Weak mean random attractors for non-local random and stochastic reaction-diffusion equations

R. Caballero[1], P. Marín-Rubio[2] and José Valero[1]

[1]Centro de Investigación Operativa, Universidad Miguel Hernández de Elche,
Avda. Universidad s/n, 03202, Elche (Alicante), Spain
E.mail: jvalero@gmail.com

[2]Dpto. Ecuaciones Diferenciales y Análisis Numérico,
Universidad de Sevilla, C/Tarfia, 41012-Sevilla, Spain
E.mail: pmr@us.es

In love memory of Prof. María José Garrido-Atienza, colleague and friend, with deep sorrow

Abstract

In this paper, we prove the existence of weak pullback mean random attractors for a non-local stochastic reaction-diffusion equation with a nonlinear multiplicative noise. The existence and uniqueness of solutions and weak pullback mean random attractors is also established for a deterministic non-local reaction-diffusion equations with random initial data.

Keywords: mean random attractor, pullback attractor, stochastic reaction-diffusion equation, non-local equation

AMS Subject Classification (2020): 35B40, 35B41, 35K55, 35K57, 35K59, 37H05, 37H30, 60H15

1 Introduction

The theory of global attractors for random dynamical systems in infinite-dimensional spaces has been developed intensively over the last 20 years. In particular, the theory of pathwise pullback random attractors [18] has been applied succesfully to stochastic equations of different types (see e.g. [4], [9], [10], [11], [17], [20], [21], [24], [25], [26], [32], [36], [41], [42], [43] among many others). However, this theory has an important limitation. Namely, it relies on a suitable change of variable which can be applied only to linear stochastic perturbations. For this reason, several authors have developed the theory of mean-square random attractors (see [23], [28], [37], [38], [39], [40]), which can be applied to equations with much more general nonlinear noises.

In this paper we apply the theory of weak mean-square random attractors developed in [37] to the following stochastic non-local reaction-diffusion problem

$$\begin{array}{l} du = (a(\|u\|_V^2)\Delta u + f(u) + h(t,x))dt + \sigma\left(u\right)dw\left(t\right) \text{ in } (\tau,\infty)\times\mathcal{O}, \\ u = 0 \quad \text{on } (\tau,\infty)\times\partial\mathcal{O}, \\ u(\tau,x) = u_\tau(x) \quad \text{for } x\in\mathcal{O}, \end{array} \tag{1}$$

where $V = H_0^1(\mathcal{O})$ and $\mathcal{O}$ is a bounded open set of $\mathbb{R}^n$. Following [37] under suitable conditions on the domain $\mathcal{O}$ and the functions f, h, σ and a we prove the existence of a weak mean-square random attractor in two situations: 1) The equation is deterministic (that is, $\sigma\left(u\right)\equiv 0$) but the initial condition u_τ is a random variable; 2) $w(t)$ is a two-sided scalar Wiener process. The existence of weak mean-square random attractors for the local problem (that is, $a\equiv 1$) was established in [37]. These results were generalized to stochastic reaction-diffusion equations generated by the p-laplacian operator [38].

Reaction-diffusion equations with non-local diffusion of the type

$$u_t - a(l(u(t)))\Delta u = f(t,u),$$

where $l : X \to \mathbb{R}$ is a suitable functional and X is the phase space (usually $L^2(\mathcal{O})$ or $H_0^1(\mathcal{O})$), appear in many applications in Physics, Biology and other sciences (see [14], [15], [16] and the references therein). When l is a linear functional of the type

$$l(u) = \int_\Omega \xi(x)u(x,t)dx,$$

where $\xi(x)$ is a given function in $L^2(\mathcal{O})$, the existence of properties of global attractors in the autonomous and non-autonomous situations have been established in [1], [5], [6], [7], [8]. The drawback of this case is that we cannot obtain a Lypaunov function for the solutions, which makes it difficult to analyze the fine structure of the attractor. However, if we consider a functional l of the type

$$l(u) = \|u\|_V^2,$$

then a Lyapunov functions exists [16]. In this situation, the existence and structure of the global attractor in both the single and set-valued frameworks have been studied in the papers [2], [3], [12], [30], [33].

In equation (1) the functional $l(u) = \|u\|_V^2$ helps us to obtain the existence and uniqueness of solutions, because the operator $u \mapsto -a(\|u\|_V^2)\Delta u$ is monotone as a map from V onto its conjugate space V^*.

This paper is organized as follows. In Section 2, we recall the main results of the theory of weak pullback mean random attractors developed in [37]. Also, we extend it by proving under suitable assumptions that the mean random attractor can be characterized using complete trajectories. In Section 3, we prove the existence of weak pullback mean random attractors for the deterministic equation (1) with random initial data. Finally, in Section 4, we obtain the existence of weak pullback mean random attractors for the stochastic equation (1).

2 Preliminaries: abstract theory of weak pullback mean random attractors

In this section we recall the main results of the theory of pullback mean random attractors developed in [37] and add some new ones about the characterization of the pullback attractor by means of complete trajectories.

2.1 Weak pullback mean attractors over probability spaces

In this subsection we provide the theory of weak $\mathcal{D}$-pullback mean random attractors for a mean random dynamical system over a probability space.

Let X be a Banach space with norm $\|\cdot\|_X$ and let $(\Omega, \mathcal{F}, P)$ be a probability space. For $p \in (1,\infty)$ llet us consider the Banach space $L^p(\Omega, \mathcal{F}; X)$ ($L^p(\Omega, X)$ for short) of Bochner integrable functions $y : \Omega \to X$ such that

$$\int_\Omega \|y\|_X^2 \, dP < \infty.$$

We denote by $\mathcal{D}$ a collection of some families $D = \{D(t)\}_{t\in\mathbb{R}}$ of non-empty bounded sets $D(t) \subset L^p(\Omega, X)$:

$$\mathcal{D} = \{D = \{D(t)\}_{t\in\mathbb{R}} : D(t) \in \beta\left(L^p(\Omega, X)\right) \text{ satisfy suitable conditions}\},$$

where $\beta\left(L^p(\Omega, X)\right)$ is the set of all non-empty bounded subsets of $L^p(\Omega, X)$. The collection $\mathcal{D}$ is said to be inclusion-closed if $D \in \mathcal{D}$ and $B(t) \subset D(t)$, $B(t) \in \beta\left(L^p(\Omega, X)\right)$, for all $t \in \mathbb{R}$, imply that $B = \{B(t)\}_{t\in\mathbb{R}} \in \mathcal{D}$.

A family $D = \{D(t)\}_{t\in\mathbb{R}}$ is said to be compact (weakly compact, bounded, etc.) if each set $D(t)$ is compact (weakly compact, bounded, etc.).

A family of maps $\Phi : \mathbb{R}^+ \times \mathbb{R} \times L^p(\Omega, X) \to L^p(\Omega, X)$ is called a mean random dynamical system if:

- $\Phi(0,\tau)$ is the identity map for all $\tau \in \mathbb{R}$;
- $\Phi(t+s,\tau) = \Phi(t, s+\tau) \circ \Phi(s,\tau)$ for all $t,s \in \mathbb{R}^+$, $\tau \in \mathbb{R}$.

Definition 1 *A family $K = \{K(t)\}_{t\in\mathbb{R}} \in \mathcal{D}$ is called a $\mathcal{D}$-pullback weakly attracting set if for all $\tau \in \mathbb{R}$, $D \in \mathcal{D}$ and any weak neighborhood $\mathcal{N}^w(K(\tau))$ of $K(\tau)$ there exists $T(\tau, D, \mathcal{N}^w(K(\tau))) > 0$ such that*

$$\Phi(t,\tau - t)D(\tau - t) \subset \mathcal{N}^w(K(\tau))$$

as soon as $t \geq T$.

We introduce the main concept of weak $\mathcal{D}$-pullback mean random attractor.

Definition 2 *A family $\mathcal{A} = \{\mathcal{A}(t)\}_{t\in\mathbb{R}} \in \mathcal{D}$ is called a weak $\mathcal{D}$-pullback mean random attractor if:*

1. *$\mathcal{A}$ is weakly compact.*
2. *$\mathcal{A}$ is $\mathcal{D}$-pullback weakly attracting.*
3. *$\mathcal{A}$ is minimal, that is, if $B \in \mathcal{D}$ is weakly compact and $\mathcal{D}$-pullback weakly attracting, then $\mathcal{A}(t) \subset B(t)$ for all $t \in \mathbb{R}$.*

It follows from this definition that a weak $\mathcal{D}$-pullback mean random attractor is unique if it exists. Further we need the concept of $\mathcal{D}$-pullback absorbing family.

Definition 3 *A family $K = \{K(t)\}_{t\in\mathbb{R}} \in \mathcal{D}$ is called a $\mathcal{D}$-pullback absorbing set if for all $\tau \in \mathbb{R}$ and $D \in \mathcal{D}$ there exists $T(\tau, D) > 0$ such that*

$$\Phi(t,\tau - t)D(\tau - t) \subset K(\tau)$$

as soon as $t \geq T$.

Theorem 4 *[37, p.2183] Let X be reflexive. Assume that $\mathcal{D}$ is an inclusion-closed collection. If the mean random dynamical system Φ possesses a weakly compact $\mathcal{D}$-pullback absorbing family $K = \{K(t)\}_{t\in\mathbb{R}} \in \mathcal{D}$, then Φ has a unique weak $\mathcal{D}$-pullback mean random attractor given by*

$$\mathcal{A}(t) = \cap_{r\geq 0} \overline{\cup_{s\geq r} \Phi(s, t-s)K(t-s)}^{w},\ t \in \mathbb{R}, \tag{2}$$

where $\overline{C}^w$ means the closure of C in the weak topology of $L^p(\Omega, X)$.

The weak $\mathcal{D}$-pullback mean random attractor is called invariant if

$$\mathcal{A}(t) = \Phi(\tau, t-\tau, \mathcal{A}(t-\tau)) \text{ for all } \tau \geq 0,\ t \in \mathbb{R}.$$

The mean random dynamical system Φ is weakly continuous if the map $\Phi(t,\tau) : L^p(\Omega, X) \to L^p(\Omega, X)$ is weakly continuous for any $t \geq 0$, $\tau \in \mathbb{R}$.

Lemma 5 *Let the conditions of Theorem 4 hold true. Assume that Φ is weakly continuous. Then the weak $\mathcal{D}$-pullback mean attractor $\mathcal{A}$ is invariant.*

Proof. Let $y \in \mathcal{A}(t)$. In view of characterization (2) and arguing as in [27, Lemma 3.3] there exist nets s_α, $y_\alpha \in \Phi(s_\alpha, t-s_\alpha)K(t-s_\alpha)$ such that $s_\alpha \to +\infty$ and $y_\alpha \to y$ weakly in $L^p(\Omega, X)$. Hence, $y_\alpha = \Phi(s_\alpha, t-s_\alpha)z_\alpha$ for some net $z_\alpha \in K(t-s_\alpha)$. Thus,

$$y_\alpha = \Phi(s_\alpha, t-s_\alpha)z_\alpha = \Phi(\tau, t-\tau)\Phi(s_\alpha - \tau, t-s_\alpha)z_\alpha = \Phi(\tau, t-\tau)x_\alpha.$$

There exists s_0 such that $x_\alpha \in K(t-\tau)$ for $s_\alpha \geq s_0$. Since $K(t-\tau)$ is bounded, passing to a subnet we have that $x_\alpha \to x$ for some x. The weak continuity of Φ implies then that $y = \Phi(\tau, t-\tau)x$. Note that

$$x_\alpha \in \Phi(s_\alpha - \tau, t-s_\alpha)K(t-s_\alpha) = \Phi(\widetilde{s}_\alpha, t-\tau-\widetilde{s}_\alpha)K(t-\tau-\widetilde{s}_\alpha),$$

so

$$x \in \cap_{r\geq 0}\overline{\cup_{s\geq r}\Phi(s,t-\tau-s)K(t-\tau-s)}^{w} = \mathcal{A}(t-\tau).$$

We have obtained that $\mathcal{A}(t) \subset \Phi(\tau,t-\tau)\mathcal{A}(t-\tau)$.

