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The museum pass problem with consortia

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Abstract

In this paper, we extend the museum pass problem to incorporate the market structure. To be more precise, we consider that museums are organized into several pass programs or consortia. Within this framework, we propose four allocation mechanisms based on the market structure and the principles of proportionality and egalitarianism. Each mechanism satisfies a distinct set of reasonable properties related to fairness and stability, which serve to axiomatically characterize them.

Keywords Museum pass · Consortia · Allocation rule · Equal treatment · Proportionality

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1 Introduction

Ginsburgh and Zang (2001) introduced the intriguing museum pass problem. The crux of this problem lies in how to fairly distribute the revenue generated from selling museum passes that grant access to a group of participating museums.[1] This issue extends beyond this particular application (see Algaba et al. (2019)) and finds relevance in various real-life contexts where bundling products into packages proves more lucrative than selling them individually (Adams and Yellen 1976). In recent years, digital streaming platforms such as Spotify or Netflix have experienced significant growth potential. These platforms typically offer users a catalog of services at a fixed subscription price, allowing customers to choose the content or service they wish to consume. The challenge arises when these platforms need to compensate content producers (artists or writers) based on the consumption of each offered service. Previous research has examined the museum pass problem, including works by Martínez and Sánchez-Soriano (2026), Bergantiños and Moreno-Ternero (2015), Casas-Méndez et al. (2011), Estévez-Fernández et al. (2012), Wang (2011), and Ginsburgh and Zang (2003). However, these studies assume the existence of a single global pass in which all museums participate, excluding other intermediate self-organizing structures.[2] Consider a city that offers tourists a pass granting access to all its monuments and museums. Typically, some of these monuments (such as cathedrals or monasteries) are owned by the local Church, while public museums coexist with private ones. Although the church or a consortium of private museums may participate in the city pass, they sometimes also offer their own passes restricted to the monuments and museums they manage. Another example could be the North American Reciprocal Museum Association (NARM) that involves museums from United States, Canada, Mexico, Bermuda, the Cayman Islands, and Mexico which allows reciprocal access to a network of more than 1,450 museums in those countries and reciprocal benefits among the institutions involved. In this context, at least three levels of management can be considered the NARM, the museums in the same country, and each museum itself. These more complex organizations must be considered, not only to allocate revenue from the sub-programs, but also to distribute the revenues obtained from selling the global pass. Beyond the museum pass literature, our work is also related to the growing literature on participation and fragmented markets, which studies how institutional structures and market segmentation affect stability and performance of market institutions (see, e.g., Peivandi and Vohra (2021) and Roth and Shorrer (2021)). While our setting is different, the presence of consortia can be interpreted as an institutional source of market fragmentation, making stability and robustness considerations particularly relevant. The novelty of this paper lies in incorporating intermediate market structures into the design of allocation mechanisms for the museum pass problem. The relationship between our model and findings and several of the previously mentioned works is elaborated upon in the final section.

[1] Casas-Méndez et al. (2011) provide a good survey of the literature on the museum pass problem.

[2] Bergantiños and Moreno-Ternero (2015) consider that pass holders may use the general pass or purchase tickets for individual museums, but intermediate structures are neglected.

Although the primary motivation for this model stems from the *museum pass problem* (as it is commonly referred to in the literature) the framework we present is versatile and can be adapted to a variety of contexts. In particular, it is well-suited for applications involving streaming services, which have experienced significant growth over the past decade. Many internet service providers and streaming platforms use soccer as a key incentive to attract subscribers, typically offering access to multiple national soccer competitions. For instance, in Spain, the company *Movistar* provides subscription packages that include not only the Spanish national competition but also the English, German, and Italian ones. These packages often cover both the top-tier leagues (such as *La Liga*, *The Premier League*, *Bundesliga*, and *Serie A*) and the second-tier leagues within each competition. Alternatively, customers may choose to subscribe to individual national competitions rather than purchasing the full package. In some cases, users can even select only the first or second division of a given league. This setup mirrors the museum pass scenario, where consumers choose among various bundles of services. Consequently, a central question arises: how should the total revenue generated from subscription sales be fairly distributed among the agents involved?

The enrichment of standard models through the incorporation of *a priori* unions (consortia, in our terminology), which broadens the range of allocation schemes, has also been explored in other areas of the literature. For example, Borm et al. (2005) in bankruptcy problems, Owen (1981) for power indices, Alonso-Meijide et al. (2010) for public good indices, Owen (1977), Alonso-Meijide et al. (2020) and Béal et al. (2021) in cooperative games, and Vázquez-Brage et al. (1997) in airport problems. We adopt the same methodological approach as these studies.

In our model, a *problem* is described by five elements: the set of *museums* that may participate in various programs, the set of *consortia* representing the market structure, the sets of *pass holders* who purchase each of the available passes, the list of *pass prices* granting access to the museums involved in each program, and the list of *consumption matrices* that indicate the museums visited by each pass holder. A *rule* is a method to distribute among the museums the revenue obtained from selling all the passes.

For our analysis, we adopt the axiomatic approach, which has a well-established tradition in economics literature dating back to Arrow (1951). Instead of directly choosing from existing rules or alternatives, this methodology advocates selecting rules based on the axioms (or properties) they satisfy. In particular, the axioms we analyze in this paper implement different notions of fairness and stability. In the first group we include: *dummy* (museums without visitors do not receive anything), *symmetry within consortia* and *symmetry between consortia*, which guarantee equitable treatment for museums and consortia with symmetric features. With regard stability, we study the property of *composition* that states that the revenue-sharing process can be conducted in multiple stages without impacting the final allocation. In addition, we consider three axioms (*splitting-proofness of museums*, *splitting-proofness of consortia*, and *the consortia property*) that guarantee that no museum or consortium may alter the allocation of other museums or consortia by artificially splitting their offer (segregating, for example, the main nave of a cathedral from its adjacent tower, or detaching different buildings of the same museum).

With regard to the rules, the ones we propose are based on four key principles for revenue distribution. First, the consortium structure of museums should be reflected in the distribution process. We do that by considering two-stage mechanisms, where revenue is initially distributed among consortia and subsequently among museums within each consortium. Second, only museums visited by a pass holder should receive a portion of the price paid for that pass, as they are the ones that attracted the pass holder's interest. The final two principles are equality and proportionality, which are widely used in practice and generally accepted as fair allocation methods. Thus, in the *egalitarian-egalitarian rule* the pass of each pass holder is firstly split among the visited consortia, and in the second stage, each consortium's allocation is further divided among the museums that comprise it. In contrast, in the *proportional-proportional rule* the pass of each pass holder is initially divided among the consortia the pass holder visits, in proportion to the consortium prices, and then, within each consortium, this share is further proportionally distributed among the visited museums based on their individual prices. The other two allocation methods we propose, the *egalitarian-proportional* and *proportional-egalitarian rules* are crossed combinations of the two previous ones, mixing egalitarianism and proportionality.

In our main results, we establish normative foundations for the four aforementioned rules. We demonstrate that, among all possible revenue-distribution mechanisms, the egalitarian-egalitarian rule is the unique method that satisfies composition, dummy, and both symmetries (Theorem 1). Interestingly, we find out that if we replace fairness (symmetries) with non-manipulability (splitting-proofness and consortia property), the proportional-proportional rule is characterized instead (Theorem 2). Given these results, one might wonder whether fairness and non-manipulability are compatible. We show that the answer is affirmative (Theorems 3 and 4).

The rest of the paper is organized as follows. In section 2 we present the model. In Section 3 we introduce the main rules we analyze. In Sect. 4 we propose several axioms that are suitable for this setting and our characterization results. Finally, in Sect. 5 we conclude with some final remarks.

2 The consortia model

Let $\mathbb{M}$ represent the set of all potential museums and let $\mathcal{M}$ be the set of all finite (non-empty) subsets of $\mathbb{M}$. Now, let $\mathbb{N}$ represent the set of all potential buyers of a museum pass. Let $\mathcal{N}$ be the set of all finite (non-empty) subsets of $\mathbb{N}$. We denote by $\mathcal{P}^M$ the set of all possible partitions of M (for any $P = \{P_1, \ldots, P_s\} \in \mathcal{P}^M$ and any $P_k, P_l \in P$ we have $P_k \cap P_l = \varnothing$ and $\bigcup_{k=1}^{s} P_k = M$). A **museum pass problem with (a priori) consortia**, or simply a **problem**, is a 5-tuple $D = (M, P, N, \pi, C)$, where[3]:

- $M = \{1, \ldots, m\} \in \mathcal{M}$ is the set of **museums**.
- $P = \{P_1, \ldots, P_s\} \in \mathcal{P}^M$ is the set of **consortia** that represent the market structure.

[3] See Example 1 for an illustration of all the elements of the problem.

- $N = \{N^{-m}, \ldots, N^{-1}, N^0, N^1, \ldots, N^s\} \in \mathcal{N}$ is the set of **pass holders**. Negative superindices refer to individual passes, zero superindex refers to the general pass, while positive superindices refer to consortium passes. Thus, for each $i \in M$, N^{-i} indicates the pass holders that buy (and visit) the individual pass for museum *i*. Similarly, for each consortium $P_t \in P$, N^t specifies the set of pass holders that buy the pass to visit all or some museums in the consortium P_t. Finally, N^0 indicates the pass holders that purchase the pass that allows the entrance to any museum in *M*. Notice that, for a given $\hat{N} \in N$, it may occur that $\hat{N} = \varnothing$ if no consumer buys the pass to enter to the corresponding museum or group of museums. For the sake of simplicity, if a consortium is formed by just one museum ($P_t = \{i\}$), we impose that $N^{-i} = \varnothing$. By doing this we avoid considering singletons twice. Moreover, we assume that the same person cannot buy two different passes; if this were the case, we would consider as many different pass holders as passes he or she had purchased.[4]
- $\pi = \{\pi^{-m}, \ldots, \pi^{-1}, \pi^0, \pi^1, \ldots, \pi^s\}$ is the set of **pass prices** where $\pi^l \in \mathbb{R}_{++}$ for all $l \in \{-m, \ldots, -1, 0, 1, \ldots, s\}$. Using the same convention as before, the first elements $\pi^{-m}, \ldots, \pi^{-1}$ indicate the pass price of each individual museum, π^0 is the price of the general pass, while $\pi^1, \ldots, \pi^s$ indicate the prices to visit the consortia. Again, for the sake of simplicity, if a consortium is formed by just one museum ($P_t = \{i\}$), we impose that $\pi^{-i} = \pi^t$.
- $C = \{C^{-m}, \ldots, C^{-1}, C^0, C^1, \ldots, C^s\}$ is the list of **consumption matrices**. The rows of all these matrices are labeled with the name of the museums and the columns with the name of the pass holders. Furthermore, all of these matrices are binary, where a 1 in cell *ia* means that museum *i* has been visited by pass holder *a*, and 0 otherwise. The first *m* matrices $\{C^{-m}, \ldots, C^{-1}\}$ refer to the consumption of individual museums. Each C^{-i} has one row (for museum *i*) and $|N^{-i}|$ columns (one for each pass holder that buys that individual pass). The matrix C^0 refers to the consumption of the global pass. It has *m* rows (one for each museum) and $|N^0|$ columns (one for each pass holder that buys the global pass). The last matrices $\{C^1, \ldots, C^s\}$ refer to the consumption of consortia. Each C^t has $|P^t|$ rows (one for each museum in the consortium P^t) and $|N^t|$ columns (one for each pass holder that buys the corresponding consortium pass). For each $\hat{C} \in C$,

$$\hat{C}_{ia} = \begin{cases} 1 & \text{if museum } i \text{ has been visited by pass holder } a \\ 0 & \text{otherwise} \end{cases}$$

For the sake of convention, if a pass has not been purchased by any pass holder ($\hat{N} = \varnothing$), the associated consumption matrix would have zero columns; in such a case we write $\hat{C} = \varnothing$. As in Ginsburgh and Zang (2003) and Bergantiños and Moreno-Ternero (2015), we assume that any pass holder visits, at least, one museum. The domain of all possible problems (M, P, N, π, C) is denoted by $\mathcal{D}$. The total revenue obtained from selling the passes is given by

[4] We assume that each pass holder only belongs to a unique $\hat{N} \in N$, that is, for any pair $\hat{N}, \overline{N} \in N$, $\hat{N} \cap \overline{N} = \varnothing$.

$$E = \sum_{i=1}^{m} |N^{-i}|\pi^{-i} + |N^0|\pi^0 + \sum_{t=1}^{s} |N^t|\pi^t,$$

where $\sum_{i=1}^{m} |N^{-i}|\pi^{-i}$ is the revenue from individual passes ($|N^{-i}|\pi^{-i}$ corresponds to the revenue generated by museum *i*), $|N^0|\pi^0$ is the revenue from the global pass, and $\sum_{t=1}^{s} |N^t|\pi^t$ is the revenue from consortia passes ($|N^t|\pi^t$ corresponds to the revenue generated by consortium *t*).

Now, we introduce some notation which we will use in the rest of the paper:

- For each $i \in M$ and each $\sigma \in \{-m, \ldots, -1, 0, 1, \ldots, s\}$, $N_i^\sigma = \{a \in N^\sigma : C_{ia}^\sigma = 1\}$ is the set of pass holders in N^σ that visit museum *i*.
- For each $i \in M$, $P^{(i)}$ is the consortium *i* belongs to, i.e., $P^{(i)} \in P$ is the unique consortium such that $i \in P^{(i)}$. Similarly, $N^{(i)}$ and $C^{(i)}$ are the set of pass holders and consumption matrix associated with consortium $P^{(i)}$.
- For each $a \in N$, $M_a = \{i \in M : \hat{C}_{ia} = 1 \text{ for some } \hat{C} \in C\}$ is the set of museum visited by pass holder *a*.
- For each $a \in N^0$, $K_a^0 = \{k \in \{1, \ldots, s\} : C_{ia}^0 = 1 \text{ for some } i \in P_k\}$ is the set of consortia visited (with a general pass) by *a*.
- For each $\hat{C} \in C$ and each $i \in M$, we define $\hat{C}_{i\cdot} = \sum_{a \in \hat{N}} \hat{C}_{ia}$ as the number of visitors of *i* in the consumption matrix $\hat{C}$.
- Let $\mathcal{D}^0 \subset \mathcal{D}$ be the subclass of problems in which pass holders only buy general passes (i.e. $(M, P, N, \pi, C) \in \mathcal{D}^0$ iff $\hat{N} = \varnothing$ for any $\hat{N} \in N \backslash N^0$).
- For each $P_t \in P$, let $\mathcal{D}^t \subset \mathcal{D}$ be the subclass of problems in which pass holders only buy the pass to access museums in the consortium *t* (i.e. $(M, P, N, \pi, C) \in \mathcal{D}^t$ iff $\hat{N} = \varnothing$ for any $\hat{N} \in N \backslash N^t$).
- For each $i \in M$, let $\mathcal{D}^{-i} \subset \mathcal{D}$ be the subclass of problems in which pass holders only buy passes to access museum *i* (i.e. $(M, P, N, \pi, C) \in \mathcal{D}^{-i}$ iff $\hat{N} = \varnothing$ for any $\hat{N} \in N \backslash N^{-i}$).

A **rule** is a mechanism to distribute the generated revenue among the museums. Formally, it is a mapping $R : \mathcal{D} \longrightarrow \mathbb{R}_+^m$ that, for each problem $D \in \mathcal{D}$, determines an allocation $R(D) \in \mathbb{R}_+^m$ such that

$$\sum_{i \in M} R_i(D) = E.$$

3 Revenue allocation mechanisms

In this section we present four natural rules that are suitable for this setting which propose different alternatives to distribute *E*. As the aggregate revenue is sum of the revenues from the general pass, the consortium passes, and the revenue from each individual museum, it is natural for the rules to operate in a similar manner and to divide the total revenue into the three contributing levels. In particular, the rules we propose are designed base on four principles when establishing the distribution of revenues. The first is that the consortium structure of museums should be reflected in the distribution process, therefore the rules should be designed in stages. The second principle is that only museums visited by a pass holder should receive part of the price paid for that pass, since they are the ones that aroused the interest of the pass holder. The last two principles are equality and proportionality when allocating the revenues. The application of one or the other of these last two principles will depend on the relationship between the museums or between the consortia, and the relevance the central planner gives to the prices of the passes. The next four proposals divide this distribution process into two stages. In the first stage, the pass of each pass holder is allocated among the visited consortia, and in the second stage, each consortium's allocation is further divided among the museums that comprise it.

The first rule we introduce is the *egalitarian-egalitarian rule*. In this rule, at the first stage the entrance fee π^0 is equally divided among the consortia the pass holder visits. At the second stage this share is further equally distributed among the visited museums within each consortium. The pass of the consortium, $\pi^{(i)}$, is also equally allocated among the visited museums, and the revenue associated with the sale of individual passes, $|N^{-i}|\pi^{-i}$, is assigned to the corresponding museum.

