Existence and characterization of attractors for a nonlocal reaction-diffusion equation with an energy functional

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Dedicated to the memory of Russell Johnson

Abstract

In this paper we study a nonlocal reaction-diffusion equation in which the diffusion depends on the gradient of the solution.

Firstly, we prove the existence and uniqueness of regular and strong solutions. Secondly, we obtain the existence of global attractors in both situations under rather weak assumptions by defining a multivalued semiflow (which is a semigroup in the particular situation when uniqueness of the Cauchy problem is satisfied). Thirdly, we characterize the attractor either as the unstable manifold of the set of stationary points or as the stable one when we consider solutions only in the set of bounded complete trajectories.

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1 Introduction

In real applications there might exist several nonlocal effects that influence the evolution of a system. For instance, usually we do not have enough information about the systems under study and its features at every point. In reality, the measurements are not made pointwise but through some local average. This is just one possible reason of introducing nonlocal terms in models. Actually, during the last decades many mathematicians have been studying nonlocal problems motivated by its various applications in physics, biology or population dynamics [13, 14, 15, 16, 17, 27].

For instance, let consider the problem of finding a function $u(t,x)$ such that

$$\left\{\begin{array}{l} u_t - a(\int_\Omega u(t,x)dx)\Delta u = g(t,u), \text{ in } \Omega\times(0,\infty), \\ u = 0 \quad \text{on } \partial\Omega\times(0,\infty), \\ u(0) = u_0 \quad \text{in } \Omega. \end{array}\right. \tag{1}$$

Here Ω is a bounded open subset in $\mathbb{R}^n$, $n \geq 1$, with smooth boundary and a is some function from $\mathbb{R}$ to $(0,+\infty)$. In such equation u could describe the density of a population subject to spreading. The diffusion coefficient a is then supposed to depend on the entire population in the domain rather than on the local density.

A wide literature with significant results about (1) have been developed during the last few decades (see for example [14, 17, 27]). However, it is possible to distinguish two basic cases of the following more general equation

$$\left\{\begin{array}{l} u_t - a(u)\Delta u = g(t,u), \quad t>0,\ x\in\Omega, \\ u = 0,\ on\ \partial\Omega\times(0,\infty), \\ u(0,x) = u_0(x) \quad x\in\Omega. \end{array}\right.$$

Some authors consider a depending on a linear functional $l(u)$, i.e.,

$$a(u) = a(l(u))$$

with

$$l(u) = \int_\Omega \Phi(x) u(x,t) dx,$$

where $\Phi(x)$ is a given function in $L^2(\Omega)$. For $g(t,u) = f(t)$ the existence and uniqueness of solutions and their asymptotic behavior are studied for example in [15, 16, 18, 32]. For $g(t,u) = f(u) + h(t)$ the existence, uniqueness and asymptotic behaviour of solutions is studied in [1, 6, 8, 9]. Moreover, the authors prove the existence of pullback attractors in $L^2(\Omega)$ and $H_0^1(\Omega)$. Extensions in this direction for equations governed by the p-laplacian operator instead of the laplacian operator Δ are given in [7, 10], whereas nonclassical diffusion equations are considered in [29].

On the other hand, it is possible to consider a function a such that $a\,(u) = a(\|u\|^2_{H_0^1})$. The existence and uniqueness of solutions of the following problem

$$\begin{cases} u_t - a(\|u\|^2_{H_0^1})\Delta u = f, \quad t > 0,\ x \in \Omega, \\ u = 0, \text{ on } \partial\Omega \times (0,\infty)\,, \\ u(0,x) = u_0(x) \quad x \in \Omega. \end{cases}$$

is proved in [32, 19], where $f \in L^2(\Omega)$, $u_0 \in H_0^1(\Omega)$ and $a = a(s)$ is a continuous function such that $0 < m \le a(s) \le M$.

By this way, in this paper the following problem is considered

$$\begin{cases} u_t - a(\|u\|^2_{H_0^1})\Delta u = f(u) + h(t), \text{ in } \Omega \times (0,\infty), \\ u = 0 \quad \text{on } \partial\Omega \times (0,\infty), \\ u(0,x) = u_0\,(x) \quad \text{in } \Omega, \end{cases} \tag{2}$$

where $h \in L^2(0,T;L^2(\Omega))$, for all $T > 0$, $a : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function such that $a\,(s) \ge m > 0$ and f is a continuous function satisfying standard dissipative and growth conditions (see (7) below).

The aim of this paper is three-fold. First, we will prove the existence of solutions for problem (2) under different assumptions on the nonlinear function f. Second, we will obtain the existence of attractors for the semiflows generated by either regular or strong solutions in the autonomous case, that is, when h does not depend on time. Third, we establish that the global attractor can be characterized by the unstable manifold of the set of stationary points. It is important to notice that the proof of this last fact requires the existence of a Lyapunov function on the attractor, and for this aim the term $a(\|u\|^2_{H_0^1})$ is crucial. In the case when $a(u) = a(l(u))$ it is not known whether such a function exists or not.

We prove the existence of strong solutions by assuming that either the function f is continuously differentiable and $f'\,(s) \le \eta$ or a more strict growth condition on f. Supposing additionaly that the function a has sublinear growth we prove the existence of regular solutions as well. Moreover, when $f'\,(s) \le \eta$ and the function $s \mapsto a\left(s^2\right)s$ is non-decreasing, uniqueness is proved.

When studying the asymptotic behaviour of solutions, new challenging difficulties arise for problem (2). For this problem we consider the autonomous situation, that is, $h \in L^2\,(\Omega)$ does not depend on t.

If uniqueness holds, then we define classical semigroups (one for regular solutions and one for strong solutions) and prove the existence of the global attractor. Under some extra assumptions on the functions a, h we are able to obtain that the global attractor is bounded in $H^2\,(\Omega)$ and $L^\infty\,(\Omega)$.

If uniqueness is not known to be true, then we have to define a (possibly) multivalued semiflow. Then the existence of the global attractor is proved for regular solutions in the topology of the space $L^2\,(\Omega)$ and for strong solutions in the topology of the space $H_0^1\,(\Omega)$ (or $H_0^1\,(\Omega) \cap L^p\,(\Omega)$), extending in this way the known results for the local problem [21].

The structure of the global attractor is an important feature as it gives us an insight into the long-term dynamics of the solutions. In the multivalued situation it is a challenging problem that has not been completely understood yet. So far in the local case several results in this direction have been obtained for reaction-diffusion equations without uniqueness [2, 5, 21, 22].

In our nonlocal problem for both situations (for regular and strong solutions) we are able under some conditions to define a Lyapunov function on the attractor and to prove that it is characterized as the unstable set of the stationary points (denoted by $M^u(\mathfrak{R})$). Also, the attractor is equal to the stable set of the stationary points when we consider solutions only in the set of bounded complete trajectories (denoted by $M^s(\mathfrak{R})$).

2 Existence of solutions

Throughout this paper we will denote by $\|\cdot\|_X$ the norm in the Banach space X.

We consider the following nonlocal reaction-diffusion equation

$$\begin{cases} u_t - a(\|u\|^2_{H^1_0})\Delta u = f(u) + h(t), \text{ in } \Omega \times (0,\infty), \\ u = 0 \quad \text{on } \partial\Omega \times (0,\infty), \\ u(0,x) = u_0(x) \quad \text{in } \Omega, \end{cases} \tag{3}$$

where Ω is a bounded open set of $\mathbb{R}^n$ with smooth boundary $\partial\Omega$.

Let us consider the following conditions on the functions a, f, h :

$$h \in L^2(0,T;L^2(\Omega)) \ \forall T > 0, \tag{4}$$

$$a \in C(\mathbb{R}^+), \ f \in C(\mathbb{R}), \tag{5}$$

$$a(s) \geq m > 0, \tag{6}$$

$$-\kappa - \alpha_2|s|^p \leq f(s)s \leq \kappa - \alpha_1|s|^p, \tag{7}$$

where $m,\ \alpha_1,\ \alpha_2 > 0$ and $\kappa \geq 0$, $p \geq 2$. Observe that then there exists $C > 0$ such that

$$|f(s)| \leq C(1 + |s|^{p-1}) \quad \forall s \in \mathbb{R}, \tag{8}$$

and that the function $\mathcal{F}(s) := \int_0^s f(r)dr$ satisfies

$$-\widetilde{\alpha}_2|s|^p - \widetilde{\kappa} \leq \mathcal{F}(s) \leq \widetilde{\kappa} - \widetilde{\alpha}_1|s|^p \tag{9}$$

for certain positive constants $\widetilde{\alpha}_i$, $i = 1, 2$, and $\widetilde{\kappa} \geq 0$, and

$$|\mathcal{F}(s)| \leq \widetilde{C}(1 + |s|^p) \quad \forall s \in \mathbb{R}. \tag{10}$$

Conditions (4)-(7) will be always assumed throughout the paper. Sometimes, some of the following additional assumptions will also be used:

$$f \in C^1(\mathbb{R}) \text{ be such that } f'(s) \leq \eta, \ \forall s \in \mathbb{R}, \tag{11}$$

$$p \leq \frac{2n-2}{n-2}, \text{ if } n \geq 3, \tag{12}$$

$$a(s) \leq M_1 + M_2 s, \ \forall s \geq 0, \tag{13}$$

$$s \mapsto a(s^2)s \text{ is non-decreasing}, \tag{14}$$

$$a(\cdot) \in C^1\left(\mathbb{R}^+\right) \text{ and } a'(s) \geq 0, \ \forall s \geq 0, \tag{15}$$

for some constants $M_1, M_2, \eta \geq 0$.

Remark 1 $a'(s) \geq 0$ *implies that (14) holds, so condition (15) is stronger than (14). Assumption (14) is used to prove uniqueness of solutions. Assumption (15) is used to obtain the* $H^2(\Omega)$ *regularity of the global attractor.*

Definition 2 *A weak solution to (3) is a function $u(\cdot)$ such that $u \in L^{\infty}(0,T;L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega)) \cap L^p(0,T;L^p(\Omega))$ for any $T>0$ and satisfies the equality*

$$\frac{d}{dt}(u,v) + a(\|u(t)\|^2_{H_0^1})(\nabla u(t), \nabla v) = (f(u(t)),v) + (h(t),v) \quad \forall v \in H_0^1(\Omega) \cap L^p(\Omega), \tag{16}$$

in the sense of scalar distributions.

Here, we denote by $(\cdot,\cdot)$ the inner product in $L^2(\Omega)$ (or $\left(L^2(\Omega)\right)^d$ for $d \in \mathbb{N}$) and also the duality product between $L^p(\Omega)$ and $L^q(\Omega)$ (where q is the conjugate exponent of p, that is, $q = p/(p-1)$). The duality between $H_0^1(\Omega)$ and $H^{-1}(\Omega)$ will be denoted by $\langle \cdot,\cdot \rangle$.

We need to guarantee that the initial condition of the problem makes sense for a weak solution. This can be achieved in a standard way assuming that the function a has an upper bound, that is, there exists $M>0$ such that

$$a(s) \le M \text{ for all } s \ge 0. \tag{17}$$

Indeed, if u is a weak solution to (3), taking into account (8) and (17) it follows that

$$u_t = a(\|u\|^2_{H_0^1})\Delta u + f(u) + h \in L^2(0,T;H^{-1}(\Omega)) + L^q(0,T;L^q(\Omega)). \tag{18}$$

Therefore, by [12, p.33] $u \in C([0,T],L^2(\Omega))$ and the initial condition makes sense when $u_0 \in L^2(\Omega)$.

For the operator $A = -\Delta$, thanks to the assumptions on the domain Ω, it is well known that $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$ [30, Proposition 6.19].

Definition 3 *A regular solution to (3) is a weak solution with the extra regularity $u \in L^{\infty}(\varepsilon,T;H_0^1(\Omega))$ and $u \in L^2(\varepsilon,T;D(A))$ for any $0<\varepsilon<T$.*

Remark 4 *Since $\dfrac{du}{dt} \in L^q(\varepsilon,T;L^q(\Omega))$ for any regular solution, in this case equality (16) is equivalent to the following one:*

$$\begin{aligned} &\int_{\varepsilon}^{T}\int_{\Omega}\frac{du(t,x)}{dt}\xi(t,x)\,dxdt - \int_{\varepsilon}^{T} a(\|u(t)\|^2_{H_0^1})\int_{\Omega}\Delta u \xi dxdt \\ &= \int_{\varepsilon}^{T}\int_{\Omega} f(u(t,x))\xi(t,x)\,dxdt + \int_{\varepsilon}^{T}\int_{\Omega} h(t,x)\xi(t,x)\,dxdt, \end{aligned} \tag{19}$$

for all $0<\varepsilon<T$ and $\xi \in L^p(0,T;L^p(\Omega))$.

Lemma 5 *Let $u \in L^p(\varepsilon,T;X)$, $\dfrac{du}{dt} \in L^q(\varepsilon,T;X')$ for all $0<\varepsilon<T$, where X is a reflexive and separable Banach space and X' denotes its dual space. Assume that $\beta \in C(\mathbb{R}^+)$ is such that $\beta \in W^{1,\infty}(\varepsilon,T;[\beta(\varepsilon),\beta(T)])$ and $0<\beta(\varepsilon)<\beta(T)$ for all $0<\varepsilon<T$. Then $w(\cdot) = u(\beta(\cdot)) \in L^p(\varepsilon,T;X)$, $\dfrac{dw}{dt} \in L^q(\varepsilon,T;X')$, for all $0<\varepsilon<T$, and*

$$\frac{dw}{dt}(t) = \frac{du}{dt}(\beta(t))\frac{d\beta}{dt}(t) \text{ for a.a. } t>0. \tag{20}$$

Proof. We fix arbitrary $0<\varepsilon<T$. There exists a sequence $u_n \in C^1([\beta(\varepsilon),\beta(T)],X)$ such that $u_n \to u$ in $L^p(\beta(\varepsilon),\beta(T);X)$ and $\dfrac{du_n}{dt} \to \dfrac{du}{dt}$ in $L^q(\beta(\varepsilon),\beta(T);X')$ [20, Chapter IV]. We define $w_n(t) = u_n(\beta(t))$. Following the same proof of [4, Corollary VIII.10] we obtain that $w_n(\cdot) \in W^{1,\infty}(\varepsilon,T;X)$ and

$$\frac{dw_n}{dt}(t) = \frac{du_n}{dt}(\beta(t))\frac{d\beta}{dt}(t) \text{ for a.a. } t>0.$$

It is clear that $w_n \to w$ in $L^p(\varepsilon,T;X)$ and $\dfrac{du_n}{dt}(\beta(\cdot)) \to \dfrac{du}{dt}(\beta(\cdot))$ in $L^q(\varepsilon,T;X')$. Passing to the limit we obtain that

$$\frac{dw}{dt}(\cdot) = \frac{du}{dt}(\beta(\cdot))\frac{d\beta}{dt}(\cdot)$$

in the sense of distributions $\mathcal{D}'(0,+\infty;X)$. As $\frac{du}{dt}(\beta(\cdot))\frac{d\beta}{dt}(\cdot)\in L^q(\varepsilon,T;X')$, $\frac{dw}{dt}\in L^q(\varepsilon,T;X')$ and (20) holds true. ■

We would like to avoid a being uniformly bounded by above (i.e. to relax assumption (17)). We can still prove the continuity in $L^2(\Omega)$ of u for regular solutions by assuming that a has at most linear growth.