Conversely, for any weak neighborhood $\mathcal{N}^w(\mathcal{A}(t))$ we have

$$\begin{aligned}\Phi(\tau,t-\tau)\mathcal{A}(t-\tau) &\subset \Phi(\tau,t-\tau)\Phi(r,t-\tau-r)\mathcal{A}(t-\tau-r)\\ &= \Phi(\tau+r,t-\tau-r)\mathcal{A}(t-\tau-r) \subset \mathcal{N}^w(\mathcal{A}(t)),\end{aligned}$$

for r large enough, so $\Phi(\tau,t-\tau)\mathcal{A}(t-\tau) \subset \mathcal{A}(t)$. ∎

The map $\phi : \mathbb{R} \to L^p(\Omega, X)$ is a complete trajectory if $\phi(t) = \Phi(t-s,s)\phi(s)$ for all $s < t$. We can characterize the attractor in terms of complete trajectories belonging to $\mathcal{D}$.

Lemma 6 *Assume that $\mathcal{D}$ is an inclusion-closed collection. Assume that Φ possesses an invariant weak $\mathcal{D}$-pullback mean attractor $\mathcal{A} \in \mathcal{D}$. Then*

$$\mathcal{A}(t) = \{\phi(t) : \phi \in \mathcal{D} \textit{ is a complete trajectory}\}. \tag{3}$$

Proof. If $\phi \in \mathcal{D}$ is a complete trajectory, then

$$\phi(t) = \Phi(s,t-s,\phi(t-s)) \text{ for all } s \geq 0,\ t \in \mathbb{R},$$

so for any $t \in \mathbb{R}$ and any weak neighborhood $\mathcal{N}^w(\mathcal{A}(t))$ there exists $T = T(t,\phi,\mathcal{N}^w(\mathcal{A}(t)))$ such that

$$\phi(t) = \Phi(s,t-s,\phi(t-s)) \in \mathcal{N}^w(\mathcal{A}(t)) \text{ for all } s \geq T,$$

which implies that $\phi(t) \in \mathcal{A}(t)$.

Conversely, let $y \in \mathcal{A}(t)$, $t \in \mathbb{R}$ be arbitrary. Since $\mathcal{A}$ is invariant, we have

$$y \in \mathcal{A}(t) = \Phi(1,t-1,\mathcal{A}(t-1)),$$

so there exists $z_1 \in \mathcal{A}(t-1)$ such that $y = \Phi(1,t-1,z_1)$. We put $\phi^1(r) = \Phi(r-t+1,t-1,z_1)$ for all $r \geq t-1$. By the invariance of the attractor, $\phi^1(r) \in \mathcal{A}(r)$ for all $r \geq t-1$. Also, $\phi^1(r) = \Phi(r-s,s,\phi^1(s))$ for any $r \geq s \geq t-1$. By the same argument, there is $z_2 \in \mathcal{A}(t-2)$ such that $z_1 = \Phi(1,t-2,z_2)$. The function $\phi^2(r) = \Phi(r-t+2,t-2,z_2)$, for all $r \geq t-2$, satisfies $\phi^2(r) \in \mathcal{A}(r)$, for all $r \geq t-2$, $\phi^2(r) = \Phi(r-s,s,\phi^2(s))$, for any $r \geq s \geq t-2$, and $\phi^2(r) = \phi^1(r)$ for all $r \geq t-1$. In this way, we construct a sequence of functions $\phi^k : [t-k,+\infty) \to L^p(\Omega,X)$ such that $\phi^k(r) \in \mathcal{A}(r)$, for all $r \geq t-k$, $\phi^k(r) = \Phi(r-s,s,\phi^k(s))$, for any $r \geq s \geq t-k$, and $\phi^k(r) = \phi^{k-1}(r)$ for all $r \geq t-k+1$. Let $\phi : \mathbb{R} \to L^p(\Omega,X)$ be the common value of the functions ϕ^k at any point $t \in \mathbb{R}$. It is clear that ϕ is a complete trajectory satisfying $\phi(t) \in \mathcal{A}(r)$ for all t. Also, $\phi \in \mathcal{D}$ because $\mathcal{A} \in \mathcal{D}$ and $\mathcal{D}$ is inclusion closed.
∎

A family of sets D is said to be backwards bounded if there is $t_0 \in \mathbb{R}$ such that $\cup_{t\leq t_0}D(t)$ is bounded. It is called bounded if $\cup_{t\in\mathbb{R}}D(t)$ is bounded.

Lemma 7 *Assume that $\mathcal{D}$ is an inclusion-closed collection and that Φ possesses an invariant weak $\mathcal{D}$-pullback mean attractor $\mathcal{A} \in \mathcal{D}$ which is backwards bounded. Then*

$$\mathcal{A}(t) = \{\phi(t) : \phi \in \mathcal{D} \textit{ is a backwards bounded complete trajectory}\}. \tag{4}$$

If, moreover, $\mathcal{D}$ contains any backwards bounded family, then

$$\mathcal{A}(t) = \{\phi(t) : \phi \textit{ is a backwards bounded complete trajectory}\}. \tag{5}$$

Proof. The first statement follows directly from (3) and the fact that $\mathcal{A}$ is backwards bounded.

The second one follows because any backwards bounded complete trajectory has to belong to $\mathcal{D}$, so the two sets defined in (4) and (5) coincide. ∎

In the same way we prove the following.

Lemma 8 *Assume that Φ possesses an invariant weak $\mathcal{D}$-pullback mean attractor $\mathcal{A} \in \mathcal{D}$ which is bounded. Then*

$$\mathcal{A}(t) = \{\phi(t) : \phi \in \mathcal{D} \text{ is a bounded complete trajectory}\}. \tag{6}$$

If, moreover, $\mathcal{D}$ contains any bounded family, then

$$\mathcal{A}(t) = \{\phi(t) : \phi \text{ is a bounded complete trajectory}\}. \tag{7}$$

2.2 Weak pullback mean attractors over filtered probability spaces

In this subsection we recall the main results of the theory of weak $\mathcal{D}$-pullback mean random attractors for mean random dynamical systems over filtered probability spaces.

As usual, we denote by $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in\mathbb{R}}, P)$ a complete filtered probability space, where $\{\mathcal{F}_t\}_{t\in\mathbb{R}}$ is an increasing right continuous family of sub-σ-algebras of $\mathcal{F}$ which contains all P-null sets. For $p \in (1, \infty)$ let $L^p(\Omega, \mathcal{F}_t; X)$ be the subspace of (class of) functions f of $L^p(\Omega, \mathcal{F}; X)$ such that f is strongly $\mathcal{F}_t$-measurable.

As before, we denote by $\mathcal{D}$ a collection of some families $D = \{D(t)\}_{t\in\mathbb{R}}$ of non-empty bounded sets $D(t) \subset L^p(\Omega, \mathcal{F}_t; X)$:

$$\mathcal{D} = \{D = \{D(t)\}_{t\in\mathbb{R}} : D(t) \in \beta\left(L^p(\Omega, \mathcal{F}_t; X)\right) \text{ satisfying suitable conditions}\}.$$

A family $D = \{D(t)\}_{t\in\mathbb{R}}$ is said to be compact (weakly compact, bounded, etc.) if each set $D(t)$ is compact (weakly compact, bounded, etc.) in $L^p(\Omega, \mathcal{F}_t; X)$.

Definition 9 *The family of maps $\Phi(t, \tau) : L^p(\Omega, \mathcal{F}_\tau; X) \to L^p(\Omega, \mathcal{F}_{t+\tau}; X)$, $t \in \mathbb{R}^+$, $\tau \in \mathbb{R}$ is called a mean random dynamical system on $L^p(\Omega, \mathcal{F}; X)$ over $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in\mathbb{R}}, P)$ if:*

- *$\Phi(0, \tau)$ is the identity map for all $\tau \in \mathbb{R}$;*
- *$\Phi(t + s, \tau) = \Phi(t.s + \tau) \circ \Phi(s, \tau)$ for all $t, s \in \mathbb{R}^+$, $\tau \in \mathbb{R}$.*

Definition 10 *The family $K = \{K(t)\}_{t\in\mathbb{R}} \in \mathcal{D}$ is called $\mathcal{D}$-pullback weakly attracting for Φ if for all $\tau \in \mathbb{R}$, $D \in \mathcal{D}$ and any weak neighborhood $\mathcal{N}^w(K(\tau))$ of $K(\tau)$ in $L^p(\Omega, \mathcal{F}_\tau; X)$ there exists $T(\tau, D, \mathcal{N}^w(K(\tau))) > 0$ such that*

$$\Phi(t, \tau - t)D(\tau - t) \subset \mathcal{N}^w(K(\tau))$$

as soon as $t \geq T$.

We introduce the main concept of weak $\mathcal{D}$-pullback mean random attractor.

Definition 11 *A family $\mathcal{A} = \{\mathcal{A}(t)\}_{t\in\mathbb{R}} \in \mathcal{D}$ is called a weak $\mathcal{D}$-pullback mean random attractor for Φ if:*

1. *$\mathcal{A}(\tau)$ is weakly compact in $L^p(\Omega, \mathcal{F}_\tau; X)$ for all $\tau \in \mathbb{R}$.*
2. *$\mathcal{A}$ is $\mathcal{D}$-pullback weakly attracting.*
3. *$\mathcal{A}$ is minimal, that is, if $B \in \mathcal{D}$ satisfies conditions 1-2, then $\mathcal{A}(t) \subset B(t)$ for all $t \in \mathbb{R}$.*

It follows from this definition that a weak $\mathcal{D}$-pullback mean random attractor is unique if it exists.

Definition 12 *The family $K = \{K(t)\}_{t\in\mathbb{R}} \in \mathcal{D}$ is called $\mathcal{D}$-pullback absorbing for Φ if for all $\tau \in \mathbb{R}$ and $D \in \mathcal{D}$ there exists $T(\tau, D) > 0$ such that*

$$\Phi(t, \tau - t)D(\tau - t) \subset K(\tau)$$

as soon as $t \geq T$.

Theorem 13 *[37, p.2188] Let X be reflexive. Assume that $\mathcal{D}$ is an inclusion-closed collection. If the mean random dynamical system Φ possesses a weakly compact $\mathcal{D}$-pullback absorbing family $K = \{K(t)\}_{t\in\mathbb{R}} \in \mathcal{D}$, then Φ has a unique weak $\mathcal{D}$-pullback mean random attractor given by*

$$\mathcal{A}(t) = \cap_{r\geq 0}\overline{\cup_{s\geq r}\Phi(s,t-s)K(t-s)}^{w},\ t\in\mathbb{R}, \tag{8}$$

where the closure is taken with respect to the weak topology of $L^p(\Omega,\mathcal{F}_t;X)$.

As before, the weak $\mathcal{D}$-pullback mean random attractor is called invariant if

$$\mathcal{A}(t) = \Phi(\tau,t-\tau,\mathcal{A}(t-\tau)) \text{ for all } \tau\geq 0,\ t\in\mathbb{R}.$$

The mean random dynamical system on $L^p(\Omega,\mathcal{F};X)$ over $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\in\mathbb{R}},P)$ Φ is weakly continuous if the map $\Phi(t,\tau): L^p(\Omega,\mathcal{F}_\tau;X)\to L^p(\Omega,\mathcal{F}_{t+\tau};X)$ is weakly continuous for any $t\geq 0$, $\tau\in\mathbb{R}$.

