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Decomposing profit change: Konüs, Bennet and Luenberger indicators

Juan Aparicio [a,b], José L. Zofío [c,d,*]

[a] Center of Operations Research, Miguel Hernandez University, Avda. de la Universidad s/n, E-03202, Elche, Spain
[b] valgrAI - Valencian Graduate School and Research Network of Artificial Intelligence, Camino de Vera s/n, E-46022, Valencia, Spain
[c] Department of Economics, Universidad Autónoma de Madrid, Ciudad Universitaria de Cantoblanco, E-28049, Madrid, Spain
[d] Erasmus Research Institute of Management, Erasmus University, NL-3062PA, Rotterdam, the Netherlands

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ABSTRACT

We introduce complementary decompositions of profit change that, relying on the duality between the profit function and the directional distance function, shed light on the different sources of profit growth including measures of technical efficiency, allocative efficiency and technological change. Our decompositions extend the literature on Konüs and Bennet quantity and price indicators to profit change. The first decomposition is '*exact*' in the sense of Diewert, by completely exhausting the sources of profit change into profit inefficiency change (including technical and allocative inefficiency change), technological change, and output and input price change. The second decomposition equates the Bennet quantity indicator to a productivity measure represented by the Luenberger indicator plus allocative inefficiency change. We deem it '*complete*' because in contrast to the existing literature, it retains the information on allocative inefficiency change while preventing the existence of residual terms capturing price variations, whose meaningful interpretation has not been addressed until now. Our proposed solution takes advantage of the flexibility of the directional distance function when choosing a suitable directional vector. All decompositions have the same structural form and therefore their components can be compared to each other *vis-à-vis*, providing alternative measures of equivalent sources of profit growth.

1. Introduction

In recent years there have been several attempts to decompose profit change into quantity and price indicators that can be interpreted in a meaningful way; particularly the possibility of exhausting such change into several terms that, based on the duality of the profit function and a measure of technical inefficiency—e.g., the directional distance function, include measures of technical efficiency change, allocative inefficiency change, and technological change, without leaving a trace of residual terms that convolute price and quantities pertaining to different time periods—for a revision of these concepts see Refs. [1,2] and, more recently, [3].

We show that such decomposition including economic efficiency terms and exhausting profit change into meaningful sources exists. Indeed, on the one hand, we propose two families of additive decompositions, one based on [4] quantity and price indicators and the other one resorting to Ref. [5] indicators, which can be related to the existing literature and among themselves. On the one hand, our favored symmetric decomposition grounded on the Konüs-based approach completes and extends the results obtained by Refs. [6,7] who decompose cost and profitability change in a multiplicative framework. Following the terminology coined by Diewert, including the Konüs--based denomination, we term our proposed decompositions *exact* because they include changes in profit inefficiency, technology, and output and input price indicators. We further improve the *exact* decompositions by resorting to the duality between the profit function and the directional distance function — [8]; which allows us to decompose the profit inefficiency change term into technical inefficiency change and allocative inefficiency change. On the other hand, based also on the directional distance function, we propose an alternative decomposition that, being related to Bennet's quantity and price indicators, includes a measure of productivity change—the so-called Luenberger productivity indicator—as initially proposed by Ref. [9]. We deem it *complete* because, contrary to existing proposals, either avoids or provides rationale for the presence of cross-period allocative terms confounding technology, prices and quantities from different periods. The two different decompositions of profit change, *exact* and *complete*, have the same structural form, i.e., they are isomorphic by decomposing into the same number of terms including technical efficiency change, allocative efficiency change, technological change, output price change and input

* Corresponding author. Department of Economics, Universidad Autónoma de Madrid, Ciudad Universitaria de Cantoblanco, E-28049, Madrid, Spain.
E-mail addresses: j.aparicio@umh.es (J. Aparicio), jose.zofio@uam.es (J.L. Zofío).

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price change. While the technical efficiency change and allocative efficiency change components are common to all decompositions, they rely on different definitions of technological change, output price change and input price change. These terms can be compared *vis-à-vis* between decompositions, representing complementary ways to analyze these sources of profit change.

Our study constitutes a common methodological framework to decompose profit change using alternative approaches, while offering a solution to previous attempts including undesirable price effects incapable of interpretation or imposing very restrictive conditions that seriously hamper the application of these methods to real-life studies. Specifically, we provide a complete relationship between profit change and productivity change that excludes the presence of price residuals, or relate both terms under rather restrictive assumptions, e.g., the way out suggested by Ref. [10] who assumes allocative efficient behavior on the part of the firms, combined with a parametric characterization of the production technology through specific functional forms. Our solution takes advantage of the flexibility of the directional distance function and endogenizes the directional vector to prevent the appearance of price terms while retaining the allocative efficiency change component. Moreover, regarding the need to assume specific functional forms, we overcome this limitation by implementing our solution through the non-parametric approach to efficiency and productivity measurement, generally known as Data Envelopment Analysis (DEA), which approximates the technology through the principle of minimum extrapolation. While the endogenization of the directional vector has been proposed in the literature, based either on economic criteria [11,12], or data-driven methods [13], to the best of our knowledge, it has never been considered in the decomposition of profit change.

From a methodological perspective, our results can be mirrored to solve similar problems encountered in the literature when decomposing profit, cost, revenue or profitability change. Regarding profit change, similar price terms can be found in Ref. [14]; who rely on the weighted additive distance function to define the corresponding Luenberger productivity indicator. [15] decompose cost (revenue) change using [16] input (output) distance function to determine the contribution of productivity change through the Malmquist index. Profitability change has been decomposed by Ref. [17] relying on the generalized (hyperbolic) distance function, which also allows the inclusion of its corresponding Malmquist productivity index in the decomposition. All these proposals struggle with the interpretation of cross-period price effects that emerge when introducing productivity indices, and we show that it is possible to either interpret them or eliminate them altogether using flexible technical inefficiency measures, as the directional distance function.

Moreover, from an empirical perspective, most applied research implements decompositions that, considering our proposal, fail to capture all the sources contributing to profit change. Our results provide much needed theoretical guidance when choosing a suitable model to study what drives economic performance. Otherwise, stakeholders like managers, investors, employees and officials, are offered a partial view of the causes underlying profit change, missing relevant information like the role played by the change in profit inefficiency—technical and allocative, productivity change (including technological change) or changes in output and inputs prices. Our results contribute to reduce the risk of misrepresenting these terms. If some elements of the decompositions are missing, it results in the over- or underestimation of the remaining terms included in the specification, which biases the conclusions drawn from the analysis and leads to erroneous decision-making.

Overall, this study continues the tradition initiated and sustained by earlier theoretical work on index (and indicator) number theory by Refs. [4,18–23] and [3,24,25].

The paper unfolds as follows. Next section 2 is devoted to the introduction of the *exact* decompositions of profit change based on the Konüs and Bennet quantity and price indicators. We start the section with the presentation of the background material necessary to construct our models, including the definitions of profit change and profit inefficiency change. Afterwards, following the Konüs approach initiated by Ref. [6]; we present a first decomposition of profit change which includes an asymmetric definition of technological change, followed by an alternative, symmetric proposal, based on [7]. Subsequently, we resort to the Bennet approach to propose counterpart decompositions of the output and input indicators of price change. Section 3 deals with the decomposition of profit inefficiency change into technical and allocative inefficiency change. This requires the presentation of the duality between the profit function and the directional distance function, as well as the definition of a normalized measure of profit inefficiency. A drawback of the *exact* decompositions is that they do not include a (pure) quantity measure of productivity change. The inclusion of this term in the form of the Luenberger productivity indicator results in our *complete* decomposition of profit change. In section 4 we show how this decomposition can be accomplished and discuss its relation to the previous *exact* decompositions. We also discuss the meaning and existence of the price effect accompanying the Luenberger indicator and show that, thanks to the flexibility of the directional distance function, it can be cancelled out, resulting in our preferred decomposition. We relate this decomposition to that proposed by Ref. [9] and the qualifications made by Ref. [10]. Implementing the *exact* and *complete* decomposition of profit change require the calculation of their terms relying on the directional distance function. We show how to calculate these models employing the DEA approach to efficiency measurement. Comparing the different proposals, we demonstrate in Section 5 that our approach prevents the inclusion of terms whose residual nature results in doubtful interpretations but does so without imposing restrictive assumptions. In section 6 we discuss the empirical relevance and implications of our new decomposition for applied research. Here we survey recent studies decomposing profit change and show how they fall short of the *complete* model and therefore could benefit from our theoretical findings. Finally, Section 7 summarizes by drawing the main conclusions and discussing limitations and future research directions.

2. Exact decompositions of profit change based on Konüs or Bennet indicators

We develop *exact* decompositions of profit change in which all terms can be interpreted in a meaningful way. In doing so we introduce to the additive framework the approach developed by Ref. [6]; for the multiplicative decomposition of cost variation—later on reformulated marginally by Ref. [1] and perfected by Ref. [7] as shown in what follows.[1] Following these authors, the new decomposition defines Konüs-type indicators of technological change and price changes (input and output). Afterwards, taking advantage of the decomposition of profit change *à la* Bennet into a quantity change indicator and price change indicators, we show that an alternative decomposition accounting for the same concepts can be obtained in a straightforward and, arguably, simpler way. We start out this section by presenting the definitions that will be used in the remainder of the paper.

2.1. Profit change and profit inefficiency change

Let us consider a panel of production units (e.g., firms) observed in the base and comparison periods: $t = 0, 1$, and transforming input vectors $x^t = (x_1^t, ..., x_M^t) \in \mathbb{R}_+^M$ into output vectors $y^t = (y_1^t, ..., y_N^t) \in \mathbb{R}_+^N$ according to the production technology T^t. We assume the standard axioms discussed in Ref. [26]; particularly convexity, closeness and free

[1] [7] introduce symmetric decompositions of cost variation that prevent the definition of unbalanced technical terms mixing price and quantities from different-base and comparison-periods.

disposability in inputs and outputs. Given the input and output market prices: $w^t \in \mathbb{R}^M_{++}$ and $p^t \in \mathbb{R}^N_{++}$, firms make *observed* profit[2]:

$$\pi^t \equiv p^t \cdot y^t - w^t \cdot x^t; \quad t = 0, 1. \tag{1}$$

Managers and stakeholders are also interested in profit change over time, corresponding to:

$$\Delta\pi \equiv \pi^1 - \pi^0 = p^1 \cdot y^1 - w \cdot ^1 x^1 - \left(p^0 \cdot y^0 - w^0 \cdot x^0\right). \tag{2}$$

Thus, if $\Delta\pi > 0$ then profit has *increased* over time, whereas if $\Delta\pi < 0$, it has *reduced*.

For each production process, observed profit can fall short from *maximum* profit, which, given the production technology, T^t, defines as:

$$\pi^t(w^t, p^t) \equiv \max_{x^t, y^t} \left\{p^t \cdot y^t - w^t \cdot x^t : (x^t, y^t) \in T^t\right\}, \tag{3}$$

where the optimal input and output quantities are denoted by x^{*t} and y^{*t}, respectively.

We contend that production processes may incur in inefficiencies, so actually observed quantities differ from the optimal ones: $(x^t, y^t) \neq (x^{*t}, y^{*t})$. Following the tradition of [27,28] or [16], these inefficiencies keep firms from reaching maximum profit. If this is the case $\pi^t(w^t,p^t) \geq \pi^t = p^t \cdot y^t - w^t \cdot x^t$, and the difference between the two naturally results in a measure of *profit inefficiency*:

$$\pi I^t \equiv \pi^t(w^t, p^t) - \pi^t \geq 0, t = 0, 1. \tag{4}$$

Therefore, observed profit in period t, π^t, will be equal or less than maximum best practice profit $\pi^t(w^t, p^t)$. Attaining maximum profit, $\pi I^t = 0$, implies that the firm not only makes the most out of the potential best practice technology, but also supplies and demands the optimal amounts of outputs and inputs given their market prices. If the firm is inefficient, then $\pi I^t > 0$, and the larger the value, the more inefficient it is. In section 3 we shed light on the sources of profit inefficiency. Particularly, the firm is technically inefficient if it falls short from attaining the production frontier (engineering or technological inefficiencies), and allocative inefficient if it demands the wrong quantities of inputs or supplies the wrong quantities of outputs under existing market prices (i.e., managerial or economic inefficiencies resulting from wrong input or outputs mixes).

Besides actual profit change (2), from a managerial perspective it is also relevant to know whether profit inefficiency has increased or reduced from the base to the comparison period; i.e., whether the firm is performing better or worse than the optimal reference benchmark in time. This can be answered by defining an indicator of profit inefficiency change:

$$\Delta\pi I \equiv \pi I^0 - \pi I^1 = \left(\pi^0\left(w^0, p^0\right) - \pi^0\right) - \left(\pi^1\left(w^1, p^1\right) - \pi^1\right). \tag{5}$$

Thus if $\Delta\pi I > 0$, then profit inefficiency has *improved* over time (the firm is less profit inefficient in the comparison period than in the base period), whereas it has *deteriorated* if $\Delta\pi I < 0$. Profit inefficiency change (5) can be equivalently expressed as observed profit change minus maximum profit change:

$$\Delta\pi I \equiv \Delta\pi - \Delta\pi(w, p) = \left(\pi^1 - \pi^0\right) - \left(\pi^1\left(w^1, p^1\right) - \pi^0\left(w^0, p^0\right)\right). \tag{6}$$

We now set out to assign meaningful interpretations to these terms and decompose profit change (2) in an *exact* way. Based on the arguments of the profit function $\pi^t(w^t, p^t)$, maximum profit depends on: a) the time period t indicating the *reference (best practice) technology* T^t available to the firm, b) the vector of *input prices* w^t that the firm faces, and c) the vector of *output prices* p^t at which the firm sells its outputs. Based on Konüs definitions of "true cost of living" price indices, as discussed by Refs. [6,7] for the multiplicative decomposition of cost change, we extend index number theory to the additive case by defining three families of additive *indicators* that measure the change in maximum profit (6) as a result of changes in the above variables between the base and comparison periods.

2.2. Exact decompositions of profit change based on Konüs indicators

The definition of Konüs indicators in the *exact* approach take as starting reference the profit maximizing benchmarks, $\pi^t(w^t,p^t)$, $t = 0, 1$, when defining the terms of technological change, output price change and input price change. In this subsection, we present two new *exact* decompositions of profit change. The first one corresponds to the asymmetric approach proposed by Ref. [6]; which [7] deem unsatisfactory as the arguments of the technological change term correspond to price information pertaining to different periods. Our second decomposition introduces a symmetric breakdown that does not suffer from the above drawback by considering the conventionally accepted definition of technological change.

2.2.1. The asymmetric approach

Our first family of measures corresponds to the *profit based Konüs* (K) *indicator of technological change*, $\tau^K(w^t, p^{t'}, 0, 1)$:

$$\tau^K\left(w^t, p^{t'}, 0, 1\right) = \pi^1\left(w^t, p^{t'}\right) - \pi^0\left(w^t, p^{t'}\right), \tag{7}$$

which captures the difference in maximum profit due to changes in the production technology T between the base period 0 and the comparison period 1, while keeping the input and output price vectors constant, although refereed to *different* periods, t and t'. Considering $t = 0, 1$; $t' = 0, 1$; $t \neq t'$, [6]; p. 61) proposed this formulation of technological change based on *mixed vectors* (w^0, p^1) and (w^1, p^0). At the time, this author justified this *asymmetric* choice of time periods on the following grounds: "*The reason for these rather odd looking choices will be explained below in more detail but basically, we make these choices in order to have cost decompositions into explanatory factors that are exact.*"

If given a reference firm in the base or comparison period, the frontier expands as a result of technological progress, then $\tau^K(w^t, p^{t'}, 0, 1) > 0$, if the frontier does no change $\tau^K(w^t, p^{t'}, 0, 1) = 0$, while a downward shift of the frontier results in $\tau^K(w^t, p^{t'}, 0, 1) < 0$. Considering the two alternative reference periods in (7), $t = 0, 1$; $t' = 0, 1$; $t \neq t'$, leads to the following indicators:

$$\tau^{K,01} \equiv \tau^K\left(w^0, p^1, 0, 1\right) = \pi^1\left(w^0, p^1\right) - \pi^0\left(w^0, p^1\right); \tag{8}$$

$$\tau^{K,10} \equiv \tau^K\left(w^1, p^0, 0, 1\right) = \pi^1\left(w^1, p^0\right) - \pi^0\left(w^1, p^0\right), \tag{9}$$

Therefore, for each choice of reference vectors of input and output prices, one obtains two alternative definitions of technological change indicators. As for the properties of these two dual indicators of technological change, they satisfy an extensive list of tests in terms of the family of price and quantity Bennet indicators in which they decompose, see Ref. [23]; pp. 326–330).[3] Among them continuity, identity, bounding, monotonicity, positivity or negativity in the change of quantities and prices, invariance to changes in units of measurement, and linear homogeneity in quantities and prices. Since both indicators are equally

[2] Notation: the inner product of two D dimensional vectors $v \equiv [v_1, \ldots, v_D]$ and $u \equiv [u_1, \ldots, u_D]$ is denoted as $v \cdot u \equiv \sum_{d=1}^{D} v_d u_d$.