Lemma 6 *Assume that conditions (4)-(7), (13) hold. Then any regular solution satisfies that $u\in C([0,T],L^2(\Omega))$ for all $T>0$. Moreover, $w(t)=u\left(\alpha^{-1}(t)\right)$, where $\alpha(t)=\int_0^t a(\|u(s)\|^2_{H_0^1})ds$, is a regular solution to the problem*

$$\begin{cases} w_t-\Delta w=\dfrac{f(w)+h(t)}{a(\|w\|^2_{H_0^1})}, & \text{in } \Omega\times(0,\infty),\\ w=0 \quad \text{on } \partial\Omega\times(0,\infty), & \\ w(0,x)=u_0(x) \quad \text{in } \Omega. & \end{cases} \tag{21}$$

Proof. Condition (13) guarantees that $a(\|u(\cdot)\|^2_{H_0^1})\in L^1(0,T)$ if $u\in L^2\left(0,T;H_0^1(\Omega)\right)$. We make the following time rescaling

$$u(t,x)=w(\alpha(t),x).$$

As $a(\|u(\cdot)\|^2_{H_0^1})\in L^1(0,T)$, the function $t\mapsto\alpha(t)$ is continuous and $\beta(\cdot)=\alpha^{-1}(\cdot)$ satisfies the conditions of Lemma 5. It is clear that the function $w(t,x)=u(\alpha^{-1}(t),x)$ belongs to the space $L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;H_0^1(\Omega))\cap L^p(0,T;L^p(\Omega))$ and also to the spaces $L^\infty(\varepsilon,T;H_0^1(\Omega))$ and $L^2(\varepsilon,T;D(A))$ for any $0<\varepsilon<T$. Moreover, $\frac{du}{dt}\in L^q(\varepsilon,T;L^q(\Omega))$ and Lemma 5 give $\frac{dw}{dt}\in L^q(\varepsilon,T;L^q(\Omega))$ and

$$\frac{dw}{dt}(t)=\frac{du}{dt}\left(\alpha^{-1}(t)\right)\frac{d}{dt}\alpha^{-1}(t)=\frac{du}{dt}\left(\alpha^{-1}(t)\right)\frac{1}{a\left(\|w(t))\|^2_{H_0^1}\right)},\ \text{for a.a. } t. \tag{22}$$

Equality (19) implies that

$$\frac{du}{dt}\left(\alpha^{-1}(t)\right)-a\left(\|u(\alpha^{-1}(t))\|^2_{H_0^1}\right)\Delta u\left(\alpha^{-1}(t)\right)=f\left(u\left(\alpha^{-1}(t)\right)\right)+h(t),\ \text{for a.a. } t>0,$$

so (22) gives

$$\frac{dw}{dt}(t)-\Delta w(t)=\frac{f(w(t))}{a(\|w(t)\|^2_{H_0^1})}+\frac{h(t)}{a(\|w(t)\|^2_{H_0^1})}\ \text{for a.a. } t>0.$$

Hence, w is a regular solution to problem (21). Since $0<\frac{1}{a(s)}\leq\frac{1}{m}$, we obtain that

$$\frac{dw}{dt}\in L^2(0,T;H^{-1}(\Omega))+L^q(0,T;L^q(\Omega)).$$

Therefore, $w\in C([0,T],L^2(\Omega))$, so that

$$u\in C([0,T],L^2(\Omega)).$$

■

Remark 7 *Under assumptions (4)-(7) any regular solution $u(\cdot)$ satisfies that $\frac{du}{dt}\in L^q(\varepsilon,T;L^q(\Omega))$ for all $0<\varepsilon<T$. Then by [12, p.33] $u\in C([\varepsilon,T],L^2(\Omega))$, $t\mapsto\|u(t)\|^2$ is absolutely continuous on $[\varepsilon,T]$ and*

$$\frac{d}{dt}\|u(t)\|^2_{L^2}=2\left(\frac{du}{dt},u\right)\ \textit{for a.a. } t>\varepsilon.$$

If the initial condition belongs to $H_0^1(\Omega)\cap L^p(\Omega)$, we can define strong solutions as well.

Definition 8 *A strong solution to (3) is a weak solution with the extra regularity* $u \in L^{\infty}(0,T;H_0^1(\Omega) \cap L^p(\Omega))$*,* $u \in L^2(0,T;D(A))$ *and* $\dfrac{du}{dt} \in L^2\left(0,T;L^2\left(\Omega\right)\right)$ *for any* $T>0$*.*

We observe that if u is a strong solution, then $u \in C([0,T],H_0^1(\Omega))$ (see [31, p.102]). Also, $u \in L^{\infty}(0,T;L^p(\Omega))$ and $u \in C([0,T],L^2(\Omega))$ imply that $u \in C_w([0,T],L^p(\Omega))$ (see [33, p.263]). Thus, an initial condition in $H_0^1(\Omega) \cap L^p(\Omega)$ makes sense. Also, the equality $f(u) = u_t - a\left(\|u\|_{H_0^1}^2\right)\Delta u - h$ implies that $f(u) \in L^2\left(0,T;L^2\left(\Omega\right)\right)$

Also, if u is a regular solution such that $\dfrac{du}{dt} \in L^2\left(\varepsilon,T;L^2\left(\Omega\right)\right)$ for all $0<\varepsilon<T$, then $u \in C((0,T],H_0^1(\Omega))$.

The phase space for regular solutions will be $L^2(\Omega)$, whereas for strong solutions we will use the space $H^1(\Omega) \cap L^p(\Omega)$ (or just $H_0^1(\Omega)$ when $H_0^1(\Omega) \subset L^p(\Omega)$).

The following results will be proved in Theorems 9, 10, 11, 12, 14:

- Conditions (4)-(7), (11), (13) imply the existence of at least one regular solution for any $u_0 \in L^2(\Omega)$. If, in addition, (14) holds, then it is the unique regular solution.
- Conditions (4)-(7), (11) imply the existence of at least one strong solution for any $u_0 \in H_0^1(\Omega) \cap L^p(\Omega)$. If, in addition, (14) holds, then it is the unique strong solution.
- Conditions (4)-(7), (12) imply the existence of at least one strong solution for any $u_0 \in H_0^1(\Omega)$.
- Conditions (4)-(7), (12), (17) imply the existence of at least one regular solution for any $u_0 \in L^2(\Omega)$.

To start with we prove the existence of regular solutions for initial conditions in $L^2(\Omega)$.

Theorem 9 *Assume that conditions (4)-(7), (11) and (13) hold. Then, for any* $u_0 \in L^2(\Omega)$ *there exists at least one regular solution to (3).*

Proof. We will prove the result by compactness and using Faedo-Galerkin approximations.

Consider a fixed value $T>0$. Let $\{w_j\}_{j\geq 1}$ be a sequence of eigenfunctions of $-\Delta$ in $H_0^1(\Omega)$ with homogeneous Dirichlet boundary conditions, which forms a special basis of $L^2(\Omega)$. If Ω is a bounded regular domain, then it is well known that $\{w_j\} \subset H_0^1(\Omega) \cap L^p(\Omega)$ and that for the set $V_n = span[w_1, \ldots, w_n]$ we have that $\cup_{n\in\mathbb{N}} V_n$ is dense in $L^2(\Omega)$ and also in $H_0^1(\Omega) \cap L^p(\Omega)$ [25]. As usual, P_n will be the orthogonal projection in $L^2(\Omega)$, that is

$$z_n := P_n z = \sum_{j=1}^{n} (z, w_j) w_j,$$

and λ_j will be the eigenvalues associated to the egienfunctions w_j. For each integer $n \geq 1$, we consider the Galerkin approximations

$$u_n(t) = \sum_{j=1}^{n} \gamma_{nj}(t) w_j,$$

which satisfy the following nonlinear ODE system

$$\begin{cases} \dfrac{d}{dt}(u_n, w_i) + a(\|u_n\|_{H_0^1}^2)(\nabla u_n, \nabla w_i) = (f(u_n), w_i) + (h, w_i) \quad \forall i = 1, \ldots, n, \\ u_n(0) = P_n u_0. \end{cases} \tag{23}$$

where $P_n u_0 \to u_0$ in $L^2(\Omega)$. Since (23) can be written in the normal form with a continuous right-hand side, this Cauchy problem possesses a solution on some interval $[0,t_n)$. We claim that for any $T>0$ such a solution can be extended to the whole interval $[0,T]$, which follows from a priori estimates in the space $L^2(\Omega)$ of the sequence $\{u_n\}$.

Multiplying by $\gamma_{ni}(t)$ and summing from $i=1$ to n, we obtain

$$\frac{1}{2}\frac{d}{dt}\|u_n(t)\|_{L^2}^2 + a(\|u_n\|_{H_0^1}^2)\|u_n(t)\|_{H_0^1}^2 = (f(u_n(t)), u_n(t)) + (h, u_n(t)) \text{ for } a.e.\ t \in (0,t_n). \tag{24}$$

Using (7) and the Young and Poincaré inequalities we deduce that

$$(f(u_n(t)), u_n(t)) \leq \kappa|\Omega| - \alpha_1 \|u_n(t)\|_{L^p}^p,$$

$$(h(t), u_n(t)) \leq \frac{m}{2}\|u_n(t)\|_{H_0^1}^2 + \frac{1}{2\lambda_1 m}\|h(t)\|_{L^2}^2.$$

Hence, from (24) it follows that

$$\frac{1}{2}\frac{d}{dt}\|u_n(t)\|_{L^2}^2 + \frac{m}{2}\|u_n(t)\|_{H_0^1}^2 + \alpha_1\|u_n(t)\|_{L^p}^p \leq \kappa|\Omega| + \frac{1}{2\lambda_1 m}\|h(t)\|_{L^2}^2 \text{ for } a.e.\ t \in (0, t_n). \tag{25}$$

Then, integrating (25) from 0 to $t \in (0, t_n)$ we deduce

$$\begin{aligned}&\frac{1}{2}\|u_n(t)\|_{L^2}^2 + \frac{m}{2}\int_0^t \|u_n(s)\|_{H_0^1}^2 ds + \alpha_1 \int_0^t \|u_n(s)\|_{L^p}^p ds \\ &\leq \kappa|\Omega|t + \frac{1}{2\lambda_1 m}\int_0^t \|h(s)\|_{L^2}^2 ds + \frac{1}{2}\|u_n(0)\|_{L^2}^2 \leq TK_2 + K_3(T) + \frac{1}{2}\|u_n(0)\|_{L^2}^2.\end{aligned} \tag{26}$$

Therefore, the sequence $\{u_n\}$ is well defined and bounded in $L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;H_0^1(\Omega))\cap L^p(0,T;L^p(\Omega))$. Also, $\{-\Delta u_n\}$ is bounded in $L^2(0,T;H^{-1}(\Omega))$.

On the other hand, by (8) it follows that

$$\int_0^T \int_\Omega |f(u(x,t))|^q dxdt \leq 2^{q-1}C^q(|\Omega|T + \int_0^T \|u(t)\|_{L^p}^p dt),$$

with $\frac{1}{p} + \frac{1}{q} = 1$. Hence, since $\{u_n\}$ is bounded in $L^p(0,T;L^p(\Omega))$, $\{f(u_n)\}$ is bounded in $L^q(0,T;L^q(\Omega))$.

On the other hand, multiplying (23) by $\lambda_i \gamma_{ni}(t)$ and summing from $i = 1$ to n, we obtain

$$\frac{1}{2}\frac{d}{dt}\|u_n\|_{H_0^1}^2 + m\|\Delta u_n\|_{L^2}^2 \leq \langle f(u_n), -\Delta u_n\rangle + (h(t), -\Delta u_n) \leq \eta\|u_n\|_{H_0^1}^2 + \frac{1}{2m}\|h(t)\|_{L^2}^2 + \frac{m}{2}\|\Delta u_n\|_{L^2}^2.$$

Integrating the previous expression between s and t, with $0 < s \leq t \leq T$, and using (11) we have

$$\frac{1}{2}\|u_n(t)\|_{H_0^1}^2 + \frac{m}{2}\int_s^t \|\Delta u_n(r)\|_{L^2}^2 dr \leq \eta\int_0^T \|u_n(r)\|_{H_0^1}^2 dr + \frac{1}{2}\|u_n(s)\|_{H_0^1}^2 + \frac{1}{2m}\int_s^t \|h(r)\|_{L^2}^2 dr. \tag{27}$$

Now, integrating in s between 0 and t, it follows that

$$t\|u_n(t)\|_{H_0^1}^2 \leq (2\eta T + 1)\int_0^T \|u_n(r)\|_{H_0^1}^2 dr + K_3(T)T.$$

Hence,

$$\|u_n(t)\|_{H_0^1}^2 \leq \frac{2\eta T + 1}{\varepsilon}\int_0^T \|u_n(r)\|_{H_0^1}^2 dr + \frac{K_3(T)T}{\varepsilon}. \tag{28}$$

for all $t \in [\varepsilon, T]$ with $\varepsilon \in (0,T)$. From the last inequality and (26) we deduce that $\{\|u_n(t)\|_{H_0^1}\}$ is uniformly bounded in $[\varepsilon, T]$ and by the continuity of the function a we get that $\{a(\|u_n(t)\|_{H_0^1}^2)\}$ is bounded in $[\varepsilon, T]$. Also, it follows that

$$\{u_n\} \text{ is bounded in } L^\infty(\varepsilon, T; H_0^1(\Omega)). \tag{29}$$

On the other hand, taking $s = \varepsilon$ and $t = T$ in (27), by (26) we obtain that

$$\{u_n\} \text{ is bounded in } L^2(\varepsilon, T; D(A)), \tag{30}$$

so $\{-\Delta u_n\}$ and $\{a(\|u_n\|_{H_0^1}^2)\Delta u_n\}$ are bounded in $L^2(\varepsilon, T; L^2(\Omega))$. Thus,

$$\left\{\frac{du_n}{dt}\right\} \text{ is bounded in } L^q(\varepsilon, T; L^q(\Omega)). \tag{31}$$

Therefore, there exists $u \in L^\infty(\varepsilon,T;H_0^1(\Omega)) \cap L^2(0,T;H_0^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega)) \cap L^2(\varepsilon,T;D(A)) \cap L^p(0,T;L^p(\Omega))$ such that $\dfrac{du}{dt} \in L^q(\varepsilon,T;L^q(\Omega))$ and a subsequence $\{u_n\}$, relabelled the same, such that

$$\begin{aligned}
u_n &\overset{*}{\rightharpoonup} u \text{ in } L^\infty(\varepsilon,T;H_0^1(\Omega)),\\
u_n &\overset{*}{\rightharpoonup} u \text{ in } L^\infty(0,T;L^2(\Omega)),\\
u_n &\rightharpoonup u \text{ in } L^2(0,T;H_0^1(\Omega)),\\
u_n &\rightharpoonup u \text{ in } L^p(0,T;L^p(\Omega)),\\
u_n &\rightharpoonup u \text{ in } L^2(\varepsilon,T;D(A)),\\
\frac{du_n}{dt} &\rightharpoonup \frac{du}{dt} \text{ in } L^q(\varepsilon,T;L^q(\Omega)),\\
f(u_n) &\rightharpoonup \chi \text{ in } L^q(0,T;L^q(\Omega)),\\
a(\|u_n\|_{H_0^1}^2) &\overset{*}{\rightharpoonup} b \text{ in } L^\infty(\varepsilon,T),
\end{aligned} \tag{32}$$

for any $0<\varepsilon<T$, where $\rightharpoonup$ means weak convergence and $\overset{*}{\rightharpoonup}$ weak star convergence.

Moreover, by (30)-(31) the Aubin-Lions Compactness Lemma gives that $u_n \to u$ in $L^2(\varepsilon,T;H_0^1(\Omega))$, so $u_n(t) \to u(t)$ in $H_0^1(\Omega)$ a.e. on (ε,T) for any $\varepsilon>0$. Consequently, there exists a subsequence $\{u_n\}$, relabelled the same, such that $u_n(t,x) \to u(t,x)$ a.e. in $\Omega\times(0,T)$. Also, we know that $P_n f(u_n) \rightharpoonup \chi$ (see [30, p.224]). Since f is continuous, it follows that $f(u_n(t,x)) \to f(u(t,x))$ a.e. in $\Omega\times(0,T)$. Therefore, in view of (32), by [26, Lemma 1.3] we have that $\chi = f(u)$.

As a consequence, by the continuity of a, we get that

$$a(\|u_n(t)\|_{H_0^1}^2) \to a(\|u(t)\|_{H_0^1}^2) \quad \text{a.e. on } (\varepsilon,T).$$

Since the sequence is bounded, by the Lebesgue theorem this convergence takes place in $L^2(\varepsilon,T)$ and $b=a(\|u\|_{H_0^1}^2)$ on (ε,T). Thus,

$$a(\|u_n\|_{H_0^1}^2)\Delta u_n \rightharpoonup a(\|u\|_{H_0^1}^2)\Delta u, \quad \text{ in } L^2(\varepsilon,T;L^2(\Omega)). \tag{33}$$

Finally, since $\{w_i\}$ is dense in $H_0^1(\Omega)\cap L^p(\Omega)$, in view of (32) and (33), we can pass to the limit in (23) and conclude that (16) holds for all $v\in H_0^1(\Omega)\cap L^p(\Omega)$.