Lemma 14 *Let the conditions of Theorem 13 hold true. Assume that Φ is weakly continuous. Then the weak $\mathcal{D}$-pullback mean attractor $\mathcal{A}$ is invariant.*

Proof. Let $y\in\mathcal{A}(t)$. In view of characterization (8) and arguing as in [27, Lemma 3.3] there exist nets s_α, $y_\alpha\in\Phi(s_\alpha,t-s_\alpha)K(t-s_\alpha)$ such that $s_\alpha\to+\infty$ and $y_\alpha\to y$ weakly in $L^p(\Omega,\mathcal{F}_t;X)$. Hence, $y_\alpha=\Phi(s_\alpha,t-s_\alpha)z_\alpha$ for some net $z_\alpha\in K(t-s_\alpha)$. Thus,

$$y_\alpha=\Phi(s_\alpha,t-s_\alpha)z_\alpha=\Phi(\tau,t-\tau)\Phi(s_\alpha-\tau,t-s_\alpha)z_\alpha=\Phi(\tau,t-\tau)x_\alpha.$$

There exists s_0 such that $x_\alpha\in K(t-\tau)$ for $s_\alpha\geq s_0$. Since $K(t-\tau)$ is bounded, passing to a subnet we have that $x_\alpha\to x$ for some x. The weak continuity of Φ implies then that $y=\Phi(\tau,t-\tau)x$. Note that

$$x_\alpha\in\Phi(s_\alpha-\tau,t-s_\alpha)K(t-s_\alpha)=\Phi(\widetilde{s}_\alpha,t-\tau-\widetilde{s}_\alpha)K(t-\tau-\widetilde{s}_\alpha),$$

so

$$x\in\cap_{r\geq 0}\overline{\cup_{s\geq r}\Phi(s,t-\tau-s)K(t-\tau-s)}^{w}=\mathcal{A}(t-\tau).$$

We have obtained that $\mathcal{A}(t)\subset\Phi(\tau,t-\tau)\mathcal{A}(t-\tau)$.

Conversely, for any weak neighborhood $\mathcal{N}^w(\mathcal{A}(t))$ we have

$$\begin{aligned}\Phi(\tau,t-\tau)\mathcal{A}(t-\tau)&\subset\Phi(\tau,t-\tau)\Phi(r,t-\tau-r)\mathcal{A}(t-\tau-r)\\&=\Phi(\tau+r,t-\tau-r)\mathcal{A}(t-\tau-r)\subset\mathcal{N}^w(\mathcal{A}(t)),\end{aligned}$$

■

A map ϕ such that $\phi(t)\in L^p(\Omega,\mathcal{F}_t;X)$ for all $t\in\mathbb{R}$ is a complete trajectory if $\phi(t)=\Phi(t-s,s)\phi(s)$ for all $s<t$. We can characterize the attractor in terms of complete trajectories belonging to $\mathcal{D}$.

Lemma 15 *Assume that $\mathcal{D}$ is an inclusion-closed collection. Assume that Φ possesses an invariant weak $\mathcal{D}$-pullback mean attractor $\mathcal{A}\in\mathcal{D}$. Then*

$$\mathcal{A}(t)=\{\phi(t):\phi\in\mathcal{D}\ \textit{is a complete trajectory}\}. \tag{9}$$

Proof. If $\phi\in\mathcal{D}$ is a complete trajectory, then

$$\phi(t)=\Phi(s,t-s,\phi(t-s)) \text{ for all } s\geq 0,\ t\in\mathbb{R},$$

so for any $t\in\mathbb{R}$ and any weak neighborhood $\mathcal{N}^w(\mathcal{A}(t))$ there exists $T=T(t,\phi,\mathcal{N}^w(\mathcal{A}(t)))$ such that

$$\phi(t)=\Phi(s,t-s,\phi(t-s))\in\mathcal{N}^w(\mathcal{A}(t)) \text{ for all } s\geq T,$$

which implies that $\phi(t)\in\mathcal{A}(t)$.

Conversely, let $y\in\mathcal{A}(t)$, $t\in\mathbb{R}$ be arbitrary. Since $\mathcal{A}$ is invariant, we have

$$y\in\mathcal{A}(t)=\Phi(1,t-1,\mathcal{A}(t-1)),$$

so there exists $z_1 \in \mathcal{A}(t-1)$ such that $y = \Phi(1, t-1, z_1)$. We put $\phi^1(r) = \Phi(r-t+1, t-1, z_1)$ for all $r \geq t-1$. By the invariance of the attractor, $\phi^1(r) \in \mathcal{A}(r)$ for all $r \geq t-1$. Also, $\phi^1(r) = \Phi(r-s, s, \phi^1(s))$ for any $r \geq s \geq t-1$. By the same argument, there is $z_2 \in \mathcal{A}(t-2)$ such that $z_1 = \Phi(t-1, t-2, z_2)$. The function $\phi^2(r) = \Phi(r-t+2, t-2, z_1)$, for all $r \geq t-2$, satisfies $\phi^2(r) \in \mathcal{A}(r)$, for all $r \geq t-2$, $\phi^2(r) = \Phi(r-s, s, \phi^2(s))$, for any $r \geq s \geq t-2$, and $\phi^2(r) = \phi^1(r)$ for all $r \geq t-1$. In this way, we construct a sequence of functions ϕ^k such that $\phi^k(r) \in L^p(\Omega, \mathcal{F}_r; X)$, $\phi^k(r) \in \mathcal{A}(r)$, for all $r \geq t-k$, $\phi^k(r) = \Phi(r-s, s, \phi^k(s))$, for any $r \geq s \geq t-k$, and $\phi^k(r) = \phi^{k-1}(r)$ for all $r \geq t-k+1$. Let ϕ be the common value of the functions ϕ^k at any point $t \in \mathbb{R}$. It is clear that ϕ is a complete trajectory satisfying $\phi(t) \in \mathcal{A}(r)$ for all t. Also, $\phi \in \mathcal{D}$ because $\mathcal{A} \in \mathcal{D}$ and $\mathcal{D}$ is inclusion closed. ∎

Lemma 16 *Assume that $\mathcal{D}$ is an inclusion-closed collection and that Φ possesses an invariant weak $\mathcal{D}$-pullback mean attractor $\mathcal{A} \in \mathcal{D}$ which is backwards bounded. Then*

$$\mathcal{A}(t) = \{\phi(t) : \phi \in \mathcal{D} \textit{ is a backwards bounded complete trajectory}\}. \tag{10}$$

If, moreover, $\mathcal{D}$ contains any backwards bounded family, then

$$\mathcal{A}(t) = \{\phi(t) : \phi \textit{ is a backwards bounded complete trajectory}\}. \tag{11}$$

Proof. The first statement follows directly from (4) and the fact that $\mathcal{A}$ is backwards bounded.

The second one follows because any backwards bounded complete trajectory has to belong to $\mathcal{D}$, so the two sets defined in (10) and (11) coincide. ∎

In the same way we prove the following.

Lemma 17 *Assume that Φ possesses an invariant weak $\mathcal{D}$-pullback mean attractor $\mathcal{A} \in \mathcal{D}$ which is bounded. Then*

$$\mathcal{A}(t) = \{\phi(t) : \phi \in \mathcal{D} \textit{ is a bounded complete trajectory}\}.$$

If, moreover, $\mathcal{D}$ contains any bounded family, then

$$\mathcal{A}(t) = \{\phi(t) : \phi \textit{ is a bounded complete trajectory}\}.$$

3 The mean random attractor for a non-local problem with random initial data

We put $H = L^2(\mathcal{O})$ with norm $\|\cdot\|$ (we will use $\|\cdot\|$ also for the norm in $\left(L^2(\mathcal{O})\right)^d$, $d \geq 1$) and $V = H_0^1(\mathcal{O})$ with norm $\|u\|_V = \|\nabla u\|$. As usual, $(\cdot,\cdot)$ is the scalar product in H^d, $d \geq 1$, and also the duality between $L^q(\mathcal{O})$ and $L^p(\mathcal{O})$, where $p \geq 2$ and q is its conjugate, that is, $1/p + 1/q = 1$. The duality between V and its dual space V^* is denoted by $\langle\cdot,\cdot\rangle_{V^*,V}$.

Let us consider the following reaction-diffusion equation

$$\begin{cases} \dfrac{\partial u}{\partial t} - a(\|u\|_V^2)\Delta u = f(u) + h(t,x) \text{ in } (\tau,\infty) \times \mathcal{O}, \\ u = 0 \quad \text{on } (\tau,\infty) \times \partial\mathcal{O}, \\ u(\tau,x) = u_\tau(x) \quad \text{for } x \in \mathcal{O}, \end{cases} \tag{12}$$

where $\mathcal{O}$ is a bounded open set of $\mathbb{R}^n$ with smooth boundary $\partial\mathcal{O}$ and the functions h, $a \in C(\mathbb{R}^+)$, $f \in C^1(\mathbb{R})$, satisfy the following assumptions:

$$f(r)r \leq -\alpha |r|^p + \beta, \tag{13}$$

$$|f(r)| \leq \gamma |r|^{p-1} + \delta, \tag{14}$$

$$f'(r) \leq \eta, \tag{15}$$

$$h \in L^2_{loc}(\mathbb{R}, H), \tag{16}$$

$$0 < m \leq a(s) \leq M, \tag{17}$$

$$s \mapsto a(s^2)s \text{ is non-decreasing,} \tag{18}$$

where $p \geq 2$, $r \in \mathbb{R}$, $s \geq 0$ and $\alpha, \beta, \gamma, \delta, \eta > 0$.

Remark 18 *Without loss of generality we can assume that $f(0) = 0$, as defining $\overline{f}(u) = f(u) - f(0)$, $\overline{h}(t) = h(t) + f(0)$ we obtain the equivalent equation*

$$\frac{\partial u}{\partial t} - a(\|u\|_V^2)\Delta u = \overline{f}(u) + \overline{h}(t,x) \text{ in } (\tau,\infty) \times \mathcal{O},$$

for which (13)-(18) hold and, additionally, $\overline{f}(0) = 0$.

We will study the existence of solutions of problem (12) for random initial data $u_\tau \in L^2(\Omega, H)$ in a probability space $(\Omega, \mathcal{F}, P)$.

For the operator $A = -\Delta$, thanks to the assumptions on the domain Ω, it is well known that $D(A) = H^2(\mathcal{O}) \cap H_0^1(\mathcal{O})$ [35, Proposition 6.19].

Definition 19 *Let be given $\tau \in \mathbb{R}$ and $u_\tau \in L^2(\Omega, H)$. A continuous mapping $u : [\tau, \infty) \to L^2(\Omega, H)$ is called a regular solution of problem (12) if*

$$u \in C([\tau,\infty), L^2(\Omega,H)) \cap L^2_{loc}(\tau,\infty; L^2(\Omega,V)) \cap L^p_{loc}(\tau,\infty; L^p(\Omega, L^p(\mathcal{O}))),$$

$$u \in L^\infty(\tau+\varepsilon, T; L^2(\Omega,V) \cap L^2(\tau+\varepsilon, T; L^2(\Omega, D(A)), \ \forall\ 0 < \varepsilon < T < \infty,$$

and u satisfies, P-a.s., the equality

$$\begin{aligned}(u(t),\xi) + \int_\tau^t a(\|u(s)\|^2_{H_0^1})(\nabla u(s), \nabla\xi)ds \\ = (u_0,\xi) + \int_\tau^t \int_{\mathcal{O}} f(u(s,x))\ \xi(x)dxds + \int_\tau^t \int_{\mathcal{O}} h(s,x)\xi(x)dxds,\end{aligned} \tag{19}$$

for every $t > \tau$ and $\xi \in V \cap L^p(\mathcal{O})$.