Egalitarian-egalitarian rule: For each $D \in \mathcal{D}$ and each $i \in M$,

$$R_i^{EE}(D) = \sum_{a \in N_i^0} \frac{1}{|M_a \cap P^{(i)}|} \frac{1}{|K_a^0|} \pi^0 + \sum_{a \in N_i^{(i)}} \frac{1}{|M_a|} \pi^{(i)} + |N^{-i}|\pi^{-i}$$

In the next rule, at the first stage the general pass π^0 is initially divided among the consortia the pass holder visits, in proportion to the consortium prices. Then, within each consortium, this share is further proportionally distributed among the visited museums based on their individual prices. The consortium pass $\pi^{(i)}$ is also proportionally allotted among the visited museums, using the individual prices as a reference.

Proportional-proportional rule

For each $D \in \mathcal{D}$ and each $i \in M$,

$$R_i^{PP}(D) = \sum_{a \in N_i^0} \frac{\pi^{-i}}{\sum_{j \in M_a \cap P^{(i)}} \pi^{-j}} \frac{\pi^{(i)}}{\sum_{t \in K_a^0} \pi^t} \pi^0 + \sum_{a \in N_i^{(i)}} \frac{\pi^{-i}}{\sum_{j \in M_a} \pi^{-j}} \pi^{(i)} + |N^{-i}|\pi^{-i}$$

The following two rules are mixtures of the previous ones. As for the *proportional-egalitarian rule*, the general entrance π^0 paid by each pass holder is firstly divided among the consortia she visits in proportion to their consortium prices, and secondly, this share is then equally distributed among the visited museums within each consortium. The *egalitarian-proportional rule* is the reverse process. The general pass is firstly divided uniformly among the visited consortia, and then in proportion to the individual prices within each consortium.

Proportional-egalitarian rule: For each $D \in \mathcal{D}$ and each $i \in M$,

$$R_i^{PE}(D) = \sum_{a \in N_i^0} \frac{1}{|M_a \cap P^{(i)}|} \frac{\pi^{(i)}}{\sum_{t \in K_a^0} \pi^t} \pi^0 + \sum_{a \in N_i^{(i)}} \frac{1}{|M_a|} \pi^{(i)} + |N^{-i}| \pi^{-i}$$

Egalitarian-proportional rule: For each $D \in \mathcal{D}$ and each $i \in M$,

$$R_i^{EP}(D) = \sum_{a \in N_i^0} \frac{\pi^{-i}}{\sum_{j \in M_a \cap P^{(i)}} \pi^{-j}} \frac{1}{|K_a^0|} \pi^0 + \sum_{a \in N_i^{(i)}} \frac{\pi^{-i}}{\sum_{j \in M_a} \pi^{-j}} \pi^{(i)} + |N^{-i}| \pi^{-i}$$

In the following example we explain in detail the elements of the model and illustrate how the aforementioned rules operate.

Example 1 Consider the problem $D \in \mathcal{D}$ with three museums, $M = \{1, 2, 3\}$, organized into two consortia $P = \{\{1, 2\}, \{3\}\}$, and eight pass holders, $N = \{N^{-3}, N^{-2}, N^{-1}, N^0, N^1, N^2\}$, where

$$N^{-3} = \varnothing, \qquad N^{-2} = \{1\}, \qquad N^{-1} = \{2, 3\}, \qquad N^0 = \{4, 5\}, \qquad N^1 = \{6, 7\}, \qquad N^2 = \{8\}.$$

The pass prices are

$$\pi^{-3} = 6, \quad \pi^{-2} = 4, \quad \pi^{-1} = 2, \quad \pi^0 = 7, \quad \pi^1 = 4, \quad \pi^2 = 6,$$

and the visits are described by the following consumption matrices

$$C^{-3} = \varnothing, \qquad C^{-2} = \begin{pmatrix} 1 \end{pmatrix}, \qquad C^{-1} = \begin{pmatrix} 1 & 1 \end{pmatrix},$$
$$C^0 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}, \qquad C^1 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \qquad C^2 = \begin{pmatrix} 1 \end{pmatrix}.$$

Therefore,

$$E = [2 \cdot 2 + 1 \cdot 4 + 0 \cdot 6] + 2 \cdot 7 + [2 \cdot 4 + 1 \cdot 6] = 36$$

Several comments are in order. First, note that the consortium P_2 consists of just one pass holder. As outlined in the model's setup, for consistency, we assume $N^{-3} = \varnothing$ (to avoid double-counting of visitors) and $\pi^{-3} = \pi^2$ (prices remain the same whether 3 is considered as a consortium of a singleton or as an individual museum). Secondly, $N^{-2} = \{1\}$ indicates that pass holder 1 has bought tickets to access museum 2 at a price of $\pi^{-2} = 4$. Similarly, pass holders 4 and 5 have purchased passes granting entry to all museums at a cost of $\pi^0 = 7$, and pass holders in $N^1 = \{6, 7\}$ have acquired combined entrance tickets to the consortium P_1, which includes museums 1 and 2. Thirdly, regarding the consumption matrices, $C^{-3} = \varnothing$ for consistency. All entries in C^{-1} and C^{-2} are set to one, based on the assumption that any pass holder visits at least one of the museums included in their pass.[5]

Below is a detailed look at how each of the four rules introduced previously works.

- Egalitarian-egalitarian rule. Regarding the global pass, the entrance fee of pass holder 4 is initially divided equally among the consortia she visits, allocating $\frac{\pi^0}{2}$ to both P_1 and P_2. Within each consortium, this share is further equally distributed among the visited museums. Consequently, museum 1 receives $\frac{1}{2} \cdot \frac{\pi^0}{2}$, museum 2 receives $\frac{1}{2} \cdot \frac{\pi^0}{2}$, and museum 3 receives $\frac{\pi^0}{2}$. A similar procedure is applied to the pass paid by pass holder 5, as well as the revenue generated from selling the passes of consortium P_1. The amount paid by pass holder 6 is allocated to museum 2, while the price $\pi^1 = 4$ paid by pass holder 7 is divided between museums 1 and 2. Since the second consortium, P_2, consists of only one museum, it receives all the generated revenue. Regarding individual passes, each museum obtains its respective profit.

$$R_1^{EE}(D) = \left[\frac{1}{2} \cdot \frac{1}{2} \cdot 7 + 0 \cdot 7\right] + \left[0 \cdot 4 + \frac{1}{2} \cdot 4\right] + [2 \cdot 2] = \frac{31}{4}$$
$$R_2^{EE}(D) = \left[\frac{1}{2} \cdot \frac{1}{2} \cdot 7 + 1 \cdot \frac{1}{2} \cdot 7\right] + \left[1 \cdot 4 + \frac{1}{2} \cdot 4\right] + [1 \cdot 4] = \frac{61}{4}$$
$$R_3^{EE}(D) = \left[1 \cdot \frac{1}{2} \cdot 7 + 1 \cdot \frac{1}{2} \cdot 7\right] + [1 \cdot 6] + [0 \cdot 6] = 13$$

- Proportional-proportional rule. The ticket paid by pass holder 4 is initially divided among the consortia she visits in proportion to their consortium prices, allocating $\frac{4}{10}\pi^0$ to P_1 and $\frac{6}{10}\pi^0$ to P_2. Within each consortium, this share is further distributed among the visited museums in proportion to their corresponding individual prices. Consequently, museum 1 receives $\frac{2}{6} \cdot \frac{4}{10} \cdot \pi^0$, museum 2 receives $\frac{4}{6} \cdot \frac{4}{10} \cdot \pi^0$, and museum 3 receives $\frac{6}{10} \cdot \pi^0$. The process for pass holder \$5\$ works similarly. Regarding the revenue generated by consortium P_1, the entrance fee paid by pass holder 6 goes entirely to museum 2, while the price $\pi^1 = 4$ paid by pass holder 7 is distributed between museums 1 and 2 in proportion to the prices of their individual passes ($\frac{2}{6}$ for museum 1 and $\frac{4}{6}$ for museum 2). The computation of the remaining revenue allocation is straightforward. Thus, the overall revenue distribution according to the proportional-proportional rule is as follows:

[5] This is a standard assumption in the literature on the museum pass problem. In the final remarks, we provide a more detailed discussion on its implications and possible relaxation.

$$R_1^{PP}(D) = \left[\frac{2}{6}\cdot\frac{4}{10}\cdot 7 + 0\cdot 7\right] + \left[0\cdot 4 + \frac{2}{6}\cdot 4\right] + [2\cdot 2] = \frac{94}{15}$$
$$R_2^{PP}(D) = \left[\frac{4}{6}\cdot\frac{4}{10}\cdot 7 + 1\cdot\frac{4}{10}\cdot 7\right] + \left[1\cdot 4 + \frac{4}{6}\cdot 4\right] + [1\cdot 4] = \frac{46}{3}$$
$$R_3^{PP}(D) = \left[1\cdot\frac{6}{10}\cdot 7 + 1\cdot\frac{6}{10}\cdot 7\right] + [1\cdot 6] + [0\cdot 6] = \frac{72}{5}$$

- Proportional-egalitarian rule. The entrance by pass holder 4 is firstly divided among the consortia she visits in proportion to their consortium prices, allocating $\frac{4}{10}\pi^0$ to P_1 and $\frac{6}{10}\pi^0$ to P_2. Within each consortium, this share is further equally distributed among the visited museums. Then, museum 1 receives $\frac{1}{2}\cdot\frac{4}{10}\cdot\pi^0$, museum 2 receives $\frac{1}{2}\cdot\frac{4}{10}\cdot\pi^0$, and museum 3 receives $\frac{6}{10}\cdot\pi^0$. A similar procedure is applied to the pass paid by pass holder 5. The distribution of revenues generated by the consortia and individual passes works as in the egalitarian-egalitarian rule.

$$R_1^{PE}(D) = \left[\frac{1}{2}\cdot\frac{4}{10}\cdot 7 + 0\cdot 7\right] + \left[0\cdot 4 + \frac{1}{2}\cdot 4\right] + [2\cdot 2] = \frac{37}{5}$$
$$R_2^{PE}(D) = \left[\frac{1}{2}\cdot\frac{4}{10}\cdot 7 + 1\cdot\frac{4}{10}\cdot 7\right] + \left[1\cdot 4 + \frac{1}{2}\cdot 4\right] + [1\cdot 4] = \frac{71}{5}$$
$$R_3^{PE}(D) = \left[1\cdot\frac{6}{10}\cdot 7 + 1\cdot\frac{6}{10}\cdot 7\right] + [1\cdot 6] + [0\cdot 6] = \frac{72}{5}$$

- Egalitarian-proportional rule. Regarding the global pass, the entrance fee of pass holder 4 is initially divided equally among the consortia she visits, allocating $\frac{\pi^0}{2}$ to both P_1 and P_2. Within each consortium, this share is further distributed among the visited museums in proportion to their corresponding individual prices. Thus, museum 1 receives $\frac{2}{6}\cdot\frac{1}{2}\cdot\pi^0$, museum 2 receives $\frac{4}{6}\cdot\frac{1}{2}\cdot\pi^0$, and museum 3 receives $\frac{1}{2}\cdot\pi^0$. The process for pass holder \$5\$ works similarly. The distribution of revenues generated by the consortia and individual passes works as in the proportional-proportional rule.

$$R_1^{EP}(D) = \left[\frac{2}{6}\cdot\frac{1}{2}\cdot 7 + 0\cdot 7\right] + \left[0\cdot 4 + \frac{2}{6}\cdot 4\right] + [2\cdot 2] = \frac{13}{2}$$
$$R_2^{EP}(D) = \left[\frac{4}{6}\cdot\frac{1}{2}\cdot 7 + 1\cdot\frac{1}{2}\cdot 7\right] + \left[1\cdot 4 + \frac{4}{6}\cdot 4\right] + [1\cdot 4] = \frac{33}{2}$$
$$R_3^{EP}(D) = \left[1\cdot\frac{1}{2}\cdot 7 + 1\cdot\frac{1}{2}\cdot 7\right] + [1\cdot 6] + [0\cdot 6] = 13$$

As the previous example illustrates, the four allocation rules introduced above generally lead to different outcomes. However, under certain assumptions, these rules may yield identical allocations. The following remark examines the specific conditions on pass prices and consortia structure that ensure pairwise equivalence among the egalitarian-egalitarian, proportional-proportional, egalitarian-proportional, and proportional-egalitarian rules.

Remark 1 Given a problem $D = (M, P, N, \pi, C) \in \mathcal{D}$:

(a) If all museum passes have the same price, and all consortia are priced equally, then the four rules yield identical outcomes. That is, if $\pi^k = \pi^t$ for all $k, t \in \{1, \ldots, s\}$ and $\pi^{-i} = \pi^{-j}$ for all $i, j \in M$ then $R^{EE}(D) = R^{PP}(D) = R^{PE}(D) = R^{EP}(D)$.
(b) If all museum passes have the same price, but consortia are not priced equally, then R^{EE} coincides with R^{PE}, and R^{PP} coincides with R^{EP}; however the allocation resulting from the first pair differs from that of the second. That is, if $\pi^k = \pi^t$ for all $k, t \in \{1, \ldots, s\}$ and there exists $i, j \in M$ such that $\pi^{-i} \neq \pi^{-j}$ then $R^{EE}(D) = R^{PE}(D)$ and $R^{PP}(D) = R^{EP}(D)$.
(c) In the dual scenario, if all consortia passes have the same price, but museums are not priced equally, then R^{EE} coincides with R^{EP}, and R^{PP} coincides with R^{PE}; however, as before, the allocation resulting from the first pair differs from that of the second. That is, if $\pi^{-i} = \pi^{-j}$ for all $i, j \in M$ and there exists $k, t \in \{1, \ldots, s\}$ such that $\pi^k \neq \pi^t$ then $R^{EE}(D) = R^{EP}(D)$ and $R^{PP}(D) = R^{PE}(D)$.
(d) If each consortium consists of a single museum, then R^{EE} coincides with R^{EP}, and R^{PP} coincides with R^{PE}; however, the two pairs do not coincide with each other. That is, if $P = \{\{1\}, \{2\}, \ldots, \{m\}\}$ then $R^{EE}(D) = R^{EP}(D)$ and $R^{PP}(D) = R^{PE}(D)$.
(e) At the other extreme, if there is a single consortium that includes all museums, then R^{EE} coincides with R^{PE}, and R^{PP} coincides with R^{EP}; however, the two pairs do not coincide with each other. That is, if $P = \{\{1, \ldots, m\}\}$ then $R^{EE}(D) = R^{PE}(D)$ and $R^{PP}(D) = R^{EP}(D)$.

The models proposed by Ginsburgh and Zang (2001) and Bergantiños and Moreno-Ternero (2015) primarily focus on the distribution of revenues generated exclusively from general passes, excluding the existence of consortia. In our terminology, their rules are only applicable within the subdomain $\mathcal{D}^0$. Nevertheless, there is a relationship between the rules we characterize in this paper and those proposed by the aforementioned authors, provided that appropriate assumptions are made and the analysis is confined to the subclass of problems $\mathcal{D}^0$. Specifically, when the consortium structure is disregarded, either because there exists a single consortium encompassing all museums ($P = \{\{1, \ldots, m\}\}$), or because each museum constitutes its own consortium ($P = \{\{1\}, \{2\}, \ldots, \{m\}\}$), the rules R^{EE}, R^{PP}, R^{EP}, and R^{PE} coincide with the Shapley and the p-Shapley rules. The details of this correspondence are provided in the remark below.

Remark 2 For the sake of comparability with Ginsburgh and Zang (2001) and Bergantiños and Moreno-Ternero (2015), we restrict our analysis to the domain $\mathcal{D}^0$. Within this domain, the following statements hold.

(a) The Shapley rule coincides with the egalitarian-egalitarian and egalitarian-proportional rules if and only if $P = \{\{1\}, \ldots, \{m\}\}$; and it coincides with the egalitarian-egalitarian and proportional-egalitarian rules if and only if $P = \{\{1, \ldots, m\}\}$.
(b) The p-Shapley rule coincides with the proportional-proportional and proportional-egalitarian rules if and only if $P = \{\{1\}, \ldots, \{m\}\}$; and it coincides with the proportional-proportional and egalitarian-proportional rules if and only if $P = \{\{1, \ldots, m\}\}$.

4 Normative foundation

In this section, we provide the normative foundations of the rules introduced earlier. We propose several axioms that articulate the ethical underpinnings and ensure the consistency and stability of the solutions previously presented. In general, there is no single criterion that determines which allocation rule should be applied. Therefore, we introduce a set of axioms that are reasonable within the context of the problem under consideration. Different combinations of these axioms give rise to specific rules, each reflecting distinct normative perspectives. This axiomatic approach clarifies the ethical and operational assumptions on which the proposed solutions rest.