[3] This additive definition of profit-based indicators of technological change is, in contrast to alternative index proposals, based on ratios, i.e., $\pi^1(w^s, p^h)/\pi^0(w^s, p^h)$, which, nevertheless, lacks desirable properties from an axiomatic (test) perspective: proportionality, monotonicity, linear homogeneity in prices, homogeneity of degree zero in prices. Hence the multiplicative definition of profit change, although defined by some authors, is considered as unsatisfactory from an index number perspective. Similar remarks can be made regarding the—forthcoming—definitions of profit base indicators of changes in output prices: $\pi^t(w^t, p^1)/\pi^t(w^t, p^0)$, and input prices: $\pi^t(w^1, p^t)/\pi^t(w^0, p^t)$, see Ref. [24]; 162), citing [21].

representative, it is possible to define the symmetric average of the two by taking the arithmetic mean (i.e., resulting in a Fisher-type indicator):

$$\tau^K \equiv \frac{1}{2}\left[\tau^{K,01} + \tau^{K,10}\right] = \frac{1}{2}\left[\tau^K\left(w^0, p^1, 0, 1\right) + \tau^K\left(w^1, p^0, 0, 1\right)\right]. \quad (10)$$

The second family of measures corresponds to the following *profit based Konüs indicators of change in output prices*, $\rho^K(w^t, p^0, p^1, t)$:

$$\rho^K\left(w^t, p^0, p^1, t\right) = \pi^t\left(w^t, p^1\right) - \pi^t\left(w^t, p^0\right), \quad (11)$$

which captures the difference in maximum profit due to changes in output prices, from p^0 to p^1, given the reference period t technology while keeping the input prices constant at period, w^t. If $\rho^K(w^t, p^0, p^1, t) > 0$, there has been an increase in output prices, while if $\rho^K(w^t, p^0, p^1, t) < 0$ a reduction is observed—$\rho^K(w^t, p^0, p^1, t) = 0$ signals that they remain constant in time. It is natural to consider either the based or comparison periods for inputs prices as reference. Depending on whether one chooses as benchmark $t = 0$ or $t = 1$, it is possible to particularize (11) in the following ways:

$$\rho^{K,0} \equiv \rho\left(w^0, p^0, p^1, 0\right) = \pi^0\left(w^0, p^1\right) - \pi^0\left(w^0, p^0\right); \quad (12)$$

$$\rho^{K,1} \equiv \rho\left(w^1, p^0, p^1, 1\right) = \pi^1\left(w^1, p^1\right) - \pi^1\left(w^1, p^0\right). \quad (13)$$

By using the base period technology and input prices, the first indicator (12) is a Laspeyres-type indicator, while the second one, (13), using the comparison period technology and input prices, is a Paasche-type indicator. Then, we can define the Fisher version as the arithmetic mean of both indicators:

$$\rho^K \equiv \frac{1}{2}\left[\rho^{K,0} + \rho^{K,1}\right] = \frac{1}{2}\left[\rho\left(w^0, p^0, p^1, 0\right) + \rho\left(w^1, p^0, p^1, 1\right)\right]. \quad (14)$$

In the same vein, we now define a new family of *profit based Konüs indicators of change in input prices*, $\varpi^K(w^0, w^1, p^t, t)$,

$$\varpi^K\left(w^0, w^1, p^t, t\right) = \pi^t\left(w^1, p^t\right) - \pi^t\left(w^0, p^t\right). \quad (15)$$

As before, this input price indicator is equal to the difference in maximum profit due to changes in input prices from the base, w^0, to the comparison period, w^1, given the technology at the reference period t and keeping the output prices constant at t, p^t. As for the values of the indicator, $\varpi^K(w^0, w^1, p^t, t) < 0$ reflects *increases* in input prices, $\varpi^K(w^0, w^1, p^t, t) > 0$ price *reductions*, and $\varpi^K(w^0, w^1, p^t, t) = 0$ price stability. Consistent with the previous case, we can single out two cases of this family of indicators corresponding respectively to the base and comparison period technologies, T^0 and T^1, and equal period output prices, p^0 or p^1. These are:

$$\varpi^{K,0} \equiv \varpi\left(w^0, w^1, p^0, 0\right) = \pi^0\left(w^1, p^0\right) - \pi^0\left(w^0, p^0\right); \quad (16)$$

$$\varpi^{K,1} \equiv \varpi\left(w^0, w^1, p^1, 1\right) = \pi^1\left(w^1, p^1\right) - \pi^1\left(w^0, p^1\right). \quad (17)$$

In this case, one obtains two alternative definitions of input price indicators for each choice of technology and reference vector of output prices. Following the output case, we can define the arithmetic mean of both indices, as an equally weighted measure of input price changes:

$$\varpi^K \equiv \frac{1}{2}\left[\varpi^{K,0} + \varpi^{K,1}\right] = \frac{1}{2}\left[\varpi\left(w^0, w^1, p^0, 0\right) + \varpi\left(w^0, w^1, p^1, 1\right)\right]. \quad (18)$$

With the above families of indicators of technological change, $\tau^K(w^t, p^{t'}, 0, 1)$ in (7), changes in output prices, $\rho^K(w^t, p^0, p^1, t)$ in (11), and input prices $\varpi^K(w^0, w^1, p^t, t)$ in (15), along with the definition of profit inefficiency change in (5), we can define pairwise *exact* decompositions of profit change from the base period to the comparison period: $\Delta\pi \equiv \pi^1 - \pi^0 = p^1 \cdot y^1 - w^1 \cdot x^1 - (p^0 \cdot y^0 - w^0 \cdot x^0)$—definition (2). These alternative decompositions depend on the different reference periods, either base, comparison, or the arithmetic mean of the two. In all three cases, profit change is decomposed additively into four terms:

- Changes in the firms' profit inefficiency; i.e., a term of the form $\Delta\pi I = \pi I^0 - \pi I^1$ as defined in (5) and (6) above.
- Exogenous increases in profit due to technological progress; i.e., a term of the form of $\tau^K(w^t, p^{t'}, 0, 1)$ as defined in (7).
- Change in output prices; i.e., a term of the form of $\rho^K(w^t, p^0, p^1, t)$ as defined in (11).
- Change in input prices; i.e., a term of the form of $\varpi^K(w^0, w^1, p^t, t)$ as defined in (15).

Based on the above explanatory terms, simple algebra shows that a first possibility to decompose profit change is the following:

$$\begin{aligned}\Delta\pi &\equiv \pi^1 - \pi^0 = p^1 \cdot y^1 - w^1 \cdot x^1 - \left(p^0 \cdot y^0 - w^0 \cdot x^0\right) \\ &= \underbrace{\left(\underbrace{\left(\pi^1 - \pi^0\right)}_{\Delta\pi} - \underbrace{\left(\pi^1\left(w^1, p^1\right) - \pi^0\left(w^0, p^0\right)\right)}_{\Delta\pi(w,p)}\right)}_{\Delta\pi I} \\ &+ \underbrace{\left(\pi^1\left(w^1, p^0\right) - \pi^0\left(w^1, p^0\right)\right)}_{\tau^{K,10} \equiv \tau(w^1, p^0, 0, 1)} + \underbrace{\left(\pi^1\left(w^1, p^1\right) - \pi^1\left(w^1, p^0\right)\right)}_{\rho^{K,1} \equiv \rho(w^1, p^0, p^1, 1)} \\ &+ \underbrace{\left(\pi^0\left(w^1, p^0\right) - \pi^0\left(w^0, p^0\right)\right)}_{\varpi^{K,0} \equiv \varpi(w^0, w^1, p^0, 0)}.\end{aligned} \quad (19)$$

Or, alternatively:

$$\begin{aligned}\Delta\pi &\equiv \pi^1 - \pi^0 = p^1 y^1 - w^1 x^1 - \left(p^0 y^0 - w^0 x^0\right) \\ &= \underbrace{\left(\underbrace{\left(\pi^1 - \pi^0\right)}_{\Delta\pi} - \underbrace{\left(\pi^1\left(w^1, p^1\right) - \pi^0\left(w^0, p^0\right)\right)}_{\Delta\pi(w,p)}\right)}_{\Delta\pi I} \\ &+ \underbrace{\left(\pi^1\left(w^0, p^1\right) - \pi^0\left(w^0, p^1\right)\right)}_{\tau^{K,01} \equiv \tau(w^0, p^1, 0, 1)} + \underbrace{\left(\pi^0\left(w^0, p^1\right) - \pi^0\left(w^0, p^0\right)\right)}_{\rho^{K,0} \equiv \rho(w^0, p^0, p^1, 0)} \\ &+ \underbrace{\left(\pi^1\left(w^1, p^1\right) - \pi^1\left(w^0, p^1\right)\right)}_{\varpi^{K,1} \equiv \varpi(w^0, w^1, p^1, 1)}.\end{aligned} \quad (20)$$

Therefore, there are two alternative ways to decompose profit change, and in the absence of any specific criterion to choose one over the other, we can rely on the previously defined arithmetic means to define a (comprehensive) third *exact* decomposition of profit change. This decomposition makes use of definitions (5), (10), (14), and (18), above:

$$\Delta\pi \equiv \pi^1 - \pi^0 = \Delta\pi I + \tau^K + \rho^K + \varpi^K = (\Delta\pi - \Delta\pi(w, p)) + \tau^K + \rho^K + \varpi^K. \quad (21)$$

This last result shows that the Konüs-based additive framework can offer an *exact* decomposition of profit change measure into the previously mentioned four indicators. For all these terms, the arithmetic mean decompositions in (21) will be more representative in general since each indicator corresponds to the mean of the individual values, which could be quite apart numerically. It is apparent that if the firm is profit efficient in both periods (or equally inefficient), then $\Delta\pi I = 0$. Also, the technological change vanishes if the production possibility set remains constant, $\tau^K = 0$. Finally, if output and input prices are stable over time, their corresponding price change indicators are null: $\rho^K = 0$, and $\varpi^K = 0$, respectively.

2.2.2. The symmetric approach

The previous decomposition includes a technological change term (7) whose functional structure combines input and output prices from different periods, i.e., the *mixed vectors* (w^0, p^1) and (w^1, p^0). However, it is possible to propose an alternative definition of profit change that includes the (symmetric) definition of technological change commonly accepted in the literature—e.g., Ref. [24]; 175), and corresponding to[4]:

$$\tau_S^K(w^t, p^t, 0, 1) \equiv \Delta\pi^{0,1}(w^t, p^t) = \pi^1(w^t, p^t) - \pi^0(w^t, p^t), t = 0, 1, \tag{22}$$

where the subscript S stands for symmetry. Then, the input price change and output price change indicators can be redefined to obtain an *exact* decomposition that also includes the firms' profit inefficiency term. Here, for brevity, we present directly the Fisher-type definitions of the technological change and the price change terms.

The symmetric indicator of the *profit based Konüs indicator of technological change,* corresponds to:

$$\begin{aligned}\tau_S^K &\equiv \frac{1}{2}\left[\tau_S^K\left(w^0, p^0, 0, 1\right) + \tau_S^K\left(w^1, p^1, 0, 1\right)\right] \\ &= \frac{1}{2}\left[\left(\pi^1\left(w^0, p^0\right) - \pi^0\left(w^0, p^0\right)\right) + \left(\pi^1\left(w^1, p^1\right) - \pi^0\left(w^1, p^1\right)\right)\right].\end{aligned} \tag{23}$$

This indicator captures the difference in maximum profit due to changes in the production technology T between the base period 0 and the comparison period 1, while keeping the input and output price vectors constant in the *same* period t. If $\tau_S^K > 0$ then there has been an expansion of the frontier, while if $\tau_S^K < 0$ a reduction is observed, and $\tau_S^K = 0$ indicates that the frontier has not changed.

We now present the symmetric versions of the *profit based Konüs indicators of output price change and input price change*, counterparts to (14) and (18), respectively.[5] Regarding the indicator of output price change we have:

$$\begin{aligned}\rho_S^K &\equiv \frac{1}{4}\left[\rho^K\left(w^0, p^0, p^1, 0\right) + \rho^K\left(w^1, p^0, p^1, 0\right) + \rho^K\left(w^0, p^0, p^1, 1\right)\right. \\ &\quad \left. + \rho^K\left(w^1, p^0, p^1, 1\right)\right] \\ &= \frac{1}{4}\left[\left(\pi^0\left(w^0, p^1\right) - \pi^0\left(w^0, p^0\right)\right) + \left(\pi^0\left(w^1, p^1\right) - \pi^0\left(w^1, p^0\right)\right)\right. \\ &\quad \left. + \left(\pi^1\left(w^0, p^1\right) - \pi^1\left(w^0, p^0\right)\right) + \left(\pi^1\left(w^1, p^1\right) - \pi^1\left(w^1, p^0\right)\right)\right].\end{aligned} \tag{24}$$

Indicator (24) considers all four combinations of output price changes from the base to the comparison period given the base and comparison period technologies T^t, $t = 0, 1$, while keeping the input prices constant at period t, w^t, $t = 0, 1$. Now, if $\rho_S^K > 0$ there has been in increase in the output prices, while $\rho_S^K < 0$ signals their reduction and $\rho_S^K = 0$ their stability.

Correspondingly, we define the indicator of input price change as:

$$\begin{aligned}\varpi_S^K &\equiv \frac{1}{4}\left[\varpi^K\left(p^0, w^0, w^1, 0\right) + \varpi^K\left(p^1, w^0, w^1, 0\right) + \varpi^K\left(p^0, w^0, w^1, 1\right)\right. \\ &\quad \left. + \varpi^K\left(p^1, w^0, w^1, 1\right)\right] \\ &= \frac{1}{4}\left[\left(\pi^0\left(w^1, p^0\right) - \pi^0\left(w^0, p^0\right)\right) + \left(\pi^0\left(w^1, p^1\right) - \pi^0\left(w^0, p^1\right)\right)\right. \\ &\quad \left. + \left(\pi^1\left(w^1, p^0\right) - \pi^1\left(w^0, p^0\right)\right) + \left(\pi^1\left(w^1, p^1\right) - \pi^1\left(w^0, p^1\right)\right)\right],\end{aligned} \tag{25}$$

which considers all four combinations of input price changes from the base to the comparison period given a reference period technology T^t, $t = 0, 1$, while keeping the output prices constant at period, p^t, $t = 0, 1$. If input prices increase from the base to the comparison period, then $\varpi_S^K < 0$, if they decrease $\varpi_S^K > 0$, while $\varpi_S^K = 0$ indicates that they remain constant.

Combining expressions (23), (24), and (25), along with profit inefficiency change (6), we can decompose profit change (2) as follows:

$$\Delta\pi \equiv \pi^1 - \pi^0 = \Delta\pi I + \tau_S^K + \rho_S^K + \varpi_S^K = (\Delta\pi - \Delta\pi(w, p)) + \tau_S^K + \rho_S^K + \varpi_S^K. \tag{26}$$

2.2.3. Exact decomposition of profit change based on Bennet indicators

Considering the four terms in which profit change can be decomposed to gain information on its sources, we now offer an alternative—but equally sound—decomposition of profit change that retains the symmetric definition of technological change but relies on Bennet indicators to characterize the output and input indicators of price change.

For convenience, let us rewrite $\Delta\pi(w,p) = \pi^1(w^1,p^1) - \pi^0(w^0,p^0)$ in the following terms:

$$\begin{aligned}\Delta\pi(w,p) &= \pi^1\left(w^1, p^1\right) - \pi^0\left(w^0, p^0\right) \\ &= \frac{1}{2}\left[\underbrace{\pi^1\left(w^0, p^0\right) - \pi^0\left(w^0, p^0\right)}_{\tau_S^K(w^0,p^0,0,1) \equiv \Delta\pi^{0,1}(w^0,p^0)} + \underbrace{\pi^1\left(w^1, p^1\right) - \pi^0\left(w^1, p^1\right)}_{\tau_S^K(w^1,p^1,0,1) \equiv \Delta\pi^{0,1}(w^1,p^1)}\right] \\ &\quad + \frac{1}{2}\left[\underbrace{\pi^0\left(w^1, p^1\right) - \pi^0\left(w^0, p^0\right)}_{\Delta\pi^0(w^0,p^0,w^1,p^1)} + \underbrace{\pi^1\left(w^1, p^1\right) - \pi^1\left(w^0, p^0\right)}_{\Delta\pi^1(w^0,p^0,w^1,p^1)}\right]\end{aligned} \tag{27}$$

By updating the technology while keeping prices constant at either period; i.e., $\tau_S^K(w^0,p^0,0,1)$ and $\tau_S^K(w^1,p^1,0,1)$, the first bracket in (27) corresponds to the symmetric (favored) definition of the profit based Konüs indicator of technological change; i.e., τ_S^K in expression (23) above. However, to complete the decomposition, rather than adopting the four alternative definitions of output and input price changes included in expressions (24) and (25), it is possible to follow the same rationale underlying the symmetric definition of technological change, and define two novel indicators of price change that keep technologies pairwise constant either at $t = 0$ or $t = 1$. These terms, presented in the last term of (27), and denoted by $\Delta\pi^0(w^0,p^0,w^1,p^1)$ and $\Delta\pi^1(w^0,p^0,w^1,p^1)$, can be expressed as Bennet indicators capturing the overall effect of price changes.