Theorem 20 *Suppose that (13)-(18) hold true. Then for every $\tau \in \mathbb{R}$ and $u_\tau \in L^2(\Omega, H)$, problem (12) has a unique regular solution $u(\cdot)$, which is continuous with respect to the initial datum u_τ in $L^2(\Omega, H)$. Moreover, it satisfies the energy equality*

$$\frac{d}{dt}\mathbb{E}(\|u(t)\|^2) + 2\mathbb{E}(a(\|u(t)\|_V^2)\,\|u(t)\|_V^2) = 2\mathbb{E}((f(u(t),u(t)) + (h(t),u(t))), \tag{20}$$

for a.a. $t > \tau$.

Proof. We will prove the result using the Faedo-Galerkin method.

Consider a fixed value $T > 0$. Let $\{e_j\}_{j\geq 1}$ be a sequence of eigenfunctions of $-\Delta$ in V with homogeneous Dirichlet boundary conditions, which forms a special basis of $L^2(\mathcal{O})$.

We need to ensure that the eigenfunctions are elements of $L^p(\mathcal{O})$. Indeed, by the Sobolev embedding theorem, we have

$$H^s(\mathcal{O}) \subset L^p(\mathcal{O}) \quad \text{for } s \geq n(p-2)/2p.$$

Taking $A = -\Delta$, we define the domain of a fractional power of A as

$$D(A^{s/2}) = \{u \in L^2(\mathcal{O}) : \sum_{j=1}^\infty \lambda_j^s (u, e_j)^2 < \infty\},$$

where λ_j is the eigenvalue associated to e_j. Also, $\{e_j\} \in D(A^{s/2})$. If we assume $\mathcal{O}$ to be a bounded C^s domain (smoothness condition on the domain), by Theorem 6.18 in [35] we have that $D(A^{s/2}) \subset H^s(\mathcal{O})$ and so $\{e_j\} \in L^p(\mathcal{O})$.

Therefore, we can consider $\{e_j\} \subset V \cap L^p(\mathcal{O})$ a basis of $L^2(\mathcal{O})$, with $s \geq \max\{n(p-2)/2p, 1\}$. By this way, $H_0^s(\mathcal{O}) \subset V \cap L^p(\mathcal{O})$ and the set $\cup_{n\in\mathbb{N}} V_n$ is dense in $L^2(\mathcal{O})$ and also in $V \cap L^p(\mathcal{O})$ [29], where $V_n = span[e_1, \ldots, e_n]$.

As usual, P_n is the orthogonal projection in H, that is

$$u_n := P_n u = \sum_{j=1}^{n}(u, e_j)e_j, \quad \forall u \in H.$$

Let $u_\tau : \Omega \mapsto H$ be a $\mathcal{F}$-measurable mapping such that $\mathbb{E}(\|u_\tau\|^2)) < \infty$. Then for every fixed $\omega \in \Omega$ and for each integer $n \geq 1$, we consider the Galerkin approximations

$$u_n(t,\omega) = \sum_{j=1}^{n} \gamma_{nj}(t,\omega)e_j,$$

which satisfy the following deterministic system parametrized by ω:

$$\begin{cases} \dfrac{d}{dt}(u_n, e_i) + a(\|u_n\|_V^2)(\nabla u_n, \nabla e_i) = (f(u_n), e_i) + (h, e_i), \ \forall i = 1, \dots, n, \\ u_n(\tau,\omega) = P_n u_\tau(\omega). \end{cases} \tag{21}$$

Using the fact that the eigenfunctions $\{e_j\}$ are orthonormal, we obtain that (21) is equivalent to the Cauchy problem

$$\begin{aligned} \frac{d\gamma_{n_j}}{dt} &= -a(\|u_n\|_V^2)\lambda_j \gamma_{n_j} + (f(u_n), e_j) + (h, e_j), \\ (u_n(\tau,\omega), e_j) &= (P_n u_\tau(\omega), e_j), \quad j = 1, \dots, n. \end{aligned} \tag{22}$$

Since the right hand side of (22) is continuous in $u_n(t)$, for every fixed $\omega \in \Omega$ and $\tau \in \mathbb{R}$ this Cauchy problem possesses a solution on some interval $[\tau, t_n), \tau < t_n < T$ [35, cf. p. 51]. In addition, for each $t \geq \tau$, $u_n(t,\omega)$ is $\mathcal{F}-$measurable with respect to $\omega \in \Omega$. Indeed, since $u_n(t,\omega)$ can be written as $u_n(t, \tau, u_\tau(\omega))$, the result follows since u_n is continuous and u_τ is measurable [22, Lemma 8.2.3].

We claim that this solution can be extended to any $[0,T]$ with $T > 0$. This will follow from a priori estimates in the space H of the sequence $\{u_n\}$.

Multiplying by $\gamma_{ni}(t,\tau,\omega)$ and summing from $i = 1$ to n, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|u_n(t,\omega)\|^2 + a(\|u_n\|_V^2)\|u_n(t,\omega)\|_V^2 \\ &= (f(u_n(t,\omega)), u_n(t,\omega)) + (h(t), u_n(t,\omega)) \end{aligned} \tag{23}$$

for $a.e.$ $t \in (0, t_n)$.
Using (13) and the Young and Poincaré inequalities we deduce that

$$(f(u_n(t,\omega)), u_n(t,\omega)) \leq \beta|\mathcal{O}| - \alpha\|u_n(t,\omega)\|_{L^p(\mathcal{O})}^p,$$

$$(h(t), u_n(t,\omega)) \leq \frac{m}{2}\|u_n(t,\omega)\|_V^2 + \frac{1}{2\lambda_1 m}\|h(t)\|^2.$$

Hence, from (23) it follows that

$$\frac{1}{2}\frac{d}{dt}\|u_n(t,\omega)\|^2 + \frac{m}{2}\|u_n(t,\omega)\|_V^2 + \alpha\|u_n(t,\omega)\|_{L^p(\mathcal{O})}^p \leq \beta|\mathcal{O}| + \frac{1}{2\lambda_1 m}\|h(t)\|^2, \tag{24}$$

for $a.e.$ $t \in (0, t_n)$.

Then, integrating (24) from τ to $t \in (\tau, t_n)$ we deduce

$$\begin{aligned} &\frac{1}{2}\|u_n(t,\omega)\|^2 + \frac{m}{2}\int_\tau^t \|u_n(s,\omega)\|_V^2 ds + \alpha\int_\tau^t \|u_n(s,\omega)\|_{L^p(\mathcal{O})}^p ds \\ &\leq \beta|\mathcal{O}|(t-\tau) + \frac{1}{2\lambda_1 m}\int_\tau^t \|h(s)\|^2 ds + \frac{1}{2}\|u_n(\tau,\omega)\|^2 \\ &\leq TK_1 + K_2(T) + \frac{1}{2}\|u_n(\tau,\omega)\|^2. \end{aligned} \tag{25}$$

Since $P_n u_\tau(\omega) \to u_\tau(\omega)$ in H, for every fixed $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $T > 0$, the sequence $\{u_n(\cdot,\omega)\}$ is well defined and bounded in $L^\infty(\tau,\tau+T;H)) \cap L^2(\tau,\tau+T;V) \cap L^p(\tau,\tau+T;L^p(\mathcal{O})$. Also, $\{-\Delta u_n\}$ is bounded in $L^2(\tau,\tau+T;V^*)$.

On the other hand, by (14) it follows that

$$\int_\tau^{\tau+T} \int_{\mathcal{O}} |f(u_n(s,\omega))|^q dx ds \leq 2^{q-1} C^q \left(|\mathcal{O}|T + \int_\tau^{\tau+T} \|u_n(s,\omega)\|_{L^p(\mathcal{O})}^p ds \right),$$

with $\frac{1}{p} + \frac{1}{q} = 1$. Hence, since $\{u_n\}$ is bounded in $L^p(\tau,\tau+T;L^p(\mathcal{O}))$, $\{f(u_n)\}$ is bounded in $L^q(\tau,\tau+T;L^q(\mathcal{O}))$.

Now, multiplying (21) by $\lambda_i \gamma_{ni}(t)$, summing from $i = 1$ to n and using the Young inequality we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|u_n(t,\omega)\|_V^2 + m\|\Delta u_n(t,\omega)\|^2 \\ &\leq (f(u_n(t,\omega)), -\Delta u_n(t,\omega)) + (h(t), -\Delta u_n(t,\omega)) \\ &\leq \eta \|u_n(t,\omega)\|_V^2 + \frac{1}{2m}\|h(t)\|^2 + \frac{m}{2}\|\Delta u_n(t,\omega)\|^2, \end{aligned} \tag{26}$$

where we have supposed by Remark 18 that $f(0) = 0$. Hence, if we apply the Uniform Gronwall Lemma to the following inequality

$$\frac{d}{dt}\|u_n(t,\omega)\|_V^2 \leq 2\eta \|u_n(t,\omega)\|_V^2 + \frac{1}{m}\|h(t)\|^2,$$

in view of (25) for $r > 0$ we obtain that

$$\|u_n(t,\omega)\|_V^2 \leq \left(\frac{2TK_1 + 2K_2(T) + \|u_n(\tau,\omega)\|^2}{mr} + K_3(T) \right) e^{2\eta r} \tag{27}$$

for $t \geq \tau + r = t_1$. Therefore,

$$\{\|u_n(\cdot,\omega)\|_V\} \text{ is uniformly bounded in } [t_1, \tau+T]$$

and by the continuity of the function a we get that

$$\{a(\|u_n(\cdot,\omega)\|_V^2)\} \text{ is bounded in } [t_1, \tau+T].$$

Also, it follows that

$$\{u_n(\cdot,\omega)\} \text{ is bounded in } L^\infty(t_1,\tau+T;V). \tag{28}$$

On the other hand, by (27) and integrating in (26) we obtain that

$$\begin{aligned} &m\int_{t_1}^T \|\Delta u_n(t,\omega)\|^2 dt \\ &\leq \frac{1}{2}\|u_n(t_1,\omega)\|_V^2 + \eta \int_{t_1}^T \|u_n(t,\omega)\|_V^2 dt + \frac{1}{2m}\int_{t_1}^T \|h(t)\|^2 dt \\ &\leq K_4(T,r)(1 + \|u_n(\tau,\omega)\|^2) + \eta \int_{t_1}^T \|u_n(t,\omega)\|_V^2 dt + \frac{1}{2m}\int_{t_1}^T \|h(t)\|^2 dt, \end{aligned} \tag{29}$$

so by (25)

$$\{u_n(\cdot,\omega)\} \text{ is bounded in } L^2(t_1,\tau+T;D(A)). \tag{30}$$

This implies that

$$\{-\Delta u_n(\cdot,\omega)\}$$

and

$$\{a(\|u_n(\cdot,\omega)\|_V^2)\Delta u_n(\cdot,\omega)\}$$

are bounded in $L^2(t_1, \tau + T; L^2(\mathcal{O}))$.