To conclude the proof, we have to check that $u(0)=u_0$. Indeed, let be $\phi \in C^1([0,T]);H_0^1(\Omega)\cap L^p(\Omega))$, with $\phi(T)=0$, $\phi(0)\neq 0$. We consider the functions $w(t)=u(\alpha^{-1}(t))$, $w_n(t)=u_n\left(\alpha_n^{-1}(t)\right)$ (here $\alpha_n(t)=\int_0^t a(\|u_n(r)\|_{H_0^1}^2 dr)$), which by Lemma 6 are regular solutions to problem (21) with initial conditions $w(0)=u_0$ and to the corresponding Galerkin approximations with initial condition $w_n(0)=u_n(0)=P_n u_0$, respectively. Since $\dfrac{dw}{dt}\in L^2(0,T;H^{-1}(\Omega))+L^q(0,T;L^q(\Omega))$, we can multiply the equation in (21) by ϕ and integrate by parts in the t variable to obtain that

$$\int_0^T \left(-\left(w(t),\phi'(t)\right)-\langle\Delta w(t),\phi(t)\rangle\right)dt = \int_0^T \left(\frac{f(w(t))+h(t)}{a(\|w(t)\|_{H_0^1}^2)},\phi(t)\right)dt + \left(w(0),\phi(0)\right), \tag{34}$$

$$\int_0^T \left(-\left(w_n(t),\phi'(t)\right)-\langle\Delta w_n(t),\phi(t)\rangle\right)dt = \int_0^T \left(\frac{P_n f(w_n(t))+P_n h(t)}{a(\|w_n(t)\|_{H_0^1}^2)},\phi(t)\right)dt + \left(w_n(0),\phi(0)\right). \tag{35}$$

We can easily obtain by the previous convergences and (6) that

$$\begin{aligned}
w_n &\rightharpoonup w \text{ in } L^2\left(0,T;H_0^1(\Omega)\right),\\
\Delta w_n &\rightharpoonup \Delta w \text{ in } L^2\left(0,T;H^{-1}(\Omega)\right),\\
\frac{P_n f(w_n(t))+P_n h(t)}{a(\|w_n(t)\|_{H_0^1}^2)} &\rightharpoonup \frac{f(w(t))+h(t)}{a(\|w(t)\|_{H_0^1}^2)} \text{ in } L^q\left(0,T;L^q(\Omega)\right).
\end{aligned}$$

Passing to the limit in (35), taking in to account (34) and bearing in mind $w_n(0) = P_n u_0 \to u_0$ we get

$$(w(0), \phi(0)) = (u_0, \phi(0)).$$

Since $\phi(0) \in H_0^1(\Omega) \cap L^p(\Omega)$ is arbitrary, we infer that $w(0) = u(0) = u_0$.

Hence, u is a regular solution to (3) satisfying $u(0) = u_0$. ■

Second, we will prove the existence of strong solutions for initial conditions in $H_0^1(\Omega) \cap L^p(\Omega)$. In this case, we do not need to impose the upper bound (13) of the function a.

Theorem 10 *Suppose that conditions (4)-(7) and (11) are fulfilled. Then, for any $u_0 \in H_0^1(\Omega) \cap L^p(\Omega)$ there exists at least a strong solution to (3).*

Proof. We consider, as in Theorem 9, the Galerkin approximations $\{u_n\}$ and an element u for which (32) holds. Under the aforementioned conditions, we will obtain that u_n converges to a strong solution to (3). In this proof it is important to observe that $P_n u_0 \to u_0$ in the spaces $H_0^1(\Omega)$ and $L^p(\Omega)$ [30, p.199 and 220]. Thus, the sequences $\|P_n u_0\|_{H_0^1}$ and $\|P_n u_0\|_{L^p}$ are bounded.

First, we multiply the equation in (23) by $\frac{du_n}{dt}$ to obtain

$$\|\frac{d}{dt}u_n(t)\|_{L^2}^2 + a(\|u_n\|_{H_0^1}^2)\frac{1}{2}\frac{d}{dt}\|u_n\|_{H_0^1}^2 = \frac{d}{dt}\int_\Omega \mathcal{F}(u_n)dx + (h(t), \frac{du_n}{dt}).$$

Introducing

$$A(s) = \int_0^s a(r)dr, \tag{36}$$

we have

$$\frac{1}{2}\|\frac{d}{dt}u_n(t)\|_{L^2}^2 + \frac{d}{dt}\left[\frac{1}{2}A(\|u_n\|_{H_0^1}^2) - \int_\Omega \mathcal{F}(u_n(t))dx\right] \leq \frac{1}{2}\|h(t)\|_{L^2}^2. \tag{37}$$

Now, integrating (37) we have

$$\frac{1}{2}\int_0^t \|\frac{d}{ds}u_n(s)\|_{L^2}^2 ds + \frac{1}{2}A(\|u_n(t)\|_{H_0^1}^2) - \int_\Omega \mathcal{F}(u_n(t))dx$$
$$\leq \frac{1}{2}A(\|u_n(0)\|_{H_0^1}^2) - \int_\Omega \mathcal{F}(u_n(0))dx + \frac{1}{2}\int_0^t \|h(s)\|_{L^2}^2 ds.$$

From (6) and (9) we get

$$\begin{aligned} &\frac{m}{2}\|u_n(t)\|_{H_0^1}^2 + \widetilde{\alpha}_1\|u_n(t)\|_{L^p}^p + \frac{1}{2}\int_0^t \|\frac{d}{ds}u_n(s)\|_{L^2}^2 ds \\ &\leq \frac{1}{2}A(\|u_n(0)\|_{H_0^1}^2) + \widetilde{\alpha}_2\|u_n(0)\|_{L^p}^p + K(T). \end{aligned} \tag{38}$$

Now, from (38) we obtain that

$$\left\{\frac{du_n}{dt}\right\} \text{ is bounded in } L^2(0,T;L^2(\Omega)), \tag{39}$$

so

$$\frac{du_n}{dt} \rightharpoonup \frac{du}{dt} \text{ in } L^2(0,T;L^2(\Omega)). \tag{40}$$

On the other hand, the embedding $H_0^1(\Omega) \subset\subset L^2(\Omega)$ and the Aubin-Lion Compactness Lemma imply that

$$u_n \to u \text{ in } L^2(0,T;L^2(\Omega)).$$

Hence,

$$u_n \to u \text{ for a.e. } (x,t) \in \Omega \times (0,T).$$

Moreover, thanks to

$$\|u_n(t_2) - u_n(t_1)\|_{L^2}^2 = \left\| \int_{t_1}^{t_2} \frac{d}{dt} u_n(s) ds \right\|_{L^2}^2 \leq \|\frac{d}{dt} u_n\|_{L^2(0,T;L^2(\Omega))}^2 \, |t_2 - t_1| \quad \forall t_1, t_2 \in [0,T],$$

(38), (39) and $H_0^1(\Omega) \subset\subset L^2(\Omega)$, the Ascoli-Arzelà theorem implies that $\{u_n\}$ converges strongly in the space $C([0,T]; L^2(\Omega))$ for all $T > 0$. Therefore, we obtain from (38) that $u_n(t) \rightharpoonup u(t)$ in $H_0^1(\Omega) \cap L^p(\Omega)$, for any $t \geq 0$, and

$$u_n \overset{*}{\rightharpoonup} u \text{ in } L^\infty(0,T; H_0^1(\Omega) \cap L^p(\Omega)). \tag{41}$$

Also, by the continuity of the function a, $\left\{ a(\|u_n(t)\|_{H_0^1}^2) \right\}$ is uniformly bounded in $[0,T]$.

Multiplying (23) by $\lambda_i \gamma_{ni}(t)$ and summing from $i = 1$ to n, we obtain

$$\frac{1}{2} \frac{d}{dt} \|u_n\|_{H_0^1}^2 + m\| - \Delta u_n\|_{L^2}^2 = (f(u_n), -\Delta u_n) + (h(t), -\Delta u_n) \leq \eta \|u_n\|_{H_0^1}^2 + \frac{1}{2m} \|h(t)\|_{L^2}^2 + \frac{m}{2} \| - \Delta u_n\|_{L^2}^2.$$

Integrating the previous expression between 0 and T it follows that

$$\frac{1}{2} \|u_n(T)\|_{H_0^1}^2 + \frac{m}{2} \int_0^T \|\Delta u_n(s)\|_{L^2}^2 ds \leq \eta \int_0^T \|u_n(t)\|_{H_0^1}^2 dt + \frac{1}{2} \|u_n(0)\|_{H_0^1}^2 + K(T). \tag{42}$$

Finally, taking into account (26), from (42) we deduce that

$$u_n \text{ is uniformly bounded in } L^2(0,T; D(A)),$$

so

$$u_n \rightharpoonup u \text{ in } L^2(0,T; D(A)). \tag{43}$$

Arguing as in Theorem 9 we also obtain that

$$\begin{aligned} u_n &\to u \text{ in } L^2\left(0,T; H_0^1(\Omega)\right), \\ a\left(\|u_n\|_{H_0^1}^2\right) &\to a\left(\|u\|_{H_0^1}^2\right) \text{ in } L^2(0,T), \\ f(u_n) &\rightharpoonup f(u) \text{ in } L^q(0,T; L^q(\Omega)), \\ a\left(\|u_n\|_{H_0^1}^2\right) \Delta u_n &\rightharpoonup a\left(\|u\|_{H_0^1}^2\right) \Delta u \text{ in } L^2(0,T; L^2(\Omega)). \end{aligned} \tag{44}$$

Therefore, we can pass to the limit to conclude that u is a strong solution.

It remains to show that $u(0) = u_0$. This can be done, in a similar way as in Theorem 9, by multiplying the equation in (3) by a function $\phi \in C^1([0,T]); H_0^1(\Omega) \cap L^p(\Omega))$, with $\phi(T) = 0$, $\phi(0) \neq 0$ for the Galerkin approximations u_n and the limit function u and integrating by parts. Then taking into account the above convergences and $P_n u_0 \to u_0$ in $L^2(\Omega)$ we obtain that $u(0) = u_0$. ■

We can still ensure the existence of strong solutions without using condition (11) by imposing extra assumptions on the parameter p. Indeed, if (12) is satisfied, then the embedding $H_0^1(\Omega) \subset L^{2(p-1)}(\Omega) \subset L^p(\Omega)$ and (8) imply that

$$||f(u(t))||_{L^2}^2 \leq 2C(1 + \int_\Omega |u(t,x)|^{2(p-1)} dx) \leq \widetilde{C} \left(1 + \|u(t)\|_{H_0^1}^{2(p-1)}\right), \tag{45}$$

so

$$f(u) \in L^2(0,T; L^2(\Omega)) \tag{46}$$

provided that $u \in L^\infty(0,T; H_0^1(\Omega))$. Moreover, $f(A)$ is bounded in $L^2(0,T; L^2(\Omega)$ if A is a bounded set of $L^\infty(0,T; H_0^1(\Omega))$.

Theorem 11 *Assume that (4)-(7) and (12) hold. Then for any $u_0 \in H_0^1(\Omega)$ there exists at least one strong solution to (3).*

Proof. Reasoning as in Theorem 10 and considering as well the Galerkin scheme, (32), (40) and (41) hold. We just need to check that (43) is also true and then repeat the same lines of Theorem 10.

Multiplying (23) by $\lambda_i \gamma_{ni}(t)$ and summing from $i = 1$ to n, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u_n\|^2_{H^1_0} + m\|\Delta u_n\|^2_{L^2} &= (f(u_n), -\Delta u_n) + (h(t), -\Delta u) \\ &\leq \frac{1}{2m}\|f(u_n)\|^2_{L^2} + \frac{m}{2}\|-\Delta u_n\|^2_{L^2} + \frac{1}{m}\|h(t)\|^2_{L^2} + \frac{m}{4}\|\Delta u_n\|^2_{L^2}.\end{aligned}$$

Integrating between 0 and T it follows that

$$\begin{aligned}&\frac{1}{2}\|u_n(T)\|^2_{H^1_0} + \frac{m}{4}\int_0^T \|\Delta u_n(s)\|^2_{L^2} ds \\ &\leq \frac{1}{2m}\int_0^T \|f(u_n(t))\|^2_{L^2} dt + \frac{1}{2}\|u_n(0)\|^2_{H^1_0} + \frac{1}{m}\int_0^T \|h(t)\|^2_{L^2} dt.\end{aligned} \tag{47}$$

In view of (41) and (45), we have that $f(u)$ is bounded in $L^2\left(0,T;L^2(\Omega)\right)$, so from (47) we get that $\{u_n\}$ is bounded in $L^2(0,T;D(A))$. Therefore,

$$u_n \rightharpoonup u \text{ in } L^2(0,T;D(A)), \tag{48}$$

as required. ■

Actually, in the case of regular solutions, we can get rid of the condition (11) as well by imposing the extra assumption (12) on the constant p.

Theorem 12 *Assume that (4)-(7), (12) and (17) hold. Then, for any $u_0 \in L^2(\Omega)$ there exists at least one regular solution to (3).*

Proof. Let $u^n_0 \in H^1_0(\Omega)$ be a sequence such that $u^n_0 \to u_0$ in $L^2(\Omega)$. By Theorem 11 there exists a strong solution $u^n(\cdot)$ of (3) with $u^n(0) = u^n_0$. Since $u^n \in L^2(0,T;D(A))$ and $\frac{du^n}{dt} \in L^2\left(0,T;L^2(\Omega)\right)$, from [31, p.102] the equality

$$\frac{d}{dt}\|u^n\|^2_{H^1_0} = 2(-\Delta u^n, u^n_t)$$

holds true for a.a. $t > 0$.

Now, multiplying (3) by u^n and using (7) it follows that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|u^n(t)\|^2_{L^2} + m\|u^n\|^2_{H^1_0} + \alpha_1\|u^n(t)\|^p_{L^p} \\ &\leq \kappa|\Omega| + \|h(t)\|_{L^2}\|u^n(t)\|_{L^2} \leq \kappa|\Omega| + \frac{1}{2m\lambda_1}\|h(t)\|^2_{L^2} + \frac{m}{2}\|u^n(t)\|^2_{H^1_0},\end{aligned} \tag{49}$$

so

$$\|u^n(t)\|^2_{L^2} \leq \|u^n(0)\|^2_{L^2} + K_1(T). \tag{50}$$

Thus, integrating in (49) between t and $t+r$ we get

$$\begin{aligned}&\|u^n(t+r)\|^2_{L^2} + m\int_t^{t+r} \|u^n(s)\|^2_{H^1_0} ds + 2\alpha_1\int_t^{t+r} \|u^n(s)\|^p_{L^p} ds \\ &\leq 2\kappa|\Omega|r + \frac{1}{m\lambda_1}\int_t^{t+r} \|h(s)\|^2_{L^2} ds + \|u^n(t)\|^2_{L^2} \leq \|u^n(0)\|^2_{L^2} + K_2(T).\end{aligned} \tag{51}$$

Also, by (9) and (17) we deduce that

$$\begin{aligned}&\int_t^{t+r}\left(\frac{1}{2}A(\|u^n(s)\|^2_{H^1_0}) - \int_\Omega \mathcal{F}(u^n(s))dx\right)ds \\ &\leq \int_t^{t+r} \frac{M}{2}\|u^n(s)\|^2_{H^1_0} ds + \widetilde{\kappa}|\Omega| r + \widetilde{\alpha}_2\int_t^{t+r} \|u^n(s)\|^p_{L^p} ds \\ &\leq K_3(T)\left(1 + \|u^n(0)\|^2_{L^2}\right),\end{aligned} \tag{52}$$

for all $n > 0$ and $t \geq 0$.