However, to show the connection between the new Bennet indicators of overall price change and the symmetrical Konüs indicator of technological change, we first expand the base and comparison period definitions of this last indicator to gain knowledge about their structure. That is,

$$\begin{aligned}\tau_S^K\left(w^0, p^0, 0, 1\right) &= \pi^1\left(w^0, p^0\right) - \pi^0\left(w^0, p^0\right) \\ &= p^0 \cdot \left(y^1\left(w^0, p^0\right) - y^0\left(w^0, p^0\right)\right) - w^0 \cdot \left(x^1\left(w^0, p^0\right)\right. \\ &\quad \left. - x^0\left(w^0, p^0\right)\right),\end{aligned} \tag{28}$$

[4] It is worth noting that an alternative decomposition of profit change would be possible following the proposal to decompose cost change by Ref. [1]; p. 283). Their decomposition marginally differs from that of [6] in that the mix period values are to be found in the quantity change indices rather than in the technological change indices. Adopting this altenative would reconcile the definition of technological change in (7) with that of expression (22), at the cost of a mix-period definition of output price changes in (11).

[5] The structural form of these indicators is equivalent to that presented by Ref. [7]; p. 1190) in their additive decomposition of cost change.

$$\tau_S^K(w^1,p^1,0,1) = \pi^1(w^1,p^1) - \pi^0(w^1,p^1) = p^1\cdot\left(y^1(w^1,p^1) - y^0(w^1,p^1)\right) - w^1\cdot\left(x^1(w^1,p^1) - x^0(w^1,p^1)\right), \tag{29}$$

In these expressions it is explicitly shown that technological change is measured as the change in the optimal quantities maximizing profit (3), under period t technology and period t' prices, i.e., $y^t(w^{t'},p^{t'})$ and $x^t(w^{t'},p^{t'})$, t, $t' = 0, 1$. Recall that these optimal quantities may consider prices and technologies belonging to the same period, $t = t'$, or cross-periods, $t \neq t'$. Taking the arithmetic average yields the Fisher-type indicator of technological change in expression (23):

$$\tau_S^K \equiv \frac{1}{2}\left[\tau_S^K(w^0,p^0,0,1) + \tau_S^K(w^1,p^1,0,1)\right] = \frac{1}{2}\left[p^0\cdot\left(y^1(w^0,p^0) - y^0(w^0,p^0)\right) + p^1\cdot\left(y^1(w^1,p^1) - y^0(w^1,p^1)\right)\right] - \frac{1}{2}\left[w^0\cdot\left(x^1(w^0,p^0) - x^0(w^0,p^0)\right) + w^1\cdot\left(x^1(w^1,p^1) - x^0(w^1,p^1)\right)\right]. \tag{30}$$

In the second line of expression of (30), we identify two indicators of output quantity change, taking as reference prices in the base period (w^0, p^0)—i.e., $y^1(w^0,p^0) - y^0(w^0,p^0)$, and prices in the comparison period (w^1, p^1), i.e., $y^1(w^1,p^1) - y^0(w^1,p^1)$, while in the third line we observe an analogous indicator of variation in input quantities. The whole expression, once output and input variations are respectively multiplied by the corresponding input and output prices, p^t and w^t, measures technological change in monetary terms over the possible period combinations of the two periods $t = 0, 1$. Note that technological change τ_S^K represents a measure of productivity change at optimal quantities defined as the monetary value of changes in optimal output quantities minus the monetary value of changes in optimal input quantities. This is equivalent to the structure of technological change in additive decompositions of cost change, corresponding to changes in the minimum cost associated with the optimal input quantities in both periods and keeping prices and output quantities constant at $t = 0, 1$—see proposition 3 in Ref. [29]; p. 22) and expression (6) in Ref. [7]; p. 1191).

We now introduce the expressions corresponding to the symmetric indicators of *overall price change* in the third line of (27).

$$\Delta\pi^0(w^0,p^0,w^1,p^1) = \pi^0(w^1,p^1) - \pi^0(w^0,p^0) = \rho_B^K(p^0,w^0,p^1,w^1,0) - \varpi_B^K(p^0,w^0,p^1,w^1,0) = \left[p^1\cdot y^0(w^1,p^1) - p^0\cdot y^0(w^0,p^0)\right] - \left[w^1\cdot x^0(w^1,p^1) - w^0\cdot x^0(w^0,p^0)\right], \tag{31}$$

$$\Delta\pi^1(w^0,p^0,w^1,p^1) = \pi^1(w^1,p^1) - \pi^1(w^0,p^0) = \rho_B^K(p^0,w^0,p^1,w^1,1) - \varpi_B^K(p^0,w^0,p^1,w^1,1) = \left[p^1\cdot y^1(w^1,p^1) - p^0\cdot y^1(w^0,p^0)\right] - \left[w^1\cdot x^1(w^1,p^1) - w^0\cdot x^1(w^0,p^0)\right]. \tag{32}$$

In each one of these expressions, we identify the variation in maximum profit (27) resulting from changes in output prices and input prices, while keeping the technology constant at either the base period $t = 0$—expression (31), or the comparison period $t = 1$— expression (32). These expressions reflect the direct change in output and input prices as well as the induced effect causing the variation in the optimal output and input quantities maximizing profit, i.e., $y^t(w^{t'},p^{t'})$ and $x^t(w^{t'},p^{t'})$, t, $t' = 0, 1$. In contrast to the axiomatic approach to index (indicator) number theory that considers that quantity vectors are independent of price vectors, the Konüs approach grounded on economic theory assumes that the quantity vectors are function of the technology and the existing market prices.

We now recall Bennet's methodology to separate the actual changes in observed output and input prices from the variation that they induce in the optimal output and input quantities—hence the subscript B in ρ_B^K and ϖ_B^K. Bennet's seminal contribution allows decomposing profit change into indicators reflecting the change in observed output and input prices and the change in output and input quantities, i.e.,

$$\Delta\pi = \pi^1 - \pi^0 = p^1\cdot y^1 - w^1\cdot x^1 - \left(p^0\cdot y^0 - w^0\cdot x^0\right) = \underbrace{\left[\underbrace{\frac{1}{2}(p^0+p^1)\cdot(y^1-y^0)}_{Q_y^B\equiv Q_y^B(p^1,y^1,p^0,y^0)} - \underbrace{\frac{1}{2}(w^0+w^1)\cdot(x^1-x^0)}_{Q_x^B\equiv Q_x^B(w^1,x^1,w^0,x^0)}\right]}_{Q^B\equiv Q^B(w^1,p^1,x^1,y^1,w^0,p^0,x^0,y^0)} + \underbrace{\left[\underbrace{\frac{1}{2}(p^1-p^0)\cdot(y^0+y^1)}_{P_y^B\equiv P_y^B(p^1,y^1,p^0,y^0)} - \underbrace{\frac{1}{2}(w^1-w^0)\cdot(x^0+x^1)}_{P_x^B\equiv P_x^B(w^1,x^1,w^0,x^0)}\right]}_{P^B\equiv P^B(w^1,p^1,x^1,y^1,w^0,p^0,x^0,y^0)}, \tag{33}$$

where $Q_y^B \equiv Q_y^B(p^1,y^1,p^0,y^0)$ and $Q_x^B \equiv Q_x^B(w^1,x^1,w^0,x^0)$ are volume indicators of output quantity change and input quantity change, while $P_y^B \equiv P_y^B(p^1,y^1,p^0,y^0)$ and $P_x^B \equiv P_x^B(p^1,y^1,p^0,y^0)$ are indicators of output price change and input price change. Adding the quantity indicators results in a quantity change indicator of total factor productivity, $Q^B \equiv Q^B(w^1,p^1,x^1,y^1,w^0,p^0,x^0,y^0)$, while jointly considering the price indicators yields a total price recovery change indicator $P^B \equiv P^B(w^1,p^1,x^1,y^1,w^0,p^0,x^0,y^0)$—see, e.g., Ref. [25]; 130) or [1]; 215).

Rearranging for analytical convenience the order of the different terms in (33), we follow Bennet's approach to express the indicators of price change (31) and (32) as follows:

$$\Delta\pi^0(w^0,p^0,w^1,p^1) = \pi^0(w^1,p^1) - \pi^0(w^0,p^0) = \rho_B^K(p^0,w^0,p^1,w^1,0) - \varpi_B^K(p^0,w^0,p^1,w^1,0) = \underbrace{\left[\frac{1}{2}(p^1-p^0)\cdot\left(y^0(w^1,p^1)+y^0(w^0,p^0)\right)\right]}_{P_{y*}^{B,0}\equiv P_y^B(p^0,p^1,y^0(w^1,p^1),y^0(w^0,p^0))} + \underbrace{\left[\frac{1}{2}(p^0+p^1)\cdot\left(y^0(w^1,p^1)-y^0(w^0,p^0)\right)\right]}_{Q_{y*}^{B,0}\equiv Q_y^B(p^0,p^1,y^0(w^1,p^1),y^0(w^0,p^0))} - \left(\underbrace{\left[\frac{1}{2}(w^1-w^0)\cdot\left(x^0(w^1,p^1)+x^0(w^0,p^0)\right)\right]}_{P_{x*}^{B,0}\equiv P_x^B(w^0,w^1,x^0(w^1,p^1),x^0(w^0,p^0))} + \underbrace{\left[\frac{1}{2}(w^0+w^1)\cdot\left(x^0(w^1,p^1)-x^0(w^0,p^0)\right)\right]}_{Q_{x*}^{B,0}\equiv Q_x^B(w^0,w^1,x^0(w^1,p^1),x^0(w^0,p^0))}\right), \tag{34}$$

$$\Delta\pi^1(w^0,p^0,w^1,p^1) = \pi^1(w^1,p^1) - \pi^1(w^0,p^0) = \rho^K(p^0,w^0,p^1,w^1,1) - \varpi^K(p^0,w^0,p^1,w^1,1) = \underbrace{\left[\frac{1}{2}(p^1-p^0)\cdot\left(y^1(w^1,p^1)+y^1(w^0,p^0)\right)\right]}_{P^{B,1}_{y*}\equiv P^B_y(p^0,p^1,y^1(w^1,p^1),y^1(w^0,p^0))}$$
$$+\underbrace{\left[\frac{1}{2}(p^0+p^1)\cdot\left(y^1(w^1,p^1)-y^1(w^0,p^0)\right)\right]}_{Q^{B,1}_{y*}\equiv Q^B_y(p^0,p^1,y^1(w^1,p^1),y^1(w^0,p^0))} - \left(\underbrace{\left[\frac{1}{2}(w^1-w^0)\cdot\left(x^1(w^1,p^1)+x^1(w^0,p^0)\right)\right]}_{P^{B,1}_{x*}\equiv P^B_x(w^0,w^1,x^1(w^1,p^1),x^1(w^0,p^0))} + \underbrace{\left[\frac{1}{2}(w^0+w^1)\cdot\left(x^1(w^1,p^1)-x^1(w^0,p^0)\right)\right]}_{Q^{B,1}_{x*}\equiv Q^B_x(w^0,w^1,x^0(w^1,p^1),x^0(w^0,p^0))}\right). \quad (35)$$

These formulations show that it is possible to decompose the contribution that changes in output and input prices make to the variation in maximum profit into actual price changes and the induced effect that they make through changes in optimal output and input quantities. Following [7]; these latter changes measure the so-called *activity effect*—also known in the literature as *size effect*. That is, the scale of operation of the firm in terms of output and input quantities. Consistent with the economic approach to index numbers, changes in optimal quantities depend on price changes, implying that they are naturally embedded in the definitions of overall price change $\Delta\pi^0(w^0,p^0,w^1,p^1)$ and $\Delta\pi^1(w^0,p^0,w^1,p^1)$.[6]

Although the structure of the above expressions coincides with that of the standard Bennet formulation (33), there are two relevant differences. First, in formulations (34) and (35) the output and input quantities are optimal and not observed—hence the superscript "*" in notation. Second, technologies are kept constant at either the base period or the comparison period, while in the standard Bennet approach technological change is embedded. One sees immediately that if there are not changes in output prices and input prices, that is $p = p^t$ and $w = w^t$ $(t = 0, 1)$, all four terms vanish. However, if only output prices remain constant, $p = p^t$, while input prices change, $w^t \neq w^{t'}$, only the direct price changes $P^{B,0}_{y*}$ and $P^{B,1}_{y*}$ cancel out, as optimal output quantities $Q^{B,0}_{y*}$ and $Q^{B,1}_{y*}$ are dependent on input prices. Likewise, if only input prices remain constant, $w = w^t$, while output prices change, $p^t \neq p^{t'}$, the indicators of direct price change $P^{B,0}_{x*}$ and $P^{B,1}_{x*}$ disappear, while $Q^{B,0}_{x*}$ and $Q^{B,1}_{x*}$ measure the monetary effect of changes of output prices in optimal input quantities.

Net of technological change, which is captured by (30), the different elements of the overall price change terms can be combined in meaningful ways. For the base period—Laspeyres—price change term, $\Delta\pi^0(w^0,p^0,w^1,p^1)$, the net change in output and input prices is given by $P^{B,0}_{y*} - P^{B,0}_{x*}$, while the net contribution of the induced changes in output and input quantities corresponds to $Q^{B,0}_{y*} - Q^{B,0}_{x*}$. This implies that the variation in output and input prices (weighted by their corresponding optimal quantities) contribute positively (negatively) to increases in maximum profit if the net effect is positive, i.e., $P^{B,0}_{y*} - P^{B,0}_{x*} > 0$ ($P^{B,0}_{y*} - P^{B,0}_{x*} < 0$). However, these changes in prices induce changes in the optimal output and input quantities. For example, if these price changes resulted in an increment (reduction) in output quantities and a reduction (increment) in input quantities, then the net quantity effects contribute to increase (reduce) maximum profit: $Q^{B,0}_{y*} - Q^{B,0}_{x*} > 0$ ($Q^{B,0}_{y*} - Q^{B,0}_{x*} < 0$).

Comparable effects can be defined considering the comparison period—Paasche—price change term: $\Delta\pi^1(w^0,p^0,w^1,p^1)$.[7]

Taking the arithmetic average of expressions (34) and (35) yields the Fisher-type expression of changes in output prices and input prices:

$$\rho^K_B \equiv \frac{1}{2}\left[\rho^K_B(p^0,w^0,p^1,w^1,0) + \rho^K_B(p^0,w^0,p^1,w^1,1)\right], \quad (36)$$

$$\varpi^K_B \equiv \frac{1}{2}\left[\varpi^K_B(p^0,w^0,p^1,w^1,0) + \varpi^K_B(p^0,w^0,p^1,w^1,1)\right], \quad (37)$$

whose interpretation in terms of firms' revenue and cost is straightforward. As for output prices, $\rho^K_B > 0$ ($\rho^K_B < 0$), indicates that output price changes increase (decrease) the revenue of the firm over time through the direct and induced effects on optimal output quantities, and therefore contribute to profit growth (reduction). On the contrary, an increase (reduction) in input prices, $\varpi^K_B > 0$ ($\varpi^K_B < 0$), increases (reduces) production costs, and consequently reduces (increases) profits.

Combining expressions (30), (36), and (37), along with profit inefficiency change (6), we can decompose profit change (2) as follows:

$$\Delta\pi \equiv \pi^1 - \pi^0 = \Delta\pi I + \tau^K_S + \rho^K_B - \varpi^K_B = (\Delta\pi - \Delta\pi(w,p)) + \tau^K_S + \rho^K_B - \varpi^K_B. \quad (38)$$

2.2.4. Comparing the exact Konüs decompositions of profit change

Our Konüs based decompositions of profit change introduced so far, either asymmetric (21), symmetric (26), or based on Bennet indicators (38), extend the economic theory approach to multiplicative index numbers based on the "true cost of living" definition by Ref. [4] to the additive approach. In doing so, hypothetical equilibria are brought into the analysis. This is standard in economic analysis by relying on the comparative statics ("*ceteris paribus*") method that contrasts two equilibria, before and after a change in an underlying exogenous variable, which in this case correspond to the technology, the output prices, and the input prices. In the multiplicative approach presented in Ref. [6]—later modified by Refs. [1,7]; this comparison is made by way of meaningful index numbers, defined as ratios capturing the change in the objective function, e.g., the cost function $C^t(y^t,w^t)$, when one of its arguments changes, thereby allowing the identification of their individual contribution to cost change. In the current profit change decomposition, the comparison is made by relying on the difference in the values of the objective function when each one of these variables changes. For instance, in the symmetric approach, adopting the base period as

[6] Unsurprisingly, the fact that price indicators capture both direct and induced changes is widely acknowledged in the literature. See, for instance, Ref. [1]; 160): "Thus, the price effect plays two roles in the analysis: as the set of variables responsible for the impact of price change, and as a subset of variables receiving the financial impact of quantity change".

[7] Note that the previous Konüs indicators of output price change and input price change included in the asymmetric decomposition of maximum profit change—i.e., expressions (11) and (15), as well as each of the four indicators of output and input price change included in the symmetric decomposition—expressions (24) and (25), reflect changes in prices and optimal quantities. However, in these terms the individual contribution of changes in prices cannot be separated from the change in optimal quantities because they cannot be expressed in terms of proper Bennet indicators.

reference, we define the hypothetical $\pi^1(w^0,p^0)$ for technological change in (22), $\pi^0(w^0,p^1)$ for output price change in (24), and $\pi^0(w^1,p^0)$ for input price change in (25). This exhausts the change in profit over time.