Thus,

$$\left\{ \frac{du_n(\cdot,\omega)}{dt} \right\} \text{ is bounded in } L^q(t_1, \tau + T; L^q(\mathcal{O})). \tag{31}$$

Therefore, there exists $u(\cdot,\omega) \in L^\infty(t_1, \tau+T; V) \cap L^2(\tau, \tau+T; V) \cap L^\infty(\tau, \tau+T; H) \cap L^2(t_1, \tau+T; D(A)) \cap L^p(\tau, \tau + T; L^p(\mathcal{O}))$ such that $\dfrac{du}{dt} \in L^q(t_1, \tau + T; L^q(\mathcal{O}))$ and a subsequence $\{u_n\}$, relabelled the same, such that (for each $\omega \in \Omega$)

$$\begin{aligned}
u_n &\overset{*}{\rightharpoonup} u \text{ in } L^\infty(t_1, \tau + T; V), \\
u_n &\overset{*}{\rightharpoonup} u \text{ in } L^\infty(\tau, \tau + T; H), \\
u_n &\rightharpoonup u \text{ in } L^2(\tau, \tau + T; V), \\
u_n &\rightharpoonup u \text{ in } L^p(\tau, \tau + T; L^p(\mathcal{O})), \\
u_n &\rightharpoonup u \text{ in } L^2(t_1, T; D(A)), \\
\frac{du_n}{dt} &\rightharpoonup \frac{du}{dt} \text{ in } L^q(t_1, \tau + T; L^q(\mathcal{O})), \\
f(u_n) &\rightharpoonup \chi \text{ in } L^q(\tau, \tau + T; L^q(\mathcal{O})), \\
a(\|u_n\|_V^2) &\overset{*}{\rightharpoonup} b \text{ in } L^\infty(t_1, \tau + T),
\end{aligned} \tag{32}$$

where $\rightharpoonup$ means weak convergence and $\overset{*}{\rightharpoonup}$ weak star convergence. Also, let $t_0 \in (\tau, \tau + T)$ be fixed. Then, there exists $v \in H$ such that

$$u_n(t_0, \omega) \rightharpoonup v \text{ in } H \tag{33}$$

for some subsequence. Moreover, by (30)-(31) the Aubin-Lions Compactness Lemma gives that

$$u_n(\cdot,\omega) \to u(\cdot,\omega) \text{ in } L^2(t_1, \tau + T; V),$$

so

$$u_n(t,\omega) \to u(t,\omega) \text{ in } V \text{ a.e. on } (t_1, \tau + T).$$

Consequently, by Corollary 1.12 in [35], there exists a subsequence $\{u_n\}$, relabelled the same, such that

$$u_n(t,\omega)(x) \to u(t,\omega)(x) \text{ a.e. in } (\tau, \tau + T) \times \mathcal{O}.$$

Since f is continuous, it follows that

$$f(u_n(t,\omega)(x)) \to f(u(t,\omega)(x)) \text{ a.e. in } (\tau, \tau + T) \times \mathcal{O}.$$

Therefore, in view of (32), by [31, Lemma 1.3] we have that $\chi = f(u)$.

As a consequence, by the continuity of a, we get that

$$a(\|u_n(\cdot,\omega)\|_V^2) \to a(\|u(\cdot,\omega)\|_V^2) \quad \text{ a.e. on } (t_1, \tau + T).$$

Since the sequence is bounded, by the Lebesgue theorem this convergence takes place in $L^2(t_1, \tau + T)$ and $b = a(\|u\|^2_{H_0^1})$ a.e. on $(t_1, \tau + T)$.

Thus,

$$a(\|u_n(\cdot,\omega)\|_V^2)\Delta u_n(\cdot,\omega) \rightharpoonup a(\|u(\cdot,\omega)\|_V^2)\Delta u(\cdot,\omega), \tag{34}$$

in $L^2(t_1, \tau + T; H)$. Also, we know [35, p.224] that

$$P_n f(u_n) \rightharpoonup \chi. \tag{35}$$

Since $\{e_i\}$ is dense in $V \cap L^p(\mathcal{O})$, in view of (32), (34) and (35) we can pass to the limit in (21) and conclude that (19) holds for all $\xi \in V \cap L^p(\mathcal{O})$.

We need to guarantee that the initial condition of the problem makes sense. If u is a weak solution to (12), taking into account (14) and (17) it follows that

$$\frac{du}{dt} = a(\|u\|_V^2)\Delta u + f(u) + h \in L^2(\tau, \tau+T; V^*) + L^q(\tau, \tau+T; L^q(\mathcal{O})). \tag{36}$$

Therefore, by [13, p.33] $u(\cdot, \omega) \in C([\tau, \tau+T], H)$, so the initial condition makes sense when $u_\tau(\omega) \in H$.

We have to check that $u(\tau, \omega) = u_\tau(\omega)$ and $u(t_0, \tau, \omega) = v$. Indeed, let be $\phi \in C^1([\tau, \tau+T]; V \cap L^p(\mathcal{O}))$, with $\phi(\tau+T) = 0$, $\phi(\tau) \neq 0$. Using (36) we can multiply the equation in (12) by ϕ and integrate by parts in the t variable to obtain that

$$\begin{aligned} &\int_\tau^{\tau+T} \Big(-\left(u\left(t,\omega\right), \phi'\left(t\right)\right) - a(\|u(t,\omega\|_V^2) \left\langle \Delta u\left(t,\omega\right), \phi\left(t\right)\right\rangle_{V^*,V} \Big)\, dt \\ &= \int_\tau^{\tau+T} \left(f(u(t,\omega)) + h(t), \phi\left(t\right)\right) dt + \left(u\left(\tau,\omega\right), \phi\left(\tau\right)\right), \end{aligned} \tag{37}$$

$$\begin{aligned} &\int_\tau^{\tau+T} \Big(-\left(u_n\left(t,\omega\right), \phi'\left(t\right)\right) - a(\|u_n(t,\omega\|_{H_0^1(\mathcal{O})}^2) \left\langle \Delta u_n\left(t,\omega\right), \phi\left(t\right)\right\rangle_{V^*,V} \Big)\, dt \\ &= \int_\tau^{\tau+T} \left(f(u_n(t,\omega)) + h(t), \phi\left(t\right)\right) dt + \left(u_n\left(\tau,\omega\right), \phi\left(\tau\right)\right). \end{aligned} \tag{38}$$

Passing to the limit in (38), taking in to account (37) and bearing in mind $u_n(\tau,\omega) = P_n u_\tau(\omega) \to u_\tau(\omega)$ we get

$$\left(u\left(\tau,\omega\right), \phi\left(\tau\right)\right) = \left(u_\tau(\omega), \phi\left(\tau\right)\right).$$

Since $\phi\left(\tau\right) \in V \cap L^p(\mathcal{O})$ is arbitrary, we infer that $u(\tau,\omega) = u_\tau(\omega)$.

In a similar way we check that

$$u(t_0, \omega) = v. \tag{39}$$

By (33) and (39) we get

$$u_n(t_0, \omega) \rightharpoonup u(t_0, \omega) \text{ in } H \tag{40}$$

Hence, $u(t,\omega)$ is a regular solution to (12) satisfying $u\left(\tau,\omega\right) = u_\tau(\omega)$, for a fixed ω. This solution is unique (which is proved exactly as in Theorem 13 in [3]), so any converging subsequence has the same limit. Hence, by (40) the whole sequence $u_n(t,\omega)$ converges weakly to $u(t,\omega)$ in H for any $t \geq \tau$ and $\omega \in \Omega$. Since $u_n(t,\omega)$ is measurable in $\omega \in \Omega$, the weak limit $u(t,\omega)$ is weakly measurable, and this implies that $\omega \mapsto u(t,\omega)$ is measurable as by the Pettis theorem strong and weak measurability are equivalent properties when the space is separable (see [19, p. 42]).

On the other hand, by (25), (27) and (29) we obtain that

$$\|u(t,\omega)\|^2 \leq \|u_\tau(\omega)\|^2 + 2TK_1 + 2K_2(T), \ \forall\ t \in [\tau, \tau+T], \tag{41}$$

$$\int_\tau^{\tau+T} \|u(s,\omega)\|_V^2 ds \leq \frac{1}{m}\|u_\tau(\omega)\|^2 + \frac{2TK_1 + 2K_2(T)}{m}, \tag{42}$$

$$\int_\tau^{\tau+T} \|u(s,\omega)\|_{L^p(\mathcal{O})}^p ds \leq \frac{1}{2\alpha}\|u_\tau(\omega)\|^2 + \frac{TK_1 + K_2(T)}{\alpha}, \tag{43}$$

$$\|u(t,\omega)\|_V^2 \leq \left(\frac{2TK_1 + 2K_2(T) + \|u_\tau(\omega)\|^2}{mr} + K_3(T) \right) e^{2\eta r}, \ \forall t \geq \tau + r, \tag{44}$$

$$\begin{aligned} \int_{\tau+r}^T \|\Delta u_n(t,\omega)\|^2 dt &\leq \frac{K_4(T,r)}{m}(1 + \|u_\tau(\omega)\|^2) + \frac{1}{2m^2}\int_{\tau+r}^T \|h(t)\|^2 dt \\ &+ \frac{\eta}{m^2}\left(\|u_\tau(\omega)\|^2 + 2TK_1 + 2K_2(T)\right), \end{aligned} \tag{45}$$

for any $r>0$, $\tau\in\mathbb{R}$, $T>0$ and $\omega\in\Omega$. Since $u_\tau\in L^2(\Omega,H)$, we have

$$\begin{aligned} & u\in L^\infty_{loc}(\tau,\infty;L^2(\Omega,H))\cap L^2_{loc}(\tau,\infty;L^2(\Omega,V))\cap L^p_{loc}(\tau,\infty;L^p(\Omega,L^p(\mathcal{O}))) \\ & \quad\cap L^\infty(\tau+\varepsilon,\tau+T;L^2(\Omega,V))\cap L^2(\tau+\varepsilon,\tau+T;L^2(\Omega,D(A))), \end{aligned}\tag{46}$$

for any $0<\varepsilon<T$. For every fixed ω, $u(\cdot,\omega)\in C([\tau,\tau+T],H)$ and in view of (41) and the Lebesgue dominated convergence theorem, we have that

$$u\in C([\tau,\infty),L^2(\Omega,H))$$

obtaining that u is a solution in the sense of Definition 19.

We will prove that the solution is unique. If u,v are two solutions, then the difference $w=u-v$ satisfies

$$\frac{dw}{dt}-a(\|u\|_V^2)\Delta u+a(\|v\|_V^2)\Delta v=f(u_1)-f(u_2),$$

so multiplying by v and using (15) we have

$$\frac{1}{2}\frac{d}{dt}\left\|v\right\|^2+\int_{\mathcal{O}}\left(-a(\|u\|_V^2)\Delta u+a(\|v\|_V^2)\Delta v\right)wdx\leq\eta\left\|v\right\|^2.$$

Since (18) implies

$$\begin{aligned} & \int_{\mathcal{O}}\left(-a(\|u\|_V^2)\Delta u+a(\|v\|_V^2)\Delta v\right)(u-v)dx \\ & =\int_{\mathcal{O}}(a(\|u\left(t\right)\|_V^2)|\nabla u|^2-a(\|u\left(t\right)\|_V^2)\nabla u\nabla v-a(\|v\left(t\right)\|_V^2)\nabla u\nabla v+a(\|v\left(t\right)\|_V^2)|\nabla v|^2)dx \\ & \geq a(\|u\left(t\right)\|_V^2)\|u\left(t\right)\|_V^2-\left(a(\|u\left(t\right)\|_V^2)+a(\|v\left(t\right)\|_V^2)\right)\|u\left(t\right)\|_V\|v\left(t\right)\|_V+a(\|v\left(t\right)\|_V^2)\|v\left(t\right)\|_V^2 \\ & =\left(a(\|u\left(t\right)\|_V^2)\|u\left(t\right)\|_V-a(\|v\left(t\right)\|_V^2)\|v\left(t\right)\|_V\right)\left(\|u\left(t\right)\|_V-\|v\left(t\right)\|_V\right)\geq 0, \end{aligned}\tag{47}$$

we obtain by the Gronwall Lemma and taking expectations that

$$\mathbb{E}\left(\|w(t)\|^2\right)\leq e^{2\eta(t-\tau)}\mathbb{E}\left(\|u(\tau)-v(\tau)\|^2\right),$$

which implies uniqueness and the continuity of solutions with respecto to the initial datum as well.