On the other hand, multiplying (3) by u^n_t we have

$$\frac{1}{2}\|u^n_t(t)\|^2_{L^2} + \frac{d}{dt}\left(\frac{1}{2}A(\|u^n(t)\|^2_{H^1_0}) - \int_\Omega \mathcal{F}(u^n(t))dx\right) \leq \frac{1}{2}\|h(t)\|^2_{L^2}, \tag{53}$$

where the fact that $t \mapsto \int_\Omega \mathcal{F}(u^n(t))dx$ is absolutely continuous on $[0,T]$ and

$$\frac{d}{dt}\int_\Omega \mathcal{F}(u^n(t))dx = \left(f\left(u^n\left(t\right)\right), \frac{du^n}{dt}\left(t\right)\right), \text{ for a.a. } t > 0,$$

is proved by regularization using the regularity of strong solutions and (45). By the Uniform Gronwall Lemma [34] we obtain

$$\frac{1}{2}A(\|u^n(t+r)\|^2_{H^1_0}) - \int_\Omega \mathcal{F}(u^n(t+r))dx \leq \frac{K_3(T)(1+\|u^n(0)\|^2_{L^2})}{r} + K_4(T), \quad \text{for all } 0 \leq t \leq t+r, \tag{54}$$

so that by (6) and (9) we obtain that

$$\|u^n(t+r)\|^2_{H^1_0} + \|u^n\left(t+r\right)\|^p_{L^p} \leq \frac{K_5(T)(1+\|u^n(0)\|^2_{L^2})}{r} + K_6(T), \tag{55}$$

for all $t \geq 0$. Therefore, the sequence $u^n(\cdot)$ is bounded in $L^\infty(r,T;H^1_0(\Omega))$ for all $0 < r < T$. Consequently, $a(\|u^n\left(\cdot\right)\|^2_{H^1_0})$ is bounded in $[r,T]$.

Integrating (53) over (r,T), from (6), (9) and (54) it follows that

$$\begin{aligned}
&\frac{1}{2}\int_r^T \|\frac{d}{dt}u^n(t)\|^2_{L^2}dt + \frac{m}{2}\|u^n(T)\|^2_{H^1_0} + \widetilde{\alpha}_1\|u^n(T)\|^p_{L^p} - \kappa|\Omega| \\
&\leq \frac{1}{2}\int_r^T \|\frac{d}{dt}u^n(t)\|^2_{L^2}dt + \frac{1}{2}A(\|u^n(T)\|^2_{H^1_0}) - \int_\Omega \mathcal{F}(u^n(T))dx \\
&\leq \frac{1}{2}\int_r^T \|h(t)\|^2_{L^2}dt + \frac{1}{2}A(\|u^n(r)\|^2_{H^1_0}) - \int_\Omega \mathcal{F}(u^n(r))dx \\
&\leq \frac{1}{2}\int_r^T \|h(t)\|^2_{L^2}dt + \frac{K_3(T)(1+\|u^n(0)\|^2_{L^2})}{r} + K_4(T).
\end{aligned} \tag{56}$$

Thus $\dfrac{du^n}{dt}$ is bounded in $L^2(r,T;L^2(\Omega))$ for all $0 < r < T$.

Taking into account (45) and (55) we infer that $f\left(u^n\right)$ is bounded in $L^2\left(r,T;L^2\left(\Omega\right)\right)$. By this way, the equality $a(\|u^n\|^2_{H^1_0})\Delta u^n = u^n_t - f(u^n) + h(t)$ implies that u^n and $a(\|u^n\|^2_{H^1_0})\Delta u^n$ are bounded in $L^2(r,T;D(A))$ and $L^2(r,T;L^2(\Omega))$, respectively, for all $0 < r < T$.

By the compact embedding $H^1_0(\Omega) \subset L^2(\Omega)$, we can apply the Ascoli-Arzelà theorem and obtain that, up to a sequence, there exists a function u such that

$$\begin{aligned}
u^n &\overset{*}{\rightharpoonup} u \text{ in } L^\infty(r,T;H^1_0(\Omega)), \\
u^n &\to u \text{ in } C([r,T],L^2(\Omega)), \\
u^n &\rightharpoonup u \text{ in } L^2(r,T;D(A)), \\
\frac{du^n}{dt} &\rightharpoonup \frac{du}{dt} \text{ in } L^2(r,T;L^2(\Omega)),
\end{aligned} \tag{57}$$

for all $0 < r < T$.

On the other hand, from (51) we infer that u^n is bounded in $L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1_0(\Omega)) \cap L^p(0,T;L^p(\Omega))$, for all $T > 0$. Therefore, there exists a subsequence u^n, relabelled the same, such that

$$\begin{aligned}
u^n &\overset{*}{\rightharpoonup} u \text{ in } L^\infty(0,T;L^2(\Omega)), \\
u^n &\rightharpoonup u \text{ in } L^2(0,T;H^1_0(\Omega)), \\
u^n &\rightharpoonup u \text{ in } L^p(0,T;L^p(\Omega)),
\end{aligned} \tag{58}$$

for all $T>0$. On the other hand, arguing as in the proof of Theorem 9 we obtain that

$$\begin{gathered}
f(u^n) \rightharpoonup f(u) \text{ in } L^q(0,T;L^q(\Omega)), \\
u^n \to u \text{ in } L^2(r,T;H_0^1(\Omega)), \\
a(\|u^n\|_{H_0^1}^2) \to a(\|u\|_{H_0^1}^2) \text{ in } L^2(0,T), \\
a(\|u^n(t)\|_{H_0^1}^2)\Delta u^n \rightharpoonup a(\|u(t)\|_{H_0^1}^2)\Delta u \quad \text{in } L^2(r,T;L^2(\Omega)).
\end{gathered}$$

Passing to the limit we obtain that $u(\cdot)$ is a regular solution.

Finally, by a similar argument as in the proof of Theorem 9 we establish that $u(0)=u_0$. ■

Remark 13 *Under the conditions of Theorem 12 any regular solution $u(\cdot)$ satisfies from (45) that $f(u) \in L^2(\varepsilon,T;L^2(\Omega))$ for all $0<\varepsilon<T$, and then $\dfrac{du}{dt} \in L^2(\varepsilon,T;L^2(\Omega))$ as well. Hence, $u \in C((0,T],H_0^1(\Omega))$ for all $T>0$.*

We finish this section by giving a sufficient condition ensuring the uniqueness of solutions.

Theorem 14 *Assume the conditions of Theorem 9 and additionally that (14) is satisfied. Then there can exists at most one regular solution to the Cauchy problem (3) for $u_0 \in L^2(\Omega)$.*

If, moreover, $M_2 = 0$ in condition (13), then there can be at most one weak solution.

Under the conditions of Theorem 10 and (14), there can exists at most one strong solution to the Cauchy problem (3) for $u_0 \in H_0^1(\Omega) \cap L^p(\Omega)$.

Proof. Suppose that u and v are two regular solutions to (3) with the same initial condition $u_0 = v_0$. Then by subtraction and multiplying by $u-v$ we get by Remark 7 that

$$\frac{1}{2}\frac{d}{dt}\|u-v\|_{L^2}^2 + \langle -a(\|u(t)\|_{H_0^1}^2)\Delta u + a(\|v(t)\|_{H_0^1}^2)\Delta v, u-v\rangle = (f(u)-f(v),u-v).$$

Let us consider

$$I = \langle -a(\|u(t)\|_{H_0^1}^2)\Delta u + a(\|v(t)\|_{H_0^1}^2)\Delta v, u-v\rangle.$$

After integrating by parts, we obtain

$$\begin{aligned}
I &= \int_\Omega (a(\|u(t)\|_{H_0^1}^2)|\nabla u|^2 - a(\|u(t)\|_{H_0^1}^2)\nabla u \nabla v - a(\|v(t)\|_{H_0^1}^2)\nabla u\nabla v + a(\|v(t)\|_{H_0^1}^2)|\nabla v|^2)dx \\
&\geq a(\|u(t)\|_{H_0^1}^2)\|u(t)\|_{H_0^1}^2 - \left(a(\|u(t)\|_{H_0^1}^2) + a(\|v(t)\|_{H_0^1}^2)\right)\|u(t)\|_{H_0^1}\|v(t)\|_{H_0^1} + a(\|v(t)\|_{H_0^1}^2)\|v(t)\|_{H_0^1}^2 \\
&= \left(a(\|u(t)\|_{H_0^1}^2)\|u(t)\|_{H_0^1} - a(\|v(t)\|_{H_0^1}^2)\|v(t)\|_{H_0^1}\right)\left(\|u(t)\|_{H_0^1} - \|v(t)\|_{H_0^1}\right) \geq 0,
\end{aligned} \tag{59}$$

where we have used (14) in the last inequality.

Hence, from (59) and $f'(s) \leq \eta$, we infer

$$\frac{1}{2}\frac{d}{dt}\|u-v\|_{L^2}^2 \leq \int_\Omega (f(u)-f(v))(u-v)dx = \int_\Omega \left(\int_v^u f'(s)ds\right)(u-v)dx \leq \eta\|u-v\|_{L^2}^2.$$

By Remark 7 it is correct to apply the Gronwall lemma over an arbitrary interval (ε,t), so

$$\|u(t)-v(t)\|_{L^2}^2 \leq \|u(\varepsilon)-v(\varepsilon)\|_{L^2}^2 \, e^{2\eta(t-\varepsilon)}, \quad t \geq 0.$$

Since Lemma 6 implies that $u,v \in C([0,T],L^2(\Omega))$, we pass to the limit as $\varepsilon \to 0$ to get

$$\|u(t)-v(t)\|_{L^2}^2 \leq \|u(0)-v(0)\|_{L^2}^2 \, e^{2\eta t}, \quad t \geq 0.$$

Hence, the uniqueness follows.

If $M_2 = 0$ in (13), then by (18) the above argument is valid for weak solutions as well.

The proof of the last statement is the same with the only difference that condition (13) is not needed.

■

3 Existence and structure of attractors

In this section we will prove the existence of global attractors for the semiflows generated by regular and strong solutions under different assumptions in the autonomous case, that is, when the function h does depend on t. We will also establish that the attractor is equal to the unstable set of the stationary points or to the stable one when we only consider solutions in the set of bounded complete trajectories.

We consider the following condition instead of (4):

$$h \in L^2(\Omega). \tag{60}$$

Throughout this section, for a metric space X with metric ρ we will denote by $dist_X(C, D)$ the Hausdorff semidistance from C to D, that is, $dist_X(C, D) = \sup_{c \in C} \inf_{d \in D} \rho(c, d)$.

It is important to observe that in the theorems of existence of solutions of the previous section we have used either assumption (11) or (12). Now, when we use condition (11) in some cases it is necessary to add a restriction on the constant p given below in (83).

We summarize the main results of this section:

- Conditions (5)-(7), (11), (17), (14) and (60) imply that the regular solutions generate a semigroup in the phase space $L^2(\Omega)$ possessing a global attractor, which is compact in $H_0^1(\Omega)$ and bounded in $L^p(\Omega)$ (Theorem 17 and Lemma 39). If, in addition, either $h \in L^\infty(\Omega)$ or $p \leq 2n/(n-2)$ for $n \geq 3$, then it is characterized by the unstable set of the stationary points (Proposition 40). Moreover, condition (15) implies that the attractor is bounded in $H^2(\Omega)$ (Proposition 19).
- Conditions (5)-(7), (17), (60) and either (12) or (11), (83) imply that the regular solutions generate a (possibly) multivalued semiflow in the phase space $L^2(\Omega)$ possessing a global attractor, which is compact in $H_0^1(\Omega)$ and $L^p(\Omega)$ and is equal to the unstable set of the stationary points (Theorems 33, 37).
- Conditions (5)-(7), (17), (60) and either (12) or (11), (83) imply that the strong solutions generate a (possibly) multivalued semiflow in the phase space $H_0^1(\Omega)$ possessing a global attractor, which is compact in $H_0^1(\Omega)$ and $L^p(\Omega)$ and is equal to the unstable set of the stationary points (Theorems 45, 48).
- Conditions (5)-(7), (11), (17), (14), (60) and (83) imply that the strong solutions generate a semigroup in the phase space $H_0^1(\Omega)$ possessing a global attractor, which is compact in $H_0^1(\Omega)$ and $L^p(\Omega)$ and is equal to the unstable set of the stationary points (Theorems 50, 53). Moreover, condition (15) implies that the attractor is bounded in $H^2(\Omega)$ (Proposition 54).
- Conditions (5)-(7), (11), (17), (14) and (60) imply that the strong solutions generate a semigroup in the phase space $H_0^1(\Omega) \cap L^p(\Omega)$ (endowed with the induced topology of $H_0^1(\Omega)$) possessing a global attractor, which is compact in $H_0^1(\Omega)$ and bounded in $L^p(\Omega)$ (Theorem 57). If, in addition, either $h \in L^\infty(\Omega)$ or $p \leq 2n/(n-2)$ for $n \geq 3$, then it is characterized by the unstable set of the stationary points (Theorem 60). Moreover, condition (15) implies that the attractor is bounded in $H^2(\Omega)$ (Proposition 61).
- In all the above situations $h \in L^\infty(\Omega)$ implies that the global attractor is bounded in $L^\infty(\Omega)$ (Theorems 18, 36, 47, 59).

3.1 Regular solutions

We split this part into three subsections.

3.1.1 The case of uniqueness

If we assume conditions (5)-(7), (11), (14), (60), then by Theorems 9 and 14 we can define the following continuous semigroup $T_r : \mathbb{R}^+ \times L^2(\Omega) \to L^2(\Omega)$:

$$T_r(t, u_0) = u(t), \tag{61}$$

where $u(\cdot)$ is the unique regular solution to (3). We denote by $\mathfrak{R}$ the set of fixed points of T_r, that is, the points z such that $T_r(t,z) = z$ for any $t \geq 0$.

We also observe that if we assume (17), then using the calculations in (52)-(55) for the Galerkin approximations of any regular solution $u(\cdot)$ one can obtain that $u \in L^{\infty}(\varepsilon, T; L^p(\Omega))$, for all $0 < \varepsilon < T$, and then $u \in C_w((0,+\infty), L^p(\Omega))$.

Our first purpose is to obtain a global attractor. We recall that the set $\mathcal{A}$ is a global compact attractor for T_r if it is compact, invariant (which means $T_r(t,\mathcal{A}) = \mathcal{A}$ for any $t \geq 0$) and it attracts any bounded set B, that is,

$$dist_{L^2}(T_r(t,B), \mathcal{A}) \to 0 \text{ as } t \to +\infty.$$

Proposition 15 *Let (5)-(7), (11), (13), (14) and (60) hold. Then the semigroup T_r has a bounded absorbing set in L^2; that is, there exists a constant K such that for any $R > 0$ there is a time $t_0 = t_0(R)$ such that*

$$\|u(t)\|_{L^2} \leq K \quad \textit{for all} \quad t \geq t_0, \tag{62}$$

where $\|u_0\|_{L^2} \leq R$, $u(t) = T_r(t,u_0)$. Moreover, there is a constant L such that

$$\int_t^{t+1} \|u(s)\|_{H_0^1}^2 ds \leq L \quad \textit{for all} \quad t \geq t_0. \tag{63}$$

Proof. Multiplying equation (3) by u and using (7) and Remark 7 we have

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|_{L^2}^2 + \frac{m}{2}\|u(t)\|_{H_0^1}^2 + \alpha_1 \|u(t)\|_{L^P}^p \leq \kappa|\Omega| + \frac{1}{2\lambda_1 m}\|h\|_{L^2}^2 = \frac{\kappa_1}{2}. \tag{64}$$

The Gronwall lemma and the inequality $\|u(t)\|_{H_0^1}^2 \geq \lambda_1 \|u(t)\|_{L^2}^2$ give

$$\|u(t)\|_{L^2}^2 \leq \|u(\varepsilon)\|_{L^2}^2 e^{-\lambda_1 m(t-\varepsilon)} + \frac{\kappa_1}{\lambda_1 m}, \text{ for any } \varepsilon > 0.$$

As $u \in C([0,T], L^2(\Omega)$ by Lemma 6, passing to the limit we have

$$\|u(t)\|_{L^2}^2 \leq \|u(0)\|_{L^2}^2 e^{-\lambda_1 m t} + \frac{\kappa_1}{\lambda_1 m}. \tag{65}$$

Hence, taking

$$t \geq t_0 \equiv \frac{1}{\lambda_1 m} \ln\left(\frac{\lambda_1 m R^2}{\kappa_1}\right)$$

we get (62) for $K = \frac{2\kappa_1}{\lambda_1 m}$. On the other hand, integrating (64) between t and $t+1$ and using (65) we obtain

$$m\int_t^{t+1} \|u(s)\|_{H_0^1}^2 ds \leq \|u(t)\|_{L^2}^2 + \kappa_1$$

and using the previous bound we get

$$\int_t^{t+1} \|u(s)\|_{H_0^1}^2 ds \leq \frac{\kappa_1}{m} + \frac{2\kappa_1}{\lambda_1 m^2}, \quad \text{for all } t \geq t_0,$$

so that (63) follows. ■

Proposition 16 *Let (5)-(7), (11), (17), (14) and (60) hold. Then there exists a bounded absorbing set in $H_0^1(\Omega) \cap L^p(\Omega)$; that is, there is a constant M such that for any $R > 0$ there is a time $t_1 = t_1(R)$ such that*

$$\|u(t)\|_{H_0^1} + \|u(t)\|_{L^p} \leq M \quad \textit{for all } t \geq t_1,$$

where $\|u_0\|_{L^2} \leq R$, $u(t) = T_r(t,u_0)$.