The first property says that the allocation is independent of the timing. Imagine the following two alternatives. One, we distribute the revenue every semester, considering N_1 and N_2 two disjoint groups of pass holders in each period. And two, we solve the problem once a year, considering the revenue generated by the whole group of pass holders, $N_1 \cup N_2$. In both cases the allocation must be the same. Similar principles have been applied, among others, by Martínez and Sánchez-Soriano (2026) Martínez and Moreno-Ternero (2022), Martínez and Sánchez-Soriano (2021) and Bergantiños and Moreno-Ternero (2015) in analogous contexts.

Composition. For each pair $(M, P, N_1, \pi, C_1), (M, P, N_2, \pi, C_2) \in \mathcal{D}$, with $N_1 \cap N_2 = \phi$

$$R(M, P, N_1 \cup N_2, \pi, (C_1, C_2)) = R(M, P, N_1, \pi, C_1) + R(M, P, N_2, \pi, C_2),$$

where (C_1, C_2) are the matrices obtained by properly concatenating C_1 and C_2.

The following two axioms embody the fundamental fairness principle that equals should be treated equally. The distinction between them lies in their scope: while the first axiom establishes the conditions under which two museums within the same consortium are considered equal, the second axiom defines when two consortia, as collective entities, should be regarded as equal. More specifically, *symmetry within consortia* asserts that, if two museums belong to the same consortium and have been visited by the same pass holders, then they must be considered as equal and therefore their awards must also be identical.

Symmetry within consortia. For each $D \in \mathcal{D}$ and each pair $i, j \in M$ such that $P^{(i)} = P^{(j)}$, if the three following conditions hold

(i) $C^0_{ia} = C^0_{ja}$ for any $a \in N^0$.

(ii) $C_{ia}^{(i)} = C_{ja}^{(j)}$ for any $a \in N^{(i)} = N^{(j)}$.
(iii) $\pi^{-i}|N^{-i}| = \pi^{-j}|N^{-j}|$.

Then, $R_i(D) = R_j(D)$.

We now explain conditions (i) to (iii) from the previous definition. Consider two museums $i, j \in M$, both belonging to the same consortium. Condition (i) says that every pass holder who purchases the general pass visits both museums.[6] Analogously, condition (ii) requires that anyone who buys the consortium pass accesses both *i* and *j*. Finally, (iii) stipulates that at the individual level, both *i* and *j* generate the same individual revenue.

As previously anticipated, the second requirement also reflects the ethical principle that equals should receive equal treatment. In this case, the notion of equality applies not to individual museums, but to consortia as collective entities. More specifically, the axiom of *symmetry between consortia* asserts that consortia deemed equal must receive equal aggregate benefits, where these benefits are defined as the total sum of the awards allocated to all museums within each consortium.[7]

Symmetry between consortia. For each $D \in \mathcal{D}$ and each pair $P_r, P_t \in P$, if the following three conditions hold

(i) $\sum_{i \in P_r} C_{ia}^0 > 0 \Leftrightarrow \sum_{i \in P_t} C_{ia}^0 > 0$ for any $a \in N^0$.

(ii) $\pi^r|N^r| = \pi^t|N^t|$
(iii) $\sum_{i \in P_r} \pi^{-i}|N^{-i}| = \sum_{j \in P_t} \pi^{-j}|N^{-j}|$

Then, $\sum_{i \in P_r} R_i(D) = \sum_{j \in P_t} R_j(D)$.

Conditions (i) to (iii) in the previous definition establish the criteria under which two consortia are considered equal, and can be described as follows. Consider two consortia $P_r, P_t \in P$. Condition (i) requires that if consortium P_r has been visited, then consortium P_t must also have been visited. Condition (ii) states that both consortia must obtain the same revenue from selling their consortium passes. And Condition (iii) requires that the total benefit obtained by the museums in P_r from selling their individual passes must equal the total benefit obtained by the museums in P_t from doing the same.

The following example illustrates the two preceding definitions using a case involving symmetric museums within a consortium and symmetric consortia.

Example 2 Consider the problem $D \in \mathcal{D}$ with seven museums, *M* = 1, 2, 3, 4, 5, 6, 7, organized into three consortia $P = \{\{1,2\}, \{3,4\}, \{5,6,7\}\}$, and thirteen pass holders, $N = \{N^{-7}, N^{-6}, N^{-5}, N^{-4}, N^{-3}, N^{-2}, N^{-1}, N^0, N^1, N^2, N^3\}$, where

$$N^{-7} = \{1,2\}, \quad N^{-6} = \varnothing, \quad N^{-5} = \varnothing, \quad N^{-4} = \varnothing, \quad N^{-3} = \{3\},$$
$$N^{-2} = \{4,5\}, \quad N^{-1} = \{6\}, \quad N^0 = \{7,8\}, \quad N^1 = \{9\}, \quad N^2 = \{10,11\}, \quad N^3 = \{12,13\}.$$

[6] Notice that, if no one buys the general pass, i.e. $C^0 = \varnothing$, the condition is vacuously satisfied.

[7] Although applied in different contexts, the principles underlying both *symmetry within consortia* and *symmetry between consortia* are well-established requirements in the literature (e.g. Alonso-Meijide et al. (2020, 2015), Calvo and Gutiérrez (2010)).

The pass prices are

$$\pi^{-7}=1.5,\quad \pi^{-6}=1,\quad \pi^{-5}=1,\quad \pi^{-4}=2,\quad \pi^{-3}=3,\quad \pi^{-2}=1,\quad \pi^{-1}=2,$$
$$\pi^{0}=4,\quad \pi^{1}=2,\quad \pi^{2}=3,\quad \pi^{3}=3,$$

and the visits are described by the following consumption matrices

$$C^{-7}=\begin{pmatrix}1 & 1\end{pmatrix},\quad C^{-5}=\varnothing,\quad C^{-4}=\varnothing,\quad C^{-3}=\varnothing,\quad C^{-2}=\begin{pmatrix}1 & 1\end{pmatrix},\quad C^{-1}=\begin{pmatrix}1\end{pmatrix},$$
$$C^{0}=\begin{pmatrix}1 & 0\\ 1 & 0\\ 1 & 1\\ 1 & 1\\ 1 & 1\\ 1 & 1\\ 1 & 0\end{pmatrix},\quad C^{1}=\begin{pmatrix}1 & 1\end{pmatrix},\quad C^{2}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},\quad C^{3}=\begin{pmatrix}1 & 1\\ 1 & 0\\ 0 & 1\end{pmatrix},$$

In problem *D*, Museums 1 and 2 belong to the same consortium and are considered symmetric, as they satisfy all three conditions of symmetry within consortia.

(i) $C^0_{1,7}=C^0_{2,7}$ and $C^0_{1,8}=C^0_{2,8}$.[8]

(ii) $C^1_{1,9}=C^1_{2,9}$.

(iii) $\pi^{-1}|N^{-1}|=2\cdot 1=1\cdot 2=\pi^{-2}|N^{-2}|$.

If a rule *R* fulfills this axiom, it must happen that $R_1(D)=R_2(D)$.

On the other hand, Consortia 2 and 3 are also symmetric in *D*, as all three conditions required for symmetry between consortia are satisfied. Indeed,

(i) $\sum_{i\in P_2} C_{i7}>0$ and $\sum_{j\in P_3} C_{j7}>0$. $\sum_{i\in P_2} C_{i8}>0$ and $\sum_{j\in P_3} C_{j8}>0$.

(ii) $\pi^2|N^2|=3\cdot 2=\pi^3|N^3|$.

(iii) $\sum_{i\in P_2}\pi^{-i}|N^{-i}|=3\cdot 1=1.5\cdot 2=\sum_{j\in P_3}\pi^{-j}|N^{-j}|$.

If a rule *R* satisfies this axiom, it must follow that the aggregate allocations received by the two consortia are equal; that is, $R_3(D)+R_4(D)=R_5(D)+R_6(D)+R_7(D)$.

The next requirement relies on a well-established principle: any museum that does not contribute to revenue should be excluded from distribution. We say that a museum $i\in M$ is *dummy* if it has no visitors, either because they do not buy any pass that includes access to *i*, or because even with a pass that grants access to *i*, they do not visit it. The axioms states that the allocation of *i* should be zero. Let $D\in\mathcal{D}$, we denote by $\mathrm{NM}(D)$ the set of dummy museums in problem *D*. More specifically,

$$\mathrm{NM}(D)=\{i\in M:\ \text{for each } \sigma\in\{-i,0,(i)\},\ \text{either } C^{\sigma}=\varnothing \text{ or } C^{\sigma}_{ia}=0\ \forall a\in N^{\sigma}\}$$

Dummy. For each $D\in\mathcal{D}$ and each $i\in M$, if $i\in\mathrm{NM}(D)$ then $R_i(D)=0$.

The following two properties establish stability criteria in a specific sense. If a museum chooses to divide into multiple entities, the other museums remain unaf-

[8] To avoid ambiguity, in numerical examples, we use commas to separate the sub-indices of the consumption matrices.

fected. For instance, imagine that the Louvre Museum artificially splits its whole collection (between painting and sculpture, for example) pretending to act as two different museums, and this choice does not alter the revenues. The next requirement states that Louvre's strategy does not impact the allocation of resources among the other museums. These same principles have already been applied to similar settings. See, for example, Slikker (2023), Knudsen and Østerdal (2012), Moulin (2007), or Ju (2003).[9]

Splitting-proofness of museums. Let $(M, P, N, \pi, C) \in \mathcal{D}$ and $i \in M$. Consider $(M', P', N', \pi', C') \in \mathcal{D}$ such that

(i) $M' = (M\backslash\{i\}) \cup \{i_1, \ldots, i_r\}$.
(ii) $P' = \left(P\backslash P^{(i)}\right) \cup \left(\left(P^{(i)}\backslash\{i\}\right) \cup \{i_1, \ldots, i_r\}\right)$.
(iii) $N' = \left(N\backslash N^{-i}\right) \cup \{N^{-i_1}, \ldots, N^{-i_r}\}$ where $N^{-i_h} = N^{-i}$ for all $h \in \{1, \ldots, r\}$.
(iv) $\pi' = \left(\pi\backslash\pi^{-i}\right) \cup \{\pi^{-i_1}, \ldots, \pi^{-i_r}\}$ where $\sum_{h=1}^{r} \pi^{-i_h} = \pi^{-i}$.
(v) $C' = \left(C\backslash \left(C^0 \cup C^{(i)} \cup C^{-i}\right)\right) \cup C'^0 \cup C'^{(i)} \cup \bigcup_{l=1}^{t}\{C^{-i_1}, \ldots, C^{-i_r}\}$ where $C'^0_{ja} = C^0_{ja}$ and $C'^0_{i_h a} = C^0_{ia}$, for all $a \in N^0$, $j \in M\backslash\{i\}$, $h \in \{1, \ldots, r\}$; $C'^{(i)}_{ja} = C^{(i)}_{ja}$, $C'^{(i)}_{i_h a} = C^{(i)}_{ia}$ and $C^{-i_h} = C^{-i}$ for all $j \in P^{(i)}\backslash\{i\}$ and $h \in \{1, \ldots, r\}$.

Then, for each $j \in M\backslash\{i\}$,

$$R_j(M, P, N, \pi, C) = R_j(M', P', N', \pi', C').$$

Beyond the technicalities of the required notation, the essence of conditions (i) through (v) in the previous axiom lies in adapting the elements of the problem to a new context in which museum *i* is replaced by its subdivisions $\{i_1, \ldots, i_r\}$. To illustrate, consider the motivating example we introduced above where museum *i* corresponds to the Louvre, which now intends to operate as several distinct galleries or newly defined museums $\{i_1, \ldots, i_r\}$, rather than as a single entity. Condition (i) stipulates that the new set of museums, M', is obtained by removing the Louvre from the original set *M* and incorporating the subdivided galleries $\{i_1, \ldots, i_r\}$. Condition (ii) adjusts the set of consortia to reflect this change. Specifically, the consortium $P^{(i)}$ to which the Louvre originally belonged is modified by removing the Louvre and including the new galleries in its place. Conditions (iii) to (v) implement analogous modifications for the sets of pass holders (at each level), the pricing of passes, and the consumption matrices. All references to the Louvre in these structures are replaced by corresponding references to the galleries $\{i_1, \ldots, i_r\}$.

The following example illustrates how the splitting-proofness property operates.

[9] Even though the theoretical appeal of the splitting and merging proofness axiom has been extensively justified in the literature, this axiom also holds practical relevance. For example, in the context of soccer subscriptions discussed in the introduction, providers may manipulate the offering by splitting the regular season into two parts: the first and second rounds. Alternatively, they might divide the content into two separate packages, one including the most popular and attractive teams, and another with the remaining teams.

Example 3 Consider the problem $D = (M, P, N, \pi, C)$ of Example 1. Take museum 1 divided into three entities $\{1_1, 1_2, 1_3\}$. Therefore, we have $D' = (M', P', N', \pi', C')$ such that

(i) $M' = \{1_1, 1_2, 1_3, 2, 3\}$, where 1 is replaced by its subdivisions $\{1_1, 1_2, 1_3\}$.
(ii) $P' = \{\{1_1, 1_2, 1_3, 2\}, \{3\}\}$. Museum 1 belongs to the first consortium. After the splitting, this consortium is formed by $\{1_1, 1_2, 1_3\}$, along with Museum 2, which was already part of the original consortium.
(iii) $N' = \{N^{-3}, N^{-2}, N^{-1_1}, N^{-1_2}, N^{-1_3}, N^0, N^1, N^2\}$ where $N^{-1_1} = N^{-1_2} = N^{-1_3} = N^{-1}$. The set of visitors of each subdivision coincides with the original set of visitors N^{-1}.
(iv) $\pi' = \{\pi^{-3}, \pi^{-2}, \pi^{-1_1}, \pi^{-1_2}, \pi^{-1_3}, \pi^0, \pi^1, \pi^2\}$ where $\pi^{-1_1} = \frac{1}{3}$, $\pi^{-1_2} = 1$ and $\pi^{-1_3} = \frac{2}{3}$. Before the splitting, the pass price of Museum 1 was $\pi^{-1} = 2$. After the splitting, the aggregate pass prices of its subdivisions must equal the original price, that is, $\pi^{-1_1} + \pi^{-1_2} + \pi^{-1_3} = 2$. Any disaggregation of prices that satisfies this condition is considered admissible.
(v) $C' = \{C^{-3}, C^{-2}, C^{-1_1}, C^{-1_2}, C^{-1_3}, C'^0, C'^1, C^2\}$ where

$$C^{-1_1} = C^{-1_2} = C^{-1_3} = C^{-1},$$

$$C'^0 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad C'^1 = \begin{pmatrix} 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}.$$

As for the consumption matrices in the new problem, they are defined so that any pass holder who visited Museum 1 now visits all of its subdivisions.

If a rule R satisfies splitting-proofness of museums, then museums not involved in the splitting must remain unaffected by the manipulation. That is, $R_2(D) = R_2(D')$ and $R_3(D) = R_3(D')$.

In line with the previous requirement, the following axiom also aim to prevent the consequences of potential manipulations. However, in this case, the focus shifts from individual museums to the consortia as collective entities. Consider a major metropolitan area (e.g. New York, London, or Paris) where multiple consortia operate, for instance, municipal and national museums. The following axiom asserts that the municipal consortium cannot manipulate the aggregate revenue distribution of the other consortia by artificially fragmenting the municipal consortium into several smaller consortia, one for each district, for instance. As in the case of splitting-proofness of consortia, beyond the technical notation, Conditions (i) to (v) in the *splitting-proofness of consortia* axiom define the reformulated problem that arises when the role of consortium P_k is replaced by its subdivided consortia across all relevant components of the model.

Splitting-proofness of consortia. Let $(M, P, N, \pi, C) \in \mathcal{D}$ and $P_k \in P$ where $P_k = \{i_1, \ldots, i_r\}$. Consider $P_{k^1}, \ldots, P_{k^t}$ such that $P_{k^l} = \{i_1^l, \ldots, i_r^l\}$ for all $l \in \{1, \ldots, t\}$; and $(M', P', N', \pi', C') \in \mathcal{D}$ be such that

(i) $M' = (M \backslash \{i_1, \ldots, i_r\}) \cup \bigcup_{l=1}^{t} \{i_1^l, \ldots, i_r^l\}$.

(ii) $P' = (P \backslash P_k) \cup \bigcup_{l=1}^{t} P_{k^l}$.