Although the three decompositions yield the same result, the different terms corresponding to technological change, and output and input prices differ numerically. Also, their construction differs, as the input price change term in the Bennet approach enters with a negative sign, in contrast to the asymmetric and symmetric decompositions where they enter additively. The reason is that in the Bennet decomposition increments in input prices results in positive values of the indicator, while in the latter approaches the value of the indicator is negative. Given the asymmetric definition of technological change (21), we believe the symmetric (26) or Bennet (38) decompositions provide a more accurate view of the contribution that the technology and input and output prices make to profit change. Between the last two, the Bennet approach can disentangle the direct effect of changes in prices from the induced effect on quantities, which is embedded in the indicators of output and input price change.

Applying these decompositions empirically requires solving for all the intermediate profit functions that make up the indices or indicators. This can be done by approximating the profit function through regression analysis techniques known generally as Stochastic Frontier Analysis, SFA, introduced by Ref. [30] —see, e.g., Ref. [31]; or mathematical programming techniques known as Data Envelopment Analysis, DEA, which constitutes the non-parametric approach to production theory introduced by Charnes et al. (1978) from an operations research perspective. For an illustration of the latter approach comparing different decompositions of cost change, asymmetric or symmetric, see Ref. [7]. We also rely on the DEA approach to develop our *complete* decomposition of profit change in Section 4.

3. Exact decompositions of profit change including efficiency criteria

The previous decompositions of profit change based on the Konüs approach, (21), (25), and Bennet approach, (38), can be further qualified by resorting to duality theory. In this section, we show that profit inefficiency change $\Delta\pi I$, (5) and (6), can be decomposed into two mutually exclusive terms corresponding to technical inefficiency change and allocative inefficiency change.

To obtain these results, we rely on the duality between the profit function and the directional distance function, *DDF*, as introduced by Chambers et al. [8,32,33], drawing from Ref. [34]. For any firm, the directional distance function defines as[8]:

$$\vec{D}^t\left(x^{t'},y^{t'};\,g_x^t,g_y^t\right)=\max_{\beta^{t,t'}}\ \left\{\beta^{t,t'}:\left(x^{t'}-\beta^{t,t'}g_x^t,y^{t'}+\beta^{t,t'}g_y^t\right)\in T^t\right\},x^t\in\mathbb{R}_+^N,y^t\in\mathbb{R}_+^M,t,t'=0,1, \tag{39}$$

where the first superscript t in $\beta^{t,t'}$ denotes the period of the reference technology, while the second one t' denotes the period in which the firm is observed. This function expands outputs and contracts inputs towards the frontier of the technology by adding and subtracting the amount represented by $\beta^{t,t'}$ times the elements of the preassigned non-zero directional vector $g^t = (g_x^t, g_y^t) \geq (0_N, 0_M)$. Therefore, the optimal solution to the above maximization problem, $\beta^{t,t'*}$, can be interpreted as a measure of technical inefficiency. If both the technology and observed output and input quantities belong to the same period, $t = t'$, then technical inefficiency is measured contemporaneously, while if $t \neq t'$ then we are measuring technical inefficiency against technologies of alternative periods (also known as mix-period technical inefficiency).

Let us assume for the time being that $t = t'$. Because the directional distance function characterizes the technology: $\vec{D}^t(x^t,y^t;g^t) \geq 0$ if and only if $(x^t,y^t) \in T^t$, the profit function (3) can be expressed in the following way: $\pi^t(w^t,p^t) = \max_{x^t,y^t}\ \{p^t\cdot y^t - w^t\cdot x^t : \vec{D}^t(x^t,y^t;g^t)\geq 0\}$. Consequently, for firm (x^t,y^t), it is observed that $\pi(w^t,p^t) \geq p^t\cdot(y^t + D_T(x^t,\, y^t;g^t)g_y^t) - w^t\cdot(x^t - \vec{D}^t(x^t,\, y^t;g^t)g_x^t) = p^t\cdot\hat{y}^t - w^t\cdot\hat{x}^t\ \hat{y}^t$; i.e., maximum profit is greater than or equal to observed profit attained at any feasible output−input vector on the boundary of T^t. Here the feasible output−input vector corresponds to the projection of the firm under evaluation (x^t,y^t) to the production frontier in the direction set by the *DDF*, i.e., $(\hat{x}^t,\hat{y}^t) = (y^t + D_T(x^t,y^t;g^t)g_y^t, x^t - \vec{D}^t(x^t,y^t;g^t)g_x^t)$. This formulation showing the relationship between the profit function and the directional distance function allows defining the following additive Fenchel-Mahler inequality [35]:

$$\frac{\pi^t(w^t,p^t)-(p^t\cdot y^t-w^t\cdot x^t)}{p^t\cdot g_y^t+w^t\cdot g_x^t}=\widetilde{\pi}^t(\widetilde{w}^t,\widetilde{p}^t)-\widetilde{\pi}^t\geq\vec{D}^t(x^t,y^t;g^t), \tag{40}$$

where $(\widetilde{w}^t,\widetilde{p}^t) = (w^t,p^t)/(p^t\cdot g_y^t + w^t\cdot g_x^t)$ are normalized prices, and therefore $\widetilde{\pi}^t(\widetilde{w}^t,\widetilde{p}^t) - \widetilde{\pi}^t$ constitutes a measure of normalized (also called *Nerlovian*) profit inefficiency equivalent to (4).[9] Note that if the directional vectors g^t are firm specific, the individual normalization factors entering the denominator in (40) would be different, and normalized profit inefficiency would not be comparable across firms. Choosing a common g^t as many authors do in empirical applications would solve the comparability problem. However, this solution may be too restrictive. Indeed, all that is required to ensure comparability among firms is the normalization factor $p^t\cdot g_y^t + w^t\cdot g_x^t$ to be common to all firms. We take advantage of this to make all profit change decompositions in this study numerically comparable and propose choosing a value for the directional vector that renders the normalization (*N*) factor equal to a positive scalar k: $p^t\cdot g_y^{Nt} + w^t\cdot g_x^{Nt} = k$, where '*N*' denotes the value of the different elements of the directional vector satisfying this normalization. In this simple way, the normalization factor is the same for all the firms.

[8] The directional distance function corresponds to the concept of shortage function introduced by Ref. [34]; p. 242, Definition 4.1). Luenberger's shortage function measures the distance of a production plan to the boundary of the production possibility set in the direction of the vector g [32]. redefine the shortage function as efficiency measure, introducing the concept of directional distance function [8]. discuss the properties of the directional distance function, including translation invariance, continuity, monotonicity and concavity.

[9] Alternative decompositions of profit inefficiency based on different characterizations of the production technology have been proposed in the literature; e.g. Ref. [36], use the weighted additive distance function, while [37] rely on the slack based directional distance function to decompose cost inefficiency, which can be easily extended to decompose profit inefficiency. Any of these definitions could be used in principle in the remaining of this section to obtain equivalent results, although the basic properties of the different terms in which profit change decomposes would depend on the preferred approach [38]. study the alternative proposals found in the literature to decompose profit inefficiency.

Additionally, given a preassigned directional vector g^t, g^{Nt} is a vector that follows the same direction than the original vector g^t but with different size (norm).

To show that profit decompositions are comparable across firms under the above normalization we recall the projection of the firm under evaluation on the frontier, $(\hat{x}^t, \hat{y}^t)$, and rewrite the value of the directional distance function—representing a measure of technical inefficiency: $TI(x^t, y^t; g^{Nt})$—as follows:

$$\vec{D}^t\left(x^t, y^t; g^{Nt}\right) = \frac{(p^t \cdot \hat{y}^t - w^t \cdot \hat{x}^t) - (p^t \cdot y^t - w^t \cdot x^t)}{p^t \cdot g_y^{Nt} + w^t \cdot g_x^{Nt}} = \tilde{\hat{\pi}}^t - \tilde{\pi}^t \equiv TI\left(x^t, y^t; g^{Nt}\right). \quad (41)$$

In this way, $\vec{D}^t(x^t, y^t; g^{Nt})$ may be interpreted as a normalized profit difference due to technical inefficiencies. The numerator in (41) is a money metric measure of inefficiency (the difference between profit evaluated at the technically efficient projection point and observed profit), which is normalized by the factor $(p^t \cdot g_y^{Nt} + w^t \cdot g_x^{Nt})$. Given that the components of the vector $g_x^{Nt} = (g_x^{Nt}, g_y^{Nt})$ are expressed in the same units of measurement than the original inputs and outputs, the factor $(p^t \cdot g_y^{Nt} + w^t \cdot g_x^{Nt})$ is expressed in monetary terms. Under this reinterpretation of the directional distance function, $\vec{D}^t(x^t, y^t; g^{Nt})$ relativizes the value of (technical) inefficiency in terms of the economic value of $(p^t \cdot g_y^{Nt} + w^t \cdot g_x^{Nt})$. This differs from the interpretation of the *DDF* in studies of technical—rather than economic—inefficiency, where it measures the distance of a firm to the frontier as the number of times the vector g^{Nt} fits between both—implying that if the values of the directional vectors were different across firms (in magnitude and/or direction) their corresponding *DDFs* would be incomparable.

In our case, by adopting the normalization factor $p^t \cdot g_y^{Nt} + w^t \cdot g_x^{Nt} = k$ we ensure that the denominator in (41) takes the same economic value across firms, making the values of the *DDFs* comparable. This is the strategy that we adopt in the last part of the study, where the directional vectors are considered endogenous. Moreover, without loss of generality, if we consider $k = 1$, the directional vector is defined in such way that the *DDF* measures technical inefficiency in monetary terms, see Ref. [11] and Kapelko et al. (2022). Under this condition we can remove the tilde '$\sim$' from our notation in (40), showing that the definitions of normalized observed profit, maximum profit and profit inefficiency coincide with those already presented in Section 2. We can then proceed to decompose profit inefficiency (4) departing from (40) under condition $p^t \cdot g_y^{Nt} + w^t \cdot g_x^{Nt} = 1$. First, the left hand side in (40) is a measure of overall profit inefficiency comparing maximum profit to observed profit in period t: $\pi I^t(w^t, x^t, p^t, y^t; g^{Nt}) \equiv \pi^t(w^t, p^t) - \pi^t$. Second, recalling the interpretation of technical inefficiency as the monetary loss caused by failing to exploit the best available technology: $\vec{D}^t(x^t, y^t; g^{Nt}) \equiv TI(x^t, y^t; g^{Nt})$ in (41), we can render (40) an equality by introducing a residual allocative efficiency term, AI_{DDF}^t, defined as[10]:

$$\begin{aligned} AI_{DDF}^t\left(w^t, x^t, p^t, y^t; g^{Nt}\right) &\equiv \frac{\pi^t(w^t, p^t) - (p^t \cdot \hat{y}^t - w^t \cdot \hat{x}^t)}{p^t \cdot g_y^{Nt} + w^t \cdot g_x^{Nt}} \\ &= \pi I^t\left(w^t, x^t, p^t, y^t; g^{Nt}\right) - TI_{DDF}^t\left(x^t, y^t; g^{Nt}\right) \\ &= \left(\pi^t(w^t, p^t) - \pi^t\right) - \vec{D}^t\left(x^t, y^t; g^{Nt}\right). \end{aligned} \quad (42)$$

This last term measures the residual loss in profit caused by failing to choose the correct mix of output and input quantities that maximize profit given their market prices. [39] provide the rationale to interpret this as management failure due to a poor estimation of market prices in the phase of production planning.

Consequently, we can decompose profit inefficiency as follows[11]:

$$\pi I^t\left(w^t, x^t, p^t, y^t; g^{Nt}\right) = TI_{DDF}^t\left(x^t, y^t; g^{Nt}\right) + AI_{DDF}^t\left(w^t, x^t, p^t, y^t; g^{Nt}\right), \quad (43)$$

From a temporal perspective, it is straightforward to define the change in profit inefficiency (5) as the sum of the change in technical and allocative inefficiencies:

$$\begin{aligned} \Delta\pi I &\equiv \pi I^0 - \pi I^1 = \left(\pi^0\left(w^0, p^0\right) - \pi^0\right) - \left(\pi^1\left(w^1, p^1\right) - \pi^1\right) \\ &= \underbrace{\vec{D}^0\left(x^0, y^0; g^{N0}\right)}_{TI_{DDF}^0} - \underbrace{\vec{D}^1\left(x^1, y^1; g^{N1}\right)}_{TI_{DDF}^1} \\ &\quad + \underbrace{\left(\left(\pi^0\left(w^0, p^0\right) - \pi^0\right) - \vec{D}^0\left(x^0, y^0; g^{N0}\right)\right)}_{AI_{DDF}^0} \\ &\quad - \underbrace{\left(\left(\pi^1\left(w^1, p^1\right) - \pi^1\right) - \vec{D}^1\left(x^1, y^1; g^{N1}\right)\right)}_{AI_{DDF}^1} \\ &= TI_{DDF}^0 - TI_{DDF}^1 + AI_{DDF}^0 - AI_{DDF}^1 = \Delta TI_{DDF} + \Delta AI_{DDF}. \end{aligned} \quad (44)$$

Substituting this result in the last terms of the above decompositions of profit change, (21), (26) and (38), sheds light on the sources of profit inefficiency; either technical or allocative. That is,

$$\Delta\pi \equiv \pi^1 - \pi^0 = \Delta\pi I + \tau^K + \rho^K + \varpi^K = \Delta TI_{DDF} + \Delta AI_{DDF} + \tau^K + \rho^K + \varpi^K, \quad (45)$$

$$\Delta\pi \equiv \pi^1 - \pi^0 = \Delta\pi I + \tau_S^K + \rho_S^K + \varpi_S^K = \Delta TI_{DDF} + \Delta AI_{DDF} + \tau_S^K + \rho_S^K + \varpi_S^K, \quad (46)$$

$$\Delta\pi = \pi^1 - \pi^0 = \Delta\pi I + \tau_S^K + \rho_B^K - \varpi_B^K = \Delta TI_{DDF} + \Delta AI_{DDF} + \tau_S^K + \rho_B^K - \varpi_B^K. \quad (47)$$

4. Complete decompositions of profit change including efficiency and productivity criteria

In this section, we introduce a final decomposition of profit change that includes technical and allocative inefficiency criteria and where the popular Luenberger indicator of productivity change introduced by Ref. [40] can be singled out in a meaningful way. In contrast to the *exact* denomination, we term this decomposition *complete*, because it comprises the Luenberger indicator, not included in the above formulations (45), (46), and (47). In doing so, we will also evaluate the pros and cons of the alternative decompositions and shed light on the interpretation of the different terms in which they decompose.

[10] Again, regarding notation, we adopt *in*efficiency rather than efficiency, as the greater the value of πI^t, TI_{DDF}^t and AI_{DDF}^t, the worse is the firm's performance. Other authors adhere to the efficiency notation by taking negative values: $TE_{DDF}^t(x^t, y^t; g) \equiv -D_T(x^t, y^t; g^{Nt})$, *e.g.*, Ref. [24].

[11] Note that in this decomposition all three terms are numerically comparable among themselves and across different firms because they are normalized by the same factor, $p^t g_y^{Nt} + w^t g_x^{Nt}$.