Finally, equality (20) is obtained multiplying the equation by $u(t)$ and taking expectations. ■

Let Φ be the mapping from $\mathbb{R}^+\times\mathbb{R}\times L^2(\Omega,H)$ to $L^2(\Omega,H)$ given by

$$\Phi(t,\tau,u_\tau)=u(t+\tau)$$

where $t\geq 0$, $\tau\in\mathbb{R}$, $u_\tau\in L^2(\Omega,H)$, and u is the unique solution to (12) with $u\left(\tau\right)=u_\tau$. By the previous result, Φ is a continuous mean random dynamical system on $L^2(\Omega,H)$.

We recall that $\lambda_1>0$ is the first eigenvalue of $-\Delta$ with homogeneous Dirichlet boundary conditions, and assume that there exists $\mu\in(0,2m\lambda_1)$ such that

$$\int_{-\infty}^0 e^{\mu s}\|h(s)\|^2ds<\infty.\tag{48}$$

Remark 21 *Following Remark 18, the function* $\overline{h}(t)=h(t)+f(0)$ *satisfies assumption (48) as well.*

According to the previous assumption, let us consider the following universe: denote $\mathcal{D}$ the class of all families of nonempty bounded subsets of $L^2(\Omega,H)$, $D=\{D(\tau):\tau\in\mathbb{R}\}$, such that

$$\lim_{\tau\to-\infty}e^{\mu\tau}\sup_{v\in D(\tau)}\|v\|^2_{L^2(\Omega,H)}=0.\tag{49}$$

Within this setting we can derive uniform estimates on the solutions to (12) that will lead to the existence of $\mathcal{D}$-pullback absorbing family in $L^2(\Omega,H)$. Namely, we have the following result.

Lemma 22 *Assume that (13)-(18) and (48) hold. Then for any $\tau \in \mathbb{R}$ and $D = \{D(t) : t \in \mathbb{R}\} \in \mathcal{D}$ there exists $T = T(\tau, D) > 0$ such that for all $t \geq T$ and $u_{\tau-t} \in D(\tau - t)$ we have*

$$E(\|u(\tau)\|^2) \leq M + Me^{-\mu\tau} \int_{-\infty}^{\tau} e^{\mu s} \|h(s)\|^2 ds,$$

where u is the unique solution satisfying $u(t - \tau) = u_{t-\tau}$ and M denotes a positive constant independent of τ and D (but dependent on μ).

Proof. From the energy equality (20) and assumptions (13), (14), (17) and the Poincaré inequality we have

$$\begin{aligned}&\frac{d}{ds}E(\|u(s)\|^2) + 2m\lambda_1 E(\|u(s)\|^2)\\ \leq& -2\alpha E(\|u(s)\|^p_{L^p(\mathcal{O})}) + 2\beta|\mathcal{O}| + \frac{1}{2m\lambda_1 - \mu}\|h(s)\|^2 + (2m\lambda_1 - \mu)E(\|u(s)\|^2) \text{ for } a.a.\ s > \tau.\end{aligned}$$

Multiplying by $e^{\mu s}$ we deduce

$$\frac{d}{ds}(e^{\mu s}E(\|u(s)\|^2)) + 2\alpha e^{\mu s}E(\|u(s)\|^p_{L^p(\mathcal{O})}) \leq 2\beta|\mathcal{O}|e^{\mu s} + \frac{1}{2m\lambda_1 - \mu}e^{\mu s}\|h(t)\|^2 \ a.a.\ s > \tau.$$

Integrating in $[\tau - t, \tau]$

$$\begin{aligned}&E(\|u(\tau)\|^2) + 2\alpha e^{-\mu\tau}\int_{\tau-t}^{\tau} e^{\mu s}E(\|u(s)\|^p_{L^p(\mathcal{O})})ds\\ \leq& e^{-\mu\tau}e^{\mu(\tau-t)}E(\|u_{\tau-t}\|^2) + \frac{1}{2m\lambda_1 - \mu}e^{-\mu\tau}\int_{\tau-t}^{\tau} e^{\mu s}\|h(s)\|^2 ds + 2\frac{\beta|\mathcal{O}|}{\mu} \quad \forall t \geq 0.\end{aligned}$$

Since $u_{\tau-t} \in D(\tau - t)$, we have that there exists $T = T(\tau, D)$ such that

$$e^{-\mu\tau}e^{\mu(\tau-t)}E(\|u_{\tau-t}\|^2) \leq 1 \quad \forall t \geq T.$$

The proof is complete. ∎

Corollary 23 *Suppose that (13)-(18) and (48) hold. Then, the family $K = \{K(\tau) : \tau \in \mathbb{R}\}$ with $K(\tau) = \{u \in L^2(\Omega, L^2(\mathcal{O})) : E(\|u\|^2) \leq R(\tau)\}$, where*

$$R(\tau) = M + Me^{-\mu\tau}\int_{-\infty}^{\tau} e^{\mu s}\|h(s)\|^2 ds,$$

belongs to $\mathcal{D}$ and is a weakly compact $\mathcal{D}$-pullback absorbing family for Φ.

This allows us to use Theorem 4 to conclude the main result of this section.

Theorem 24 *Suppose that (13)-(18) and (48) hold. Then, the continuous mean random dynamical system Φ defined through the solutions to problem (12) has the unique weak $\mathcal{D}$-pullback mean random attractor $\mathcal{A} = \{\mathcal{A}(\tau) : \tau \in \mathbb{R}\} \in \mathcal{D}$.*

In general, the radius $R(\tau)$ can be unbounded as $\tau \to \pm\infty$. However, under an additional assumption on the function $h(t)$ we are able to force it to be bounded in either one or both directions. The following result is straightforward to check.

Lemma 25 *Suppose that (13)-(18) and (48) hold. If, additionally,*

$$\sup_{t \leq t_0} e^{-\mu t}\int_{-\infty}^{t} e^{\mu r}\|h(r)\|^2 dr < \infty$$

for some $t_0 \in \mathbb{R}$, then $\sup_{\tau \leq t} R(\tau) < \infty$ for any $t \in \mathbb{R}$. Hence, the union $\cup_{\tau \leq t}\mathcal{A}(\tau)$ is bounded for any $t \in \mathbb{R}$.

If

$$\sup_{t \in \mathbb{R}} e^{-\mu t}\int_{-\infty}^{t} e^{\mu r}\|h(r)\|^2 dr < \infty,$$

then $\sup_{\tau \in \mathbb{R}} R(\tau) < \infty$. Hence, the union $\cup_{\tau \in \mathbb{R}}\mathcal{A}(\tau)$ is bounded.

Corollary 26 *If h does not depend on time, that is, $h(t) \equiv h_0 \in H$, then the union $\cup_{\tau \in \mathbb{R}}\mathcal{A}(\tau)$ is bounded.*

4 The mean random attractor for a stochastic non-local problem

We consider now the following stochastic nonlocal reaction-diffusion equation

$$\begin{cases} du = (a(\|u\|_V^2)\Delta u + f(u) + h(t,x))dt + \sigma(u)\,dw(t) \text{ in } (\tau,\infty)\times\mathcal{O}, \\ u = 0 \quad \text{on } (\tau,\infty)\times\partial\mathcal{O}, \\ u(\tau,x) = u_\tau(x) \quad \text{for } x\in\mathcal{O}, \end{cases} \tag{50}$$

where $\mathcal{O}$ is a bounded open set of $\mathbb{R}^n$ with smooth boundary $\partial\mathcal{O}$, $w(t)$ is a two-sided scalar Wiener process with respect to the filtration $\{\mathcal{F}_t\}_{t\in\mathbb{R}}$ and $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\in\mathbb{R}},P)$ is a complete filtered probability space such that $\{\mathcal{F}_t\}_{t\in\mathbb{R}}$ is a right continuous family of sub-σ-algebras of $\mathcal{F}$ that contains all P-null sets. The integral is understood in the Itô sense.

The functions $f\in C^1(\mathbb{R})$, $a\in C(\mathbb{R}^+)$, $\sigma:\mathbb{R}\to\mathbb{R}$ and h satisfy:

$$f'(r) \le \gamma_1, \tag{51}$$

$$|f(r)| \le \gamma_2(1+|r|), \tag{52}$$

$$f(r)\,r \le \gamma_3 + \gamma_4 r^2, \tag{53}$$

$$0 < m \le a(s) \le M,\ \forall s\ge 0, \tag{54}$$

$$s \mapsto a(s^2)s \text{ is non-decreasing}, \tag{55}$$

$$h\in L^2(\tau,T;H),\ \text{for all } \tau<T, \tag{56}$$

$$\sigma \text{ is globally Lipschitz (with constant } C_\sigma), \tag{57}$$

for some $\gamma_i, m, M>0$ and all $s\ge 0$, $r\in\mathbb{R}$. Additionally, we will need to assume that

$$\gamma_4 + C_\sigma^2 < \frac{m\lambda_1}{2}. \tag{58}$$

Although some of these conditions are the same as in Section 3, for clarity of exposition we prefer to write them here again.

Let $\widetilde{f}: H\to H$ be the Nemitsky operator given by $\widetilde{f}(u)(x) = f(u(x))$ for almost all $x\in\mathcal{O}$. We define the operator $B:\mathbb{R}\times V\to V^*$ given by

$$\langle B(t,u),v\rangle_{V^*,V} = -a(\|u\|_V^2)(\nabla u,\nabla v) + \left(\widetilde{f}(u)+h(t),v\right).$$

It is straigthforward to see using Lebesgue's theorem that the operator $\widetilde{f}$ is continuous.

In the same way, let $\widetilde{\sigma}: H\to H$ be the Nemitsky operator given by $\widetilde{\sigma}(u)(x) = \sigma(u(x))$ for almost all $x\in\mathcal{O}$. It is clear that $\widetilde{\sigma}$ is globally Lipschitz as well. Indeed,

$$\begin{aligned} \|\widetilde{\sigma}(u)-\widetilde{\sigma}(v)\| &= \left(\int_{\mathcal{O}} (\sigma(u(x))-\sigma(v(x)))^2 dx\right)^{\frac{1}{2}} \\ &\le C_\sigma \left(\int_{\mathcal{O}} (u(x)-v(x))^2 dx\right)^{\frac{1}{2}} = C_\sigma \|u-v\|. \end{aligned} \tag{59}$$

Under conditions (51)-(57) the following lemmas hold.

Lemma 27 *B is hemicontinuous.*

Proof. Since $a,\widetilde{f}$ are continuous, for any $u,v,z\in V$ we have that the function

$$\lambda \mapsto \langle B(t,u+\lambda z),v\rangle_{V^*,V} = a(\|u+\lambda z\|_V^2)\left((\nabla u,\nabla v)+\lambda(\nabla z,\nabla v)\right) + \left(\widetilde{f}(u+\lambda z)+h(t),v\right)$$

is continuous. Hence, B is hemicontinuous. ∎

Lemma 28 *B is weakly monotone, that is, there exists $c > 0$ such that*

$$2 \langle B(t,u) - B(t,v), u - v\rangle_{V^*,V} + \|\sigma(u) - \sigma(v)\|_H^2 \leq c\|u - v\|^2 \ \ \forall u, v \in V.$$

Proof. By (51) we obtain that

$$\begin{aligned}\left(\widetilde{f}(u) - \widetilde{f}(v), u - v\right) &= \int_{\mathcal{O}} f'(\alpha(x) u(x) + (1 - \alpha(x)) v(x)) (u(x) - v(x))^2 \, dx \\ &\leq \gamma_1 \|u - v\|^2 .\end{aligned}$$

Hence, the result follows from (47) and (59). ∎

Lemma 29 *B is coercive, that is, there are $c_1 \geq 0$, $c_2 > 0$ such that*

$$2 \langle B(t,u), u\rangle_{V^*,V} + \|\widetilde{\sigma}(u)\|_H^2 \leq c_1 \|u\|^2 - c_2 \|u\|_V^2 + g(t),$$

where $g \in L^1(\tau, T)$ for all $\tau < T$.