(iii) $N' = \left(N \backslash \left(N^k \cup \bigcup_{j=i_1}^{i_r} N^{-j}\right)\right) \cup \bigcup_{l=1}^{t} N^{k_l} \cup \bigcup_{l=1}^{t} \{N^{-i_1^l}, \ldots, N^{-i_r^l}\}$ where $N^{k_l} = N^k$ and $N^{-i_h^l} = N^{-i_h}$ for all $l \in \{1, \ldots, t\}$, $h \in \{1, \ldots, r\}$.

(iv) $\pi' = \left(\pi \backslash \left(\pi^k \cup \bigcup_{j=i_1}^{i_r} \pi^{-j}\right)\right) \cup \bigcup_{l=1}^{t} \pi^{k_l} \cup \bigcup_{l=1}^{t} \{\pi^{-i_1^l}, \ldots, \pi^{-i_r^l}\}$ where $\sum_{l=1}^{t} \pi^{k_l} = \pi^k$ and $\sum_{l=1}^{t} \pi^{-i_h^l} = \pi^{-i_h}$ for all $h \in \{1, \ldots, r\}$.

(v) $C' = \left(C \backslash \left(C^0 \cup C^k \cup \bigcup_{j=i_1}^{i_r} C^{-j}\right)\right) \cup C'^0 \cup \bigcup_{l=1}^{t} C^{k_l} \cup \bigcup_{l=1}^{t} \{C^{-i_1^l}, \ldots, C^{-i_r^l}\}$ where $C'^0_{ja} = C^0_{ja}$, $C'^0_{i_h^l a} = C^0_{i_h a}$, for all $a \in N^0, j \in M \backslash \{i_1, \ldots, i_r\}, h \in \{1, \ldots, r\}, l \in \{1, \ldots, t\}$; and $C^{k_l} = C^k$ and $C^{-i_h^l} = C^{-i_h}$ for all $l \in \{1, \ldots, t\}$, $h \in \{1, \ldots, r\}$.

Then, for each $P_r \in P \backslash \{P_k\}$

$$\sum_{j \in P_r} R_j(M, P, N, \pi, C) = \sum_{j \in P_r} R_j(M', P', N', \pi', C')$$

Similarly to Example 3, the example below illustrates the functioning of splitting-proofness of consortia.

Example 4 Consider the problem $D = (M, P, N, \pi, C)$ of Example 1. Take consortium 1 divided into two consortia $\{1^1, 1^2\}$. Therefore, we have $D' = (M', P', N', \pi', C')$ such that

(i) $M' = \{1^1, 2^1, 1^2, 2^2, 3\}$. In the original problem, Consortium 1 was formed by Museums 1 and 2. When this consortium is split into two new consortia, each of them includes copies of both museums, resulting in two consortia that each contain two museums.

(ii) $P' = \{\{1^1, 2^1\}, \{1^2, 2^2\}, \{3\}\}$. The new set of consortia consists of the two subdivisions of Consortium 1, together with the original Consortium 2, which is composed solely of Museum 3.

(iii) $N' = \{N^{-3}, N^{-2^2}, N^{-2^1}, N^{-1^2}, N^{-1^1}, N^0, N^{1^1}, N^{1^2}, N^2\}$ where $N^{-2^2} = N^{-2^1} = N^{-2}$, $N^{-1^2} = N^{-1^1} = N^{-1}$ and $N^{1^1} = N^{1^2} = N^1$. The set of visitors for each museum within each subdivision of the consortium coincides with the original set of visitors.

(iv) $\pi' = \{\pi^{-3}, \pi^{-2^2}, \pi^{-2^1}, \pi^{-1^2}, \pi^{-1^1}, \pi^0, \pi^{1^1}, \pi^{1^2}, \pi^2\}$ where $\pi^{-1^1} = \frac{4}{3}$, $\pi^{-1^2} = \frac{2}{3}$, $\pi^{-2^1} = 2$, $\pi^{-2^2} = 2$, $\pi^{1^1} = 1$ and $\pi^{1^2} = 3$. As in Example 1, the pass prices are divided in such a way that the aggregate prices of the subdivisions equal the original prices.

(v) $C' = \{C^{-3}, C^{-2^2}, C^{-2^1}, C^{-1^2}, C^{-1^1}, C'^0, C^{1^1}, C^{1^2}, C^2\}$ where $C^{-2^2} = C^{-2^1} = C^{-2}$, $C^{-1^2} = C^{-1^1} = C^{-1}$, $C^{1^1} = C^{1^2} = C^1$ and

$$C'^0 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Again, similarly to Example 1, we assume that pass holders replicate their original consumption patterns across all subdivisions.

If a rule *R* satisfies splitting-proofness for consortia, then any consortium not involved in the splitting must remain unaffected by the manipulation. Specifically, the single museum in Consortium 2 must receive the same share of revenue as before the split, that is, $R_3(D) = R_3(D')$.

The last properties also pertain to the stability of the allocation with respect to potential manipulations. In the previous two axioms, a museum (or a consortium) was split into several museums (or consortia). The manipulation referred to by the following property is an intermediate case, where a consortium intends to operate as a single museum. More specifically, consider the family of problems $\mathcal{D}^0$ (where pass holders only acquire the general pass). If each consortium acts as a single museum, the axiom requires that the allocation of this single museum coincides with the aggregated allotment of the consortium in the original problem.

Consortia consistency. For each $(M, P, N, \pi, C) \in \mathcal{D}^0$ and each $P_k \in P$,

$$\sum_{i \in P_k} R_i(M, P, N, \pi, C) = R_k \left(\{1, ..., s\}, \hat{P}^s, \hat{N}, \hat{\pi}, \hat{C}\right),$$

where $\left(\{1, ..., s\}, P^s, \hat{N}, \hat{\pi}, \hat{C}\right) \in \mathcal{D}^0$ is such that $\hat{P}^s = \{\{1\}, ..., \{s\}\}$, $\hat{N} = \{\emptyset, \dots, \emptyset, N^0, \emptyset, \dots, \emptyset\}$, $\hat{\pi} = \{\hat{\pi}^{-s}, \dots, \hat{\pi}^{-1}, \pi^0, \hat{\pi}^1, \dots, \hat{\pi}^s\}$ with $\hat{\pi}^{-t} = \hat{\pi}^t = \pi^t$ for any $t \in \{1, ..., s\}$, and $\hat{C} = \{\emptyset, \dots, \emptyset, \hat{C}^0, \emptyset, \dots, \emptyset\}$ with, for any $t \in \{1, \dots, s\}$ and $a \in \hat{N}^0$

$$\hat{C}^0_{ta} = \begin{cases} 1 & \text{if } \exists i \in P_t : C_{ia} = 1 \\ 0 & \text{otherwise.} \end{cases}$$

For each $D \in \mathcal{D}^0$, we will refer to $\left(\{1, ..., s\}, \hat{P}^s, \hat{N}, \hat{\pi}, \hat{C}\right)$ as the *reduced problem* associated to *D*.

Splitting-proofness of museums, splitting-proofness of consortia, and consortia consistency require the rule to be immune to specific (and distinct) types manipulations. Remarks 4 and 5 on the tightness of the characterization results prove that these three axioms are logically independent of one another.

Next, we present our main results, in which we characterize the rules introduced in the previous section through various combinations of the axioms outlined above. For sake of exposition, the proofs are deferred to Appendix A.

Theorem 1 *A rule satisfies composition, dummy, symmetry within consortia, and symmetry between consortia, if and only if it is the egalitarian-egalitarian rule.*

Theorem 2 *A rule satisfies composition, dummy, splitting-proofness of museums, splitting-proofness of consortia, and consortia consistency if and only if it is the proportional-proportional rule.*

Theorem 3 *A rule satisfies composition, dummy, symmetry within consortia, splitting-proofness of consortia, and consortia consistency if and only if it is the proportional-egalitarian rule.*

Theorem 4 *A rule satisfies composition, dummy, symmetry between consortia, and splitting-proofness of museums if and only if it is the egalitarian-proportional rule.*

As observed in the previous results, the properties of composition and dummy are present in all cases. In conjunction with the other requirements, the former axioms imply that the allocation scheme fundamentally depends on how the profit generated by each individual pass holder is distributed taking into account her consumption behavior. The remaining axioms in the characterizations determine the specific manner in which the revenue from each individual pass holder is divided.

Symmetry within consortia appears in the characterizations of both the egalitarian-egalitarian and proportional-egalitarian rules. Broadly speaking, this is because, once the share of revenues allocated to each consortium is determined, the combination of this axiom and the dummy principle lead to an egalitarian treatment of museums within each consortium. This feature is precisely one of the defining characteristics of the egalitarian-egalitarian and proportional-egalitarian schemes. Symmetry between consortia, along with the dummy principle, plays a dual role in the allocation between consortia in Theorems 1 and 4, where the egalitarian treatment is applied to consortia rather than to individual museums.

With regard to the non-manipulability requirements, both the splitting-proofness of museums and the splitting-proofness of consortia are grounded in the same underlying principle. The former prevents manipulations through splitting by individual museums, while the latter guards against similar strategies employed by entire consortia. Although not solely by themselves, these non-manipulability axioms are closely related to proportionality criteria.[10] Thus, the application of splitting-proofness to museums, as in Theorems 2 and 3, is key to ensuring a proportional distribution of revenues among museums. Similarly, the application of splitting-proofness to consortia plays an analogous role at the consortium level in Theorem 2. This is

[10] Ju et al. (2007), Ju (2003), and de los Frutos (1999) provide good examples of how non-manipulability, combined with other requirements, leads to proportional mechanisms.

Table 1 Properties satisfied by each rule

Property	R^{EE}	R^{PP}	R^{PE}	R^{EP}
Composition	Y	Y	Y	Y
Dummy	Y	Y	Y	Y
Symmetry within consortia	Y	N	Y	N
Symmetry between consortia	Y	N	N	Y
Splitting-proofness of museums	N	Y	N	Y
Splitting-proofness of consortia	N	Y	Y	N
Consortia consistency	Y	Y	Y	Y

why, for instance, in the proportional-egalitarian rule characterized in that theorem, revenue is first distributed proportionally among the consortia.

Table 1 summarizes the properties satisfied by each of the four rules characterized in this section.

The independence of the axioms in Theorems 1 to 4 is proved in Remarks 3 to 6 in the appendix. These remarks demonstrate that, beyond their individual implications, none of the axioms nor any subset or combination of them is sufficient to fully characterize any of the rules introduced in the previous section. The distinction among the four mechanisms lies in how symmetry and non-manipulability, whether applied to museums or consortia, are combined (always in conjunction with the axioms of composition and dummy).

5 Final remarks

We have extended the fundamental museum pass problem by incorporating a more sophisticated market structure, allowing museums to be organized into multiple programs or consortia. Within this framework, we have proposed four different rules to distribute the overall revenue that is obtained from selling all the passes. These rules integrate the principles of egalitarianism and proportionality (relative to either consortium prices or individual prices) through a two-stage mechanism. Initially, revenue is distributed among consortia, followed by distribution among museums within each consortium. In accordance with this approach, four distinct rules are defined: the egalitarian-egalitarian, the proportional-proportional, the egalitarian-proportional, and the proportional-egalitarian rules. We employ an axiomatic methodology to establish the normative foundations of these allocation methods. The axioms examined in this paper are categorized into two groups: fairness and stability. Appropriate combinations of these principles characterize the four rules. In particular, the egalitarian-egalitarian rule is characterized by composition, dummy, symmetry within consortia, and symmetry between consortia (Theorem 1). In Theorem 2 we show that replacing both symmetry requirements with splitting-proofness of museums, splitting-proofness of consortia, and consortia consistency univocally identifies the proportional-proportional rule. Furthermore, we find that integrating fairness and non-manipulability requirements is not only feasible but also leads to precise revenue distribution methods. Thus, composition, dummy, symmetry within consortia, splitting-proofness of consortia and consortia consistency characterize the proportional-egalitarian rule (Theorem 3). Dually, composition, dummy, symmetry between consortia, and splitting-proofness of museums characterize the egalitarian-proportional rule (Theorem 4).

It is worth noting that, although all four rules satisfy consortia consistency, this axiom is only essential for the characterizations of the proportional-proportional and proportional-egalitarian rules (Theorems 2 and 3). While it is possible to identify rules that satisfy the properties other than consortia consistency in Theorems 2 and 3, the resulting families lack a clear formulation. Moreover, beyond the technical details, their practical interpretation remains ambiguous.

Considering Theorems 1 to 4 together, one can interpret that, when additional requirements such as the dummy or composition axioms are imposed, a trade-off emerges between fairness (captured by the two symmetry criteria) and non-manipulability (captured by splitting-proofness). This naturally raises the question: Which of these two principles should a policy-maker prioritize? This is a challenging question that obviously may depend on the particular situation the model is applied to. In the axiomatic literature (e.g. Thomson (2023)), there is limited discussion on which axioms should be prioritized, especially when they are not easily comparable through logical implications or, as in this case, when they entail a clear trade-off between fairness and robustness against manipulation. The decision is left to the central planner, presenting only the feasible options constrained by the axioms and their implications. That said, and perhaps going beyond the usual caution exercised in axiomatic work, we argue that symmetry should be prioritized over non-manipulability. The symmetry axiom (both within and across consortia) simply requires that equals be treated equally. This is such a fundamental principle of equity that its violation is ethically difficult to justify, even if, in doing so, we allow for the possibility that museums may manipulate the revenue distribution. From a purely philosophical standpoint (which may not be as evident from an economic perspective) the requirement of non-manipulability implicitly assumes that individuals will exploit opportunities to manipulate outcomes if given the chance. In our framework, such manipulation would occur through the artificial splitting of museums or consortia. This type of behavior is arguably costly or, at the very least, relatively easy to detect. While it is certainly desirable to design allocation mechanisms that are immune to such manipulations, if achieving this immunity comes at the expense of fairness, one might reasonably argue in favor of accepting the risk of manipulation in order to preserve equity. On the other hand, we acknowledge that there are alternative institutional settings in which non-manipulability must take precedence over fairness, particularly when manipulations are highly likely (either because they do not involve complex strategic behavior or because they are costless, for instance) or when their social implications significantly outweigh the consequences of reduced fairness.

In addition to the characterization of the egalitarian-egalitarian rule, this method finds further justification through its connection to cooperative games. Several authors have proposed different solution concepts for cooperative games with a priori unions (e.g. Alonso-Meijide et al. (2020) and Lorenzo-Freire (2016)). Among those, the *Owen value* (Owen (1977)) stands out as the most prominent one. Interestingly, the egalitarian-egalitarian rules coincides with the Owen value of the game (M, v, P) where M is the set of museums, P is the set of consortia, and v is such that, for each $S \subseteq M$,

$$v(S) = \pi^0|\{a \in N^0; M_a^0 \subseteq S\}| + \sum_{k=1}^{s} \pi^k|\{a \in N^k; M_a^k \subseteq S\}| + \sum_{k=-m}^{-1} \pi^k|\{a \in N^k; M_a^k \subseteq S\}|$$

The models in Ginsburgh and Zang (2001) and Bergantiños and Moreno-Ternero (2015) address revenue distribution from general passes, without considering consortia. In our framework, their rules apply only to the subdomain $\mathcal{D}^0$. However, under suitable assumptions (when either a single consortium includes all museums $(P = \{\{1, \ldots, m\}\})$ or each museum forms its own consortium $(P = \{\{1\}, \{2\}, \ldots, \{m\}\})$ our rules R^{EE}, R^{PP}, R^{EP}, and R^{PE} coincide with the Shapley and p-Shapley rules.

Finally, we acknowledge our model may have several potential extensions. Perhaps one of the most straightforward relates to the assumption that the set of consortia must be a partition of the set of museums. This premise implies that a museum can only belong to one consortium, aside from participating in the general pass. We believe that, provided the global and individual passes remain feasible, this assumption could be relaxed to incorporate more complex market structures. The mechanisms behind the proposed rules could be adapted to this new framework. The core principles of fairness and stability that support the axioms would remain intact, although their formal statements and definitions would need to be reformulated. However, we caution that this could significantly increase the complexity of the notation in the definitions and proofs, which could make the document more difficult to read without introducing truly innovative ideas.

Continuing with the potential extensions, the choice of one rule over another becomes particularly significant when museums strategically determine their pass prices or decide whether to join a consortium. Our focus here has been to delimit the set of admissible rules from a normative perspective. A potential direction for future research is to examine the strategic implications of these rules. This could involve developing a model in which museums and consortia endogenously determine their pricing strategies within a competitive environment and decide how to organize themselves into consortia.[11]

Appendix A. Proofs

For clarity of presentation and to avoid redundant reasoning in the proofs of the characterization results, we first prove some technical lemmas.