Let us consider the standard Bennet decomposition of profit change into Bennet's quantity and price indicators (33) for a firm evaluated in two successive periods, $t = 0,\ 1$, with corresponding quantities and prices: (x^t, y^t) and (w^t, p^t). Also, given a normalization factor $(p^t g_y^t + w^t g_x^t)$, let us recall the definitions of normalized observed profit and maximum profit, denoted by $\widetilde{\pi}^t$ and $\widetilde{\pi}(\widetilde{w}^t, \widetilde{p}^t)$, respectively. Then, mirroring (33), normalized profit change can be decomposed as follows—see expression (8) in Ref. [10]; p. 177):

$$\Delta\widetilde{\pi} = \widetilde{\pi}^1 - \widetilde{\pi}^0 = \widetilde{p}^1\cdot y^1 - \widetilde{w}^1\cdot x^1 - \left(\widetilde{p}^0\cdot y^0 - \widetilde{w}^0\cdot x^0\right)$$
$$= \underbrace{\left[\underbrace{\frac{1}{2}(\widetilde{p}^0 + \widetilde{p}^1)\cdot(y^1 - y^0)}_{\widetilde{Q}_y^B \equiv \widetilde{Q}_y^B(\widetilde{p}^1, y^1, \widetilde{p}^0, y^0)} - \underbrace{\frac{1}{2}(\widetilde{w}^0 + \widetilde{w}^1)\cdot(x^1 - x^0)}_{\widetilde{Q}_x^B \equiv \widetilde{Q}_x^B(\widetilde{w}^1, x^1, \widetilde{w}^0, x^0)}\right]}_{\widetilde{Q}^B(\widetilde{w}^1, \widetilde{p}^1, x^1, y^1, \widetilde{w}^0, \widetilde{p}^0, x^0, y^0)}$$
$$+ \underbrace{\left[\underbrace{\frac{1}{2}(\widetilde{p}^1 - \widetilde{p}^0)\cdot(y^0 + y^1)}_{\widetilde{P}_y^B \equiv \widetilde{P}_y^B(\widetilde{p}^1, y^1, \widetilde{p}^0, y^0)} - \underbrace{\frac{1}{2}(\widetilde{w}^1 - \widetilde{w}^0)\cdot(x^0 + x^1)}_{\widetilde{P}_x^B \equiv \widetilde{P}_x^B(\widetilde{w}^1, x^1, \widetilde{w}^0, x^0)}\right]}_{\widetilde{P}^B(\widetilde{w}^1, \widetilde{p}^1, x^1, y^1, \widetilde{w}^0, \widetilde{p}^0, x^0, y^0)}, \tag{48}$$

where $\widetilde{Q}^B \equiv \widetilde{Q}^B(\widetilde{w}^1, \widetilde{p}^1, x^1, y^1, \widetilde{w}^0, \widetilde{p}^0, x^0, y^0)$ and $\widetilde{P}^B \equiv \widetilde{Q}^B(\widetilde{w}^1, \widetilde{p}^1, x^1, y^1, \widetilde{w}^0, \widetilde{p}^0, x^0, y^0)$ represent the normalized Bennet quantity and price indicators between the base and comparison periods.[12]

The Bennet quantity indicator $\widetilde{Q}^B$ is to be decomposed according to technical (quantity) and allocative (price) criteria. Recalling the (normalized) definition of profit inefficiency change (6), it can be shown that[13]:

$$\widetilde{Q}^B = \Delta\widetilde{\pi}I + \left[\widetilde{\pi}^1\left(\widetilde{w}^1, \widetilde{p}^1\right) - \widetilde{\pi}^0\left(\widetilde{w}^0, \widetilde{p}^0\right)\right] - \widetilde{P}^B, \tag{49}$$

where $\Delta\widetilde{\pi}I = \widetilde{\pi}I^0 - \widetilde{\pi}I^1 = [(\widetilde{\pi}^0(\widetilde{w}^0, \widetilde{p}^0) - \widetilde{\pi}^0) - (\widetilde{\pi}^1(\widetilde{w}^1, \widetilde{p}^1) - \widetilde{\pi}^1)]$. Considering our favored definition of the normalization factor, $p^t\cdot g_y^{Nt} + w^t\cdot g_x^{Nt} = 1$, we can, once again, drop the tilde '∼', which renders expressions (33) and (48) equal. Afterwards, substituting in the above expression the profit inefficiency decomposition into technical and allocative inefficiencies as presented in (44), $\Delta\pi I = \Delta TI_{DDF} + \Delta AI_{DDF}$, one obtains:

$$\begin{aligned} Q^B &= \Delta TI_{DDF} + \Delta AI_{DDF} + \left[\pi^1\left(w^1, p^1\right) - \pi^0\left(w^0, p^0\right)\right] - P^B \\ &= \left[\overrightarrow{D}^0\left(x^0, y^0; g^{N0}\right) - \overrightarrow{D}^1\left(x^1, y^1; g^{N1}\right)\right] + \left[\left(\left(\pi^0\left(w^0, p^0\right) - \pi^0\right)\right.\right. \\ &\quad \left.\left. - \overrightarrow{D}^0\left(x^0, y^0; g^{N0}\right)\right) - \left(\left(\pi^1\left(w^1, p^1\right) - \pi^1\right) - \overrightarrow{D}^1\left(x^1, y^1; g^{N1}\right)\right)\right] \\ &\quad + \left[\pi^1\left(w^1, p^1\right) - \pi^0\left(w^0, p^0\right)\right] - P^B\left(w^1, p^1, x^1, y^1, w^0, p^0, x^0, y^0\right). \end{aligned} \tag{50}$$

The term in the last row of the above expression, corresponding to maximum profit change minus the Bennet price indicator, is equal to:

$$\begin{aligned} \left[\pi^1\left(w^1, p^1\right) - \pi^0\left(w^0, p^0\right)\right] - P^B &= \tau_{DDF}\left(T^0, T^1, x^0, y^0, x^1, y^1; g^{'0}, g^{'1}\right) \\ &\quad + PE_{DDF}\left(T^0, T^1, x^0, y^0, w^0, p^0, x^1, y^1, w^1, p^1; g^{'0}, g^{'1}\right) \\ &= \frac{1}{2}\left[\overrightarrow{D}^1\left(x^0, y^0; g^{N0}\right) - \overrightarrow{D}^0\left(x^0, y^0; g^{N0}\right)\right. \\ &\quad \left. + \overrightarrow{D}^1\left(x^1, y^1; g^{N1}\right) - \overrightarrow{D}^0\left(x^1, y^1; g^{N1}\right)\right] \\ &\quad + \frac{1}{2}\left[\left(\pi\left(w^1, p^1\right) - \left(p^1 y^0 - w^1 x^0\right)\right) - \left(\pi\left(w^0, p^0\right)\right.\right. \\ &\quad \left. - \left(p^0 y^0 - w^0 x^0\right)\right) - \left(\overrightarrow{D}^1\left(x^0, y^0; g^{N0}\right)\right. \\ &\quad \left.\left. - \overrightarrow{D}^0\left(x^0, y^0; g^{N0}\right)\right)\right] + \frac{1}{2}\left[\left(\pi\left(w^1, p^1\right) - \left(p^1 y^1\right.\right.\right. \\ &\quad \left.\left. - w^1 x^1\right)\right) - \left(\pi\left(w^0, p^0\right) - \left(p^0 y^1 - w^0 x^1\right)\right) \\ &\quad \left. - \left(\overrightarrow{D}^1\left(x^1, y^1; g^{N1}\right) - \overrightarrow{D}^0\left(x^1, y^1; g^{N1}\right)\right)\right], \end{aligned} \tag{51}$$

where $\tau_{DDF} \equiv \tau_{DDF}(T^0,\ T^1,\ x^0,\ y^0 x^1,\ y^1; g^{0*},\ g^{1*}) = \frac{1}{2}[\overrightarrow{D}^1(x^0, y^0; g^{N0}) - \overrightarrow{D}^0(x^0, y^0; g^{N0}) + \overrightarrow{D}^1(x^1, y^1; g^{N1}) - \overrightarrow{D}^0(x^1, y^1; g^{N1})]$ is the conventional additive measure of technological change presented in Ref. [40]; and $PE_{DDF} \equiv PE_{DDF}(T^0, T^1, x^0, y^0, w^0, p^0, x^1, y^1, w^1, p^1; g^{N0}, g^{N1})$ is a remaining term that [9]; p. 180) name 'price effect'. These authors contend that "*the residual shift of the profit boundary attributed to the relative price change is captured by PE_{DDF}, which represents the impact of the change in relative output–input prices on the maximum profit. In fact, this effect is deflated by the impact of the pure technical effect,* τ_{DDF}".[14] Indeed, τ_{DDF} can be singled out as part of PE_{DDF} in (51), rearranging the expression in the following terms:

$$PE_{DDF} = \frac{1}{2}\left[\underbrace{\left(\pi^1\left(w^1, p^1\right) - \left(p^1 y^0 - w^1 x^0\right)\right)}_{\pi I^{H1,0}} - \underbrace{\left(\pi^0\left(w^0, p^0\right) - \pi^0\right)}_{\pi I^0}\right] + \frac{1}{2}\left[\underbrace{\left(\pi^1\left(w^1, p^1\right) - \pi^1\right)}_{\pi I^1} - \underbrace{\left(\pi^0\left(w^0, p^0\right) - \left(p^0 y^1 - w^0 x^1\right)\right)}_{\pi I^{H0,1}}\right] - \tau_{DDF}. \tag{52}$$

In the first line of the above expression, we find the structure of a *hypothetical cross-period* profit inefficiency change, $(\pi(w^1, p^1) - (p^1 y^0 - w^1 x^0)) - \pi I^0$, comparing the profit inefficiency that would have attained the base period firm under comparison period prices, $\pi I^{H1,0} = \pi(w^1, p^1) - (p^1\cdot y^0 - w^1\cdot x^0)$, with profit ineffiency in the base period, πI^0. Considering that the terms of this expression are reversed if compared to (5), we define the following term $\Delta\pi I^{H1,0} = \pi I^0 - \pi I^{H1,0} = \pi I^0 - (\pi(w^1, p^1) - (p^1\cdot y^0 - w^1\cdot x^0))$. Analogously, in the second line of (52) we also find once again the structure of a second hypothetical cross-period profit inefficiency change, $\pi I^1 - (\pi^0(w^0,\ p^0) - (p^0\cdot y^1 - w^0\cdot x^1))$, this time comparing actual profit inefficiency in the comparison period, πI^1, to the profit efficiency that would have attained the comparison period firm under based period prices, $\pi I^{H0,1} = \pi^0(w^0,\ p^0) - (p^0\cdot y^1 - w^0\cdot x^1)$. Reversing the terms we denote this second expression as $\Delta\pi I^{H0,1} = \pi I^{H0,1} - \pi I^1 = (\pi^0(w^0, p^0) - (p^0\cdot y^1 - w^0\cdot x^1)) - \pi I^1$. We provide rationale for these terms in the following section.

Alternatively, it is possible to rewrite the 'price effect' decomposition in terms of allocative inefficiencies, i.e.,

[12] [10] denotes the Bennet indicators $\widetilde{Q}^B$ and $\widetilde{P}^B$ by $DTFPROD^B(1,0)$ and $DTPR^B(1,0)$, respectively, where (1,0) stands for the array of quantities and prices: $(\widetilde{w}^1, \widetilde{p}^1, x^1, y^1, \widetilde{w}^0, \widetilde{p}^0, x^0, y^0)$.

[13] The Bennet quantity indicator is what [9]; eq. (6)) define as "the profit-Luenberger productivity indicator πL"; i.e., $\pi L \equiv \widetilde{Q}^B$.

[14] Again, notation has been changed to match the unified framework presented here; i.e., $\tau_{DDF}(T^0, T^1, x^0, y^0, x^1, y^1; g^{N0}, g^{N1})$ is denoted as $TC(T^0, T^1, x^0, y^0, x^1, y^1; g^0, g^1)$ by Refs. [9,10].

$$PE_{DDF} = \frac{1}{2}\Bigg[\Big(\pi^1(w^1,p^1) - (p^1y^0 - w^1x^0) - \vec{D}^1(x^0,y^0;g^{N1})\Big) - \underbrace{\Big((\pi^0(w^0,p^0) - \pi^0) - \vec{D}^0(x^0,y^0;g^{N0})\Big)}_{AI^0}\Bigg] + \frac{1}{2}\Bigg[\underbrace{\Big((\pi^1(w^1,p^1) - \pi^1) - \vec{D}^1(x^1,y^1;g^{N1})\Big)}_{AI^1} - \Big(\pi^0(w^0,p^0) - (p^0y^1 - w^0x^1) - \vec{D}^0(x^1,y^1;g^{N0})\Big)\Bigg]. \quad (53)$$

4.1. Entering the Luenberger indicator of productivity change

We now recall the definition of the *Luenberger indicator of productivity change*, combining technical inefficiency change, ΔTI_{DDF}, and technological change, τ_{DDF}—see Chambers et al. [32,]:

$$\begin{aligned} L_{DDF}(x^0,y^0,x^1,y^1;g^{N0},g^{N1}) &= \tau_{DDF} + \Delta TI_{DDF} \\ &= \frac{1}{2}\Big[\vec{D}^1(x^0,y^0;g^{N1}) - \vec{D}^0(x^0,y^0;g^{N0}) \\ &\quad + \vec{D}^1(x^1,y^1;g^{N1}) - \vec{D}^0(x^1,y^1;g^{N0})\Big] \\ &\quad + \Big[\vec{D}^0(x^0,y^0;g^{N0}) - \vec{D}^1(x^1,y^1;g^{N1})\Big] \\ &= \frac{1}{2}\Big[\Big(\vec{D}^0(x^0,y^0;g^{N0}) - \vec{D}^0(x^1,y^1;g^{N0})\Big) \\ &\quad + \Big(\vec{D}^1(x^0,y^0;g^{N1}) - \vec{D}^1(x^1,y^1;g^{N1})\Big)\Big]. \end{aligned} \quad (54)$$

Making use of the Luenberger indicator $L_{DDF} \equiv L_{DDF}(x^0,y^0,x^1,y^1;g^{N0},g^{N1})$, we can rewrite the Bennet quantity indicator (49) in the following terms (see also (50) and (51)):

$$\begin{aligned} Q^B = L_{DDF} + \Delta AI_{DDF} + PE_{DDF} &= \Delta\pi I + \tau_{DDF} + PE_{DDF} \\ &= \Delta TI_{DDF} + \Delta AI_{DDF} + \tau_{DDF} + PE_{DDF}. \end{aligned} \quad (55)$$

Coming back to the decomposition of profit change, and considering this last expression of the Bennet quantity indicator, we can define normalized profit change (48) including the 'price effect' as follows:

$$\Delta\pi = \pi^1 - \pi^0 = p^1y^1 - w^1x^1 - (p^0y^0 - w^0x^0) = Q^B + P^B = \Delta\pi I + \tau_{DDF} + PE_{DDF} + P^B. \quad (56)$$

Finally, identifying in the Bennet price change indicator (48) the corresponding indicators of output price change, $\rho^B = \frac{1}{2}(p^1 - p^0)\cdot(y^0 + y^1)$, and input price change, $\varpi^B = \frac{1}{2}(w^1 - w^0)\cdot(x^0 + x^1)$—with $P^B = \rho^B - \varpi^B$, we obtain a simpler decomposition of profit change whose structural form is the same that the one obtained for the *exact* decomposition (47) based on Bennet indicators of price change (but uncapable of informing about the change in optimal output and input quantities), except for the additional price effect term, PE_{DDF}; i.e.,

$$\begin{aligned} \Delta\pi = \pi^1 - \pi^0 &= \Delta\pi I + \tau_{DDF} + \rho^B - \varpi^B + PE_{DDF} \\ &= \Delta TI_{DDF} + \Delta AI_{DDF} + \tau_{DDF} + \rho^B - \varpi^B + PE_{DDF}, \end{aligned} \quad (57)$$

where the Luenberger indicator of productivity change can be singled out as $L_{DDF} = \Delta TI_{DDF} + \tau_{DDF}$.

4.2. Is there a meaningful 'price effect' worth considering?

Because the *exact* decomposition of profit change including our Bennet indicators of price change (47) has the same structural form that the decomposition including the technical inefficiency change indicator and the allocative inefficiency change indicator, we realize that the additional price effect term PE_{DDF} in the *complete* decomposition (57) is a byproduct of the introduction of the Luenberger indicator. Including a technological change term based solely on quantities—as opposed to the *exact* decompositions where technological change includes price information as weights, results in the appearance of PE_{DDF} as a residual. Whether or not there is value added in PE_{DDF}, so we can gain further insights on the sources of profit change, depends on our ability to provide a compelling interpretation of this term. However, the interpretation of PE_{DDF} has not been thoroughly addressed in the existing literature. The reason is that it is difficult to provide rationale to the terms including the hypothetical cross-period profit inefficiencies in (52). Nevertheless, we anticipate that in Section 4.2.3 below we provide a meaningful interpretation within production theory by showing that (52) is equivalent to the Luenberger indicator of technological change under a choice of directional vector that makes PE_{DDF} nil.

The relevance of discussing the 'price effect' goes beyond the *exact* and *complete* decompositions of profit change studied here. The reason is that terms with the same structure can be found not only in similar proposals decomposing profit efficiency change in the additive framework involving Luenberger-type indicators, but also in the equivalent multiplicative cost, revenue, or profitability change decompositions. An example of the former is [14]; who decompose the Bennet quantity indicator in a similar fashion to (55); i.e., into a Luenberger (quantity) productivity indicator based on the weighted additive distance function, and a component "*measuring (price) allocative inefficiency change over time*"—see expression (19) on page 208 in Ref. [14]; and whose structure precisely corresponds to the 'price effect' as presented in (53). In multiplicative decompositions of cost (revenue) change and profitability change, equal 'price' terms involving cross-period allocative inefficiencies can be found in Ref. [15] (expression (19) on page 402), who decompose cost efficiency into an input-oriented Malmquist productivity index and a 'price or allocative inefficiency' term [17]. (expression (25) on page 130)—later reintroduced by Ref. [41] (expression (29) on page 27), decompose profitability change into analogous Malmquist productivity indices and a 'price or allocative inefficiency' term.

Unsurprisingly, cross-period price efficiency terms appear as residuals in all sorts of decompositions including productivity indicators and indices based on quantities, with authors either trying to make sense out of them, or simply disregarding their existence as approximation errors. We discuss in what follows three different ways to handle the price effect term PE_{DDF}. The first possibility explores meaningful interpretations of the different *hypothetical cross-period* profit inefficiency components of PE_{DDF}, once they are compared to actual profit inefficiency change. This method gives rise to yet an equivalent decomposition of profit change that merges profit inefficiency change and price effects. The new decomposition offers further insight to managers on how their firms are faring over time given the changing technological and market conditions. The second one, proposed by Ref. [10]; dismisses PE_{DDF} altogether, and, based on rather strong theoretical assumptions, deems it a residual showing up as measurement error in empirical studies (which should be understood as deviations from the theoretical assumptions). Because these assumptions are clearly at odds with reality, e.g., allocative efficiency is assumed away, it seems that a better solution is needed from an empirical perspective. Our third and preferred proposal overcomes this restrictive setting and, making use of the flexibility of the directional distance function, overcomes the problem posed by the presence of the 'price effect' by nullifying it. Here we show that if the directional vectors (g^{N0}, g^{N1}) are chosen appropriately, then the different components of the price effect cancel out among themselves, resulting in $PE_{DDF} = 0$. This makes the *exact* and *complete* decompositions of profit change previously defined to have the same structural form, rendering them fully comparable one-to-one.