Proof. The following calculations are formal but can be justified by the Galerkin approximations. Arguing as in (52)-(55) we obtain the existence of a constant C such that

$$\|T_r(1,u(0))\|_{H_0^1}^2+\|T_r(1,u(0))\|_{L^p}^p\leq C(1+\|u(0)\|_{L^2}^2).$$

Hence, the semigroup property $T_r(t+1,u_0)=T_r(1,T_r(t,u_0))$ and (62) imply that

$$\|T_r(t+1,u_0)\|_{H_0^1}^2+\|T_r(t+1,u_0)\|_{L^p}^p\leq C(1+K^2)\ \forall t\geq t_0(R),$$

if $\|u_0\|_{L^2}\leq R$, which proves the statement. ■

Theorem 17 *Let (5)-(7), (11), (17), (14) and (60). Then the equation (3) has a connected global compact attractor $\mathcal{A}_r$, which is bounded in $H_0^1(\Omega)\cap L^p(\Omega)$.*

Proof. Since a bounded set in $H_0^1(\Omega)$ is relatively compact in $L^2(\Omega)$ which is a connected space, the result follows from Theorem 10.5 in [30] and Proposition 16. ■

We will also obtain the boundedness of the attractor in the spaces $L^\infty(\Omega)$ and $H^2(\Omega)$.

First, we recall that a function $\phi:\mathbb{R}\to L^2(\Omega)$ is a complete trajectory of the semigroup T_r if $\phi(t)=T_r(t-s,\phi(s))$ for any $t\geq s$. ϕ is bounded if the set $\cup_{s\in\mathbb{R}}\phi(s)$ is bounded. It is well known [24] that the global attractor is characterized by

$$\mathcal{A}_r=\{\phi(0):\phi\text{ is a bounded complete trajectory}\}.\tag{66}$$

Theorem 18 *Let (5)-(7), (11), (17), (14) and (60) hold. Then the global attractor $\mathcal{A}_r$ is bounded in $L^\infty(\Omega)$, provided that $h\in L^\infty(\Omega)$.*

Proof. We define $v_+=\max\{v,0\},\ v_-=-\max\{-v,0\}$. We multiply equation (3) by $(u-M)_+$ for some appropriate constant M and integrate over Ω to obtain

$$\frac{1}{2}\frac{d}{dt}\int_\Omega|(u-M)_+|^2dx+a(\|u(t)\|_{H_0^1}^2)\int_\Omega|\nabla(u-M)_+|^2dx=\int_\Omega(f(u(t))+h)(u-M)_+dx,$$

where we have used the equality $\frac{1}{2}\frac{d}{dt}\int_\Omega|(u-M)_+|^2dx=(u_t,(u-M)_+)$, which is proved by regularization.

Since $h\in L^\infty(\Omega)$, by (7) we deduce that

$$(f(u)+h)u\leq\widetilde{\kappa}-\widetilde{\alpha}|u|^p.$$

It follows that

$$f(u)+h\leq 0\quad\text{ when }\quad u\geq(\frac{\widetilde{\kappa}}{\widetilde{\alpha}})^{1/p}=M.$$

Therefore, we have

$$(f(u)+h)(u-M)_+\leq 0.$$

Thus, by (6) and the the Poincaré inequality, we deduce that

$$\frac{d}{dt}\int_\Omega|(u-M)_+|^2dx\leq-2m\lambda_1\int_\Omega|(u-M)_+|^2dx.$$

Using the Gronwall inequality, we have

$$\int_\Omega|(u(t)-M)_+|^2dx\leq e^{-2m\lambda(t-\tau)}\int_\Omega|(u(\tau)-M)_+|^2dx.$$

For any $y\in\mathcal{A}_r$ there is by (66) a bounded complete trajectory ϕ such that $\phi(0)=y$. Then taking $t=0$ and $\tau\to-\infty$ in the last inequality, we obtain $y(x)=\phi(0,x)\leq M$, for a.a. $x\in\Omega$. The same arguments can be applied to $(u-M)_-$, which shows that

$$\|y\|_{L^\infty}\leq M,\quad\forall y\in\mathcal{A}_r.$$

■

If we assume (15), then it is possible to show that the global attractor is more regular.

Proposition 19 *Let (5)-(7), (11), (17) and (60) hold. If, additionally, (15) is satisfied, then there exists an absorbing set in $H^2(\Omega)$ and the global attractor is bounded in $H^2(\Omega)$.*

Proof. We will prove the existence of an absorbing set in $H^2(\Omega)$. The boundedness of the global attractor in this space follows then immediately. We proceed formally, but the estimates can be justified via Galerkin approximations.

Let $u(t) = T_r(t, u_0)$ with $\|u_0\|_{L^2} \leq R$. First, we differentiate the equation with respect to t

$$u_{tt} - a'(\|u\|^2_{H^1_0}) \frac{d}{dt} \|u\|^2_{H^1_0} \Delta u - a(\|u\|^2_{H^1_0}) \Delta u_t = f'(u) u_t.$$

Multiplying by u_t we get

$$\frac{1}{2} \frac{d}{dt} \|u_t\|^2_{L^2} + \frac{1}{2} a'(\|u\|^2_{H^1_0}) (\frac{d}{dt} \|u\|^2_{H^1_0})^2 + a(\|u\|^2_{H^1_0}) \|u_t\|^2_{H^1_0} = \int_\Omega f'(u)(u_t)^2 dx. \tag{67}$$

By (6), $a'(s) \geq 0$ and $f'(s) \leq \eta$ we obtain

$$\frac{1}{2} \frac{d}{dt} \|u_t\|^2_{L^2} + m \|u_t\|^2_{H^1_0} \leq \eta \|u_t\|^2_{L^2}. \tag{68}$$

Second, multiplying (3) by u_t and reordering terms, we obtain

$$\frac{d}{dt} \left(\frac{a(\|u\|^2_{H^1_0})}{2} \|u\|^2_{H^1_0} - \int_\Omega \mathcal{F}(u) dx - \int_\Omega h(x) u dx \right) + \|u_t\|^2_{L^2} = \frac{a'(\|u\|^2_{H^1_0})}{2} \|u\|^2_{H^1_0} \frac{d}{dt} \|u\|^2_{H^1_0}. \tag{69}$$

Proposition 16 implies that

$$a'(\|z\|^2_{H^1_0}) \leq \gamma := sup_{|s| \leq M} a'(s^2)$$

if z belongs to the absorbing set in $H^1_0(\Omega)$. On the other hand, multiplying the equation by $-\Delta u$ and using Proposition 16, we obtain

$$\frac{d}{dt} \|u\|^2_{H^1_0} + m \|\Delta u(t)\|^2_{L^2} \leq 2\eta \|u(t)\|^2_{H^1_0} + \frac{1}{m} \|h\|^2_{L^2} \leq K_1 \quad \forall t \geq t_1(R).$$

Hence, by (69) and Proposition 16, it follows

$$\frac{d}{dt} \left(\frac{a(\|u\|^2_{H^1_0})}{2} \|u\|^2_{H^1_0} - \int_\Omega \mathcal{F}(u) dx - \int_\Omega h(x) u dx \right) + \|u_t\|^2_{L^2} \leq \frac{\gamma}{2} K_1 M^2, \quad \forall t \geq t_1(R). \tag{70}$$

Multiplying both sides of the inequality $f'(s) \leq \eta$ by s and integrating between 0 and s, we obtain

$$s f(s) \leq \mathcal{F}(s) + \frac{s^2}{2} \eta, \quad \forall s \in \mathbb{R}. \tag{71}$$

Moreover, integrating $f'(s) \leq \eta$ twice between 0 and s, we infer

$$\mathcal{F}(s) \leq \frac{\eta}{2} s^2 + Cs, \quad \forall s \in \mathbb{R}. \tag{72}$$

Now, we multiply (3) by u and integrate between t and $t+1$ to obtain

$$\frac{1}{2} \|u(t+1)\|^2_{L^2} + \int_t^{t+1} \left(a(\|u\|^2_{H^1_0}) \|u(s)\|^2_{H^1_0} - \int_\Omega f(u) u dx - \int_\Omega h(x) u dx \right) ds = \frac{1}{2} \|u(t)\|^2_{L^2}. \tag{73}$$

From (71), (73) and Proposition 15 it follows

$$\int_t^{t+1} \left(\frac{a(\|u\|^2_{H^1_0})}{2} \|u\|^2_{H^1_0} - \int_\Omega \mathcal{F}(u) dx - \int_\Omega h(x) u dx \right) ds \leq \frac{1}{2} \|u(t)\|^2_{L^2} + \frac{\eta}{2} \int_t^{t+1} \|u\|^2_{L^2} ds \leq \widetilde{L} \quad \forall t \geq t_0$$

The last inequality allows us to apply the Uniform Gronwall Lemma to (70) in order to obtain

$$\frac{a(\|u\|^2_{H^1_0})}{2}\|u\|^2_{H^1_0} - \int_\Omega \mathcal{F}(u)dx - \int_\Omega h(x)udx \leq \widetilde{L} + \frac{\gamma}{2}K_1M^2 \quad \forall t \geq t_1 + 1. \tag{74}$$

Using (6) and (72) we get

$$\frac{a(\|u\|^2_{H^1_0})}{2}\|u\|^2_{H^1_0} - \int_\Omega \mathcal{F}(u)dx - \int_\Omega h(x)udx \geq -\frac{\eta}{2}\|u\|^2_{L^2} - \widetilde{C}\|u\|_{L^2}. \tag{75}$$

Now, integrating (70) from t to $t+1$, using (74), (75), by Proposition 15 we have

$$\int_t^{t+1} \|u_s\|^2_{L^2}ds \leq \widetilde{L} + \gamma K_1M^2 + \frac{\eta}{2}K^2 + \widetilde{C}K = \rho_1, \quad \forall t \geq t_1 + 1. \tag{76}$$

Hence, the last equation allow us to apply to (68) the Uniform Gronwall Lemma [34] to obtain

$$\|\frac{du}{dt}(t)\|^2_{L^2} \leq \rho_2, \quad \forall t \geq t_1 + 2. \tag{77}$$

Finally, we multiply (3) by $-\Delta u$ and use (6) to obtain

$$\frac{m}{2}\|\Delta u\|^2_{L^2} \leq \eta\|u\|^2_{H^1_0} + \frac{1}{m}\|h\|^2_{L^2} + \frac{1}{m}\|u_t\|^2_{L^2}.$$

Thus, by Proposition 16 and (77), we deduce that

$$\|u(t)\|^2_{H^2} \leq \rho_3 \quad \forall t \geq t_1 + 2.$$

■

3.1.2 Abstract theory of attractors for multivalued semiflows

Prior to studying the case of non-uniqueness, we recall some well-known results concerning the structure of attractors for multivalued semiflows.

Consider a metric space (X, d) and a family of functions $\mathcal{R} \subset \mathcal{C}(\mathbb{R}_+; X)$. Denote by $P(X)$ the class of nonempty subsets of X. Then we define the multivalued map $G : \mathbb{R}_+ \times X \to P(X)$ associated with the family $\mathcal{R}$ as follows

$$G(t, u_0) = \{u(t) : u(\cdot) \in \mathcal{R}, u(0) = u_0\}. \tag{78}$$

In this abstract setting, the multivalued map G is expected to satisfy some properties that fit in the framework of multivalued dynamical systems. The first concept is given now.

Definition 20 *A multivalued map $G : \mathbb{R}_+ \times X \to P(X)$ is a multivalued semiflow (or m-semiflow) if $G(0, x) = x$ for all $x \in X$ and $G(t + s, x) \subset G(t, G(s, x))$ for all $t, s \geq 0$ and $x \in X$.*
If the above is not only an inclusion, but an equality, it is said that the m-semiflow is strict.

Once a multivalued semiflow is defined, we recall the following concepts.

Definition 21 *A map $\gamma : \mathbb{R} \to X$ is called a complete trajectory of $\mathcal{R}$ (resp. of G) if $\gamma(\cdot + h)\mid_{[0,\infty)} \in \mathcal{R}$ for all $h \in \mathbb{R}$ (resp. if $\gamma(t + s) \in G(t, \gamma(s))$ for all $s \in \mathbb{R}$ and $t \geq 0$).*
A point $z \in X$ is a fixed point of $\mathcal{R}$ if $\varphi(\cdot) \equiv z \in \mathcal{R}$. The set of all fixed points will be denoted by $\mathfrak{R}_{\mathcal{R}}$.
A point $z \in X$ is a stationary point of G if $z \in G(t, z)$ for all $t \geq 0$.

Definition 22 *Given an m-semiflow G a set $B \subset X$ is said to be negatively (positively) invariant if $B \subset G(t, B)$ $(G(t, B) \subset B)$ for all $t \geq 0$, and strictly invariant (or, simply, invariant) if it is both negatively and positively invariant.*
The set B is said to be weakly invariant if for any $x \in B$ there exists a complete trajectory γ of $\mathcal{R}$ contained in B such that $\gamma(0) = x$. We observe that weak invariance implies negative invariance.

Definition 23 *A set $\mathcal{A} \subset X$ is called a global attractor for the m-semiflow G if it is negatively invariant and it attracts all bounded subsets, i.e., $dist_X(G(t,B),\mathcal{A}) \to 0$ as $t \to +\infty$.*

Remark 24 *When $\mathcal{A}$ is compact, it is the minimal closed attracting set [28, Remark 5].*

In order to obtain a detailed characterization of the internal structure of a global attractor, we introduce an axiomatic set of properties on the set $\mathcal{R}$.

(K1) For any $x \in X$ there exists at least one element $\varphi \in \mathcal{R}$ such that $\varphi(0) = x$.

(K2) $\varphi_\tau(\cdot) := \varphi(\cdot + \tau) \in \mathcal{R}$ for any $\tau \geq 0$ and $\varphi \in \mathcal{R}$ (translation property).

(K3) Let $\varphi_1, \varphi_2 \in \mathcal{R}$ be such that $\varphi_2(0) = \varphi_1(s)$ for some $s > 0$. Then, the function φ defined by

$$\varphi(t) = \begin{cases} \varphi_1(t) & 0 \leq t \leq s, \\ \varphi_2(t-s) & s \leq t, \end{cases}$$

belongs to $\mathcal{R}$ (concatenation property).

(K4) For any sequence $\{\varphi^n\} \subset \mathcal{R}$ such that $\varphi^n(0) \to x_0$ in X, there exist a subsequence $\{\varphi^{n_k}\}$ and $\varphi \in \mathcal{R}$ such that $\varphi^{n_k}(t) \to \varphi(t)$ for all $t \geq 0$.

Remark 25 *If in assumption (K1), for every $x \in X$, there exists a unique $\varphi \in \mathcal{R}$ such that $\varphi(0) = x$, then the set $\{\varphi \in \mathcal{R} : \varphi(0) = x\}$ consists of a single trajectory φ, and the equality $G(t,x) = \varphi(t)$ defines a classical semigroup $G : \mathbb{R}^+ \times X \to X$.*

It is immediate to observe [11, Proposition 2] or [23, Lemma 9] that $\mathcal{R}$ fulfilling (K1) and (K2) gives rise to an m-semiflow G through (78), and if besides (K3) holds, then this m-semiflow is strict. In such a case, a global bounded attractor, supposing that it exists, is strictly invariant [28, Remark 8].