Lemma 1 *Let R and R' be two rules that satisfy composition. If R and R' coincide in all the subclasses of problems $\mathcal{D}^{-m}, \ldots, \mathcal{D}^{-1}, \mathcal{D}^{o}, \mathcal{D}^{1}, \ldots, \mathcal{D}^{s}$, then they also coincide in the general domain of problems $\mathcal{D}$.*

Proof Let $(M, P, N, \pi, C) \in \mathcal{D}$, and let $i \in M$. In application of *composition*,

[11] We are grateful to the anonymous referees for suggesting this line of research.

$$R_i\left(M,P,N,\pi,C\right)=\sum_{i=1}^{m}R_i\left(M,P,\overline{N}^{-i},\pi,\overline{C}^{-i}\right)+R_i\left(M,P,\overline{N}^{0},\pi,\overline{C}^{0}\right)$$
$$+\sum_{t=1}^{s}R_i\left(M,P,\overline{N}^{t},\pi,\overline{C}^{t}\right),$$

where each $\overline{N}^{\sigma}$ with $\sigma\in\{-i,0,t\}$ is the set of pass holders that only purchase (individual, global or consortium) passes to access σ. Notice that $\left(M,P,\overline{N}^{-i},\pi,\overline{C}^{-i}\right)\in\mathcal{D}^{-i}$, $\left(M,P,\overline{N}^{0},\pi,\overline{C}^{0}\right)\in\mathcal{D}^{0}$, and $\left(M,P,\overline{N}^{t},\pi,\overline{C}^{t}\right)\in\mathcal{D}^{t}$. By assumption, both *R* and R' coincide in these subdomains. Therefore,

$$\begin{aligned}R_i\left(M,P,N,\pi,C\right)&=\sum_{i=1}^{m}R'_i\left(M,P,\overline{N}^{-i},\pi,\overline{C}^{-i}\right)+R'_i\left(M,P,\overline{N}^{0},\pi,\overline{C}^{0}\right)+\sum_{t=1}^{s}R'_i\left(M,P,\overline{N}^{t},\pi,\overline{C}^{t}\right)\\&=R'_i\left(M,P,N,\pi,C\right)\end{aligned}$$

□

Lemma 2 *If a rule R satisfies dummy, then, for each* $i\in M$ *and each* $(M,P,N,\pi,C)\in\mathcal{D}^{-i}$,

$$R_i(M,P,N,\pi,C)=\pi^{-i}|N^{-i}|$$

Proof Let $i\in M$, and let $(M,P,N,\pi,C)\in\mathcal{D}^{-i}$. Notice that, any $j\in M\backslash\{i\}$ is dummy in this problem. The *dummy* principle requires that $R_j(M,P,N,\pi,C)=0$. By definition of rule, it must then occur that $R_i(M,P,N,\pi,C)=\pi^{-i}|N^{-i}|$. □

Lemma 3 *Let* $(M,P,\{a\},\pi,C^{(a)})\in\mathcal{D}^0$ *be a problem with just one pass holder purchasing the general pass, and let R be a rule that satisfies splitting-proofness of museums and dummy. If* $\{i,j\}\subseteq M$ *are such that* $P^{(i)}=P^{(j)}$, $C_{ia}^{(a)}=C_{ja}^{(a)}$ *and* $\pi^{-i}=\pi^{-j}$, *then*

$$R_i\left(M,P,\{a\},P,\pi,C^{(a)}\right)=R_j\left(M,P,\{a\},P,\pi,C^{(a)}\right).$$

Proof Consider the problem $D=\left(M,P,\{a\},\pi,C^{(a)}\right)\in\mathcal{D}^0$ and two museums $\{i,j\}\subseteq N$ that fulfills the requirements in the statement, with $P_k=P^{(i)}=P^{(j)}$. If $C_{ia}^{(a)}=C_{ja}^{(a)}=0$, *dummy* implies that $R_i(D)=R_j(D)=0$. If, instead, $C_{ia}^{(a)}=C_{ja}^{(a)}=1$, we can split museum *i* into $\{i_1,i_2\}$ obtaining the problem $D^i=\left(M',P',\{a\},\pi',C'^{(a)}\right)$ where:

- $M' = (M\backslash\{i\}) \cup \{i_1, i_2\}$.
- $P' = (P\backslash P_k) \cup P'_k$ where $P'_k = (P_k\backslash\{i\}) \cup \{i_1, i_2\}$.
- $\pi' = \left(\pi\backslash\pi^{-i}\right) \cup \{\pi'^{-i_1}, \pi'^{-i_2}\}$ where $\pi'^{-i_1} = \pi'^{-i_2} = \frac{\pi^{-i}}{2}$.
- $C'^{(a)}$ is such that $C'^{(a)}_{i_1 a} = C'^{(a)}_{i_2 a} = C^{(a)}_{ia}$, and $C'^{(a)}_{ja} = C^{(a)}_{ja}$ for any $j \in M\backslash\{i\}$

Splitting-proofness of museums requires that

$$R_i(D) = R_{i_1}(D^i) + R_{i_2}(D^i).$$

Similarly, we can split museum j into $\{i_1, i_2\}$ obtaining the problem $D^j = \left(\overline{M}, \overline{P}, \{a\}, \overline{\pi}, \overline{C}^{(a)}\right)$. Again, by *splitting-proofness of museums*,

$$R_j(D) = R_{i_1}(D^j) + R_{i_2}(D^j).$$

Now, in the problem D^i we again split the museum *j* into $\{j_1, j_2\}$ obtaining the problem $D^{ij} = \left(\overline{M}', \overline{P}', \{a\}, \overline{\pi}', \overline{C}'^{(a)}\right)$ where:

- $\overline{M}' = (M'\backslash\{j\}) \cup \{j_1, j_2\}$.
- $\overline{P}' = (P'\backslash P'_k) \cup \overline{P}'_k$ where $\overline{P}'_k = (P'_k\backslash\{j\}) \cup \{j_1, j_2\}$.
- $\overline{\pi}' = \left(\pi'\backslash\pi'^{-j}\right) \cup \{\overline{\pi}'^{-j_1}, \overline{\pi}'^{-j_2}\}$ where $\overline{\pi}'^{-j_1} = \overline{\pi}'^{-j_2} = \frac{\pi^{-j}}{2}$.
- $\overline{C}'^{(a)}$ is such that $\overline{C}'^{(a)}_{j_1 a} = C''^{(a)}_{j_2 a} = C'^{(a)}_{ja}$, and $\overline{C}'^{(a)}_{la} = C'^{(a)}_{la}$ for any $l \in M'\backslash\{j\}$

  In application of *splitting-proofness of museums*, $R_{i_1}(D^i) = R_{i_1}(D^{ij})$ and $R_{i_2}(D^i) = R_{i_2}(D^{ij})$. Then,

$$R_i(D) = R_{i_1}(D^i) + R_{i_2}(D^i) = R_{i_1}(D^{ij}) + R_{i_2}(D^{ij})$$

Using an analogous argument for the problem D^j, we split museum *i* into $\{j_1, j_2\}$ and obtain that

$$R_j(D) = R_{i_1}(D^j) + R_{i_2}(D^j) = R_{i_1}(D^{ji}) + R_{i_2}(D^{ji})$$

As $D^{ij} = D^{ji}$, we have that $R_{i_1}(D^{ij}) = R_{i_1}(D^{ji})$ and $R_{i_2}(D^{ij}) = R_{i_2}(D^{ji})$. And therefore, $R_i(D) = R_j(D)$. $\square$

Lemma 4 *Let $(M, P, \{a\}, \pi, C^{(a)}) \in \mathcal{D}^0$ be a problem with just one pass holder purchasing the general pass, and let R be a rule that satisfies splitting-proofness of consortia, dummy and consortia consistency. If $P_k, P_r \in P$ are such that $\sum_{i\in P_k} C^{(a)}_{ia} = 0 \Leftrightarrow \sum_{j\in P_r} C^{(a)}_{ja} = 0$ and $\pi^k = \pi^r$, then*

$$\sum_{i\in P_k} R_i\left(M, P, \{a\}, P, \pi, C^{(a)}\right) = \sum_{j\in P_r} R_j\left(M, P, \{a\}, P, \pi, C^{(a)}\right).$$

Proof Consider the problem $D = \left(M, P, \{a\}, \pi, C^{(a)}\right) \in \mathcal{D}^0$ and its associated reduced problem $D^s = \left(\{1, \ldots, s\}, \hat{P}^s, \{a\}, \hat{\pi}, \hat{C}^{(a)}\right) \in \mathcal{D}^0$. Let $\{k, r\} \subseteq \{1, \ldots, s\}$ be a pair of consortia such that $\hat{C}^{(a)}_{ka} = \hat{C}^{(a)}_{ra}$ and $\pi^k = \pi^r$. Reasoning as in Lemma 3 with *splitting-proofness of consortia* instead of *splitting-proofness of museums*, we obtain that $R_k(D^s) = R_r(D^s)$. Finally, by *consortia consistency*, we conclude that

$$\sum_{i \in P_k} R_i \left(M, P, \{a\}, P, \pi, C^{(a)}\right) = R_k\left(D^s\right) = R_r\left(D^s\right) = \sum_{j \in P_r} R_j \left(M, P, \{a\}, P, \pi, C^{(a)}\right).$$

□

Lemma 5 *Let $(M, P, \{a\}, \pi, C^{(a)}) \in \mathcal{D}^0$ be a problem with just one pass holder purchasing the general pass, and let R be a rule that satisfies splitting-proofness of consortia. If $P_k, P_r \in P$ are such that $k, r \in K^0_a$ and $\pi^k \geq \pi^r$, then*

$$R_k\left(\{1, ..., s\}, \hat{P}^s, \hat{N}, \hat{\pi}, \hat{C}\right) \geq R_r\left(\{1, ..., s\}, \hat{P}^s, \hat{N}, \hat{\pi}, \hat{C}\right).$$

where $\left(\{1, ..., s\}, \hat{P}^s, \hat{N}, \hat{\pi}, \hat{C}\right) \in \mathcal{D}^0$ is the corresponding reduced problem.

Proof Let $(M, P, \{a\}, \pi, C^{(a)}) \in \mathcal{D}^0$ be a problem with just one pass holder purchasing the general pass. Consider its reduced problem $D = \left(\{1, ..., s\}, \hat{P}^s, \hat{N}, \hat{\pi}, \hat{C}\right)$. Let $k, r \in K^0_a$ with $\pi^k \geq \pi^r$. Let D' be the problem in which consortium *k* splits into two new consortia k' and k'' such that

$$\pi^{k'} = \pi^{-k'} = \pi^r \text{ and } \pi^{k''} = \pi^{-k''} = \pi^k - \pi^r$$

As *R* satisfies *splitting-proofness of consortia*

$$R_{k'}\left(D'\right) \leq R_{k'}\left(D'\right) + R_{k''}\left(D'\right) = R_k\left(D\right).$$

By Lemma 4

$$R_{k'}\left(D'\right) = R_r\left(D'\right) = R_r\left(D\right).$$

Therefore, $R_r\left(D\right) \leq R_k\left(D\right)$. □

Lemma 6 *Let $(M, P, \{a\}, \pi, C^{(a)}) \in \mathcal{D}^\sigma$ be a problem with just one pass holder where $\sigma \in \{0, k\}$ with $k \in \{1, \ldots, s\}$, and let R be a rule that satisfies splitting-proofness of museums. If $i, j \in P^k$ are such that $C^{(a)}_{ia} = C^{(a)}_{ja}$ and $\pi^{-i} \geq \pi^{-j}$, then*

$$R_i\left(M,P,\{a\},P,\pi,C^{(a)}\right)\geq R_j\left(M,P,\{a\},P,\pi,C^{(a)}\right).$$

Proof Let $D=(M,P,\{a\},\pi,C^{(a)})\in\mathcal{D}^\sigma$ be a problem with just one pass holder where $\sigma\in\{0,k\}$ with $k\in\{1,\ldots,s\}$. Consider $i,j\in P^k$ such that $C^{(a)}_{ia}=C^{(a)}_{ja}$ and $\pi^{-i}\geq\pi^{-j}$. Let D' be the problem where museum *i* splits into two new museums i' and i'' such that

$$\pi^{-i'}=\pi^{-j}\text{ and }\pi^{-i''}=\pi^{-i}-\pi^{-j}.$$

As *R* satisfies *splitting-proofness of museums*

$$R_{i'}\left(D'\right)\leq R_{i'}\left(D'\right)+R_{i''}\left(D'\right)=R_i\left(D\right).$$

By Lemma 3

$$R_{i'}\left(D'\right)=R_j\left(D'\right)=R_j\left(D\right).$$

Therefore, $R_j\left(D\right)\leq R_i\left(D\right)$. □

Proof of Theorem 1

We start by showing that the *egalitarian-egalitarian rule* satisfies the axioms in the statement.

- Composition. Let $(M,P,N,\pi,C),(M,P,N',\pi,C')\in\mathcal{D}$ such that $N\cap N'=\emptyset$. Let $i\in M$. It follows that

$$\begin{aligned}R_i^{EE}(M,P,N\cup N',\pi\cup\pi,C\cup C')=&\sum_{a\in N_i^0\cup N_i'^0}\frac{1}{|M_a\cap P^{(i)}|}\frac{1}{|K_a^0|}\pi^0\\&+\sum_{a\in N_i^{(i)}\cup N_i'^{(i)}}\frac{1}{|M_a|}\pi^{(i)}+|N^{-i}|\pi^{-i}+|N'^{-i}|\pi^{-i}\\&=\sum_{a\in N_i^0}\frac{1}{|M_a\cap P^{(i)}|}\frac{1}{|K_a^0|}\pi^0+\sum_{a\in N_i^{(i)}}\frac{1}{|M_a|}\pi^{(i)}+|N^{-i}|\pi^{-i}\\&+\sum_{a\in N_i'^0}\frac{1}{|M_a\cap P^{(i)}|}\frac{1}{|K_a^0|}\pi^0+\sum_{a\in N_i'^{(i)}}\frac{1}{|M_a|}\pi^{(i)}+|N'^{-i}|\pi^{-i}\\&=R_i^{EE}(M,P,N,\pi,C)+R_i^{EE}(M,P,N',\pi,C').\end{aligned}$$

- Dummy. Let $D\in\mathcal{D}$ and $i\in\,\mathrm{NM}\,(D)$. Since $N_i^0=N_i^{(i)}=N^{-i}=\varnothing$, then

$$R_i^{EE}(D) = \sum_{a \in N_i^0} \frac{1}{|M_a \cap P^{(i)}|} \frac{1}{|K_a^0|} \pi^0 + \sum_{a \in N_i^{(i)}} \frac{1}{|M_a|} \pi^{(i)} + |N^{-i}| \pi^{-i} = 0.$$

- Symmetry within consortia. Let $D \in \mathcal{D}$. Let $i, j \in M$ that satisfy the conditions in the definition of the property. Then

$$\begin{aligned} R_i^{EE}(D) &= \sum_{a \in N_i^0} \frac{1}{|M_a \cap P^{(i)}|} \frac{1}{|K_a^0|} \pi^0 + \sum_{a \in N_i^{(i)}} \frac{1}{|M_a|} \pi^{(i)} + |N^{-i}| \pi^{-i} \\ &= \sum_{a \in N_j^0} \frac{1}{|M_a \cap P^{(j)}|} \frac{1}{|K_a^0|} \pi^0 + \sum_{a \in N_i^{(j)}} \frac{1}{|M_a|} \pi^{(j)} + |N^{-j}| \pi^{-j} \\ &= R_j^{EE}(D). \end{aligned}$$

- Symmetry between consortia. Let $D \in \mathcal{D}$. Let $P_r, P_t \in P$ that satisfy the conditions in the definition of the property. Then

$$\begin{aligned} \sum_{i \in P_r} R_i^{EE}(D) &= \sum_{i \in P_r} \sum_{a \in N_i^0} \frac{1}{|M_a \cap P^{(i)}|} \frac{1}{|K_a^0|} \pi^0 + \sum_{i \in P_r} \sum_{a \in N_i^{(i)}} \frac{1}{|M_a|} \pi^{(i)} + \sum_{i \in P_r} |N^{-i}| \pi^{-i} \\ &= \sum_{a \in N^0 : r \in K_a^0} \frac{1}{|K_a^0|} \pi^0 + \pi^r |N^r| + \sum_{j \in P_t} |N^{-j}| \pi^{-j} \\ &= \sum_{a \in N^0 : t \in K_a^0} \frac{1}{|K_a^0|} \pi^0 + \pi^t |N^t| + \sum_{j \in P_t} |N^{-j}| \pi^{-j} \\ &= \sum_{j \in P_t} \sum_{a \in N_j^0} \frac{1}{|M_a \cap P^{(j)}|} \frac{1}{|K_a^0|} \pi^0 + \sum_{j \in P_t} \sum_{a \in N_j^{(j)}} \frac{1}{|M_a|} \pi^{(j)} + \sum_{j \in P_r} |N^{-j}| \pi^{-j} \\ &= \sum_{j \in P_t} R_j^{EE}(D). \end{aligned}$$

Now, we prove the converse. Let *R* be a rule that satisfies the properties in the statement, and let $(M, P, N, \pi, C) \in \mathcal{D}$. We divide the proof into several steps.

