{a_t, t \in T\}^\perp$ (the orthogonal subspace to $\{a_t, t \in T\}$). Since $a'_t x = a'_t y$ for all $t \in T$, $y \in \text{bd } \mathcal{F}(\bar{b})$ and

$$T(x) = T(y).$$

Moreover, applying Lemma 3, we have that the regularity condition (12) is held at both x and y . Hence, Proposition 3(ii) yields

$$(28) \quad \text{clm } \mathcal{F}(\bar{b}, x) = \text{clm } \mathcal{F}(\bar{b}, y).$$

Let us denote

$$C = \mathcal{F}(\bar{b}) \cap \text{span}\{a_t, t \in T\},$$

which satisfies $\text{extr } C \neq \emptyset$. If $y \in \text{extr } C = \mathcal{E}(\bar{b})$, we are done. Otherwise, if $y \in C \setminus \text{extr } C$, we can apply Proposition 4 and conclude the existence of $\tilde{x} \in \text{extr } C$, $\tilde{z} \in C$, and $\mu \in]0, 1[$ such that $y = (1 - \mu)\tilde{x} + \mu\tilde{z}$. Observe that

$$T(y) \subset T(\tilde{x}),$$

since $a'_t y = b_t$ implies $(1 - \mu)a'_t \tilde{x} + \mu a'_t \tilde{z} = b_t$, which entails $a'_t \tilde{x} = a'_t \tilde{z} = b_t$ (because both $\tilde{x}, \tilde{z} \in \mathcal{F}(\bar{b})$). So, we conclude with the desired inequality

$$\text{clm } \mathcal{F}(\bar{b}, y) \leq \text{clm } \mathcal{F}(\bar{b}, \tilde{x}),$$

which together with (28) yields

$$\text{clm } \mathcal{F}(\bar{b}, x) \leq \text{clm } \mathcal{F}(\bar{b}, \tilde{x}), \text{ with } \tilde{x} \in \mathcal{E}(\bar{b}).$$

In this way, the first equality of the current theorem is established. Finally, observe that the second one comes from Theorem 3. $\square$

4.1. On the finite case. This subsection gathers some specifics on finite linear systems. Thus, throughout this subsection we assume that T is finite, in which case, for a fixed $(\bar{b}, \bar{x}) \in \text{gph } \mathcal{F}$, $\mathcal{D}(\bar{x})$ is also finite and clearly

$$\cup_{D \in \mathcal{D}(\bar{x})} \text{conv}\{a_t, t \in D\} = \text{end } \partial f(\bar{x})$$

is a closed set; moreover, $\mathcal{E}(\bar{b})$ is also finite and $\text{clm } \mathcal{F}(\bar{b}, \bar{x})$ and $\text{Hof } \mathcal{F}(\bar{b})$ can be computed through the implementable computations:

$$\begin{aligned}\text{clm } \mathcal{F}(\bar{b}, \bar{x}) &= \max_{D \in \mathcal{D}(\bar{x})} d_*(0_n, \text{conv} \{a_t, t \in D\})^{-1}, \\ \text{Hof } \mathcal{F}(\bar{b}) &= \max_{x \in \mathcal{E}(\bar{b})} \text{clm } \mathcal{F}(\bar{b}, x).\end{aligned}$$

In addition, as a consequence of Theorem 6, we can write

$$\text{Hof } \mathcal{F} = \max_{b \in \text{dom } \mathcal{F}} \text{Hof } \mathcal{F}(b) = \max_{b \in \text{dom } \mathcal{F}} \max_{x \in \mathcal{E}(b)} \text{clm } \mathcal{F}(b, x).$$

Indeed, if the maximum in (9) in Theorem 1 is attained at $J \subset T$ such that $\text{rank } A_J = \text{rank } A$ and $\{a_t, t \in J\}$ is linearly independent, we have

$$\text{Hof } \mathcal{F} = \text{Hof } \mathcal{F}(b^J) = \text{clm } \mathcal{F}(b^J, 0_n),$$

where b^J is defined as $b_t^J = 0$ if $t \in J$ and $b_t^J = 1$ otherwise.

Finally, we observe that Proposition 3(i) admits a refinement in this finite case, which is written in the following result.

PROPOSITION 5. *Let $x^1, x^2 \in \text{bd } \mathcal{F}(\bar{b})$ such that $T(x^1) \subset T(x^2)$. Then $\mathcal{D}(x^1) \subset \mathcal{D}(x^2)$.*

Proof. Given $D \in \mathcal{D}(x^1)$, let us see that $D \in \mathcal{D}(x^2)$. First, consider $d := x^1 - x^2$ and observe that

$$\begin{cases} a'_t d = 0, & t \in T(x^1), \\ a'_t d = a'_t x^1 - a'_t x^2 < \bar{b}_t - \bar{b}_t = 0, & t \in T(x^2) \setminus T(x^1). \end{cases}$$

Now, recalling (11), the fact that $D \in \mathcal{D}(x^1)$ ensures the existence of $\bar{d} \in \mathbb{R}^n$ such that

$$\begin{cases} a'_t \bar{d} = 1, & t \in D, \\ a'_t \bar{d} < 1, & t \in T(x^1) \setminus D. \end{cases}$$