Several properties concerning fixed points, complete trajectories and global attractors are summarized in the following results [21].

Lemma 26 *Let (K1)-(K2) be satisfied. Then every fixed point (resp. complete trajectory) of $\mathcal{R}$ is also a fixed point (resp. complete trajectory) of G.*

If $\mathcal{R}$ fulfills (K1)-(K4), then the fixed points of $\mathcal{R}$ and G coincide. Besides, a map $\gamma : \mathbb{R} \to X$ is a complete trajectory of $\mathcal{R}$ if and only if it is continuous and a complete trajectory of G.

The standard well-known result in the single-valued case for describing the attractor as the union of bounded complete trajectories (see [24]) reads in the multivalued case as follows.

Theorem 27 *Consider $\mathcal{R}$ satisfying (K1) and (K2) and either (K3) or (K4). Assume that G possesses a compact global attractor $\mathcal{A}$. Then*

$$\mathcal{A} = \{\gamma(0) : \gamma \in \mathbb{K}\} = \cup_{t \in \mathbb{R}} \{\gamma(t) : \gamma \in \mathbb{K}\}, \tag{79}$$

where $\mathbb{K}$ denotes the set of all bounded complete trajectories in $\mathcal{R}$. Hence, $\mathcal{A}$ is weakly invariant.

We finish this section by stating a general result about the existence of attractors. We recall that the map $t \mapsto G(t,x)$ is upper semicontinuous if for any $x \in X$ and any neighborhood $O(G(t,x))$ in X there exists $\delta > 0$ such that if $d(y,x) < \delta$, then $G(t,y) \subset O$.

Theorem 28 *[28, Theorem 4 and Remark 8] Let the map $t \mapsto G(t,x)$ be upper semicontinuous with closed values. If there exists a compact attracting set K, that is,*

$$dist_X(G(t,B),K) \to 0, \text{ as } t \to +\infty,$$

for any bounded set B, then G possesses a global compact attractor $\mathcal{A}$, which is the minimal closed attracting set. If, moreover, G is strict, then $\mathcal{A}$ is invariant.

We observe that, although in the papers [28], [21] the space X is assumed to be complete, the results are true in a non-complete space.

3.1.3 The case of non-uniqueness

If we do not assume the additional assumptions on the function $a(\cdot)$ of Section 3.1.1 ensuring uniqueness of the Cauchy problem, we have to define a multivalued semiflow.

We have two possibilities: either to consider the conditions of Theorem 9 with an extra growth assumption or to use the conditions of Theorem 12.

If we assume conditions (5)-(7), (12), (17) and (60), then by Theorem 12 for any $u_0 \in L^2(\Omega)$ there exists at least one regular solution and (45) implies that $f(u) \in L^2(\varepsilon, T; L^2(\Omega))$ for any regular solution, so $\frac{du}{dt} \in L^2(\varepsilon, T; L^2(\Omega))$ as well. In this case, as $H_0^1(\Omega) \subset L^p(\Omega)$, we have that $u \in C((0,+\infty), H_0^1(\Omega)) \subset C((0,+\infty), L^p(\Omega))$.

If we assume conditions (5)-(7), (11), (13) and (60) as well, then we known by Theorem 9 that for any $u_0 \in L^2(\Omega)$ there exists at least one regular solution.

In order to obtain the necessary estimates leading to the existence of a global attractor, we need to ensure that

$$\frac{du}{dt} \in L^2(\varepsilon, T; L^2(\Omega)), \text{ for all } 0 < \varepsilon < T, \tag{80}$$

holds, as by [31, p.102] we obtain that

$$\frac{d}{dt}\|u\|_{H_0^1}^2 = 2(-\Delta u, u_t) \text{ for a.a. } t. \tag{81}$$

and $u \in C((0,+\infty), H_0^1(\Omega))$.

We note that the set of regular solutions of that kind is non-empty if we assume (17), as using inequalities (52)-(56) in the proof of Theorem 9 we prove that the regular solution satisfies (80).

We also observe that we can force all the regular solutions to satisfy $\frac{du}{dt} \in L^2(\varepsilon, T; L^2(\Omega))$ with an additional assumption on the constant p, which is weaker than (12). This is achieved by obtaining that $f(u) \in L^2(\varepsilon, T; L^2(\Omega))$, which can be done by using an interpolation inequality. Indeed, for $u \in L^\infty(\varepsilon, T; H_0^1(\Omega)) \cap L^2(\varepsilon, T; D(A))$ we have the interpolation inequality

$$\|u\|^{2(\gamma+1)}_{L^{2(\gamma+1)}(\varepsilon,T;L^{2(\gamma+1)}(\Omega))} \le \|u\|^{2\gamma}_{L^\infty(\varepsilon,T;L^{p_1}(\Omega))}\|u\|^2_{L^2(\varepsilon,T;L^{p_2}(\Omega))}, \tag{82}$$

where $\gamma = \frac{4}{n-2},\ p_1 = \frac{2n}{n-2},\ p_2 = \frac{2n}{n-4}$, provided that $n > 4$; $\gamma < 2,\ p_1 = 4,\ p_2 = \frac{4}{2-\gamma}$ if $n = 4$; $\gamma = 3,\ p_1 = 6,\ p_2 = +\infty$ if $n = 3$; and $\gamma \ge 0$ is arbitrary for $n = 1, 2$. We have used the embeddings $H_0^1(\Omega) \subset L^{p_1}(\Omega)$, $H^2(\Omega) \subset L^{p_2}(\Omega)$ and [35, Lemma II.4.1, p. 72]. Thus, (8) implies that $f(u) \in L^2(\varepsilon, T; L^2(\Omega))$ if

$$p \le \gamma + 2 \tag{83}$$

and also that

$$\|f(u)\|^2_{L^2(\varepsilon,T;L^2(\Omega))} = \int_\varepsilon^T \int_\Omega |f(u(x,t))|^2 dxdt \le C_1 + C_2 \int_\varepsilon^T \int_\Omega |u(x,t)|^{2(\gamma+1)} dxdt. \tag{84}$$

Condition (83) also implies $H_0^1(\Omega) \subset L^p(\Omega)$, so $u \in C((0,+\infty), L^p(\Omega))$.

Another necessary property to obtain estimates is the fact that $t \mapsto \int_\Omega \mathcal{F}(u(t))dx$ is absolutely continuous on $[\varepsilon, T]$ for all $0 < \varepsilon < T$ and

$$\frac{d}{dt}\int_\Omega \mathcal{F}(u(t))dx = \left(f(u(t)), \frac{du}{dt}(t)\right), \text{ for a.a. } t > 0. \tag{85}$$

This can be proved by regularization in both situations by using the regularity of regular solutions and either (45) or (84).

Therefore, under either the conditions of Theorem 9 with the extra assumption (83) or the conditions of Theorem 12 we define the set

$$\mathcal{R} = K_r^+ := \{u(\cdot) : u \text{ is a regular solution of (3)}\}.$$

We define the (possibly multivalued) map $G_r : \mathbb{R}^+ \times L^2(\Omega) \to P(L^2(\Omega))$ by

$$G_r(t, u_0) = \{u(t) : u \in K_r^+ \text{ and } u(0) = u_0\}.$$

With respect to the axiomatic properties $(K1) - (K4)$ given above, we observe that obviously $(K1)$ is true, and $(K2)$ can be proved easily using equality (19). Therefore, G_r is a multivalued semiflow by the results of the previous section. In this case we are not able to prove $(K3)$, so G_r could be non-strict. Further we will prove that $(K4)$ holds true.

Lemma 29 *Let us assume (5)-(7), (17) and (60). Additionally, assume one of the following assumptions:*

1. *(11) and (83) hold;*
2. *(12) is true.*

Given a sequence $\{u^n\} \subset K_r^+$ such that $u^n(0) \to u_0$ weakly in $L^2(\Omega)$, there exists a subsequence of $\{u^n\}$ (relabeled the same) and $u \in K_r^+$, satisfying $u(0) = u_0$, such that

$$u^n(t) \to u(t) \text{ strongly in } H_0^1(\Omega) \quad \forall t > 0.$$

Proof. We take an arbitrary $T > 0$. Arguing as in the proof of Theorem 9 we obtain the existence of a subsequence of u^n such that

$$\begin{aligned} \{u^n\} &\text{ is bounded in } L^\infty(0, T; L^2(\Omega)), \\ \{u^n\} &\text{ is bounded in } L^p(0, T; L^p(\Omega)), \\ \{f(u^n)\} &\text{ is bounded in } L^q(0, T; L^q(\Omega)). \end{aligned} \tag{86}$$

The only difference is that we obtain inequality (26) in an arbitrary interval $[\varepsilon, T]$ and then pass to the limit as $\varepsilon \to 0$ (see the proof of Proposition 15).

Since $\dfrac{du^n}{dt} \in L^2(\varepsilon, T; L^2(\Omega))$, for any $\varepsilon > 0$, we have that $u \in C((0, T], H_0^1(\Omega))$ and we know that (81), (85) are true. Therefore, arguing as in the proofs of Theorems 9 and 12 and using (84) and (45) there exists $u \in L^\infty(\varepsilon, T; L^2(\Omega)) \cap L^2(0, T; H_0^1(\Omega))$ and a subsequence $\{u^n\}$, relabelled the same, such that

$$\begin{aligned} u_n &\overset{*}{\rightharpoonup} u \text{ in } L^\infty(0, T; L^2(\Omega)) \\ u_n &\overset{*}{\rightharpoonup} u \text{ in } L^\infty(\varepsilon, T; H_0^1(\Omega)) \\ u_n &\rightharpoonup u \text{ in } L^2(0, T; H_0^1(\Omega)) \\ u_n &\rightharpoonup u \text{ in } L^p(0, T; L^p(\Omega)) \\ u_n &\rightharpoonup u \text{ in } L^2(\varepsilon, T; D(A)), \\ \frac{du_n}{dt} &\rightharpoonup \frac{du}{dt} \text{ in } L^2(\varepsilon, T; L^2(\Omega)) \\ f(u_n) &\rightharpoonup f(u) \text{ in } L^q(0, T; L^q(\Omega)), \\ f(u_n) &\rightharpoonup f(u) \text{ in } L^2(\varepsilon, T; L^2(\Omega)), \\ a(\|u_n\|_{H_0^1}^2)\Delta u_n &\rightharpoonup a(\|u\|_{H_0^1}^2)\Delta u \text{ in } L^2(\varepsilon, T; L^2(\Omega)). \end{aligned} \tag{87}$$

In view of (87), the Aubin-Lions Compactness Lemma gives

$$u_n \to u \text{ in } L^2(\varepsilon, T; H_0^1(\Omega)). \tag{88}$$

Since the sequence $\{u^n\}$ is equicontinuous in $L^2(\Omega)$ on $[\varepsilon, T]$ and bounded in $C([\varepsilon, T], H_0^1(\Omega))$, by the compact embedding $H_0^1(\Omega) \subset L^2(\Omega)$ and the Ascoli-Arzelà theorem, a subsequence fulfills

$$u^n \to u \text{ in } C([\varepsilon, T], L^2(\Omega)),$$

$$u^n(t) \rightharpoonup u(t) \text{ in } H_0^1(\Omega) \quad \forall t \in [\varepsilon, T].$$

By a similar argument as in the proof of Theorem 9 we establish that $u \in K_r^+$, $u(0) = u_0$.

Finally, we shall prove that $u^n(t) \to u(t)$ in $H_0^1(\Omega)$ for all $t \in [\varepsilon, T]$.

Multiplying (3) by u_t^n and using (36), (81), and (85) we obtain

$$\frac{1}{2}\left\|\frac{du^n}{dt}\right\|_{L^2}^2 + \frac{d}{dt}\left(\frac{1}{2}A(\|u^n(t)\|_{H_0^1}^2 - \int_\Omega \mathcal{F}(u^n(t))dx\right) \leq \frac{1}{2}\|h\|_{L^2}^2 = D.$$

Thus,

$$\frac{1}{2}A(\|u^n(t)\|_{H_0^1}^2) - \int_\Omega \mathcal{F}(u^n(t))dx \leq \frac{1}{2}A(\|u^n(s)\|_{H_0^1}^2) - \int_\Omega \mathcal{F}(u^n(s))dx + D(t-s),\ t \geq s \geq \varepsilon > 0.$$

The same inequality is valid for the limit function $u(\cdot)$. We observe that the map $y \longmapsto \int_\Omega \mathcal{F}(y(x))dx$ is continuous in the topology of $H_0^1(\Omega)$, which follows easily from $H_0^1(\Omega) \subset L^p(\Omega)$ and (10) using Lebesgue's theorem. Hence, the functions $J_n(t) = \frac{1}{2}A(\|u^n(t)\|_{H_0^1}^2) - \int_\Omega \mathcal{F}(u^n(t))dx - Dt$, $J(t) = \frac{1}{2}A(\|u(t)\|_{H_0^1}^2) - \int_\Omega \mathcal{F}(u(t))dx - Dt$ are continuous and non-increasing in $[\varepsilon, T]$. Moreover, from (88) we deduce that $J_n(t) \to J(t)$ for a.e. $t \in (\varepsilon, T)$. Take $\varepsilon < t_m < T$ such that $t_m \to T$ and $J_n(t_m) \to J(t_m)$ for all m. Then

$$J_n(T) - J(T) \leq J_n(t_m) - J(T) \leq |J_n(t_m) - J(t_m)| + |J(t_m) - J(T)|.$$

For any $\delta > 0$ there exist $m(\delta)$ and $N(m(\delta))$ such that $J^n(T) - J(T) \leq \delta$ if $n \geq N$. Then $\limsup J_n(T) \leq J(T)$, so $\limsup \|u^n(T)\|_{H_0^1}^2 \leq \|u(T)\|_{H_0^1}^2$ (see the explanation below). As $u^n(T) \to u(T)$ weakly in $H_0^1(\Omega)$ implies $\liminf \|u^n(T)\|_{H_0^1}^2 \geq \|u(T)\|_{H_0^1}^2$, we obtain

$$\|u^n(T)\|_{H_0^1}^2 \to \|u(T)\|_{H_0^1}^2,$$

so that $u^n(T) \to u(T)$ strongly in $H_0^1(\Omega)$.