(i) Let $i \in M$, and let $D \in \mathcal{D}^{-i}$. By Lemma 2, $R_j(D) = 0 = R_j^{EE}(D)$ for any $j \in M \backslash \{i\}$, and then $R_i(D) = \pi^{-i} |N^{-i}| = R_i^{EE}(D)$ by definition of rule. Thus, *R* coincides with the egalitarian-egalitarian rule in any subclass of problems $\mathcal{D}^{-i}$.

(ii) Let $D^0 = (M, P, \{a\}, \pi, C^{(a)}) \in \mathcal{D}^0$ be a problem with just one pass holder purchasing the general pass, and let $i \in M$ with $i \in P_k$. Notice that, if $k \notin K_a^0$, then *dummy* implies that $R_i(D^0) = 0 = R_i^{EE}(D^0)$. If $k \in K_a^0$, in application of *symmetry between consortia*, all visited consortia obtain the same aggregate award. And thus,

$$\sum_{j \in P_k} R_j(D^0) = \frac{\pi^0}{|K_a^0|}$$

Within consortium P_k we have two types of museums: visited ($j \in P_k$ such that $C_{ja}^{(a)} = 1$) and non-visited ($j \in P_k$ such that $C_{ja}^{(a)} = 0$). *Dummy* requires that any non-visited museum gets zero, while *symmetry within consortia* implies that visited museum gets equal share of $\frac{\pi^0}{|K_a^0|}$. Hence, as $i \in P_k$, if $C_{ia}^{(a)} = 0$ then $R_i(D^0) = 0 = R_i^{EE}(D^0)$. And, if $C_{ia}^{(a)} = 1$, then $R_i(D^0) = \frac{\pi}{|K_a^0|} \frac{1}{|M_a \cap P^{(i)}|} = R_i^{EE}(D^0)$. Therefore, in any case,

$$R_i(D^0) = R_i^{EE}(D^0)$$

Now, let $D = (M, P, N, \pi, C) \in \mathcal{D}^0$ without any restriction on the cardinality of N^0. By *composition*, it follows that, for each $i \in M$,

$$\begin{aligned} R_i(D) &= \sum_{a \in N^0} R_i\left(M, P, \{a\}, \pi, C^{(a)}\right) \\ &= \sum_{a \in N^0} R_i^{EE}\left(M, P, \{a\}, \pi, C^{(a)}\right) \\ &= R_i^{EE}(D) \end{aligned}$$

Therefore, R and the egalitarian-egalitarian rule coincide in $\mathcal{D}^0$.

(iii) Let $P_t \in P$, and let $D^t = (M, P, \{a\}, \pi, C^{(a)}) \in \mathcal{D}^t$ be a problem with just one of those pass holders. If $i \in \mathrm{NM}(D^t)$, *dummy* implies that $R_i(D^t) = 0 = R_i^{EE}(D^t)$. In application of *symmetry within consortia*, all the other museums in $M \setminus \mathrm{NM}(D^t)$ get an equal share of π^t. And thus, if i is not dummy, $R_i(D^t) = \frac{\pi^t}{|M_a|} = R_i^{EE}(D^t)$. Now, let $D = (M, P, N, \pi, C) \in \mathcal{D}^t$ without any restriction on the cardinality of N^t. *Composition* guarantees that, for each $i \in M$,

$$R_i(D) = \sum_{a \in N^t} R_i\left(M, P, \{a\}, \pi, C^{(a)}\right) = R_i^{EE}(D)$$

Thus, R coincides with the egalitarian-egalitarian rule in any subclass of problems $\mathcal{D}^t$.Finally, in application of Lemma 1, $R = R^{EE}$. □

Remark 3 The axioms of Theorem 1 are independent.

(a) Let R^1 de defined as follows. For each $i \in M$

$$R_i^1(D) = \sum_{a\in N_i^0} \frac{1}{|M_a \cap P^{(i)}|}\frac{1}{|K_a^0|}\pi^0 + \frac{\sum_{a\in N^{(i)}} C_{ia}}{\sum_{j\in P^{(i)}}\sum_{a\in N^{(i)}} C_{ja}}|N^{(i)}|\pi^{(i)} + |N^{-i}|\pi^{-i}.$$

The rule R^1 satisfies symmetry within consortia, symmetry between consortia, dummy, but not composition.

(b) The rule R^{EP} satisfies composition, symmetry between consortia, dummy, but not symmetry within consortia.
(c) The rule R^{PE} satisfies composition, symmetry within consortia, dummy, but not symmetry between consortia.
(d) Let R^2 be defined as follows. For each $i \in M$

$$R_i^2(D) = \frac{E}{|P||P^{(i)}|}.$$

The rule R^2 satisfies composition, symmetry within consortia, symmetry between consortia, but not dummy.

Proof of Theorem 2

- We start by showing that the *proportional-proportional rule* satisfies the axioms in the statement.Composition. Let $(M,P,N,\pi,C),(M,P,N',\pi,C') \in \mathcal{D}$ such that $N \cap N' = \emptyset$. Let $i \in M$. It follows that

$$\begin{aligned}
R_i^{PP}(M,P,N\cup N',\pi,C\cup C') &= \sum_{a\in N_i^0\cup N_i'^0} \frac{\pi^{-i}}{\sum_{j\in M_a\cap P^{(i)}}\pi^{-j}}\frac{\pi^{(i)}}{\sum_{t\in K_a^0}\pi^t}\pi^0 \\
&+ \sum_{a\in N_i^{(i)}\cup N_i'^{(i)}} \frac{\pi^{-i}}{\sum_{j\in M_a}\pi^{-j}}\pi^{(i)} + |N^{-i}|\pi^{-i} + |N'^{-i}|\pi^{-i} \\
&= \sum_{a\in N_i^0} \frac{\pi^{-i}}{\sum_{j\in M_a\cap P^{(i)}}\pi^{-j}}\frac{\pi^{(i)}}{\sum_{t\in K_a^0}\pi^t}\pi^0 \\
&+ \sum_{a\in N_i^{(i)}} \frac{\pi^{-i}}{\sum_{j\in M_a}\pi^{-j}}\pi^{(i)} + |N^{-i}|\pi^{-i} \\
&+ \sum_{a\in N_i'^0} \frac{\pi^{-i}}{\sum_{j\in M_a\cap P^{(i)}}\pi^{-j}}\frac{\pi^{(i)}}{\sum_{t\in K_a^0}\pi^t}\pi^0 \\
&+ \sum_{a\in N_i'^{(i)}} \frac{\pi^{-i}}{\sum_{j\in M_a}\pi^{-j}}\pi^{(i)} + |N'^{-i}|\pi^{-i} \\
&= R_i^{PP}(M,P,N,\pi,C) + R_i^{PP}(M,P,N',\pi,C').
\end{aligned}$$

- Dummy. Let $D \in \mathcal{D}$ and $i \in \mathrm{NM}(D)$. Then

$$R_i^{PP}(D) = \sum_{a \in N_i^0} \frac{\pi^{-i}}{\sum_{j \in M_a \cap P^{(i)}} \pi^{-j}} \frac{\pi^{(i)}}{\sum_{t \in K_a^0} \pi^t} \pi^0 + \sum_{a \in N_i^{(i)}} \frac{\pi^{-i}}{\sum_{j \in M_a} \pi^{-j}} \pi^{(i)} + |N^{-i}| \pi^{-i} = 0.$$

- Consortia consistency. Let $D \in \mathcal{D}^0$ and $P_k \in P$,

$$\sum_{i \in P_k} R_i^{PP}(D) = \sum_{i \in P_k} \sum_{a \in N_i^0} \frac{\pi^{-i}}{\sum_{j \in M_a \cap P^{(i)}} \pi^{-j}} \frac{\pi^k}{\sum_{t \in K_a^0} \pi^t} \pi^0 = \sum_{a \in N^0 : k \in K_a^0} \frac{\pi^k}{\sum_{t \in K_a^0} \pi^t} \pi^0$$
$$= R_k^{PP} \left(\{1, ..., s\}, \hat{P}^s, \hat{N}, \hat{\pi}, \hat{C} \right).$$

- Splitting-proofness of museums. Let $(M, P, N, \pi, C) \in \mathcal{D}^0$ and $i \in M$. Consider $(M', P', N', \pi', C') \in \mathcal{D}$ as it is described in the definition of the property.[12] Then, for each $j \in M \backslash \{i\}$, if $j \notin P^{(i)}$,

$$R_j^{PP}(D) = \sum_{a \in N_j^0} \frac{\pi^{-j}}{\sum_{l \in M_a \cap P^{(j)}} \pi^{-l}} \frac{\pi^{(j)}}{\sum_{t \in K_a^0} \pi^t} \pi^0 + \sum_{a \in N_j^{(j)}} \frac{\pi^{-j}}{\sum_{l \in M_a} \pi^{-l}} \pi^{(j)} + |N^{-j}| \pi^{-j}$$
$$= R_j^{PP}(M', P', N', \pi', C').$$

If $j \in P^{(i)}$,

$$R_j^{PP}(D) = \sum_{a \in N_j^0} \frac{\pi^{-j}}{\sum_{l \in M_a \cap P^{(j)}} \pi^{-l}} \frac{\pi^{(j)}}{\sum_{t \in K_a^0} \pi^t} \pi^0 + \sum_{a \in N_j^{(j)}} \frac{\pi^{-j}}{\sum_{l \in M_a} \pi^{-l}} \pi^{(j)} + |N^{-j}| \pi^{-j}$$
$$= \sum_{a \in N_j^0} \frac{\pi^{-j}}{\sum_{l \in M'_a \cap P'^{(j)}} \pi^{-l}} \frac{\pi^{(j)}}{\sum_{t \in K_a^0} \pi^t} \pi^0 + \sum_{a \in N_j^{(j)}} \frac{\pi^{-j}}{\sum_{l \in M'_a} \pi^{-l}} \pi^{(j)} + |N^{-j}| \pi^{-j}$$
$$= R_j^{PP}(M', P', N', \pi', C')$$

where $P'^{(j)} = \left(P^{(i)} \backslash \{i\}\right) \cup \{i_1, \ldots, i_r\}$ and for each $a \in N_j^\sigma$ with $\sigma \in \{0, (j)\}$, $M'_a = \{l \in M' : C'^{\sigma}_{la} = 1\}$. Therefore, if $i \in M_a$ then $i_h \in M'_a$ for all $h \in \{1, \ldots, r\}$. Since $\pi^{-i} = \sum_{h=1}^r \pi^{-i_h}$ we have $\sum_{l \in M_a} \pi^{-l} = \sum_{l \in M'_a} \pi^{-l}$.

- Splitting-proofness of consortia. Let $(M, P, N, \pi, C) \in \mathcal{D}$, and consider $(M', P', N', \pi', C') \in \mathcal{D}$ as it is set in the definition of the property. Then, for each $P_r \in P \backslash \{P_k\}$

[12] For sake of exposition and brevity, we do not replicate here the tedious definitions of the elements of (M', P', N', π', C'), which are already described in the statement of the axiom.

$$\sum_{j\in P_r} R_j^{PP}(M,P,N,\pi,C) =$$

$$\sum_{j\in P_r}\left(\sum_{a\in N_j^0}\frac{\pi^{-j}}{\sum_{l\in M_a\cap P^{(j)}}\pi^{-l}}\frac{\pi^{(j)}}{\sum_{t\in K_a^0}\pi^t}\pi^0+\sum_{a\in N_j^{(j)}}\frac{\pi^{-j}}{\sum_{l\in M_a}\pi^{-l}}\pi^{(j)}+|N^{-j}|\pi^{-j}\right)=$$

$$\sum_{a\in N^0:r\in K_a^0}\frac{\pi^{(r)}}{\sum_{t\in K_a^0}\pi^t}\pi^0+|N^{(r)}|\pi^{(r)}+\sum_{j\in P_r}|N^{-j}|\pi^{-j}=$$

$$\sum_{a\in N^0:r\in K_a'^0}\frac{\pi^{(r)}}{\sum_{t\in K_a^0}\pi^t}\pi^0+|N^{(r)}|\pi^{(r)}+\sum_{j\in P_r}|N^{-j}|\pi^{-j}=$$

$$\sum_{j\in P_r} R_j^{PP}(M',P',N',\pi',C').$$

where if $k\in K_a^0$ then $k^h\in K_a^0$ for all $h\in\{1,\dots,t\}$. Since $\pi^k=\sum_{h=1}^t\pi^{k^h}$ we have $\sum_{t\in K_a^0}\pi^t=\sum_{t\in K_a'^0}\pi^t$. By other way, if $k\notin K_a^0$ then $K_a^0=K_a'^0$.Now, we prove the converse. Let R be a rule that satisfies the properties in the statement, and let $(M,P,N,\pi,C)\in\mathcal{D}$. We divide the proof into several steps. (i) Let $i\in M$, and let $(M,P,N,\pi,C)\in\mathcal{D}^{-i}$. By Lemma 2, $R_i(M,P,N,\pi,C)=\pi^{-i}|N^{-i}|=R_i^{PP}(M,P,N,\pi,C)$. Thus, R and R^{PP} coincide in any subclass of problems $\mathcal{D}^{-i}$.

(ii) Let $D^0=(M,P,\{a\},\pi,C^{(a)})\in\mathcal{D}^0$ be a problem with just one pass holder purchasing the general pass, and let $D_q^0=(\{1,\dots,s\},\hat{P}^s,\{a\},\hat{\pi},\hat{C}^{(a)})$ be the associated problem as it is indicated in the definition of consortia consistency. As R satisfies *dummy*, for all $k\in\{1,\dots,s\}$ such that $\hat{C}_{ka}^{(a)}=0$, we have that $R_k\left(D_q^0\right)=0$. Let us consider K_a^0 the set of consortia visited by the pass holder a. By definition of rule, there must exist at least one consortium $k\in K_a^0$ such that $R_k\left(D_q^0\right)=b>0$. Then, we split the consortium k into h new consortia such that $\pi^{k_i}=\pi^{-k_i}=\frac{\pi^k}{h}$ for each $i\in\{1,\dots,h\}$. Let $D_q'^0$ be the problem where the consortium k splits into h new consortia. Lemma 4 applied to $D_q'^0$ implies that $R_{k_i}\left(D_q'^0\right)=R_{k_j}\left(D_q'^0\right)$ for all $i,j\in\{1,\dots,h\}$. By *splitting-proofness of consortia* $R_k\left(D_q^0\right)=\sum_{i=1}^h R_{k_i}\left(D_q'^0\right)=hR_{k_1}\left(D_q'^0\right)$. Therefore, $R_{k_1}\left(D_q'^0\right)=\frac{b}{h}$. Now let us consider another consortia $r\in K_a^0$. We can split the consortium r into $h_r=\left\lfloor\frac{\pi^r}{\frac{\pi^k}{h}}\right\rfloor+1$ consortia[13] such that $\pi^{r_i}=\pi^{-r_i}=\frac{\pi^k}{h}$ for each $i\in\{1,\dots,h_r-1\}$, and $\pi^{r_{h_r}}=\pi^{-r_{h_r}}=\pi^r-\sum_{i=1}^{h_r-1}\pi^{r_i}$. Let $D_q''^0$ be the problem where the consortium r splits into h_r new consortia.

[13] As h can be as arbitrarily large as we need, we can always assume that $\left\lfloor\frac{\pi^r}{\frac{\pi^k}{h}}\right\rfloor\geq 1$.