For every $\alpha > 0$, we consider a new vector $d_\alpha := \bar{d} + \alpha d$; observe that

$$\begin{cases} a'_t d_\alpha = a'_t \bar{d} + \alpha a'_t d = 1, & t \in D, \\ a'_t d_\alpha = a'_t (\bar{d} + \alpha d) < 1, & t \in T(x^1) \setminus D. \end{cases}$$

Since $a'_t d < 0$ for $t \in T(x^2) \setminus T(x^1)$, we can choose α large enough (any $\alpha > \max_{t \in T(x^2) \setminus T(x^1)} \frac{a'_t \bar{d} - 1}{-a'_t d}$ will do) to make $a'_t (\bar{d} + \alpha d) < 1$ for all $t \in T(x^2) \setminus T(x^1)$. This proves $D \in \mathcal{D}(x^2)$. $\square$

The following example shows that the previous proposition does not hold in the semi-infinite framework.

Example 5. Let us consider the system, in $\mathbb{R}^2$ endowed with the Euclidean norm, given by

$$\sigma(\bar{b}) := \left\{ (1 + t \cos t) x_1 + (t \sin t) x_2 \leq 0, t \in [0, \frac{\pi}{2}] \right\},$$

and take $x^1 = (0, -1)'$ and $x^2 = (0, 0)'$. Then $T(x^1) = \{0\} \subset [0, \frac{\pi}{2}] = T(x^2)$. We have

$$\{0\} \in \mathcal{D}(x^1) \setminus \mathcal{D}(x^2).$$

To check that $\{0\} \notin \mathcal{D}(x^2)$ observe that the system, in the variable $d = (d_1, d_2)' \in \mathbb{R}^2$,

$$\{d_1 = 1, \quad (1 + t \cos t)d_1 + (t \sin t)d_2 < 1, \quad t \in]0, \frac{\pi}{2}]\} \}$$

is inconsistent.

5. Conclusions and perspectives. We have analyzed different properties oriented to quantify the global, semilocal, and local Hoffman behavior of set-valued mappings between metric spaces, where by “semilocal” we mean the study of the whole image set with respect to parameter perturbations (a similar use of this term can be found, for instance, in [25, Definition 2.1]), yielding to the known Lipschitz upper semicontinuity when the study is concentrated around a nominal parameter. Local properties, such as calmness, are focused on the behavior of the multifunction around a fixed element of its graph. The corresponding moduli are analyzed. Both Hoffman stability (5) and uniform calmness (17) constitute intermediate steps between calmness and global Hoffman properties. All these semilocal properties are shown to be equivalent (and with the same rate/modulus) for convex-graph multifunctions taking closed values in a reflexive Banach space (Theorem 4). This is the case of the feasible set mapping, $\mathcal{F}$, associated with a continuous linear semi-infinite inequality system parameterized with respect to the right-hand side. At this moment, let us comment that paper [5] analyzes the upper Lipschitz behavior of the optimal set mapping, $\mathcal{F}^{op}$, in finite linear programming, which does not have a convex graph. Appealing to a certain concept of directional convexity introduced in that paper, [5] establishes a counterpart for the optimal set mapping of formula

$$\text{Lipusc } \mathcal{F}(\bar{b}) = \sup_{x \in \mathcal{F}(\bar{b})} \text{clm } \mathcal{F}(\bar{b}, x).$$

However, it is shown there that the Hoffman and Lipschitz upper semicontinuity moduli do not coincide when applied to $\mathcal{F}^{op}$ at a nominal parameter.

For this feasible set mapping we succeed in giving the following formula for the global Hoffman constant (Theorem 5), which extends to the current semi-infinite framework some previous results for finite systems:

$$\text{Hof } \mathcal{F} = \sup_{\substack{J \subset T \text{ compact} \\ 0_n \notin \text{conv}\{a_t, t \in J\}}} d_*(0_n, \text{conv}\{a_t, t \in J\})^{-1}.$$

With respect to the semilocal measure, $\text{Hof } \mathcal{F}(\bar{b})$, when confined to locally polyhedral systems (which includes finite systems), Theorem 6 provides a point-based formula involving exclusively some feasible points and the nominal data a_t 's and $\bar{b}_t$'s:

$$(29) \quad \text{Hof } \mathcal{F}(\bar{b}) = \sup_{x \in \mathcal{E}(\bar{b})} \sup_{D \in \mathcal{D}(x)} d_*(0_n, \text{conv}\{a_t, t \in D\})^{-1},$$

where $\mathcal{E}(\bar{b})$ is defined as in (27). When T is finite (and hence $\mathcal{E}(\bar{b})$ and each $\mathcal{D}(x)$ also are), the previous expression yields a specially computable procedure. It provides

an alternative approach to the one given in [2] via points outside the feasible set:

$$\text{Hof } \mathcal{F}(\bar{b}) = \sup_{x \notin \mathcal{F}(\bar{b})} d_*(0_n, \text{conv} \{a_t, t \in J_{\bar{b}}(x)\})^{-1}.$$

The problem of finding an expression for $\text{Hof } \mathcal{F}(\bar{b})$ in the line of (29) for not locally polyhedral systems remains as open problem. A crucial step here is to extend Theorem 3 about the calmness modulus (traced out from [16]) to more general semi-infinite systems.

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