In order to finish the proof rigorously, we have to justify that $\limsup J_n(T) \leq J(T)$ implies the inequality $\limsup \|u^n(T)\|_{H_0^1}^2 \leq \|u(T)\|_{H_0^1}^2$. First, we observe that by (10) we have

$$\left|\int_\Omega \mathcal{F}(u_n(T,x))dx\right| \leq C\int_\Omega (1 + |u_n(T,x)|^p)\,dx,$$

so the boundedness of $u_n(T)$ in $L^p(\Omega)$ implies that $-\int_\Omega \mathcal{F}(u_n(T,x))dx < \infty$. Also, (9) gives $-\mathcal{F}(u_n(T,x)) \geq -\widetilde{\kappa}$, so by Fatou's lemma we obtain

$$\begin{aligned}\liminf\ \left(-\int_\Omega \mathcal{F}(u_n(T,x))dx\right) &\geq \int_\Omega \liminf\,(-\mathcal{F}(u_n(T,x)))\,dx \\ &= -\int_\Omega \mathcal{F}(u(T,x))dx,\end{aligned}$$

where we have used that $\mathcal{F}(u_n(T,x) \to \mathcal{F}(u(T,x))$ for a.a. $x \in \Omega$. By contradiction let us assume that $\limsup \|u_n(T)\|_{H_0^1} > \|u(T)\|$. Then using the continuity of the function $A(s)$ we have

$$\begin{aligned}&\limsup\left(\frac{1}{2}A\left(\|u^n(T)\|_{H_0^1}^2\right) - \int_\Omega \mathcal{F}(u_n(T,x))dx\right) \\ &\geq \limsup\frac{1}{2}\int_0^{\|u^n(T)\|_{H_0^1}^2} a(s)\,ds + \liminf\left(-\int_\Omega \mathcal{F}(u_n(T,x))dx\right) \\ &\geq \frac{1}{2}\int_0^{\limsup\|u^n(T)\|_{H_0^1}^2} a(s)\,ds - \int_\Omega \mathcal{F}(u_n(T,x))dx \\ &> \frac{1}{2}\int_0^{\|u(T)\|_{H_0^1}^2} a(s)\,ds - \int_\Omega \mathcal{F}(u_n(T,x))dx,\end{aligned}$$

which is a contradiction with $\limsup J_n(T) \leq J(T)$. ■

Corollary 30 *Assume the conditions of Lemma 29. Then the set K_r^+ satisfies condition $(K4)$.*

Proposition 31 *Assume the conditions of Lemma 29. The multivalued semiflow G_r is upper semicontinuous for all $t \geq 0$, that is, for any neighborhood $O(G_r(t,u_0))$ in $L^2(\Omega)$ there exists $\delta > 0$ such that if $\|u_0 - v_0\| < \delta$, then $G_r(t,v_0) \subset O$. Also, it has compact values.*

Proof. We argue by contradiction. Assume that there exists $t \geq 0, u_0 \in L^2(\Omega)$, a neighbourhood $O(G_r(t,u_0))$ and a sequence $\{y_n\}$ which fulfills that each $y_n \in G_r(t,u_0^n)$, where u_0^n converges strongly to u_0 in $L^2(\Omega)$, and $y_n \notin O(G_r(t,u_n))$ for all $n \in \mathbb{N}$. Since $y_n \in G_r(t,u_0^n)$ for all n, there exists $u^n \in K_r^+$, $u^n(0) = u_0^n$, such that $y_n = u^n(t)$. Now, since $\{u_0^n\}$ is a convergent sequence of initial data, making use of Lemma 29 there exists a subsequence of $\{u^n\}$ which converges to a function $u \in K_r^+$. Hence, $y_n \to y \in G_r(t,u_0)$. This is a contradiction because $y_n \notin O(G_r(t,u_0))$ for any $n \in \mathbb{N}$. ■

Proposition 32 *Assume the conditions of Lemma 29. Then there exists an absorbing set B_1 for G_r, which is compact in $H_0^1(\Omega)$ and $L^p(\Omega)$.*

Proof. Reasoning as in Proposition 15, we obtain an absorbing set B_0 in $L^2(\Omega)$.

Let $K > 0$ be such that $\|y\| \leq K$ for all $y \in B_0$. Since $\dfrac{du}{dt} \in L\left(\varepsilon, T; L^2(\Omega)\right)$ and (85) holds, we are allowed to multiply (3) by u_t, use (81) and argue as in (52)-(55) to obtain the existence of a constant C such that

$$\|u(1)\|_{H_0^1}^2 + \|u(1)\|_{L^p}^p \leq C(1 + \|u(0)\|_{L^2}^2), \tag{89}$$

for any regular solution $u(\cdot)$ with initial condition $u(0)$.

For any $u_0 \in L^2(\Omega)$ with $\|u_0\|_{L^2} \leq R$ and any $u \in K_r^+$ such that $u(0) = u_0$, the semiflow property $G_r(t+1,u_0) \subset G_r(1, G_r(t,u_0))$ and $G_r(t,u_0) \subset B_0$, if $t \geq t_0(R)$, imply that

$$\|u(t+1)\|_{H_0^1}^2 + \|u(t+1)\|_{L^p}^p \leq C(1 + K^2) \ \forall t \geq t_0(R).$$

Then there exists $M > 0$ such that the closed ball B_M in $H_0^1(\Omega)$ centered at 0 with radius M is absorbing for G_r.

By Lemma 29 the set $B_1 = \overline{G_r(1, B_M)}$ is an absorbing set which is compact in $H_0^1(\Omega)$. The embedding $H_0^1(\Omega) \subset L^p(\Omega)$ implies that it is compact in $L^p(\Omega)$ as well. ■

Theorem 33 *Assume the conditions of Lemma 29. Then the multivalued semiflow G_r possesses a global compact attractor $\mathcal{A}_r$. Moreover, for any set B bounded in $L^2(\Omega)$ we have*

$$dist_{H_0^1}(G_r(t,B), \mathcal{A}_r) \to 0 \quad \text{as } t \to \infty. \tag{90}$$

Also $\mathcal{A}_r$ is compact in $H_0^1(\Omega)$ and $L^p(\Omega)$.

Proof. From Propositions 31 and 32 we deduce that the multivalued semiflow G_r is upper semicontinuous with closed values and the existence of an absorbing which is compact in $H_0^1(\Omega)$ and $L^p(\Omega)$. Therefore, by Theorem 28 the existence of the global attractor and its compactness in $H_0^1(\Omega)$ and $L^p(\Omega)$ follow.

The proof of (90) is analogous to that in Theorem 29 in [21]. ■

The set of all complete trajectories of K_r^+ (see Definition 21) will be denoted by $\mathbb{F}_r$. Moreover, we write $\mathbb{K}_r$ as the set of all complete trajectories which are bounded in $L^2(\Omega)$, and $\mathbb{K}_r^1$ as the ones bounded in $H_0^1(\Omega)$.

Lemma 34 *Assume the conditions of Lemma 29. Then the sets defined above coincide, that is, $\mathbb{K}_r = \mathbb{K}_r^1$.*

Proof. Let $\gamma(\cdot) \in \mathbb{K}_r$. Then there is C such that $\|\gamma(t)\|_{L^2} \leq C$ for any $t \in \mathbb{R}$. Let $u_\tau(\cdot) = \gamma(\cdot + \tau)$ for any τ, which is a regular solution. Since $\dfrac{du}{dt} \in L^2(\varepsilon, T; L^2(\Omega))$, for any $\varepsilon > 0$, the equality (81) holds true. Also, (85) is satisfied. Therefore, we can multiply the equation in (3) by u_t and apply again similar arguments as in Theorem 12 to deduce that

$$\|u(t+r)\|_{H_0^1}^2 \leq \frac{K_1(T)(1 + \|u(0)\|_{L^2}^2)}{r} + K_2(T) \text{ for any } 0 < r < T. \tag{91}$$

Denote $B_\gamma = \cup_{t\in\mathbb{R}}\gamma(t)$. Therefore,

$$B_\gamma \subset G_r(1, B_\gamma)$$

and (91) implies that B_γ is bounded in $H_0^1(\Omega)$, so $\gamma(\cdot) \in \mathbb{K}_r^1$.
The other inclusion is obvious. ■

In view of Corollary 30 and Theorem 27, the global attractor is characterized in terms of bounded complete trajectories:

$$\begin{aligned}\mathcal{A}_r &= \{\gamma(0) : \gamma(\cdot) \in \mathbb{K}_r\} = \{\gamma(0) : \gamma(\cdot) \in \mathbb{K}_r^1\} \\ &= \bigcup_{t\in\mathbb{R}}\{\gamma(t) : \gamma(\cdot) \in \mathbb{K}_r\} = \bigcup_{t\in\mathbb{R}}\{\gamma(t) : \gamma(\cdot) \in \mathbb{K}_r^1\}.\end{aligned} \tag{92}$$

The set $\mathfrak{R}_{K_r^+}$ was defined in the previous section as the set of fixed points of K_r^+, which means that $z \in \mathfrak{R}_{K_r^+}$ if the function $u(\cdot)$ defined by $u(t) = z$, for all $t \geq 0$, belongs to K_r^+. This set can be characterized as follows.

Lemma 35 *Assume the conditions of Lemma 29. Let $\mathfrak{R}$ be the set of $z \in H^2(\Omega) \cap H_0^1(\Omega)$ such that*

$$-a(\|z\|^2_{H_0^1})\Delta z = f(z) + h \text{ in } L^2(\Omega). \tag{93}$$

Then $\mathfrak{R}_{K_r^+} = \mathfrak{R}$.

Proof. If $z \in \mathfrak{R}_{K_r^+}$, then $u(t) \equiv z \in K_r^+$. Thus, $u(\cdot)$ satisfies (19) and $\dfrac{du}{dt} = 0$ in $L^2\left(0,T;L^2(\Omega)\right)$, so (93) is satisfied.

Let $z \in \mathfrak{R}$. Then the map $u(t) \equiv z$ satisfies (93) for any $t \geq 0$ and $\dfrac{du}{dt} = 0$ in $L^2\left(0,T;L^2(\Omega)\right)$, so (19) holds true. ■

The following result is proved exactly as Theorem 18.

Theorem 36 *Assume the conditions of Lemma 29. Then the global attractor $\mathcal{A}$ is bounded in $L^\infty(\Omega)$, provided that $h \in L^\infty(\Omega)$.*

We are now ready to obtain the characterization of the global attractor.

Theorem 37 *Assume the conditions of Lemma 29. Then it holds that*

$$\mathcal{A}_r = M_r^u(\mathfrak{R}) = M_r^s(\mathfrak{R}),$$

where

$$M_r^s(\mathfrak{R}) = \{z : \exists\gamma(\cdot) \in \mathbb{K}_r,\ \gamma(0) = z,\ \ dist_{\ L^2(\Omega)}(\gamma(t), \mathfrak{R}) \to 0,\ t \to +\infty\}, \tag{94}$$

$$M_r^u(\mathfrak{R}) = \{z : \exists\gamma(\cdot) \in \mathbb{F}_r,\ \gamma(0) = z,\ \ dist_{\ L^2(\Omega)}(\gamma(t), \mathfrak{R}) \to 0,\ t \to -\infty\}. \tag{95}$$

Remark 38 *In the definition of $M_r^u(\mathfrak{R})$ we can replace $\mathbb{F}_r$ by $\mathbb{K}_r$. Also, as the global attractor $\mathcal{A}$ is compact in $H_0^1(\Omega)$, in the definitions of $M_r^s(\mathfrak{R})$ and $M_r^u(\mathfrak{R})$, it is equivalent to write $H_0^1(\Omega)$ instead of $L^2(\Omega)$.*

Proof. We consider the function $E : \mathcal{A}_r \to \mathbb{R}$

$$E(y) = \frac{1}{2}A(\|y\|^2_{H_0^1}) - \int_\Omega \mathcal{F}(y(x))dx - \int_\Omega h(x)\,y(x)\,dx, \tag{96}$$

where $A(r) = \int_0^r a(s)ds$. We observe that $E(y)$ is continuous in $H_0^1(\Omega)$. Indeed, the maps $y \mapsto \frac{1}{2}A(\|y\|^2_{H_0^1})$, $y \mapsto \int_\Omega h(x)\,y(x)\,dx$ are obviously continuous in $H_0^1(\Omega)$. On the other hand, both conditions (12) and (83) imply that $H_0^1(\Omega) \subset L^p(\Omega)$, so making use of the Lebesgue theorem the continuity of $y \mapsto \int_\Omega \mathcal{F}(y(x))dx$ follows as well.

Since $\frac{du}{dt} \in L^2\left(\varepsilon, T; L^2\left(\Omega\right)\right)$ and (85) holds for any $u \in K_r^+$ and $0 < \varepsilon < T$, we obtain the energy equality

$$\int_s^t \|\frac{d}{dr}u(r)\|_{L^2}^2 dr + E(u(t)) = E(u(s)) \quad \text{for all } t \geq s > 0. \tag{97}$$

Hence, $E\left(u\left(t\right)\right)$ is non-increasing and, by (6) and (9), bounded from below. Thus, $E(u(t)) \to l$, as $t \to +\infty$, for some $l \in \mathbb{R}$.

Let $x \in \mathcal{A}_r$ and $\gamma\left(0\right) = x$, where $\gamma \in \mathbb{K}_r$. We reason by contradiction, so let suppose that there exists $\varepsilon > 0$ and a sequence $\gamma(t_n)$, $t_n \to +\infty$, such that

$$dist\ _{L^2(\Omega)}(\gamma(t_n), \mathfrak{R}) > \varepsilon.$$

In view of Theorem 33, $\mathcal{A}_r$ is compact in $H_0^1(\Omega)$, so we can take a converging subsequence (relabeled the same) such that $\gamma(t_n) \to y$ in $H_0^1(\Omega)$, where $t_n \to +\infty$. Since the function $E : H_0^1(\Omega) \to \mathbb{R}$ is continuous, it follows that $E(y) = l$. We obtain a contradiction by proving that $y \in \mathfrak{R}$. In view of Lemma 29, there exists $v \in K_r^+$ and a subsequence $v_n\left(\cdot\right) = \gamma(\cdot + t_n)$ such that $v(0) = y$ and $v_n(t) \to v(t) = z$ in $H_0^1(\Omega)$ for $t > 0$. Thus, $E(v_n(t)) \to E(z)$ implies that $E(z) = l$. Also, $v(\cdot)$ satisfies the energy equality for all $0 \leq s \leq t$, so that

$$l + \int_0^t \|v_r\|_{L^2}^2 dr = E(z) + \int_0^t \|v_r\|_{L^2}^2 dr = E(v(0)) = E(y) = l.$$

Therefore, $\frac{dv}{dt}(t) = 0$ for a.a. t, and then by Lemma 35 we have $y \in \mathfrak{R}_{K_r^+} = \mathfrak{R}$. As a consequence, $\mathcal{A}_r \subset M_r^s(\mathfrak{R})$. The converse inclusion follows from (92).

For the second equality we observe that for any $\gamma \in \mathbb{F}_r$ the energy equality (97) is satisfied for all $-\infty < s \leq t$. Let $x \in \mathcal{A}_r$ and let $\gamma \in \mathbb{K}_r = \mathbb{K}_r^1$ (cf. Lemma 34) be such that $\gamma(0) = x$. Since the second term of the energy function is bounded from above by (9), $E(\gamma(t)) \to l$, as $t \to -\infty$, for some $l \in \mathbb{R}$. We reason as before, so let suppose that there exists $\varepsilon > 0$ and a sequence $\gamma(-t_n)$, $t_n \to \infty$, such that

$$dist\ _{L^2(\Omega)}(\gamma(-t_n), \mathfrak{R}) > \varepsilon,$$

and we have that $\gamma(-t_n) \to y$ in $H_0^1(\Omega)$, $E(y) = l$. Moreover, for a fixed $t > 0$, there exists $v \in K_r^+$ and a subsequence of $v_n(\cdot) = \gamma(\cdot - t_n)$ (relabeled the same) such that $v(0) = y$ and $v_n(t) \to v(t) = z$ in $H_0^1(\Omega)$. Therefore, $E(v_n(t)) \to E(z)$ implies that $E(z) = l$ and reasoning as before we get a contradiction since it follows that $y \in \mathfrak{R}$. Hence, $\mathcal{A}_r \subset M_r^u(\mathfrak{R})$ and the converse inclusion follows from (92). ■

We can improve the regularity of the global attractor of the semigroup T_r of Section 3.1.1 and obtain its characterization

Lemma 39 *Let the conditions of Theorem 17 hold. Then the global attractor $\mathcal{A}_r$ of the semigroup T_r is compact in $H_0^1\left(\Omega\right)$, bounded in $L^p\left(\Omega\right)$ and the convergence takes place in the topology of $H_0^1\left(\Omega\right)$, that is,*

$$dist_{H_0^1(\Omega)}(T_r(t, B), \mathcal{A}) \to 0, \ \text{ as } t \to +\infty,$$

for any set B bounded in $L^2\left(\Omega\right)$.

Proof. The estimates of Lemma 29 can be justified for T_r via Galerkin approximations, so in this case we do not need to impose assumption (83) in order to use (85). Thus, the proof follows the same lines as in Proposition 32 and Theorem 33. ■

Proposition 40 *Let the conditions of Theorem 17 hold. Also, assume one of the following conditions:*

1. $h \in L^{\infty}\left(\Omega\right)$;
2. $p \leq \frac{2n}{n-2}$ *if* $n \geq 3$.

Then the global attractor $\mathcal{A}_r$ can be characterized as follows:

$$\mathcal{A}_r = M_r^u(\mathfrak{R}) = M_r^s(\mathfrak{R}),$$

where $M_r^s(\mathfrak{R})$, $M_r^u(\mathfrak{R})$ are defined in (94)-(95).