What is more, in this last case, i.e., when PE_{DDF} becomes nil, under the usual assumptions of profit maximizing firms in periods 0 and 1 (see Ref. [42], which implies $AI^0 = AI^1 = 0$, we get an interpretation of the Luenberger indicator as a Bennet quantity change indicator (see the first part of (55)). Indeed, to obtain the above result we do not need to assume profit maximizing firms in both periods but nil allocative inefficiency change over time, i.e., $AI^0 = AI^1$. This interpretation of the Luenberger indicator is important because, unlike the Malmquist index, the value of the Luenberger indicator has not been clearly understood so far. Only [24] reached a similar result in the literature but under stronger conditions (he needed to assume that the technologies in the two assessed periods are characterized by quadratic distance functions with time-invariant second order coefficients and additional allocative efficiency in the followed direction of projection onto the frontiers – see Theorem 7.2 in Ref. [24]. Lastly, the equivalence between the Luenberger indicator, which does not use market price information in its definition, and the Bennet quantity change indicator, which uses market prices as weights, is parallel to the existing relationship between the Malmquist index and the Törnqvist index established by Ref. [42].

4.2.1. Making sense of the components of the 'price effect'

The different components of the price effect PE_{DDF} can be given a meaningful interpretation if we evaluate this term in light of its definition in (52) or, equivalently, (53). Here we interpret PE_{DDF} by looking at the cross-period profit inefficiencies, while similar reasonings can be applied to the cross-period allocative profit inefficiency terms once technological change τ_{DDF} is included in the expressions of profit inefficiency We show the relevance of the hypothetical cross-period profit inefficiencies by substituting expression (52)–or (53)– in the Bennet quantity indicator (56), yielding:

$$\begin{aligned} Q^B &= L_{DDF} + \Delta AI_{DDF} + PE_{DDF} = \Delta TI_{DDF} + \Delta AI_{DDF} + \tau_{DDF} + PE_{DDF} \\ &= \Delta\pi I + \frac{1}{2}\left[-\Delta\pi I^{H1,0} - \Delta\pi I^{H0,1}\right] \\ &= \frac{1}{2}\left[\left(\pi^1(w^1,p^1) - (p^1\cdot y^0 - w^1\cdot x^0)\right) - \pi I^0\right] + \frac{1}{2}\left[\pi I^1 - \left(\pi^0(w^0,p^0)\right.\right. \\ &\quad \left.\left. - (p^0\cdot y^1 - w^0\cdot x^1)\right)\right] \\ &= \frac{1}{2}\left[\Delta\pi I + \left(\pi^1(w^1,p^1) - (p^1\cdot y^0 - w^1\cdot x^0)\right) - \pi I^0\right] + \frac{1}{2}\left[\Delta\pi I + \pi I^1\right. \\ &\quad \left. - \left(\pi^0(w^0,p^0) - (p^0\cdot y^1 - w^0\cdot x^1)\right)\right] \\ &= \frac{1}{2}\left[\left(\pi^1 - (p^1\cdot y^0 - w^1\cdot x^0)\right) + \left((p^0\cdot y^1 - w^0\cdot x^1) - \pi^0\right)\right]. \end{aligned} \tag{58}$$

This expression of Q^B leaves out the terms based on distance functions by focusing on the hypothetical cross-period profit inefficiency changes that constitute PE_{DDF}. The second and third lines in the above expression show that Q^B is driven by the comparison of $-\Delta\pi I^{H1,0} = (\pi^1(w^1,p^1) - (p^1\cdot y^0 - w^1\cdot x^0)) - \pi I^0$ and $-\Delta\pi I^{H0,1} = \pi I^1 - (\pi^0(w^0,p^0) - (p^0\cdot y^1 - w^0\cdot x^1))$, to actual profit efficiency change, $\Delta\pi I$. The fourth line shows that Q^B can be finally rewritten as the weighted sum of $\pi^1 - (p^1\cdot y^0 - w^1\cdot x^0)$ and $(p^0\cdot y^1 - w^0\cdot x^1) - \pi^0$. Profit decreases over time if both terms are negative: $\pi^1 - (p^1\cdot y^0 - w^1\cdot x^0) < 0$ and $(p^0\cdot y^1 - w^0\cdot x^1) - \pi^0 < 0$, or the sum of the two is negative. Conversely, profit will increase if $\pi^1 - (p^1\cdot y^0 - w^1\cdot x^0) > 0$ and $(p^0 y^1 - w^0 x^1) - \pi^0 > 0$, or the sum of the two is positive.

To interpret these values we start with the first term presented in the third line of (58), $\Delta\pi I + (\pi^1(w^1,p^1) - (p^1\cdot y^0 - w^1\cdot x^0)) - \pi I^0$, or $\Delta\pi I - \Delta\pi I^{H1,0}$. One can see some merit in knowing to what extent a firm would be more or less profit (in)efficient in the future by correctly forecasting upcoming technological changes and prices. Indeed, such prospective simulation is something that all firms perform in reality, because it allows them to evaluate their reactions to changing conditions in the technological landscape and the input and output markets. If an industry is prone to technological progress and markets' equilibria are unstable because of aggregate demand and supply shifts, then this exercise is even mandatory, simply because firms need to anticipate these changing conditions. However, this evaluation can only be performed *a posteriori*, by confronting the hypothetical profit inefficiency change that the base period firm would had experienced under the comparison period prices to actual profit inefficiency change, i.e., $(\pi^1(w^1,p^1) - (p^1\cdot y^0 - w^1\cdot x^0)) - \pi I^0$ versus $\Delta\pi I = \pi I^0 - \pi I^1$. Note that this is precisely the first expression in the third line of (58), whose sign we evaluate; i.e., $\frac{1}{2}[\Delta\pi I + (\pi^1(w^1,p^1) - (p^1\cdot y^0 - w^1\cdot x^0)) - \pi I^0] \gtreqless 0$, which is equivalent to determine if $\Delta\pi I = \pi I^0 - \pi I^1 \gtreqless \pi I^0 - (\pi^1(w^1,p^1) - (p^1\cdot y^0 - w^1\cdot x^0))$. The worst case scenario would be $\pi I^0 - \pi I^1 < \pi I^0 - (\pi^1(w^1,p^1) - (p^1\cdot y^0 - w^1\cdot x^0))$ because then hypothetical profit inefficiency improvement, $\Delta\pi I^{H1,0}$, is larger than that actually observed, implying that the firm is unable to adapt its production process to the new technological and market conditions, to the extent that doing nothing: $(x^1,y^1) = (x^0,y^0)$, would have been a better option. This can be shown by cancelling out πI^0 in the previous expression, yielding $\pi I^1 (= \pi^1(w^1,p^1) - (p^1\cdot y^1 - w^1\cdot x^1)) > \pi^1(w^1,p^1) - (p^1\cdot y^0 - w^1\cdot x^0)$ or, equivalently, $p^1\cdot y^1 - w^1\cdot x^1 < p^1\cdot y^0 - w^1\cdot x^0$; i.e., hypothetical profit is larger than final profit in the comparison period. This negative contribution to profit change precisely corresponds to the first term in the fourth line of (58), i.e., $\pi^1 - (p^1\cdot y^0 - w^1\cdot x^0) < 0$. Conversely, if firms anticipate ever changing environments, one expects the opposite result: $\pi I^0 - \pi I^1 > (\pi^1(w^1,p^1) - (p^1\cdot y^0 - w^1\cdot x^0)) - \pi I^0$, equivalent to $p^1\cdot y^1 - w^1\cdot x^1 > p^1\cdot y^0 - w^1\cdot x^0$, or $\pi^1 - (p^1\cdot y^0 - w^1\cdot x^0) > 0$—a positive contribution to profit change, signaling that the firm does not end up in a worse economic situation.

In the same vein, we can compare the value of $\Delta\pi I + \pi I^1 - (\pi^0(w^0,p^0) - (p^0\cdot y^1 - w^0\cdot x^1))$—the second term in the third line of (58)—to actual profit inefficiency change in order to assess the ability of managers to anticipate future technological and market conditions. As before, rearranging this expression, if $\pi I^0 - \pi I^1 < (\pi^0(w^0,p^0) - (p^0\cdot y^1 - w^0\cdot x^1)) - \pi I^1$ (or $\Delta\pi I < \Delta\pi I^{H0,1}$), actual profit inefficiency improvement is even smaller than the hypothetical profit inefficiency change that the comparison period firm would had experienced under the based period prices, $\Delta\pi I^{H0,1}$. Hence, the firm, by shifting its production plan away from that of the base period—i.e., from (x^0,y^0) to (x^1,y^1)—finds itself in a worse situation with respect to the new technology T^1 and market prices (w^1,p^1) than with respect to their base period counterparts. This result can be summarized by cancelling out πI^1 in the above expression, yielding $\pi I^0 (= \pi^0(w^0,p^0) - (p^0\cdot y^0 - w^0\cdot x^0)) < (\pi^0(w^0,p^0) - (p^0\cdot y^1 - w^0\cdot x^1))$, or, equivalently, $p^0\cdot y^0 - w^0\cdot x^0 > p^0\cdot y^1 - w^0\cdot x^1$, and we find that hypothetical profit is less than observed profit in the base period. This negative contribution to profit change is equivalent to the one presented in the second term of the fourth line of (58), i.e., $(p^0\cdot y^1 - w^0\cdot x^1) - \pi^0 < 0$. Conversely, if $\pi I^0 - \pi I^1 > (\pi^0(w^0,p^0) - (p^0\cdot y^1 - w^0\cdot x^1)) - \pi I^1$, the firm's real performance is better than the hypothetical one. Now the firm, by changing its production plan to (x^1,y^1), has correctly anticipated the new technology T^1 and market conditions (w^1,p^1). Once again, cancelling out πI^1 in the above expression yields $\pi I^0 (= \pi^0(w^0,p^0) - (p^0\cdot y^0 - w^0\cdot x^0)) > \pi^0(w^0,p^0) - (p^0\cdot y^1 - w^0\cdot x^1)$—the second term in the fourth line of (58), implying that $p^0\cdot y^1 - w^0\cdot x^1 > p^0\cdot y^0 - w^0\cdot x^0$, the firm makes more profit using the comparison period production plan than it made in the base period.

Thanks to expression (58) and the discussion above comparing the change in cross-period and actual profit inefficiencies, we know that the Bennet quantity indicator Q^B measures the ability of the firm to anticipate future technological and market conditions, making a positive contribution to profit change. We contend that the information provided

Table 1

Exact and *complete* decompositions of profit change, $\Delta\pi$, including efficiency criteria. Exogenous (g^{N0}, g^{N1}), $p^t \cdot g_y^{Nt} + w^t \cdot g_x^{Nt} = 1$.

Decomposition of Profit change[a] $\Delta\pi$			Exp.	Profit inefficiency change	Technol. change	Output price change	Input price change	Price effect
Exact	Konüs	Asymm.	(45)	$\Delta\pi I = \Delta TI_{DDF} + \Delta AI_{DDF}$	τ^K	ρ^K	ϖ^K	–
		Symm.	(46)	$\Delta\pi I = \Delta TI_{DDF} + \Delta AI_{DDF}$	τ_S^K	ρ_S^K	ϖ_S^K	–
	Bennet[b]	Symm.	(47)	$\Delta\pi I = \Delta TI_{DDF} + \Delta AI_{DDF}$	τ_S^K	ρ_B^K	ϖ_B^K	–
Complete[c]	Bennet		(57)	$\Delta\pi I = \Delta TI_{DDF} + \Delta AI_{DDF}$	τ_{DDF}	ρ^B	ϖ^B	PE_{DDF}

Notes.

[a] Profit change and all its components are expressed in monetary units. This implies that in the complete decomposition the firms' normalization factors satisfy $p^t \cdot g_y^{Nt} + w^t \cdot g_x^{Nt} = 1$, $t = 0,1$.

[b] The price change indicators ρ_B^K and ϖ_B^K can be decomposed into Bennet indicators capturing the direct change in output and input prices: $P_{y^*}^{B,t}$ and $P_{x^*}^{B,t}$, $t = 0, 1$, and the induced effect in the optimal output and input quantities: $Q_{y^*}^{B,t}$ and $Q_{x^*}^{B,t}$, $t = 0, 1$.

[c] The Luenberger productivity index can be recovered as $L_{DDF} = \Delta TI_{DDF} + \tau_{DDF}$.

by these terms are useful to managers when studying the economic effects of changes in their production processes. Lastly, We can also provide an analogous explanation to PE_{DDF} including the technological change indicator as presented in expression (53), which subtracts the corresponding distance functions, and compares hypothetical cross-period allocative profit inefficiency changes: $(\pi(w^1, p^1) - (p^1 \cdot y^0 - w^1 \cdot x^0) - \vec{D}^1(x^0, y^0; g'^0)) - AI^0$ and $AI^1 - (\pi(w^0, p^0) - (p^0 \cdot y^1 - w^0 \cdot x^1) - \vec{D}^0(x^1, y^1; g'^1))$, to actual allocative profit inefficiency change, $-\Delta AI = AI^1 - AI^0$.

4.2.2. Dismissing the 'price effect' as measurement error

Departing from expression (50) and the definition of technological change τ_{DDF}, [10]; eq. (12)) solves for the 'price effect' in (51):

$$PE_{DDF} = \left[\pi^1\left(w^1, p^1\right) - \pi^0\left(w^0, p^0\right)\right] - P^B - \tau_{DDF}, \tag{59}$$

concluding that whatever the value of PE_{DDF}, it can be dismissed as "*a remainder, incapable of interpretation*". He argues that from maximum profit change $\pi^1(w^1, p^1) - \pi^0(w^0, p^0)$ above (driven by firms that are profit efficient and, therefore, technical and allocative efficient), the Bennet price indicator P^B removes, to a certain extent (as the quantity weights are different), the price change component, while in turn τ_{DDF} removes technological change. Consequently, PE_{DDF} is a remainder without independent interpretation.

[10] reinforces the above argument dismissing the price effect by recalling previous results that rule out the existence of allocative inefficiency in a purely theoretical setting that considers shadow prices.[15] Theorem 7.2 in Ref. [24]; p. 175) stresses that "*if the technologies in the two periods are characterized by quadratic distance functions with time-invariant second order coefficients and the firm is allocative efficient in the g-direction, that is,* $AI_{DDF}^1(\tilde{\overline{w}}^1, x^1, \tilde{\overline{p}}^1, y^1; g^{N1}) = AI_{DDF}^0(\tilde{\overline{w}}^0, x^0, \tilde{\overline{p}}^0, y^0; g^{N0}) = 0$, *then* $L_{DDF} = \tilde{\overline{Q}}^B$. *Viewed from this perspective expression (59) must be seen simply as approximation error*".[16] This can be clearly seen in expression (53) where PE_{DDF} is expressed in terms of allocative inefficiencies. Assuming that firms are allocative efficient implies that $\Delta AI_{DDF} = PE_{DDF} = 0$, and we immediately obtain that $\tilde{\overline{Q}}^B = L_{DDF}$ in (55). Note that we use the notation '$-$' above prices, indicating that under the above theoretical assumptions these correspond to shadow prices, for which firms are allocative efficient by definition. However, it is certain that the conditions required for this result will hardly be met in real life, because firms incur allocative inefficiencies in their everyday production and market processes. Indeed, the deviations from the theoretical conditions making ΔAI_{DDF} and PE_{DDF} different from zero, emerge in empirical studies that use market prices instead of shadow prices; i.e., ΔAI_{DDF} and PE_{DDF} reflect the deviation of (theoretical) shadow prices from (real) market prices.

Hence, we conclude that based on the interpretability of expression (58) and the existence of allocative inefficiency, dismissing ΔAI_{DDF} and PE_{DDF} as residuals capturing measurement errors or deviations from the stated theoretical conditions would be unsatisfactory in empirical studies. Therefore, we show in the following section that making use of the flexibility of the directional distance function we can accomplish a decomposition of the Bennet quantity indicator Q^B that retains allocative inefficiency change ΔAI_{DDF}—whose interpretation is undisputed—while cancelling PE_{DDF}, and resulting in $Q^B = L_{DDF} + \Delta AI_{DDF}$. Hence, this proposal relaxes the restrictive framework that forces the equality between the Bennet quantity indicator and the Luenberger productivity indicator in the decomposition of profit change (57). The solution that we next propose is possible in our context due to the natural flexibility of the DDF. This same strategy cannot be implemented in other frameworks based on Shephard's distance functions [15] or the weighted additive distance function [14] because the directions are given (input or output) or cannot be endogenized.