Proof. In view of (54), (53) and (57) we have the inequalities

$$\begin{aligned} -a(\|u\|_V^2)(\nabla u, \nabla u) &\leq -m\|u\|_V^2, \\ \left(\widetilde{f}(u), u\right) &\leq \gamma_3 |\mathcal{O}| + \gamma_4 \|u\|^2, \\ \|\widetilde{\sigma}(u)\|_H^2 &\leq (\|\widetilde{\sigma}(0)\|_H + C_\sigma \|u\|)^2, \\ (h(t), u) \leq \|h(t)\| \|u\| &\leq \frac{1}{2}\|h(t)\|^2 + \frac{1}{2}\|u\|^2, \end{aligned}$$

which imply the result by putting $g(t) = 2\gamma_3 |\mathcal{O}| + 2\|\widetilde{\sigma}(0)\|_H^2 + \|h(t)\|^2$, $c_1 = 2\gamma_4 + 1 + 2C_\sigma$, $c_2 = 2m$. ∎

Lemma 30 *The operator B is bounded, that is, there are $d_1, d_2 \geq 0$ such that*

$$\|B(t,u)\|_{V^*} \leq d_1 + d_2 \|u\|_V + g(t) \ \forall u \in V,$$

where $g \in L^2(\tau, T)$ for all $\tau < T$.

Proof. It follows from (54) and (52) that

$$\begin{aligned}\|B(t,u)\|_{V^*} &\leq M\|u\|_V + \left\|\widetilde{f}(u) + h(t)\right\|_{V^*} + \\ &\leq M\|u\|_V + \widetilde{c}\left(\left\|\widetilde{f}(u)\right\| + \|h(t)\|\right) \\ &\leq M\|u\|_V + \widetilde{c}\gamma_2\sqrt{2|\mathcal{O}| + 2\|u\|^2} + \widetilde{c}\|h(t)\| \\ &\leq d_1 + d_2\|u\|_V + g(t).\end{aligned}$$

∎

We will focus first on the existence and uniqueness of solutions to problem (50).

Definition 31 *For $\tau \in \mathbb{R}$ and $u_\tau \in L^2(\Omega, \mathcal{F}_\tau; H)$, an H-valued $\{\mathcal{F}_t\}_{t\in\mathbb{R}}$-adapted stochastic process u is called a solution to problem (50) if $u \in C([\tau, +\infty), H) \cap L^2_{loc}(\tau, +\infty; V)$ P-a.s. and satisfies the equality*

$$\begin{aligned}&(u(t), \xi) + \int_\tau^t a(\|u\|_V^2)(\nabla u, \nabla \xi) ds \\ &= (u_\tau, \xi) + \int_\tau^t (\widetilde{f}(u(s)) + h(s), \xi) ds + \int_\tau^t (\widetilde{\sigma}(u(s)), \xi) dw(s), \ \forall \ \tau < t,\end{aligned}$$

P-almost sure for all $\xi \in V$.

From Lemmas 27-30 and Theorems 4.2.4 and 4.2.5 in [34] we obtain that a unique solution exists and that it satisfies the Itô formula.

Lemma 32 *Assume (51)-(57). Then for any $u_\tau \in L^2(\Omega, \mathcal{F}_\tau; H)$ there exists a unique solution u to problem (50) which, moreover, satisfies*

$$\mathbb{E}\left(\sup_{t\in[\tau,\tau+T]} \|u(t)\|^2\right) < \infty, \tag{60}$$

$$\begin{aligned} &\|u(t)\|^2 + 2\int_\tau^t a(\|u\|_V^2)\,\|\nabla u\|^2\,ds \\ &= \|u(\tau)\|^2 + 2\int_\tau^t \left(\widetilde{f}(u(s)) + h(s), u(s)\right) ds + \int_\tau^t \|\widetilde{\sigma}(u(s))\|^2\,ds + 2\int_\tau^t (\widetilde{\sigma}(u(s)), u(s))dw(s)\ P\text{-a.s.}, \end{aligned} \tag{61}$$

for all $\tau < T$.

By using (60), $u \in C([\tau,+\infty), H)$ P-a.s. and the Lebesgue theorem we obtain that $u \in C([\tau,\infty), L^2(\Omega, H))$, so we define the map $\Phi : \mathbb{R}^+ \times \mathbb{R} \times L^2(\Omega, H) \to L^2(\Omega, H)$ by

$$\Phi(t,\tau,u_\tau) = u(t+\tau),$$

where u is the unique solution to (50) with $u(\tau) = u_\tau$. By the uniqueness of solutions, this family of mappings is a mean random dynamical system on $L^2(\Omega, \mathcal{F}; H)$ over $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in\mathbb{R}}, \mathbb{P})$ in the sense of Definition 9.

We start with an a priori estimate.

Lemma 33 *Assume (51)-(58). Then there are constants $K_1, K_2 > 0$ such that for any $u_\tau \in L^2(\Omega, \mathcal{F}_\tau; H)$ the solution u satisfies the estimate*

$$\mathbb{E}(\|u(t)\|^2) \le e^{-\omega_0(t-\tau)}\mathbb{E}(\|u_\tau\|^2) + \frac{K_1}{\omega_0} + K_2\int_\tau^t e^{-\omega_0(t-r)}\,\|h(r)\|^2\,dr, \text{ for all } \tau < t,$$

for any $0 < \omega_0 < m\lambda_1 - \gamma_4 - C_\sigma^2$.

Proof. Taking expectations in (61) we have

$$\begin{aligned} &\mathbb{E}\left(\|u(r)\|^2\right) + 2\int_\tau^r \mathbb{E}\left(a(\|\nabla u\|^2)\,\|\nabla u\|^2\right) ds \\ &= \mathbb{E}\left(\|u(\tau)\|^2\right) + 2\int_\tau^r \mathbb{E}\left(\widetilde{f}(u(s)) + h(s), u(s)\right) ds + \int_\tau^r \mathbb{E}\left(\|\widetilde{\sigma}(u(s))\|^2\right) ds \quad \text{for } r \ge \tau. \end{aligned}$$

Thus, for a.a. $r > \tau$,

$$\frac{d}{dr}\mathbb{E}\left(\|u(r)\|^2\right) + 2\mathbb{E}\left(a(\|\nabla u\|^2)\,\|\nabla u(r)\|^2\right) = 2\mathbb{E}\left(\widetilde{f}(u(r)) + h(r), u(r)\right) + \mathbb{E}\left(\|\widetilde{\sigma}(u(r))\|^2\right).$$

We estimate each term by using (53), (56) and (57):

$$\left(\widetilde{f}(u(r)), u(r)\right) \le \gamma_3\,|\mathcal{O}| + \gamma_4\,\|u(r)\|^2,$$

$$(h(r), u(r)) \le \varepsilon\left(m\lambda_1 - \gamma_4 - C_\sigma^2\right)\|u(r)\|^2 + \frac{1}{4\varepsilon\left(m\lambda_1 - \gamma_4 - C_\sigma^2\right)}\,\|h(r)\|^2,$$

$$\begin{aligned} \|\widetilde{\sigma}(u(s))\|^2 &\le (\|\widetilde{\sigma}(0)\|_H + C_\sigma\,\|u\|)^2 \\ &\le 2\,\|\widetilde{\sigma}(0)\|_H^2 + 2C_\sigma^2\,\|u\|^2, \end{aligned}$$

where $\varepsilon \in (0,1)$. Hence,

$$\begin{aligned}
&\frac{d}{dr}\mathbb{E}\left(\|u(r)\|^2\right) + 2(1-\varepsilon)(m\lambda_1 - \gamma_4 - C_\sigma^2)\mathbb{E}\left(\|u(r)\|^2\right) \\
&\leq 2C_\sigma^2 \|\widetilde{\sigma}(0)\|_H^2 + \frac{1}{2\varepsilon\,(m\lambda_1 - 2\gamma_4 - 2C_\sigma^2)} \|h(r)\|^2 + 2\gamma_3\,|\mathcal{O}| \\
&= K_1 + K_2\,\|h(r)\|^2 .
\end{aligned}$$

Thus, by the Gronwall lemma,

$$\mathbb{E}(\|u(t)\|^2) \leq e^{-\omega_0(t-\tau)}\mathbb{E}(\|u_\tau\|^2) + \frac{K_1}{\omega_0} + K_2 \int_\tau^t e^{-\omega_0(t-r)}\,\|h(r)\|^2\,dr,$$

for any $0 < \omega_0 < m\lambda_1 - \gamma_4 - C_\sigma^2$. ■

Further, let us consider the following condition: for some $0 < \omega_0 < m\lambda_1 - \gamma_4 - C_\sigma^2$ the function h satisfies that

$$\int_{-\infty}^t e^{\omega_0 r}\,\|h(r)\|^2\,dr < \infty \text{ for all } t \in \mathbb{R}. \tag{62}$$

Fixing the constant ω_0 from (62) we denote by $\mathcal{D}$ the collection of all families of non-empty bounded subsets $D = \{D(\tau) : \tau \in \mathbb{R}\}$, $D(\tau) \subset L^2(\Omega, \mathcal{F}_\tau; H)$, such that

$$\lim_{\tau \to -\infty} e^{\omega_0 \tau}\,\|D(\tau)\|_+^2 = 0,$$

where $\|D(\tau)\|_+ = \sup_{y \in D(\tau)} \|y\|_{L^2(\Omega, \mathcal{F}_\tau; H)}$.

Lemma 34 *Assume (51)-(58) and also condition (62). Then for any $t \in \mathbb{R}$ and $D = \{D(\tau) : \tau \in \mathbb{R}\} \in \mathcal{D}$ there exists $T = T(t, D)$ such that if $s \geq T$, then every solution u with initial condition at time $\tau = t - s$ given by $u_{t-s} \in D(t-s)$ satisfies*

$$\mathbb{E}(\|u(t)\|^2) \leq M(1 + e^{-\omega_0 t}\int_{-\infty}^t e^{\omega_0 r}\,\|h(r)\|^2\,dr) =: R_0(t), \tag{63}$$

where $M > 0$ is a constant which depends on ω_0.

Proof. Since $u_{t-s} \in D(t-s)$, we have

$$e^{-\omega_0 s}\mathbb{E}(\|u_{t-s}\|^2) \leq e^{-\omega_0 t}e^{\omega_0(t-s)}\,\|D(t-s)\|_+^2 \to 0 \text{ as } s \to +\infty,$$

so there exists $T(t, D)$ for which

$$e^{-\omega_0 s}\mathbb{E}(\|u_{t-s}\|^2) \leq 1 \text{ if } s \geq T.$$

From the estimate in Lemma 33 we obtain that

$$\mathbb{E}(\|u(t)\|^2) \leq 1 + \frac{K_1}{\omega_0} + K_2 e^{-\omega_0 t}\int_{-\infty}^t e^{\omega_0 r}\,\|h(r)\|^2\,dr, \text{ for } s \geq T,$$

proving the result. ■

We define now the family of bounded closed convex sets $K_0 = \{K_0(t) : \tau \in \mathbb{R}\}$ given by

$$K_0(t) = \{u \in L^2(\Omega, \mathcal{F}_t; H) : \mathbb{E}(\|u\|^2) \leq R_0(t)\},$$

where $R_0(t)$ is the function in (63). It is clear that the sets $K_0(t)$ are weakly compact. We will prove that this family is $\mathcal{D}$-pullback absorbing and that $K_0 \in \mathcal{D}$.