Lemma 4 applied to $D_q''^0$ implies that $R_{r_i}\left(D_q''^0\right) = R_{k_1}\left(D_q''^0\right)$ for each $i \in \{1, \ldots, h_r - 1\}$ and by *splitting-proofness of consortia*

$$R_{k_1}\left(D_q''^0\right) = R_{k_1}\left(D_q'^0\right) = \frac{b}{h}.$$

Therefore,

$$R_r\left(D_q^0\right) = R_r\left(D_q'^0\right) = \sum_{i=1}^{h_r - 1} R_{r_i}\left(D_q''^0\right) + R_{r_{h_r}}\left(D_q''^0\right) = (h_r - 1)\frac{b}{h} + R_{r_{h_r}}\left(D_q''^0\right).$$

Now, by Lemma 5 we have that $R_{r_{h_r}}\left(D_q''^0\right) \leq R_{r_1}\left(D_q''^0\right) = \frac{b}{h}$. Finally,

$$\frac{R_k\left(D_q^0\right)}{R_r\left(D_q^0\right)} = \frac{b}{(h_r - 1)\frac{b}{h} + R_{r_{h_r}}\left(D_q''^0\right)} \leq \frac{b}{\left\lfloor \frac{\pi^r}{\frac{\pi^k}{h}} \right\rfloor \frac{b}{h}},$$

and, by other hand,

$$\frac{R_k\left(D_q^0\right)}{R_r\left(D_q^0\right)} = \frac{b}{(h_r - 1)\frac{b}{h} + R_{r_{h_r}}\left(D_q''^0\right)} \geq \frac{b}{\left\lfloor \frac{\pi^r}{\frac{\pi^k}{h}} \right\rfloor \frac{b}{h} + \frac{b}{h}}.$$

Therefore,

$$\frac{1}{\left\lfloor \frac{\pi^r}{\frac{\pi^k}{h}} \right\rfloor \frac{1}{h} + \frac{1}{h}} \leq \frac{R_k\left(D_q^0\right)}{R_r\left(D_q^0\right)} \leq \frac{1}{\left\lfloor \frac{\pi^r}{\frac{\pi^k}{h}} \right\rfloor \frac{1}{h}}.$$

Given that $\lim\limits_{h \to +\infty} \left\lfloor \frac{\pi^r}{\frac{\pi^k}{h}} \right\rfloor \frac{1}{h} = \frac{\pi^r}{\pi^k}$ then $\frac{R_k\left(D_q^0\right)}{R_r\left(D_q^0\right)} = \frac{\pi^k}{\pi^r}$. Now, this can be done for any pair of consortia in K_a^0, therefore we have that for every $t \in \{1, \ldots, s\}$

$$R_t\left(D_q^0\right) = \begin{cases} 0 & \text{if } t \text{ is dummy in } D_q^0 \\ \dfrac{\pi^t}{\sum_{r \in K_a^0} \pi^r} \pi^0 & \text{otherwise} \end{cases}$$

Now, consider again the museum pass problem D^0. For each partition $P_t \in P$, *consortia consistency* requires that

$$\sum_{i \in P_t} R_i\left(D^0\right) = R_t\left(D_q^0\right). \tag{1}$$

Let us now consider a consortium $P_t \in P$ such that $M_a \cap P_t \neq \varnothing$ and, therefore, $R_t\left(D_q^0\right) > 0$. Since R satisfies *dummy* for all $i \in P_t$ such that $C_{ia}^{(a)} = 0$ then $R_i\left(D^0\right) = 0$. Let us consider $M_a \cap P_t$ the set of museums in the consortium P_t visited by the pass holder a. By *consortium consistency*, it is glaringly obvious that there must exist at least a museum $i \in M_a \cap P_t$ such that $R_i\left(D^0\right) = c > 0$. Then, we split the museum i into h new museums such that $\pi^{-i_l} = \frac{\pi^{-i}}{h}$ for each $l \in \{1, \ldots, h\}$. Let D'^0 be the problem where the museum i splits into h new museums. Lemma 3 applied to D'^0 implies that $R_{i_l}\left(D'^0\right) = R_{i_r}\left(D'^0\right)$ for each pair $l, r \in \{1, \ldots, h\}$. By *splitting-proofness of museums*, $R_i\left(D^0\right) = \sum_{l=1}^{h} R_{i_l}\left(D'^0\right) = hR_{i_1}\left(D'^0\right)$. Therefore, $R_{i_1}\left(D'^0\right) = \frac{c}{h}$. Now let us consider another museum $j \in M_a \cap P_t$. We can split the museum j into $h_j = \left\lfloor \frac{\pi^{-j}}{\frac{\pi^{-i}}{h}} \right\rfloor + 1$ museums[14] such that for each $l \in \{1, \ldots, h_j - 1\}$, $\pi^{-j_l} = \frac{\pi^{-i}}{h}$ and $\pi^{-j_{h_j}} = \pi^{-j} - \sum_{l=1}^{h_j-1} \pi^{-j_l}$. Let D''^0 be the problem where the museum j splits into h_j new museums. Lemma 3 applied to D''^0 implies that $R_{j_l}\left(D''^0\right) = R_{i_1}\left(D''^0\right)$ for each $l \in \{1, \ldots, h_j - 1\}$ and by *splitting-proofness of museums*

$$R_{i_1}\left(D''^0\right) = R_{i_1}\left(D'^0\right) = \frac{c}{h}$$

and

$$R_j\left(D^0\right) = R_j\left(D'^0\right) = \sum_{l=1}^{h_j-1} R_{j_l}\left(D''^0\right) + R_{j_{h_j}}\left(D''^0\right) = \left(h_j - 1\right)\frac{c}{h} + R_{j_{h_j}}\left(D''^0\right).$$

On one hand, by Lemma 6, we have that $R_{j_{h_j}}\left(D''^0\right) \leq R_{j_1}\left(D''^0\right) = \frac{c}{h}$. Finally,

$$\frac{R_i\left(D^0\right)}{R_j\left(D^0\right)} = \frac{c}{\left(h_j - 1\right)\frac{c}{h} + R_{j_{h_j}}\left(D''^0\right)} \leq \frac{c}{\left(h_j - 1\right)\frac{c}{h}}$$

On the other hand,

$$\frac{R_i\left(D^0\right)}{R_j\left(D^0\right)} = \frac{c}{\left(h_j - 1\right)\frac{c}{h} + R_{j_{h_j}}\left(D''^0\right)} \geq \frac{c}{\left(h_j - 1\right)\frac{c}{h} + \frac{c}{h}}.$$

Therefore,

[14] As h can be as arbitrarily large as we need, we can always assume that $\left\lfloor \frac{\pi^{-j}}{\frac{\pi^{-i}}{h}} \right\rfloor \geq 1$.

$$\frac{1}{\left\lfloor \frac{\pi^{-j}}{\frac{\pi^{-i}}{h}} \right\rfloor \frac{1}{h} + \frac{1}{h}} \leq \frac{R_i\left(D^0\right)}{R_j\left(D^0\right)} \leq \frac{1}{\left\lfloor \frac{\pi^{-j}}{\frac{\pi^{-i}}{h}} \right\rfloor \frac{1}{h}}.$$

Given that $\lim_{h\to+\infty} \left\lfloor \frac{\pi^{-j}}{\frac{\pi^{-i}}{h}} \right\rfloor \frac{1}{h} = \frac{\pi^{-j}}{\pi^{-i}}$ then $\frac{R_i\left(D^0\right)}{R_j\left(D^0\right)} = \frac{\pi^{-i}}{\pi^{-j}}$. Now, since this can be done for any pair of museums in P_t, together with Eq. 1, we have for each $i \in P_t$

$$R_i\left(D^{0s}\right) = \begin{cases} 0 & \text{if } i \text{ is dummy in } D^0 \\ \frac{\pi^{-i}}{\sum_{j\in M_a\cap P_t} \pi^{-j}} \frac{\pi^t}{\sum_{r\in K_a^0} \pi^r} & \text{otherwise} \end{cases}$$

Therefore, $R\left(D^0\right)$ and $R^{PP}\left(D^0\right)$ coincide. Now, let $(M, P, N, \pi, C) \in \mathcal{D}^0$ without any restriction on the cardinality of N^0. By *composition*, it follows that, for each $i \in M$,

$$\begin{aligned} R_i\left(M, P, N, \pi, C\right) &= \sum_{a\in N^0} R_i\left(M, P, \{a\}, \pi, C^{(a)}\right) \\ &= \sum_{a\in N^0} R_i^{PP}\left(M, P, \{a\}, \pi, C^{(a)}\right) \\ &= R_i^{PP}\left(M, P, N, \pi, C\right) \end{aligned}$$

Therefore, R and R^{PP} coincide in $\mathcal{D}^0$.

(iii) Let $P_t \in P$, and let $D^t = (M, P, \{a\}, \pi, C^{(a)}) \in \mathcal{D}^t$ be a problem with just one of those pass holders. Let $i \in P_t$. If $i \in \mathrm{NM}\left(D^t\right)$, *dummy* implies that $R_i(D^t) = 0 = R_i^{PP}(D^t)$. Now, following similar arguments to those used in (ii) above, we have that

$$R_i\left(D^t\right) = \frac{\pi^{-i}}{\sum_{j\in M_a} \pi^{-j}}\pi^t = R_i^{PP}(D^t)$$

As in case (ii), by *composition* we extend the coincidence between R and R^{PP} to the whole subclass D^t.In application of Lemma 1, R and R^{PP} coincide in the whole domain $\mathcal{D}$. □

Remark 4 The axioms of Theorem 2 are independent.

(a) Let R^3 be defined as follows. For each $i \in M$

$$R_i^3(D) = \sum_{a\in N_i^0} \frac{\pi^{-i}}{\sum_{j\in M_a\cap P^{(i)}} \pi^{-j}} \frac{\pi^{(i)}}{\sum_{t\in K_a^0} \pi^t}\pi^0 + \frac{\pi^{-i}\left\{1_{\exists a\in N^{(i)}:C_{ia}=1}\right\}}{\sum_{j\in P^{(i)}} \pi^{-j}\left\{1_{\exists a\in N^{(i)}:C_{ja}=1}\right\}}|N^{(i)}|\pi^{(i)} + |N^{-i}|\pi^{-i}$$

The rule R^3 satisfies dummy, splitting-proofness of museums, splitting-proofness of consortia, consortia consistency, but not composition.

(b) The rule R^{EP} satisfies composition, dummy, splitting-proofness of museums, consortia consistency, but not splitting-proofness of consortia.
(c) The rule R^{PE} satisfies composition, dummy, splitting-proofness of consortia, consortia consistency, but not splitting-proofness of museums.
(d) Let R^4 be defined as follows. For each $i \in M$

$$R_i^4(D) = \sum_{a \in N_i^0} \frac{\pi^{-i}}{\sum_{j \in M_a \cap P^{(i)}} \pi^{-j}} \frac{\pi^{(i)}}{\sum_{t \in K_a^0} \pi^t} \pi^0 + \sum_{a \in N^{(i)}} \frac{\pi^{-i}}{\sum_{j \in P^{(i)}} \pi^{-j}} \pi^{(i)} + |N^{-i}| \pi^{-i}$$

The rule R^4 satisfies composition, splitting-proofness of museums, splitting-proofness of consortia, consortia consistency, but not dummy.

(e) Let R^5 be defined as follows. For all $i \in M$

$$R_i^5(D) = \sum_{a \in N_i^0} \frac{\pi^{-i}}{\sum_{j \in M_a} \pi^{-j}} \pi^0 + \sum_{a \in N_i^{(i)}} \frac{\pi^{-i}}{\sum_{j \in M_a} \pi^{-j}} \pi^{(i)} + |N^{-i}| \pi^{-i}$$

The rule R^5 satisfies composition, splitting-proofness of museums, splitting-proofness of consortia, dummy, but not consortia consistency.

Proof of Theorem 3

We start by showing that the *proportional-egalitarian rule* satisfies the axioms in the statement.

- Composition. Let $(M, P, N, \pi, C), (M, P, N', \pi, C') \in \mathcal{D}$ such that $N \cap N' = \emptyset$. Let $i \in M$. It follows that

$$\begin{aligned}
R_i^{PE}(M, P, N \cup N', \pi, C \cup C') &= \sum_{a \in N_i^0 \cup N_i'^0} \frac{1}{|M_a \cap P^{(i)}|} \frac{\pi^{(i)}}{\sum_{t \in K_a^0} \pi^t} \pi^0 \\
&\quad + \sum_{a \in N_i^{(i)} \cup N_i'^{(i)}} \frac{1}{|M_a|} \pi^{(i)} + |N^{-i}| \pi^{-i} + |N'^{-i}| \pi^{-i} \\
&= \sum_{a \in N_i^0} \frac{1}{|M_a \cap P^{(i)}|} \frac{\pi^{(i)}}{\sum_{t \in K_a^0} \pi^t} \pi^0 + \sum_{a \in N_i^{(i)}} \frac{1}{|M_a|} \pi^{(i)} + |N^{-i}| \pi^{-i} \\
&\quad + \sum_{a \in N_i'^0} \frac{1}{|M_a \cap P^{(i)}|} \frac{\pi^{(i)}}{\sum_{t \in K_a^0} \pi^t} \pi^0 + \sum_{a \in N_i'^{(i)}} \frac{1}{|M_a|} \pi^{(i)} + |N'^{-i}| \pi^{-i} \\
&= R_i^{PE}(M, P, N, \pi, C) + R_i^{PE}(M, P, N', \pi, C').
\end{aligned}$$

- Dummy. Let $D \in \mathcal{D}$ and $i \in$ NM (D). Then

$$R_i^{PE}(D) = \sum_{a \in N_i^0} \frac{1}{|M_a \cap P^{(i)}|} \frac{\pi^{(i)}}{\sum_{t \in K_a^0} \pi^t} \pi^0 + \sum_{a \in N_i^{(i)}} \frac{1}{|M_a|} \pi^{(i)} + |N^{-i}| \pi^{-i} = 0.$$

- Consortia consistency. Let $D \in \mathcal{D}^0$ and $P_k \in P$,

$$\sum_{i \in P_k} R_i^{PE}(D) = \sum_{i \in P_k k} \sum_{a \in N_i^0} \frac{1}{|M_a \cap P^{(i)}|} \frac{\pi^{(i)}}{\sum_{t \in K_a^0} \pi^t} \pi^0 = \sum_{a \in N^0 : k \in K_a^0} \frac{\pi^k}{\sum_{t \in K_a^0} \pi^t} \pi^0$$
$$= R_k^{PE} \left(\{1, ..., s\}, \hat{P}^s, \hat{N}, \hat{\pi}, \hat{C} \right).$$

- Symmetry within consortia. Let $D \in \mathcal{D}$ and let $i, j \in M$ such that $P^{(i)} = P^{(j)}$ and satisfy the conditions in the definition of the property. Then

$$\begin{aligned} R_i^{PE}(D) &= \sum_{a \in N_i^0} \frac{1}{|M_a \cap P^{(i)}|} \frac{\pi^{(i)}}{\sum_{t \in K_a^0} \pi^t} \pi^0 + \sum_{a \in N_i^{(i)}} \frac{1}{|M_a|} \pi^{(i)} + |N^{-i}| \pi^{-i} \\ &= \sum_{a \in N_j^0} \frac{1}{|M_a \cap P^{(j)}|} \frac{\pi^{(j)}}{\sum_{t \in K_a^0} \pi^t} \pi^0 + \sum_{a \in N_j^{(j)}} \frac{1}{|M_a|} \pi^{(j)} + |N^{-j}| \pi^{-j} \\ &= R_j^{PE}(D). \end{aligned}$$

- Splitting-proofness of consortia. Let $(M, P, N, \pi, C) \in \mathcal{D}$ and $P_k \in P$ where $P_k = \{i_1, \ldots, i_r\}$, and consider $(M', P', N', \pi', C') \in \mathcal{D}$ as it is set in the definition of the property. Then, for each $P_r \in P \backslash \{P_k\}$

$$\sum_{j \in P_r} R_j^{PE}(M, P, N, \pi, C) =$$
$$\sum_{j \in P_r} \left(\sum_{a \in N_j^0} \frac{1}{|M_a \cap P^{(j)}|} \frac{\pi^{(j)}}{\sum_{t \in K_a^0} \pi^t} \pi^0 + \sum_{a \in N_j^{(j)}} \frac{1}{|M_a|} \pi^{(j)} + |N^{-j}| \pi^{-j} \right) =$$
$$\sum_{a \in N^0 : k \in K_a^0} \frac{\pi^{(k)}}{\sum_{t \in K_a^0} \pi^t} \pi^0 + |N^{(r)}| \pi^{(r)} + \sum_{j \in P_r} |N^{-j}| \pi^{-j} =$$
$$\sum_{a \in N^0 : k \in K_a'^0} \frac{\pi^{(k)}}{\sum_{t \in K_a^0} \pi^t} \pi^0 + |N^{(r)}| \pi^{(r)} + \sum_{j \in P_r} |N^{-j}| \pi^{-j} =$$
$$\sum_{j \in P_r} R_j^{PE}(M', P', N', \pi', C').$$

where if $k \in K_a^0$ then $k^h \in K_a^0$ for all $h \in \{1, \ldots, t\}$. Since $\pi^k = \sum_{h=1}^t \pi^{k^h}$ we have $\sum_{t \in K_a^0} \pi^t = \sum_{t \in K_a'^0} \pi^t$. By other way, if $k \notin K_a^0$ then $K_a^0 = K_a'^0$.Now, we prove the converse. Let *R* be a rule that satisfies the properties in the statement, and let $(M, P, N, \pi, C) \in \mathcal{D}$. We divide the proof into several steps.