4.2.3. Cancelling the 'price effect' term

From (52) it is clear that what is required for suppressing PE_{DDF} is that the different components of this expression cancel out, which in turn implies that the two following conditions are satisfied simultaneously: $\pi(w^1, p^1) - (p^1 \cdot y^0 - w^1 \cdot x^0) - \pi I^0 = \tau_{DDF}^0 = \vec{D}^1(x^0, y^0; g_y^{N0}) - \vec{D}^0(x^0, y^0; g_y^{N0})$ and $\pi I^1 - (\pi(w^0, p^0) - (p^0 \cdot y^1 - w^0 \cdot x^1)) = \tau_{DDF}^1 = \vec{D}^1(x^1, y^1; g_y^{N1}) - \vec{D}^0(x^1, y^1; g_y^{N1})$, where τ_{DDF}^0 and τ_{DDF}^1 are Luenberger indicators of technological change considering the firm observed in the base or comparison period. It is possible to rely on the flexibility of the directional distance function to ensure that these conditions are meet by endogenizing the directional vectors g^{N0} and g^{N1}. We denote the directional vectors that satisfy these two conditions with the superscript '*', (g^{N0*}, g^{N1*}), reflecting the optimality of these values in terms of the decomposition of profit change (57). Here we show how we can determine the optimal directional vectors from an empirical perspective relying on Data Envelopment Analysis, DEA, techniques. According to

[15] This implies that we are considering shadow prices, i.e., those solving the duality problem $\vec{D}^t(x^t, y^t; g) = \min_{\tilde{\overline{w}}, \tilde{\overline{p}}} \pi(\tilde{\overline{w}}^t, \tilde{\overline{p}}^t) - (\tilde{\overline{p}}^t \cdot y^t - \tilde{\overline{w}}^t \cdot x^t)$, where $(\tilde{\overline{w}}^t, \tilde{\overline{p}}^t) = (\overline{w}^t, \overline{p}^t)/(\overline{p}^t \cdot g_y^t + \overline{w}^t \cdot g_x^t)$ and the directional vector g^t is chosen so that firms are allocative efficient, i.e., $\pi(\tilde{\overline{w}}^t, \tilde{\overline{p}}^t) = \tilde{\overline{p}}^t \cdot (y^t + D_T(x^t, y^t; g^{Nt}) g_y^{Nt}) - \tilde{\overline{w}}^t \cdot (x^t - \vec{D}^t(x^t, y^t; g^{Nt}) g_x^{Nt})$, $t = 0,1$.

[16] Once again notation has been changed in the cite to match that adopted in this study.

the principle of minimum extrapolation, DEA is capable of approximating the production technologies in the base and comparison periods, T^t, $t = 0,1$, from the set of $j = 1,\ldots,J$ firms; i.e.,

$$T^t = \left\{ (x^t, y^t) : \sum_{j=1}^{J} \lambda_j^t x_{jm}^t \leq x_m^t, m = 1, \ldots, M; \ \sum_{j=1}^{J} \lambda_j^t y_{jn}^t \geq y_n^t, n = 1, \ldots, N; \ \sum_{j=1}^{J} \lambda_j^t = 1, \lambda_j^t \geq 0, j = 1, \ldots, J \right\}, \tag{60}$$

Denoting by 'o' the firm whose profit change is being evaluated, (x_o^t, y_o^t), $t = 0,1$, we can determine the directional distance functions required to decompose the Bennet quantity indicator Q^B in (55), along with the elements of the directional vectors $(g_x^{Nt*}, g_y^{Nt*}) \in \mathbb{R}^M \times \mathbb{R}^N$, $t = 0,1$, that render $PE_{DDF} = 0$, by solving the following Bilevel program:

$$\begin{array}{llllr}
\max\limits_{g,\beta,\lambda} & \sum\limits_{t=0,1} \left(\sum\limits_{m=1}^{M} g_{x_m}^{Nt} + \sum\limits_{n=1}^{N} g_{y_n}^{Nt} \right) & & & (61.1) \\
s.t. & & & & \\
 & w^t \cdot g_{x_m}^{Nt} + p^t \cdot g_{y_n}^{Nt} = 1, & & t = 0,1 & (61.2) \\
 & \pi(w^1,p^1) - (p^1 \cdot y_o^0 - w^1 \cdot x_o^0) - (\pi(w^0,p^0) - (p^0 \cdot y_o^0 - w^0 \cdot x_o^0)) = \underbrace{\beta^{1,0} - \beta^{0,0}}_{\tau_{DDF}^0}, & & & (61.3) \\
 & \pi(w^1,p^1) - (p^1 \cdot y_o^1 - w^1 \cdot x_o^1) - (\pi(w^0,p^0) - (p^0 \cdot y_o^1 - w^0 \cdot x_o^1)) = \underbrace{\beta^{1,1} - \beta^{0,1}}_{\tau_{DDF}^1}, & & & (61.4) \\
 & \max\limits_{\beta,\lambda} \ \beta^{t,t'} & & t = 0,1, \quad t' = 0,1 & (61.5) \\
 & \quad s.t. & & & \\
 & \sum\limits_{j=1}^{n} \lambda_j^{t,t'} x_{mj}^{t} \leq x_{mo}^{t'} - \beta^{t,t'} g_{x_m}^{Nt'}, & m = 1,\ldots,M & t = 0,1, \quad t' = 0,1 & (61.6) \\
 & \sum\limits_{j=1}^{n} \lambda_j^{t,t'} y_{nj}^{t} \geq y_{no}^{t'} + \beta^{t,t'} g_{y_n}^{Nt'}, & n = 1,\ldots,N & t = 0,1, \quad t' = 0,1 & (61.7) \\
 & \sum\limits_{j=1}^{n} \lambda_j^{t,t'} = 1, & & & (61.8) \\
 & \lambda_j^{t,t'} \geq 0, & j = 1,\ldots,J, & t = 0,1 \quad t' = 0,1 & (61.9) \\
 & g_{x_m}^{Nt} \in R, & & m = 1,\ldots,M, \quad t = 0,1 & (61.10) \\
 & g_{y_n}^{Nt} \in R, & & n = 1,\ldots,N, \quad t = 0,1. & (61.11) \\
 & \beta^{t,t'} \in R, & & t = 0,1, \quad t' = 0,1 & (61.12)
\end{array} \tag{61}$$

where, following the notation in (39), $\beta^{1,0} = \overrightarrow{D}^1(x^0, y^0; g_y^{N0})$, $\beta^{0,0} = \overrightarrow{D}^0(x^0, y^0; g_y^{N0})$, $\beta^{1,1} = \overrightarrow{D}^1(x^1, y^1; g_y^{N1})$, and $\beta^{0,1} = \overrightarrow{D}^0(x^1, y^1; g_y^{N1})$.

A Bilevel formulation refers to a mathematical programming model with at least one of the constraints being an optimization problem itself. Model (61) consists of a structure based on two sub-levels: (61.1) and (61.5). Both levels are intertwined in a way that the higher-level decision problem (61.1) sets parameters influencing the lower-level decision problems (61.5), which, in turn, also affect the outcome of the higher-level problem. In our context, the first constraint of the higher-level decision problem (61.2) normalizes the two considered directional vectors to make the different terms of the Konüs and Bennet profit change decompositions numerically comparable. The second and third constraints of the higher-level optimization model, (61.3) and (61.4), ensure that the term PE_{DDF} in (53) is zero. Notice that, in these two constraints, the values of the directional distance functions, $\beta^{t,t'}$, reflect contemporary and cross-period technical inefficiency evaluations, where the firm observed in period t', $t' = 0,1$, is projected onto the efficient frontier of the technology observed in the same or alternate period t, $t = 0,1$. These values depend, in turn, on the directional vectors (g_x^{Nt}, g_y^{Nt}), $t = 0,1$, that our model seeks to determine. The following constraints in model (61.5) constitute the lower-level decision problem. Each one corresponds to the optimization model associated with the determination of the value of one of the four considered directional distance functions, i.e., $\beta^{t,t'}$ for all $t, t' = 0,1$. Notice that each of these lower-level problems incorporate the production technologies T^0 and T^1 through constraints (61.6) to (61.9) as defined in (60). The technologies are employed in the calculation of the directional distance functions $\beta^{0,0} = \overrightarrow{D}^0(x^0, y^0; g_y^{N0})$ and $\beta^{1,1} = \overrightarrow{D}^1(x^1, y^1; g_y^{N1})$—where firms observed in the base and comparison periods $t, t' = 0,1$ are evaluated against their own contemporary technologies $t = t'$, and the two additional cross-period evaluations of the firms observed in each of these two periods against the alternate periods $t \neq t'$, i.e., $\beta^{1,0} = \overrightarrow{D}^1(x^0, y^0; g_y^{N0})$, and $\beta^{0,1} = \overrightarrow{D}^0(x^1, y^1; g_y^{N1})$, respectively. Regarding the objective function of the higher-level problem, among all possible feasible solutions of the set of constraints described above, the higher-level model selects one that maximizes the sum of the components of the two directional vectors, as one of the simplest possible criteria to be considered.

As for computational issues, different techniques have been proposed to solve Bilevel Programming problems (see, for example, [43,44]). One usual strategy consists in reformulating the bi-level model as a single-level program. This reformulation leads to a standard program, on which we can apply available solvers. The approach that we follow is based on transforming model (61) into a single optimization problem by applying the Karush-Kuhn-Tucker (KKT) optimality conditions of the lower-level problems [45]. Following these authors, program (61) can be transformed into:

$$
\begin{array}{llll}
\max\limits_{g,\beta,\lambda} & \sum\limits_{t=0,1}\left(\sum\limits_{m=1}^{M} g_{x_m}^{Nt}+\sum\limits_{n=1}^{N} g_{y_n}^{Nt}\right) & & \\
s.t. & & & \\
 & w^t\cdot g_{x_m}^{Nt}+p^t\cdot g_{y_n}^{Nt}=1, & t=0,1 & (62.1)\\
 & \pi(w^1,p^1)-(p^1\cdot y_o^0-w^1\cdot x_o^0)-(\pi(w^0,p^0)-(p^0\cdot y_o^0-w^0\cdot x_o^0))=\underbrace{\beta^{1,0}-\beta^{0,0}}_{\tau_{DDF}^0}, & & (62.2)\\
 & \pi(w^1,p^1)-(p^1\cdot y_o^1-w^1\cdot x_o^1)-(\pi(w^0,p^0)-(p^0\cdot y_o^1-w^0\cdot x_o^1))=\underbrace{\beta^{1,1}-\beta^{0,1}}_{\tau_{DDF}^1}, & & (62.3)\\
 & \sum\limits_{j=1}^{n}\lambda_j^{t,t'}x_{mj}^{t'}=x_{mo}^{t'}-\beta^{t,t'}g_{x_m}^{Nt}-s_m^{t,t'-}, & m=1,\dots,M,\quad t=0,1,\quad t'=0,1 & (62.4)\\
 & \sum\limits_{j=1}^{n}\lambda_j^{t,t'}y_{nj}^{t'}=y_{no}^{t'}+\beta^{t,t'}g_{y_n}^{Nt}+s_n^{t,t'+}, & n=1,\dots,N,\quad t=0,1,\quad t'=0,1 & (62.5)\\
 & \sum\limits_{j=1}^{n}\lambda_j^{t,t'}=1, & t=0,1,\quad t'=0,1 & (62.6)\\
 & \sum\limits_{m=1}^{M}\nu_m^{t,t'}+\sum\limits_{n=1}^{N}\mu_n^{t,t'}=1, & t=0,1,\quad t'=0,1 & (62.7)\\
 & \sum\limits_{m=1}^{M}\nu_m^{t,t'}x_{mj}^{t}-\sum\limits_{n=1}^{N}\mu_n^{t,t'}y_{nj}^{t}+\psi^{t,t'}-\tau_j^{t,t'}=0, & j=1,\dots,J,\quad t=0,1,\quad t'=0,1 & (62.8)\\
 & \tau_j^{t,t'}\lambda_j^{t,t'}=0, & j=1,\dots,J,\quad t=0,1,\quad t'=0,1 & (62.9)\\
 & \nu_m^{t,t'}s_m^{t,t'-}=0, & m=1,\dots,M,\quad t=0,1,\quad t'=0,1 & (62.10)\\
 & \mu_n^{t,t'}s_n^{t,t'+}=0, & n=1,\dots,N,\quad t=0,1,\quad t'=0,1 & (62.11)\\
 & \lambda_j^{t,t'}\geq 0, & j=1,\dots,J,\quad t=0,1,\quad t'=0,1 & (62.12)\\
 & g_{x_m}^{Nt}\in R, & m=1,\dots,M,\quad t=0,1, & (62.13)\\
 & g_{y_n}^{Nt}\in R, & n=1,\dots,N,\quad t=0,1 & (62.14)\\
 & \beta^{t,t'}\in R, & t=0,1,\quad t'=0,1 & (62.15)\\
 & \nu_m^{t,t'},s_m^{t,t'-}\geq 0, & m=1,\dots,M,\quad t=0,1,\quad t'=0,1, & (62.16)\\
 & s_n^{t,t'+},\mu_n^{t,t'}\geq 0, & n=1,\dots,N,\quad t=0,1,\quad t'=0,1, & (62.17)\\
 & \psi^{t,t'}\in R, & t=0,1,\quad t'=0,1, & (62.18)\\
 & \tau_j^{t,t'}\geq 0, & j=1,\dots,J,\quad t=0,1,\quad t'=0,1, & (62.19)
\end{array}
\qquad (62)
$$

Constraints (62.9)–(62.11) are not linear. However, these sorts of conditions are easy to implement through Special Ordered Sets (SOS) [46]. SOS is a mathematical structure that allows specifying that a pair of variables cannot take strictly positive values at the same time through branching optimality strategies.

For each $j=1,\dots,J$ firm, program (62) provides all the information needed to decompose the Bennet quantity indicator, according to expression (55), into $Q^B=L_{DDF*}+\Delta AI_{DDF*}=\Delta\pi I+\tau_{DDF*}=\Delta TI_{DDF*}+\Delta AI_{DDF*}+\tau_{DDF*}$, where the price effect PE_{DDF}— resulting from a choice of directional vectors differing from the optimal one given by (62), $(g^{N0},g^{N1})\neq(g^{N0*},g^{N1*})$—disappears. Moreover, looking at expression (52) we realize that the Luenberger indicator of technological change under the optimal direction (g^{N0*},g^{N1*}) coincides with the term including the changes in hypothetical cross-period profit inefficiencies, which is now given a full meaning in production theory:

$$
\begin{aligned}
\tau_{DDF*}&=\frac{1}{2}\left[\vec{D}^{1}(x^0,y^0;g^{N1*})-\vec{D}^{0}(x^0,y^0;g^{N0*})+\vec{D}^{1}(x^1,y^1;g^{N1*})\right.\\
&\quad\left.-\vec{D}^{0}(x^1,y^1;g^{N0*})\right]\\
&=\frac{1}{2}\left[(\pi^1(w^1,p^1)-(p^1y^0-w^1x^0))-\pi I^0\right]+\frac{1}{2}\left[\pi I^1-(\pi^0(w^0,p^0)\right.\\
&\quad\left.-(p^0y^1-w^0x^1))\right]
\end{aligned}
\qquad (63)
$$

Therefore, program (62) ensures that the Bennet quantity indicator can be exhaustively decomposed into the quantity based Luenberger productivity indicator that accounts for inefficiency change and

Table 2
Exact and *complete* decompositions of profit change, $\Delta\pi$, including efficiency criteria. Endogenous (g^{N0*}, g^{N1*}).

Decomposition of Profit change[a] $\Delta\pi$			Exp.	Profit inefficiency change	Technol. change	Output price change	Input price change
Exact	Konüs	Asymm.	(45)	$\Delta\pi I = \Delta TI_{DDF*} + \Delta AI_{DDF*}$	τ^K	ρ^K	ϖ^K
		Symm.	(46)	$\Delta\pi I = \Delta TI_{DDF*} + \Delta AI_{DDF*}$	τ^K_S	ρ^K_S	ϖ^K_S
	Bennet[b]	Symm.	(47)	$\Delta\pi I = \Delta TI_{DDF*} + \Delta AI_{DDF*}$	τ^K_S	ρ^K_B	ϖ^K_B
Complete[c,d]	Bennet		(57)	$\Delta\pi I = \Delta TI_{DDF*} + \Delta AI_{DDF*}$	τ_{DDF}	ρ^B	ϖ^B

Notes.
[a] Profit change and all its components are expressed in monetary units. This implies that in the *complete* decomposition the firms' normalization factors satisfy $p^t \cdot g_y^{Nt*} + w^t \cdot g_x^{Nt*} = 1$, $t = 0,1$.
[b] The price change indicators ρ^K_B and ϖ^K_B can be decomposed into Bennet indicators capturing the direct change in output and input prices: $P^{B,t}_{y^*}$ and $P^{B,t}_{x^*}$, $t = 0$, 1, and the induced effect in the optimal output and input quantities: $Q^{B,t}_{y^*}$ and $Q^{B,t}_{x^*}$, $t = 0$, 1.
[c] The Luenberger productivity index can be recovered as $L_{DDF*} = \Delta TI_{DDF*} + \tau_{DDF*}$.
[d] $PE_{DDF*} = 0$ under the optimal endogenous directional vector (g^{N0*}, g^{N1*}).

technological change, and a price related indicator that can be meaningfully interpreted as allocative inefficiency change. This decomposition passes on to that of profit change (57), which rightly accounts for the effects of changes in output prices and input prices, ρ^B and ϖ^B, thereby reaching our goal.

Our proposal summarized above has the advantage of making all the *exact* and *complete* decompositions of profit change equivalent in structure, while relaxing the conditions imposed by Ref. [10] that require specific functional forms and an allocative efficient behavior on the part of the firms.