Lemma 35 *Assume the conditions of Lemma 34. Then K_0 is a weakly compact $\mathcal{D}$-pullback absorbing family which belongs to $\mathcal{D}$.*

Proof. In view of Lemma 34, for any $t \in \mathbb{R}$ and $D = \{D(\tau) : \tau \in \mathbb{R}\} \in \mathcal{D}$ there exists $T = T(t, D)$ such that if $s \geq T$, then
$$\Phi(s, t-s, D(t-s)) \subset K_0(t),$$
so K_0 is a weakly compact $\mathcal{D}$-pullback absorbing family. Finally, we see that

$$e^{\omega_0 \tau} R_0(\tau) = M(e^{\omega_0 \tau} + \int_{-\infty}^{\tau} e^{\omega_0 r} \|h(r)\|^2 \, dr) \to 0 \text{ as } \tau \to -\infty,$$

so $K_0 \in \mathcal{D}$. ■

From Lemma 35 and Theorem 13 we deduce the main result concerning the existence of the weak $\mathcal{D}$-pullback attractor.

Theorem 36 *Assume the conditions of Lemma 34. Then the mean random dynamical system Φ has a unique weak $\mathcal{D}$-pullback mean random attractor $\mathcal{A}_0 = \{\mathcal{A}(\tau) : \tau \in \mathbb{R}\} \in \mathcal{D}$.*

As in the previous section, under an additional assumption on the function $h(t)$ we can prove that the radius $R_0(t)$ and the weak $\mathcal{D}$-pullback mean random attractor are bounded in either one or both directions.

Lemma 37 *Under the conditions of Lemma 34, if*

$$\sup_{t \leq t_0} e^{-\omega_0 t} \int_{-\infty}^{t} e^{\omega_0 r} \|h(r)\|^2 \, dr < \infty$$

for some $t_0 \in \mathbb{R}$, then $\sup_{t \leq \bar{t}} R_0(t) < \infty$ for any $\bar{t} \in \mathbb{R}$. Hence, the union $\cup_{t \leq \bar{t}} \mathcal{A}(t)$ is bounded for any $\bar{t} \in \mathbb{R}$.

If

$$\sup_{t \in \mathbb{R}} e^{-\omega_0 t} \int_{-\infty}^{t} e^{\omega_0 r} \|h(r)\|^2 \, dr < \infty,$$

then $\sup_{t \in \mathbb{R}} R_0(t) < \infty$. Hence, the union $\cup_{t \in \mathbb{R}} \mathcal{A}(t)$ is bounded.

Corollary 38 *If h does not depend on time, that is, $h(t) \equiv h_0 \in H$, then the union $\cup_{t \in \mathbb{R}} \mathcal{A}(t)$ is bounded.*

Acknowledgments.

This work has been partially supported by the Spanish Ministry of Science, Innovation and Universities, project PGC2018-096540-B-I00, by the Spanish Ministry of Science and Innovation, project PID2019-108654GB-I00, and by the Junta de Andalucía and FEDER, projects US-1254251, P18-FR-4509.

This manuscript is dedicated to the memory of our colleague in the department and research group and friend María José Garrido-Atienza who unfortunately passed away in January 2021, on her 49th birthday. She was a very kind, brave, warm, vital, joyful, plenty of life energy, enthusiastic person, and a source of happiness in any meeting she participated. We will miss her a lot and will keep her forever in our hearts. Our big hug to her family, with deep love and sorrow.

References

[1] Anh, C.T., Tinh, L.T., Toi, V.M. (2018). Global attractors for nonlocal parabolic equations with a new class of nonlinearities. *J. Korean Math. Soc.,* 55: 531-551.

[2] Caballero, R., Carvalho, A.N., Marín-Rubio, P., Valero J. (2021). About the structure of attractors for a nonlocal Chafee-Infante problem. *Mathematics* 9: 353.

[3] Caballero, R., Marín-Rubio, P., Valero J. (2021). Existence and characterization of attractors for a nonlocal reaction-diffusion equation with an energy functional. *J. Dynam. Differential Equations (to appear). DOI:* 10.1007/s10884-020-09933-5.

[4] Caraballo, T., Garrido-Atienza, M.J., Schmalfuß, B., Valero, J. (2010). Asymptotic behaviour of a stochastic semilinear dissipative functional equation without uniqueness of solutions. *Discrete Contin. Dyn. Syst. Ser. B.* 14: 439-455.

[5] Caraballo, T., Herrera-Cobos, M., Marín-Rubio, P. (2015). Long-time behavior of a non-autonomous parabolic equation with nonlocal diffusion and sublinear terms. *Nonlinear Anal.* 121: 3-18.

[6] Caraballo, T., Herrera-Cobos, M., Marín-Rubio, P. (2018). Time-dependent attractors for non-autonomous non-local reaction-diffusion equations. *Proc. Roy. Soc. Edinburgh Sect. A.* 148A: 957-981.

[7] Caraballo, T., Herrera-Cobos, M., Marín-Rubio, P. (2018). Robustness of time-dependent attractors in H^1-norm for nonlocal problems, *Discrete Contin. Dyn. Syst. Ser. B.* 23: 1011-1036.

[8] Caraballo, T., Herrera-Cobos, M., and Marín-Rubio, P. (2018). Asymptotic behaviour of nonlocal p-Laplacian reaction-diffusion problems. *J. Math. Anal. Appl.* 459, 997-1015.

[9] Caraballo, T., Langa, J.A., Robinson, J.C. (1998). Upper semicontinuity of attractors for small random perturbations of dynamical systems. *Comm. Partial Differential Equations.* 23: 1557-1581.

[10] Caraballo, T., Langa, J.A., Valero, J. (2002). Global attractors for multivalued random dynamical systems, *Nonlinear Anal.* 48: 805-829.

[11] Caraballo, T., Morillas, F., Valero, J. (2012). Attractors of stochastic lattice dynamical systems with a multiplicative noise and non-Lipschitz nonlinearities. *J. Differential Equations.* 253: 667-693.

[12] Carvalho, A.N., Moreira, E.M. (2021). Stability and hyperbolicity of equilibria for a scalar nonlocal one-dimensional quasilinear parabolic problem. *J. Differential Equations* (to appear).

[13] Chepyzhov V.V., Vishik, M.I. (2002). *Attractors for equations of mathematical physics.* Providence: Americal Mathematical Society.

[14] Chipot, M., Lovat, B. (1997). Some remarks on nonlocal elliptic and parabolic problems, *Nonlinear Anal.* 30: 461-627.

[15] Chipot, M., Lovat, B. (1999). On the asymptotic behaviour of some nonlocal problems, *Positivity.* 3: 65-81.

[16] Chipot, M., Valente, V., Vergara Caffarelli, G. (2003). Remarks on a nonlocal problem involving the Dirichlet energy. *Rend. Sem. Mat. Univ. Padova.* 110: 199-220.

[17] Crauel, H., Debussche, A., Flandoli, F. (1997). Random Attractors. *J. Dynam. Differential Equations.* 9: 307-341.

[18] Crauel, H., Flandoli, F. (1997). Attractors for random dynamical systems. *Probab. Theory Relat. Fields.* 100**:** 365-393.

[19] Diestel, J., Uhl, J.J. (1977). *Vector measures.* Providence, Rhode Island: American Mathematical Society.

[20] Fan, X., Chen, H. (2013). Attractors for the stochastic reaction-diffusion equation driven by linear multiplicative noise with a variable coefficient. *J. Math. Anal. Appl.* 398: 715-728.

[21] Flandoli, F., Schmalfuß, B. (1996). Random attractors for the 3D stochastic Navier-Stokes equation with multiplicative noise. *Stochastics Stochastics Rep.* 59: 21-45.

[22] Aubin, J.P., Frankowska, H. (1990). *Set-valued analysis*, Boston: Birkhäusser.

[23] Gu, A. (2021). Weak pullback mean random attractors for non-autonomous p-Laplacian equations. *Discrete Contin. Dyn. Syst. Ser. B.* 26: 3863-3878.

[24] X. Han (2012). Random attractors for second order stochastic lattice dynamical systems with multiplicative noise in weighted spaces. *Stoch. Dyn.* 12: 20 pp.

[25] Imkeller, P., Schmalfuß, B. (2001). The conjugacy of stochastic and random differential equations and the existence of global attractors. *J. Dynam. Differential Equations.* 13: 215-249.

[26] Jia, X., Ding, X., Gao, J. (2018). Random attractors for stochastic retarded reaction-di ffusion equations with multiplicative white noise on unbounded domains. *Open Math.* 16**:** 862-884.

[27] Kapustyan, A.V., Melnik, V.S., Valero, J. (2003). Attractors of multivalued dynamical processes generated by phase-field equations. *Internat. J. Bifur. Chaos Appl. Sci. Engrg.* 13: 1969-1984.

[28] Kloeden, P.E., Lorentz, T. (2012). Mean-square random dynamical systems. *J. Differential Equations*. 253: 1422-1438.

[29] Ladyzhenskaya, O. A., Solonnikov, V. A., Ural'tseva, N. N. (1967). *Linear and quasilinear equations of parabolic type.* Moscow: Nauka.

[30] Li, Y., Carvalho, A.N., Luna, T.L.M., Moreira, E.M. (2020). A non-autonomous bifurcation problem for a non-local scalar one-dimensional parabolic equation. *Commun. Pure Appl. Anal.* 19: 5181-5196.

[31] Lions, J. L. (1969). *Quelques méthodes de résolution des problèmes aux limites non linéaires.* Paris: Gauthier-Villar.

[32] Marín-Rubio, P., Robinson, J. (2003). Attractors for the stochastic 3D Navier-Stokes equations. *Stoch. Dyn.* 3: 279-297.

[33] Moreira, E.M., Valero, J. (2021). Structure of the attractor for a non-local Chafee-Infante problem. arXiv:2107.04334v1.

[34] Prévôt, C., Röckner, M. (2007). *A concise course on stochastic differential equations.* Berlin: Springer.

[35] Robinson, J.C. (2001). *Infinite-dimensional dynamical systems: an introduction to dissipative parabilic PDEs and the theory of global attractors.* Cambridge, UK: Cambridge University Press.

[36] Wang, B. (2009). Random attractors for the stochastic FitzHugh Nagumo system on unbounded domains. *Nonlinear Anal.* 71: 2811-2828.

[37] Wang, B. (2019). Weak pullback attractors for mean random dynamical systems in Bochner spaces. *J. Dynamics Differential Equations.* 31**:** 2177-2204.

[38] Wang, B. (2021). Mean-square random invariant manifolds for stochastic differential equations. *Discrete Contin. Dyn. Syst.* 41: 1449-1468.

[39] Wang, J., Zhu, X., Kloeden, P.E. (2019). Compactness in Lebesgue-Bochner spaces of random variables and the existence of mean-square random attractors. *Stoch. Dyn.* 19: 1950032.

[40] Wu, F., Kloeden, P.K. (2013). Mean-square random attractors of stochastic delay differential quations with random delay. *Discrete Contin. Dyn. Syst. Ser. B.* 18: 1715-1734.

[41] Yan, W., Li, Y., Ji, Sh. (2010). Random attractors for first order stochastic retarded lattice dynamical systems. *J. Math. Phys.* 51: 17pp.

[42] Zhao, W. (2013). H^1-random attractors for stochastic reaction–diffusion equations with additive noise. *Nonlinear Anal.* 84: 61-72.

[43] Zhou, S., Yin, F., Ouyang, Z. (2005). Random attractor for damped noninear wave equations with white noise. *SIAM J. Appl. Dyn. Syst.* 4: 883-903.