(i) Let $i \in M$, and let $(M, P, N, \pi, C) \in \mathcal{D}^{-i}$. By Lemma 2, $R_i(M, P, N, \pi, C) = \pi^{-i}|N^{-i}| = R_i^{PE}(M, P, N, \pi, C)$. Thus, R and R^{EP} coincide in any subclass of problems $\mathcal{D}^{-i}$.
(ii) Let $D^0 = (M, P, \{a\}, \pi, C^{(a)}) \in \mathcal{D}^0$ be a problem with just one pass holder purchasing the general pass, and let $D^{0s} = (\{1, \ldots, s\}, \hat{P}^s, \{a\}, \hat{\pi}, \hat{C}^{(a)})$ its associated quotient problem. Following arguments similar to those used in Case (ii) of the proof of Theorem 2, together with Lemma 4 and Lemma 5, we have

$$R_t\left(D^{0s}\right) = R_t^{PE}\left(D^{0s}\right) = \begin{cases} 0 & \text{if } t \text{ is dummy in } D^{0s} \\ \dfrac{\pi^t}{\sum_{r \in K_a^{0s}} \pi^r} \pi^0 & \text{otherwise} \end{cases}$$

Now, consider again the museum pass problem D^0. For each partition $P_t \in P$, the *consortia consistency* requires that

$$\sum_{i \in P_t} R_i\left(D^0\right) = R_t\left(D^{0s}\right) \tag{2}$$

Within consortium *t*, any museum $i \in P_t$ is either dummy or symmetric to any other non-dummy museum. *Dummy* and *symmetry within consortia* together imply that all dummy museums obtain zero, while the others receive equal shares. That is, if $\alpha \in \mathbb{R}_+$ denotes that equal share, it must hold that

$$\sum_{i \in P_t} R_i\left(D^0\right) = \alpha |M_a \cap P_t|$$

If we combine the previous expression with Eq. 2 we obtain that

$$\alpha = \frac{1}{|M_a \cap P_t|} \cdot R_t\left(D^{0s}\right)$$

Therefore, if $i \in M$ is dummy, then $R_i(D^0) = 0 = R^{PE}(D^0)$. Otherwise,

$$R_i(D^0) = \frac{1}{|M_a \cap P^{(i)}|} \cdot \frac{\pi^{(i)}}{\sum_{r \in K_a^{0s}} \pi^r} \pi^0 = R_i^{PE}(D^0).$$

Now, let $(M, P, N, \pi, C) \in \mathcal{D}^0$ without any restriction on the cardinality of N^0. By *composition*, it follows that, for each $i \in M$,

$$
\begin{aligned}
R_i\left(M, P, N, \pi, C\right) &= \sum_{a \in N^0} R_i\left(M, P, \{a\}, \pi, C^{(a)}\right) \\
&= \sum_{a \in N^0} R_i^{PP}\left(M, P, \{a\}, \pi, C^{(a)}\right) \\
&= R_i^{PE}\left(M, P, N, \pi, C\right)
\end{aligned}
$$

Therefore, R and R^{PE} coincide in $\mathcal{D}^0$.

(iii) Let $P_t \in P$, and let $(M, P, N, \pi, C) \in \mathcal{D}^t$. By an argument analogous to Case (iii) in Theorem 1, we can obtain that, for each $i \in P_t$,

$$
R_i(M, P, N, \pi, C) = \sum_{a \in N_i^t} \frac{1}{|M_a|}\pi^t = R_i^{PE}(M, P, N, \pi, C)
$$

In application of Lemma 1, R and R^{PE} coincide in the whole domain $\mathcal{D}$. □

Remark 5 The axioms of Theorem 3 are independent.

(a) Let R^6 be defined as follows. For each $i \in M$

$$
R_i^6(D) = \sum_{a \in N_i^0} \frac{1}{|M_a \cap P^{(i)}|} \frac{\pi^{(i)}}{\sum_{t \in K_a^0} \pi^t}\pi^0 + \frac{\sum_{a \in N^{(i)}} C_{ia}}{\sum_{j \in P^{(i)}} \sum_{a \in N^{(i)}} C_{ja}}|N^{(i)}|\pi^{(i)} + |N^{-i}|\pi^{-i}
$$

The rule R^6 satisfies dummy, symmetry within consortia, splitting-proofness of consortia and consortia consistency, but not composition.

(b) The rule R^{EE} satisfies composition, dummy, symmetry within consortia, consortia consistency, but not splitting-proofness of consortia.
(c) The rule R^{PP} satisfies composition, dummy, splitting-proofness of consortia, consortia consistency, but not symmetry within consortia.
(d) Let R^7 be defined as follows. For each $i \in M$

$$
R_i^7(D) = \sum_{a \in N_i^0} \frac{1}{|M_a \cap P^{(i)}|} \frac{\pi^{(i)}}{\sum_{t \in K_a^0} \pi^t}\pi^0 + \sum_{a \in N^{(i)}} \frac{1}{|P^{(i)}|}\pi^{(i)} + |N^{-i}|\pi^{-i}
$$

The rule R^7 satisfies composition, symmetry within consortia, splitting-proofness of consortia, consortia consistency, but not dummy.

(e) Let R^8 be defined as follows. For each $i \in M$

$$
R_i^8(D) = \sum_{a \in N_i^0} \frac{1}{|M_a \cap P^{(i)}|} \frac{\sum_{j \in P^{(i)}} \pi^{-j}}{\sum_{t \in K_a^0} \sum_{j \in P^t} \pi^{-j}}\pi^0 + \sum_{a \in N_i^{(i)}} \frac{\pi^{-i}}{\sum_{j \in M_a} \pi^{-j}}\pi^{(i)} + |N^{-i}|\pi^{-i}
$$

The rule R^8 satisfies satisfies composition, symmetry within consortia, splitting-proofness of consortia, dummy, but not consortia consistency.

Proof of Theorem 4

We start by showing that the *egalitarian-proportional rule* satisfies the axioms in the statement.

- Composition. Let $(M, P, N, \pi, C), (M, P, N', \pi, C') \in \mathcal{D}$ such that $N \cap N' = \emptyset$. Let $i \in M$. It follows that

$$\begin{aligned}
R_i^{EP}(M, P, N \cup N', \pi, C \cup C') &= \sum_{a \in N_i^0 \cup N_i'^0} \frac{\pi^{-i}}{\sum_{j \in M_a \cap P^{(i)}} \pi^{-j}} \frac{1}{|K_a^0|} \pi^0 \\
&+ \sum_{a \in N_i^{(i)} \cup N_i'^{(i)}} \frac{\pi^{-i}}{\sum_{j \in M_a} \pi^{-j}} \pi^{(i)} + |N^{-i}|\pi^{-i} + |N'^{-i}|\pi^{-i} \\
&= \sum_{a \in N_i^0} \frac{\pi^{-i}}{\sum_{j \in M_a \cap P^{(i)}} \pi^{-j}} \frac{1}{|K_a^0|} \pi^0 + \sum_{a \in N_i^{(i)}} \frac{\pi^{-i}}{\sum_{j \in M_a} \pi^{-j}} \pi^{(i)} + |N^{-i}|\pi^{-i} \\
&+ \sum_{a \in N_i'^0} \frac{\pi^{-i}}{\sum_{j \in M_a \cap P^{(i)}} \pi^{-j}} \frac{1}{|K_a^0|} \pi^0 + \sum_{a \in N_i'^{(i)}} \frac{\pi^{-i}}{\sum_{j \in M_a} \pi^{-j}} \pi^{(i)} + |N'^{-i}|\pi^{-i} \\
&= R_i^{EP}(M, P, N, \pi, C) + R_i^{EP}(M, P, N', \pi, C').
\end{aligned}$$

- Dummy. Let $D \in \mathcal{D}$ and $i \in \mathrm{NM}(D)$. Then

$$R_i^{EP}(D) = \sum_{a \in N_i^0} \frac{\pi^{-i}}{\sum_{j \in M_a \cap P^{(i)}} \pi^{-j}} \frac{1}{|K_a^0|} \pi^0 + \sum_{a \in N_i^{(i)}} \frac{\pi^{-i}}{\sum_{j \in M_a} \pi^{-j}} \pi^{(i)} + |N^{-i}|\pi^{-i} = 0.$$

- Symmetry between consortia. Let $D \in \mathcal{D}$ and let $P_r, P_t \in P$ that satisfy the conditions in the definition of the property. Then

$$\begin{aligned}
\sum_{i \in P_r} R_i^{EP}(D) &= \sum_{i \in P_r} \sum_{a \in N_i^0} \frac{\pi^{-i}}{\sum_{j \in M_a \cap P^{(i)}} \pi^{-j}} \frac{1}{|K_a^0|} \pi^0 + \sum_{i \in P_r} \sum_{a \in N_i^{(i)}} \frac{\pi^{-i}}{\sum_{j \in M_a} \pi^{-j}} \pi^{(i)} + \sum_{i \in P_r} |N^{-i}|\pi^{-i} \\
&= \sum_{a \in N^0 : r \in K_a^0} \frac{1}{|K_a^0|} \pi^0 + \pi^r |N^r| + \sum_{j \in P_t} |N^{-j}|\pi^{-j} \\
&= \sum_{a \in N^0 : t \in K_a^0} \frac{1}{|K_a^0|} \pi^0 + \pi^t |N^t| + \sum_{j \in P_t} |N^{-j}|\pi^{-j} \\
&= \sum_{i \in P_t} \sum_{a \in N_i^0} \frac{\pi^{-i}}{\sum_{j \in M_a \cap P^{(i)}} \pi^{-j}} \frac{1}{|K_a^0|} \pi^0 + \sum_{i \in P_t} \sum_{a \in N_i^{(i)}} \frac{\pi^{-i}}{\sum_{j \in M_a} \pi^{-j}} \pi^{(i)} + \sum_{i \in P_t} |N^{-i}|\pi^{-i} \\
&= \sum_{j \in P_t} R_j^{EP}(M, P, N, \pi, C).
\end{aligned}$$

- Splitting-proofness of museums. Let $D \in \mathcal{D}^0$ and $i \in M$. Consider $(M', P', N', \pi', C') \in \mathcal{D}$ as it is described in the definition of the property. Then, for each $j \in M \backslash \{i\}$, if $j \notin P^{(i)}$,

$$
\begin{aligned}
R_j^{EP}(D) &= \sum_{a\in N_j^0} \frac{\pi^{-j}}{\sum_{l\in M_a\cap P^{(j)}} \pi^{-l}} \frac{1}{|K_a^0|}\pi^0 + \sum_{a\in N_j^{(j)}} \frac{\pi^{-j}}{\sum_{l\in M_a} \pi^{-l}}\pi^{(j)} + |N^{-j}|\pi^{-j} \\
&= R_j^{EP}(M', P', N', \pi', C').
\end{aligned}
$$

If $j \in P^{(i)}$,

$$
\begin{aligned}
R_j^{EP}(D) &= \sum_{a\in N_j^0} \frac{\pi^{-j}}{\sum_{l\in M_a\cap P^{(j)}} \pi^{-l}} \frac{1}{|K_a^0|}\pi^0 + \sum_{a\in N_j^{(j)}} \frac{\pi^{-j}}{\sum_{l\in M_a} \pi^{-l}}\pi^{(j)} + |N^{-j}|\pi^{-j} \\
&= \sum_{a\in N_j^0} \frac{\pi^{-j}}{\sum_{l\in M_a'\cap P'^{(j)}} \pi^{-l}} \frac{1}{|K_a^0|}\pi^0 + \sum_{a\in N_j^{(j)}} \frac{\pi^{-j}}{\sum_{l\in M_a'} \pi^{-l}}\pi^{(j)} + |N^{-j}|\pi^{-j} \\
&= R_j^{EP}(M', P', N', \pi', C')
\end{aligned}
$$

where $P'^{(j)} = \left(P^{(i)}\backslash\{i\}\right) \cup \{i_1, \ldots, i_r\}$ and for each $a \in N_j^\sigma$ with $\sigma \in \{0, (j)\}$, $M_a' = \{l \in M' : C_{la}'^{\sigma} = 1\}$. Therefore, if $i \in M_a$ then $i_h \in M_a'$ for all $h \in \{1, \ldots, r\}$. Since $\pi^{-i} = \sum_{h=1}^r \pi^{-i_h}$ we have $\sum_{l\in M_a} \pi^{-l} = \sum_{l\in M_a'} \pi^{-l}$.Now, we prove the converse. Let *R* be a rule that satisfies the properties in the statement, and let $(M, P, N, \pi, C) \in \mathcal{D}$. We divide the proof into several steps.

(i) Let $i \in M$, and let $(M, P, N, \pi, C) \in \mathcal{D}^{-i}$. By Lemma 2, $R_i(M, P, N, \pi, C) = \pi^{-i}|N^{-i}| = R_i^{EP}(M, P, N, \pi, C)$. Thus, R and R^{EP} coincide in any subclass of problems $\mathcal{D}^{-i}$.
(ii) Let $D^0 = (M, P, \{a\}, \pi, C^{(a)}) \in \mathcal{D}^0$ be a problem with just one pass holder purchasing the general pass. As in case (ii) in Theorem 1, *dummy* and *symmetry between consortia* together imply that each consortium gets an equal share of π^0, i.e. for $P_t \in P$,

$$\sum_{i\in P_t} R_i(D^0) = \frac{\pi^0}{|K_a^0|}$$

By applying a similar reasoning to that in Case (ii) of the proof of Theorem 2, together with Lemmas 3 and 6, we have that, for each $i \in P_t$,

$$R_i\left(D^0\right) = \frac{\pi^{-i}}{\sum_{j\in M_a\cap P_t} \pi^{-j}} \frac{\pi^0}{|K_a^0|} = R^{EP}(D^0)$$

By *composition* we can extend the previous equivalence to the whole subclass $\mathcal{D}^0$. Indeed, let $(M, P, N, \pi, C) \in \mathcal{D}^0$, then, for each $i \in M$,

$$
\begin{aligned}
R_i\left(M, P, N, \pi, C\right) &= \sum_{a \in N^0} R_i\left(M, P, \{a\}, \pi, C^{(a)}\right) \\
&= \sum_{a \in N^0} R_i^{EP}\left(M, P, \{a\}, \pi, C^{(a)}\right) \\
&= R_i^{EP}\left(M, P, N, \pi, C\right)
\end{aligned}
$$

Therefore, R and R^{EP} coincide in $\mathcal{D}^0$.

(iii) Let $P_t \in P$, and let $D = (M, P, N, \pi, C^{(a)}) \in \mathcal{D}^t$. As R satisfies *dummy* and *splitting-proofness of museums*, we can construct a reasoning similar to Case (iii) in Theorem 2 to obtain that, for each $i \in P_t$,

$$R_i(D) = \sum_{a \in N^t} \frac{\pi^{-i}}{\sum_{j \in M_a} \pi^{-j}} \pi^t = R_i^{EP}(D)$$

In application of Lemma 1, R and R^{EP} coincide in the whole domain $\mathcal{D}$. □

Remark 6 The axioms of Theorem 4 are independent.

(a) Let R^9 be defined as follows. For each $i \in M$

$$R_i^9(D) = \sum_{a \in N_i^0} \frac{\pi^{-i}}{\sum_{j \in M_a \cap P^{(i)}} \pi^{-j}} \frac{1}{|K_a^0|} \pi^0 + \frac{\pi^{-i}\left\{1_{C_{ia}=1: a \in N^{(i)}}\right\}}{\sum_{j \in P^{(i)}} \pi^{-j}\left\{1_{C_{ja}=1: a \in N^{(i)}}\right\}} |N^{(i)}| \pi^{(i)} + |N^{-i}| \pi^{-i}$$

The rule R^9 satisfies dummy, symmetry between consortia, and splitting-proofness of museums, but not composition.

(b) The rule R^{EE} satisfies composition, dummy, symmetry between consortia, but not splitting-proofness of museums.
(c) The rule R^{PP} satisfies composition, dummy, splitting-proofness of museums, but not symmetry between consortia.
(d) Let R^{10} be define as follows. For each$i \in M$

$$R_i^{10}(D) = \sum_{a \in N^0} \frac{\pi^{-i}}{\sum_{j \in P^{(i)}} \pi^{-j}} \frac{1}{|K_a^0|} \pi^0 + \sum_{a \in N_i^{(i)}} \frac{\pi^{-i}}{\sum_{j \in M_a} \pi^{-j}} \pi^{(i)} + |N^{-i}| \pi^{-i}$$

The rule R^{10} satisfies satisfies composition, splitting-proofness of museums, symmetry between consortia, but not dummy.

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Declarations

Conflict of interest The authors state that there is no Conflict of interest.

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