5. Discussion

Determining the sources of profit change over time is of paramount importance for firms' managers and stakeholders. In this study we provide a series of—Laspeyres, Paasche and Fisher—*exact* decompositions that, based on the existing literature on Konüs and Bennet quantity and price indicators, allows us to decompose profit change in five mutually exclusive terms: technical efficiency change, allocative efficiency change, technological change, output price change and input price change. We start our study by proposing three *exact* decompositions of profit change based on the Konüs approach, namely (21), (26) and (38). The first two differ in the definition of their technological change component, depending on whether it has an asymmetric structure with respect to its output price and input price arguments, which are dated on different periods, $t \neq t'$, rather than the same period, as the symmetric approach does. Because technological change differs, the remaining terms of output price change and input price change are also different, to accommodate the discrepancy. The third decomposition includes symmetric terms of technological change but redefines the output and input price indicators in terms of Bennet's indicators. This last decomposition retains the desirable—symmetric—definition of technological change, while allowing to differentiate between the direct effects of price changes and the induced contribution of optimal output and input quantities to the variation in maximum profit. All these decompositions rely on comparative (*ceteris paribus*) statics to compare alternative equilibria, before and after a change in each one of the underlying variables takes place, which in this case corresponds to changes in the technology, the output prices, and the input prices.

The *exact* decompositions identify as one relevant source of profit change the change of profit inefficiency, $\Delta\pi I$, but do not qualify this in terms of technical and allocative criteria. To account for this possibility, we follow the literature on efficiency analysis and decompose this term into technical efficiency change and allocative inefficiency change, $\Delta\pi I = \Delta TI_{DDF} + \Delta AI_{DDF}$. Our approach relies on the directional distance function as measure of technical inefficiency, $TI^t_{DDF}(x^t, y^t; g^{Nt}) \equiv \overrightarrow{D}^t(x^t, y^t; g^{Nt})$. The three resulting formulations decomposing profit inefficiency change, (45), (46) and (47), are numerically comparable with their previous counterparts thanks to the homogenization of the normalizing factor to $p^t \cdot g_y^{Nt} + w^t \cdot g_x^{Nt} = 1$. This ensures that the values of the technical efficiency change and allocative efficiency change terms are expressed in monetary units, and therefore add up to pecuniary profit inefficiency change $\Delta\pi I$ in (21), (26) and (38).

After enhancing the *exact* decompositions of profit change with technical and allocative inefficiency criteria, we show that it is possible to define a *complete* profit change decomposition based on the Bennet indicators, which includes a well-known measure of productivity change, namely the Luenberger productivity indicator. The Luenberger indicator, combining information related to technical efficiency change and technological change: $L_{DDF} = \Delta TI_{DDF} + \tau_{DDF}$, enters the Bennet quantity index and results in the appearance of a price term, PE_{DDF}; i.e., $Q^B = L_{DDF} + \Delta AI_{DDF} + PE_{DDF}$. This unintended consequence prevents the direct comparison of the *complete* decomposition of profit change (57) with its *exact* counterparts. Table 1 summarizes the *exact* and *complete* profit change decompositions introduced in this study, considering their Fisher-type formulation. We notice that, indeed, all profit change decompositions have the same structure except for the *complete* one (57), which includes the price effect PE_{DDF}.

As similar price effects to PE_{DDF} show up in alternative decompositions of profit, cost, revenue, or profitability change, we offer rationale for the different cross-period profit inefficiency change terms included in PE_{DDF} in order to facilitate their meaningful interpretation. However, since our goal is to make all our decompositions comparable, which requires that they are isomorphic by having the same structural form, we show that we can select a set of directional vectors that renders the price effect null. This is made possible by endogenizing the directional vector in the calculation of the directional distance functions entering the profit change model. Our proposal not only eliminates PE_{DDF} under the optimal directional vector, (g^{N0*}, g^{N1*}), rendering all decomposition comparable regarding their structural form, but also numerically comparable in monetary terms given $p^t \cdot g_y^{Nt*} + w^t \cdot g_x^{Nt*} = 1$. Table 2 presents the different decompositions of profit change under (g^{N0*}, g^{N1*}), resulting in alternative values for the technical and allocative components of profit inefficiency change: $\Delta\pi I = \Delta TI_{DDF*} + \Delta AI_{DDF*}$.

The comparison of Tables 1 and 2 suggests that non-null values of PE_{DDF} can be linked to the existence of cross-period profit inefficiency changes related to a subjective (exogenous) choice of the directional vectors in the model evaluating technical inefficiency. Theorem 7.2 in Ref. [24]; 175) state the duality conditions that render the Bennet quantity indicator equivalent to the Luenberger productivity indicator,

$Q^B = L_{DDF}$, namely the consideration of shadow prices, implying that firms are allocative efficient by definition, i.e., $\Delta AI_{DDF} = 0$, and the characterization of the production technology in both periods through the quadratic form with time-invariant second order coefficients. Our model offers a better solution by retaining allocative inefficiency, i.e., $Q^B = L_{DDF*} + \Delta AI_{DDF*}$, showing that the price information (weights) contained in the Bennet quantity index, can be meaningfully summarized as allocative inefficiency change. Also, its implementation through non-parametric Data Envelopment Analysis techniques—program (62)—is more flexible with respect to the characterization of the production technology than the stated parametric form.

6. Empirical relevance and implications

The decomposition of profit change informs stakeholders, including managers, employees, and officials about the economic and technological progress of individual firms and industries at the aggregate level. To be credible, the different terms entering the decomposition must have a meaningful interpretation. Otherwise, they misrepresent firms' performance in time, which may lead to wrong decision-making when anticipating future conditions about current and future (potential) production technologies as well as output and input market prices.

To show the potential of our contribution from an empirical perspective, in this section we review three representative studies that propose incomplete decompositions of profit change when compared to the *complete* model and discuss how they could benefit from our results. Also, the surveyed studies rely on DEA methods to decompose profit change, which eases the adoption of the proposed method to endogenize the directional distance function, doing away with cross-period price effects. Specifically, we comment on how these studies differ or fall short of including all relevant terms included in our final decomposition of profit change, i.e.,

$$\Delta\pi = \pi^1 - \pi^0 = \Delta\pi I + \tau_{DDF^*} + \rho^B - \varpi^B = \Delta TI_{DDF^*} + \Delta AI_{DDF^*} + \tau_{DDF^*} + \rho^B - \varpi^B = L_{DDF^*} + \Delta AI_{DDF^*} + \rho^B - \varpi^B = Q^B + P^B. \tag{64}$$

First, [9] study the economic performance of 31 Taiwanese banks from 2006 to 2010 using DEA methods. They model the technology through the so-called intermediation approach whereby financial institutions, through labor and capital, collect deposits from savers to produce loans and other earning assets for borrowers. These authors calculate and decompose the Bennet quantity indicator using the directional distance function as measure of technical efficiency, adopting the average of the observed input and output quantities across all periods as directional vector: $(\overline{x}, \overline{y})$. Using our notation their decomposition is: $Q^B = L_{DDF} + \Delta AI_{DDF} + PE_{DDF} = \Delta TI_{DDF} + \Delta AI_{DDF} + \tau_{DDF} + PE_{DDF}$ (see their eq. (6) thru (9) on pages 178–179). We immediately see that these authors, by focusing on the Bennet quantity indicator: 1) do not study the most relevant concept reflecting economic performance, i.e., profit change $\Delta\pi$; 2) miss the information contained in the Bennet price indicator, $P^B = \rho^B - \varpi^B$, which is an integral part of the decomposition of profit change $\Delta\pi$ identifying the contribution of changes in output and input prices; and 3) include the residual 'price effect' PE_{DDF} in the decomposition of Q^B. In contrast, our complete decomposition (64) provides informs about profit change, $\Delta\pi$, the Bennet price indicator, P^B, and does not include terms related to price effects since they are all captured in this latter term, i.e., $PE_{DDF^*} = 0$. We then conclude that the analysis of [9] is incomplete because it does not study the change in profits of the Taiwanese banks, ignores how changes in input and output prices contribute to this change, and the only information about prices that they include is hard to interpret. For instance, in their study, the price effect PE_{DDF} is the main contributor to the growth of Q^B (see Table 5 of [9] on page 184), yet these authors fail to discuss what is driving these changes and their implications from the perspective of the Taiwanese financial sector.

Second, [47] rely on regression analysis to capture the effect that productivity change— including technological change, technical efficiency change and scale efficiency change—and congestion effects have on changes in the operating profit of 35 Chinese e-commerce firms between the first quarter of 2011 and the third quarter of 2013. These authors identify employees, cost and assets as inputs, and revenue and market share as outputs. In their approach, productivity change corresponds to an output-oriented productivity index which is calculated, along with its components, using DEA techniques. Afterwards, these values are included as regressors in a panel data model. Leaving aside congestion effects that would require modelling inputs and outputs as weakly disposable when calculating the directional distance functions, as well as scale efficiency change, which is theoretically unrelated to profit change, the regression specification adopted by Ref. [47] (see their eq. (18) on page 40) could be amended so as to make it theoretically grounded in our complete model (64). This implies that besides technological change and efficiency change, authors could have included as regressors some measures of allocative inefficiency ΔAI and output and input price changes, ρ^B and ϖ^B. Therefore, our theoretical results can guide empirical modeling of profit change to ensure that all relevant components are included in regression analyses like that perform by Ref. [47].

Finally, [48] studies the effect of organizational differences (either stated-owned commercial banks, joint-stock commercial banks or city commercial banks) on the profit change of 43 Chinese banks over the period 2010–2014. As [9], this author follows the intermediation approach and collects data for the same set of inputs and outputs. Here, profit change is defined according to expression (48), but rather than using prices and quantities of both periods to weight their change, prices of the comparison period are used to weight the change in quantities, and quantities of the base period are used to weight the change in prices, i.e., $\Delta\widetilde{\pi} = \widetilde{\pi}^1 - \widetilde{\pi}^0 = [\widetilde{p}^1 \cdot (y^1 - y^0) - \widetilde{w}^1 \cdot (x^1 - x^0)] + [y^0(\widetilde{p}^1 - \widetilde{p}^0) - x^0(\widetilde{w}^1 - \widetilde{w}^0)]$ (see their eq. (5) on page 3). By adopting the Paasche viewpoint for the change in quantities (termed 'quantity effect') and the Laspeyres viewpoint for the change in prices (termed 'price effect'), this decomposition falls short from the arithmetic mean (Fisher) formulation presented in (48), and therefore it is impossible to link profit change to the Bennet's indicators of quantity and price change. Note also that profit change corresponds to its normalized expression, so it is not measure in monetary units, which makes its interpretation harder from the perspective of the stakeholders. Subsequently, the quantity effect term $\widetilde{p}^1 \cdot (y^1 - y^0) - \widetilde{w}^1 \cdot (x^1 - x^0)$ is decomposed into a 'productivity effect' defined as the (proportional) change in outputs divided by the change in inputs, and a 'quantity margin effect', which [48] interprets as the contribution of input quantity changes to profit changes. As for the 'price effect', $y^0(\widetilde{p}^1 - \widetilde{p}^0) - x^0(\widetilde{w}^1 - \widetilde{w}^0)$, it is decomposed into a 'price recovery effect' defined as the (proportional) change in output prices divided by the change in input prices, and a 'price margin effect', which is now interpreted in terms of the degree of competition in the market or, equivalently, in terms of the market power of the firms, which implies that firms are price setters in the output and input markets. This is in stark contrast to the assumptions made in the foregoing sections where prices are exogenous and firms react to their variation by changing the amounts of outputs supplied and inputs demanded. We see that adopting the *complete* model (64) one gains far more information on the causes of profit change than relying on the model adopted by Ref. [48]. In this latter model the productivity change term depends on prices (as opposed to the Luenberger indicator), allocative inefficiency ΔAI cannot be recovered, while the role played by the changes in input and output prices cannot be disentangled into separate terms, as it is possible with the Bennet price indicator, $P^B = \rho^B - \varpi^B$. Finally, in the *complete* decomposition, the different terms have a much straightforward interpretation than the terms included in the partial decomposition of profit change. In this regard, studies as that carried out by Ref. [48] could benefit from the application of decomposition (64) in practice.

7. Conclusions

This study offers for the first time a comprehensive presentation of different possibilities to decompose profit change. These alternative models are based on various proposals existing in the literature to decompose cost, revenue and profitability change. We revise the disadvantages of the so-called *exact* decompositions, which may include asymmetric terms of technological change, do not recognize the changes in technical and allocative inefficiencies as sources of profit inefficiency change, and ignore the role that productivity change plays in profit change. To solve these drawbacks, we introduce a *complete* decomposition that includes symmetric terms, decomposes profit inefficiency change and embeds the Luenberger productivity indicator. As a relevant contribution we show that our *complete* decomposition of profit change can be related to Bennet's indicators measuring how changes in the output and input prices induce, in turn, changes in the optimal output and input quantities.

Decomposing profit inefficiency in all the above *exact* and *complete* decompositions requires the adoption of a specific technical inefficiency measure. As shown by Ref. [38], the literature proposes different possibilities to decompose profit inefficiency into technical and allocative inefficiency; e.g., the (enhanced) Russell graph measure, the weighted additive measure, the Hölder distance function, the directional distance function, etc. However, when considering the change in value of these measures over time to define a term of productivity change that can be embedded in the profit change decomposition, an untended price effect without a meaningful interpretation emerges. We show that this can be prevented by adopting the directional distance function as measure of technical inefficiency. The flexibility that this measure offers to endogenize the directional vector when defining the Luenberger indicator allows us to dismiss undesirable price terms by nullifying them. Thanks to our study academics and practitioners can evaluate the suitability of the existing decompositions of profit change, and inform managerial decision-making strategies aimed at improving the economic performance of firms. In this matter, although we do not provide an empirical application—which, nevertheless, constitutes a natural venue of future research, in the previous section we have emphasized how empirical studies can benefit from our results by offering guidance on the different sources of profit change.

Our study also presents limitations. Here we would like to highlight two relevant shortcomings related to the use of Data Envelopment Analysis (DEA) techniques when implementing the *complete* decomposition. While the suggested model corresponds to the standard directional distance function formulation in DEA, which does not require the assumption of a specific mathematical expression for the efficient frontier and deals with the multi-output production scenario in a natural way, it is deterministic. This means that it assumes that the gap between any observation and the efficient frontier exclusively corresponds to technical inefficiency, thereby failing to incorporate stochastic variations affecting quantities and prices. We are aware that, nowadays, more complex alternatives allowing for stochastic processes exist in the literature, either as parametric methods [30] or even within non-parametric techniques [49], but the analysis of these possibilities is beyond the scope of this study. A second drawback is related to the possible existence of multiple profit maximizing benchmarks. In this scenario, once again related to DEA as it generates deterministic piece-wise linear production functions, certain elements of the decompositions shown in this paper could take different values. In particular, we are referring to the Luenberger indicator of total factor productivity change and the total price recovery change indicator, which are based on optimal input and output quantity values of the profit maximizing benchmark(s), in expressions (34) and (35). However, it can be checked that the remaining components of the alternative decompositions analyzed in this paper do not suffer from this weakness.

Extending our approach to parametric techniques such as stochastic frontier analysis or non-parametric stochastic DEA methods is a fruitful line of future research as it will help solve the two previous drawbacks. For example, in the parametric case, and following [50], one can estimate the directional distance function adopting a stochastic specification of the quadratic production function. In the case of stochastic DEA, technical efficiency is measured adopting either statistical axioms or distributional assumptions that allow for a random (estimator of the) reference technology, see Ref. [51]. In both cases the model could handle deviations from the benchmark, either technical or allocative, as random deviations, including noise, measurement, and specifications errors, which in practice would prevent the emergence of multiple optima in empirical applications.

Author statement

Juan Aparicio: Methodology, Writing- Original draft preparation, Writing- Reviewing and Editing **Jose L. Zofío**: Conceptualization, Methodology, Writing- Original draft preparation, Writing- Reviewing and Editing.

Data availability

No data was used for the research described in the article.

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Dr. Juan Aparicio is a professor in the Department of Statistics, Mathematics and Information Technology at the Miguel Hernandez University of Elche (UMH), Spain, and the head of the Center of Operations Research. He has been Co-Chair (with Prof. Knox Lovell) of the Santander Chair on Efficiency and Productivity. His research interest includes efficiency and productivity analysis combined with Machine Learning and Data Science. He has published and co-edited several books focusing on performance evaluation and benchmarking using data envelopment analysis, as well as publishing approximately 100 scientific articles in different international journals. He is associate editor of Omega; The International Journal of Management Science and Journal of Productivity Analysis.

Dr. José Luis Zofío is a professor of Economics at the Universidad Autónoma de Madrid, and former Chair of the Department of Economics. He is also Visiting Professor at Erasmus University and Wageningen University and Research, and Visiting Fellow to the Erasmus Research Institute of Management ERIM, where he collaborates with several academics from the Rotterdam School of Management. His research interests are related to measurement theory in economics, in particular the use of index numbers for efficiency and productivity analysis, as well as spatial economics and trade theory. From an empirical perspective he undertakes multidisciplinary research, publishing numerous articles in top fields journals and book chapters related to environmental economics, transportation, innovation, and regional science. Finally, he has made several contributions in the field computational economics, with several packages for Matlab and Julia that focus on data envelopment analysis and regression methods.