Proof. We recall that a function $E : \mathcal{A} \to \mathbb{R}$ is a Lyapunov functional if E is continuous (with respect to the topology of $H_0^1(\Omega)$), for any $u_0 \in \mathcal{A}$ the map $t \mapsto E(T_r(t, u_0))$ is non-increasing and $E(T_r(\tau, u_0)) = E(u_0)$, for some $\tau > 0$, implies that $u(\cdot)$ is a fixed point. We estate that the function E given in (96) is a Lyapunov functional for the semigroup T_r.

We prove that $E(y)$ is continuous. First, the maps $y \mapsto \frac{1}{2}A(\|y\|^2_{H_0^1})$, $y \mapsto \int_\Omega h(x)\, y(x)\, dx$ are obviously continuous in $H_0^1(\Omega)$. Second, if $h \in L^\infty(\Omega)$, taking into account that $\mathcal{A}$ is bounded in $L^\infty(\Omega)$ by Theorem 18, it follows that

$$\left| \int_\Omega \mathcal{F}(y_1) - \mathcal{F}(y_2) dx \right| = \left| \int_\Omega \int_{y_2(x)}^{y_1(x)} f(s) ds dx \right| \leq \int_\Omega C_1 |y_1(x) - y_2(x)| dx \leq C_2 \|y_1 - y_2\|_{L^2},$$

so $y \mapsto \int_\Omega \mathcal{F}(y(x)) dx$ is continuous as well. In the case of the second condition, this result follows from the embedding $H_0^1(\Omega) \subset L^p(\Omega)$ and the Lebesgue theorem.

Multiplying the equation in (3) by u_t we obtain the energy inequality

$$\int_s^t \|\frac{d}{dr} u(r)\|^2_{L^2} dr + E(u(t)) \leq E(u(s)), \qquad \text{for all } t \geq s,$$

if $u(\cdot)$ is a bounded complete trajectory of T_r. This calculation is rigorous when $h \in L^\infty(\Omega)$ as the boundedness of the solutions in $L^\infty(\mathbb{R}; L^\infty(\Omega))$ implies by regularization that (85) is true. Under the second condition, the calculations are formal but can be justified via Galerkin approximations. Hence, $E(u(t))$ is non-increasing as a function of t. Also, if $E(u(\tau)) = E(u_0)$, then $\|\frac{du}{dt}(t)\|^2_{L^2} = 0$ for a.a. $0 < t < \tau$, so u must be a fixed point.

The result follows then from [3, p.160]. ■

3.2 Strong solutions

We split this part into two cases.

3.2.1 Attractor in the phase space $H_0^1(\Omega)$

If we assume conditions (5)-(7), (60) and that either p satisfies (12) or that (11) is satisfied, then we know by Theorems 10 and 11 that for any $u_0 \in H_0^1(\Omega) \cap L^p(\Omega)$ there exists at least one strong solution $u(\cdot)$.

In the first case, $H_0^1(\Omega) \subset L^p(\Omega)$ implies that $H_0^1(\Omega) \cap L^p(\Omega) = H_0^1(\Omega)$. This is also true in the second case if we assume additionally that (83) holds true. Under such assumptions we define then the set

$$\mathcal{R} = K_s^+ := \{u(\cdot) : u \text{ is a strong solution of (3) with } u(0) \in H_0^1(\Omega)\}.$$

We define the (possibly multivalued) map $G_s : \mathbb{R}^+ \times H_0^1(\Omega) \to P(H_0^1(\Omega))$ by

$$G_s(t, u_0) = \{u(t) : u \in K_s^+ \text{ and } u(0) = u_0\}.$$

With respect to the axiomatic properties $(K1) - (K4)$ given above, property $(K1)$ is obviously true, and $(K2) - (K3)$ can be proved easily using equality (19). Therefore, G_s is a strict multivalued semiflow by the results of Section 3.1.2.

We shall obtain a similar result as in Lemma 29.

Lemma 41 *Let assume conditions (5)-(7), (60). Additionally, assume one of the following assumptions:*

1. *(11) and (83) hold;*

2. (12) is true.

Given a sequence $\{u^n\} \subset K_s^+$ such that $u^n(0) \to u_0$ weakly in $H_0^1(\Omega)$, there exists a subsequence of $\{u^n\}$ (relabeled the same) and $u \in K_s^+$, satisfying $u(0) = u_0$, such that

$$u^n(t) \to u(t) \quad \text{in } H_0^1(\Omega),\ \forall t > 0.$$

Proof. Since $\dfrac{du^n}{dt} \in L^2\left(0,T;L^2\left(\Omega\right)\right)$ and (85) hold, we can use (81) and multiplying (3) by u_t and integrating between s and t we obtain

$$\int_s^t \|\frac{d}{dr}\|u(r)\|_{L^2}^2 dr + E(u(t)) = E(u(s)) \quad \text{ for all } t \geq s \geq 0,$$

where E was defined in (96). Therefore, by (6) and (9) we have that

$$\int_0^t \|\frac{d}{dr}u(r)\|_{L^2}^2 dr + \frac{m}{4}\|u(t)\|_{H_0^1}^2 + \widetilde{\alpha}_1\|u(t)\|_{L^p}^p \leq \frac{1}{2}A(\|u(0)\|_{H_0^1}^2) + \widetilde{\alpha}_2\|u(0)\|_{L^p}^p + K_1\left\|u\left(0\right)\right\|_{L^2}^2 + K_2 \quad (98)$$

holds for all $t > 0$.

In the first case, multiplying by $-\Delta u$, integrating over $(0,T)$ and using (98) it follows that

$$\frac{1}{2}\|u(T)\|_{H_0^1}^2 + \frac{m}{2}\int_0^T \|\Delta u(s)\|_{L^2}^2 ds \leq \eta\int_0^T \|u(s)\|_{H_0^1}^2 ds + \frac{1}{2}\|u(0)\|_{H_0^1}^2 + K_3 \leq K_4\left(T\right), \quad (99)$$

for all $T > 0$. In the second case, combining (98) with (45) the boundedness of $f\left(u^n\right)$ in $L^2\left(0,T;L^2\left(\Omega\right)\right)$ follows for any $T > 0$. Hence, the equality

$$a\left(\|u\|_{H_0^1}^2\right)\Delta u = \frac{du^n}{dt} - f\left(u^n\right) - h$$

and (6) imply that u^n is bounded in $L^2\left(0,T;D(A)\right)$.

Thus, the sequence $\{u^n\}$ is bounded in $L^\infty(0,T;H_0^1(\Omega))\cap L^2(0,T;D(A))$ and $\dfrac{du^n}{dt}$, $f\left(u^n\right)$ are bounded in $L^2(0,T;L^2(\Omega))$, for all $T > 0$. Therefore, there is u such that

$$u^n \overset{*}{\rightharpoonup} u \text{ in } L^\infty(0,T;H_0^1(\Omega)),$$

$$u^n \rightharpoonup u \text{ in } L^2(0,T;D(A)),$$
$$u_t^n \rightharpoonup u_t \text{ in } L^2(0,T;L^2(\Omega)),.$$

Arguing in a similar way as in Theorem 9 we have

$$u_n \to u \text{ in } L^2(0,T;H_0^1(\Omega)),$$

$$u_n(t,x) \to u(t,x) \text{ a.e. on } (0,T)\times\Omega,$$
$$f\left(u^n\right) \rightharpoonup f\left(u\right) \text{ in } L^2(0,T;L^2(\Omega)),$$
$$a(\|u_n\|_{H_0^1}^2)\Delta u_n \rightharpoonup a(\|u\|_{H_0^1}^2)\Delta u \text{ in } L^2(0,T;L^2(\Omega)).$$

Hence, we can pass to the limit and obtain that $u \in K_s^+$. Following the same lines of Theorem 10 we check that $u\left(0\right) = u_0$.

Moreover, arguing as in Lemma 29 we obtain

$$u^n(t) \to u(t) \text{ in } H_0^1(\Omega) \text{ for all } t > 0.$$

■

Corollary 42 *Assume the conditions of Lemma 41. Then the set K_s^+ satisfies condition $(K4)$.*

Using Lemma 41 and reasoning as before the following result holds.

Proposition 43 *Assume the conditions of Lemma 41. Then the map $G_s(t,\cdot)$ is upper semicontinuous for all $t\geq 0$ with compact values.*

Proposition 44 *Assume the conditions of Lemma 41 and (17). Then there exists an absorbing set B_1 for G_s, which is compact in $H_0^1(\Omega)$ and $L^p(\Omega)$.*

Proof. The proof follows the same lines of that in Proposition 32 but using Lemma 41. ■

From these results and Theorem 28 we obtain the existence of the global attractor.

Theorem 45 *Assume the conditions of Lemma 41 and (17). Then the multivalued semiflow G_s possesses a global compact invariant attractor $\mathcal{A}_s$, which is compact in $L^p(\Omega)$.*

Lemma 46 *Assume the conditions of Lemma 41 and (17). Then $\mathcal{A}_s=\mathcal{A}_r$, where $\mathcal{A}_r$ is the global attractor in Theorem 33.*

Proof. Since $G_s(t,u_0)\subset G_r(t,u_0)$ for all $u_0\in H_0^1(\Omega)$, it is clear that $\mathcal{A}_r$ is a compact attracting set. Hence, the minimality of the global attractor gives $\mathcal{A}_s\subset\mathcal{A}_r$.

Let $z\in\mathcal{A}_r$. Since $z=\gamma(0)$, where $\gamma\in\mathbb{K}_r^1$, and $\gamma\mid_{[s,+\infty)}$ is a strong solution of (3) for any $s\in\mathbb{R}$, we get that $z\in G_s(t_n,\gamma(-t_n))$ for $t_n\to+\infty$. Hence,

$$dist\left(z,\mathcal{A}_s\right)\leq dist\left(G_s(t_n,\gamma(-t_n)),\mathcal{A}_s\right)\to 0 \text{ as } n\to\infty,$$

so $z\in\mathcal{A}_s$. ■

The set of all complete trajectories of K_s^+ (see Definition 21) will be denoted by $\mathbb{F}_s$. Let $\mathbb{K}_s$ be the set of all complete trajectories which are bounded in $H_0^1(\Omega)$.

In view of Theorem 27, the global attractor is characterized in terms of bounded complete trajectories:

$$\mathcal{A}_s=\{\gamma(0):\gamma(\cdot)\in\mathbb{K}_s\}=\bigcup_{t\in\mathbb{R}}\{\gamma(t):\gamma(\cdot)\in\mathbb{K}_s\}. \tag{100}$$

In the same way as in Lemma 35 we obtain that $\mathfrak{R}_{K_s^+}=\mathfrak{R}$.

Reasoning as in Theorem 18 we obtain the following result.

Theorem 47 *Assume the conditions of Lemma 41 and (17). Then the global attractor $\mathcal{A}_s$ is bounded in $L^\infty(\Omega)$, provided that $h\in L^\infty(\Omega)$.*

Following the same procedure of Theorem 37 we can prove an analogous characterization of the global attractor.

Theorem 48 *Assume the conditions of Lemma 41 and (17).Then it holds that*

$$\mathcal{A}_s=M_s^u(\mathfrak{R})=M_s^s(\mathfrak{R}),$$

where

$$M_s^s(\mathfrak{R})=\{z:\exists\gamma(\cdot)\in\mathbb{K}_s,\ \gamma(0)=z,\ \ dist_{\ H_0^1(\Omega}(\gamma(t),\mathfrak{R})\to 0,\ t\to+\infty\}, \tag{101}$$

$$M_s^u(\mathfrak{R})=\{z:\exists\gamma(\cdot)\in\mathbb{F}_s,\ \gamma(0)=z,\ \ dist_{H_0^1(\Omega}(\gamma(t),\mathfrak{R})\to 0,\ t\to-\infty\}. \tag{102}$$

Remark 49 *In the definition of $M_s^u(\mathfrak{R})$ we can replace $\mathbb{F}_r$ by $\mathbb{K}_r$.*

Let us consider now the particular situation when G_s is single-valued semigroup. Under the conditions (5)-(7), (11), (60), (83), if we assume additionally that (14) is satisfied, then by Theorem 14 for any $u_0\in H_0^1(\Omega)$ there exists a unique strong solution $u(\cdot)$. Then we can define the following semigroup $T_s:\mathbb{R}^+\times H_0^1(\Omega)\to H_0^1(\Omega)$:

$$T_s(t,u_0)=u(t),$$

where $u(\cdot)$ is the unique strong solution to (3). We recall also that $u\in C([0,T],H_0^1(\Omega))$ for any $T>0$. Also, by Lemma 41 if $u_0^n\to u_0$ weakly in $H_0^1(\Omega)$, then $T_s(t,u_0^n)\to T(t,u_0)$ in $H_0^1(\Omega)$ for all $t>0$.

Since $T_s=G_s$, by Theorems 45, 47, 48 and Lemma 46 we obtain the following results.

Theorem 50 *Assume the conditions (5)-(7), (11), (17), (60), (83) and (14). Then the semigroup T_s possesses a global invariant attractor $\mathcal{A}_s$, which is compact in $H_0^1(\Omega)$ and $L^p(\Omega)$.*

Lemma 51 *Under the conditions of Theorem 50, $\mathcal{A}_s = \mathcal{A}_r$, where $\mathcal{A}_r$ is the attractor of Theorem 17.*

Theorem 52 *Assume the conditions of Theorem 50. Then the global attractor $\mathcal{A}_s$ is bounded in $L^\infty(\Omega)$ provided that $h \in L^\infty(\Omega)$.*

As before, we denote by $\mathfrak{R}$ the set of fixed points of T_s. Also, the global attractor is the union of all bounded complete trajectories

$$\mathcal{A}_s = \{\phi(0) : \phi \text{ is a bounded complete trajectory of } T_s\}.$$

Theorem 53 *Assume the conditions of Theorem 50. Then the global attractor $\mathcal{A}_s$ can be characterized as follows*

$$\mathcal{A}_s = M_s^u(\mathfrak{R}) = M_s^s(\mathfrak{R}),$$

where the sets $M_s^u(\mathfrak{R})$, $M_s^s(\mathfrak{R})$ are defined in (101)-(102).

In this case we can obtain additionally that the attractor is bounded in $H^2(\Omega)$.

Proposition 54 *Assume the conditions of Theorem 50 and also that (15) holds true. Then $\mathcal{A}_s$ is bounded in $H^2(\Omega)$.*

Proof. The proof follows the same lines as in Proposition 19, so we omit it. ■

3.2.2 Attractor in the phase space $H_0^1(\Omega) \cap L^p(\Omega)$

We consider the metric space $X = H_0^1(\Omega) \cap L^p(\Omega)$ endowed with the induced topology of the space $H_0^1(\Omega)$.

If we assume conditions (5)-(7), (11), (14) and (60), then by Theorems 10 and 14 for any $u_0 \in H_0^1(\Omega) \cap L^p(\Omega)$ there exists a unique strong solution $u(\cdot)$. Then we can define the following semigroup $T_s : \mathbb{R}^+ \times X \to X$:

$$T_s(t, u_0) = u(t),$$

where $u(\cdot)$ is the unique strong solution to (3). We recall also that $u \in C([0,T], H_0^1(\Omega)) \cap C_w([0,T], L^p(\Omega))$ for any $T > 0$.

Lemma 55 *Assume conditions (5)-(7), (11), (14) and (60). If $u_0^n \to u_0$ weakly in $H_0^1(\Omega) \cap L^p(\Omega)$, then $T_s(t, u_0^n) \to T_s(t, u_0)$ strongly in $H_0^1(\Omega)$ and weakly in $L^p(\Omega)$ for any $t > 0$.*

Proof. Repeating the same proof of Lemma 41 we obtain that $T_s(t, u_0^n) \to T_s(t, u_0)$ strongly in $H_0^1(\Omega)$ for all $t > 0$. We observe that in this case the estimates are justified via Galerkin approximations, so we do not need condition (83) in order to provide property (85).

Finally, by the Ascoli-Arzelà theorem we deduce

$$u^n \to u \text{ in } C([0,T], L^2(\Omega))$$

and combining this with (98) we infer that

$$u^n(t) \rightharpoonup u(t) \text{ in } L^p(\Omega) \ \forall t \geq 0.$$

■

Proposition 56 *Assume the conditions of Lemma 55 and (17). Then there exists an absorbing set B_1 for T_s, which is compact in $H_0^1(\Omega)$ and bounded $L^p(\Omega)$.*

Proof. Following the same lines of that in Proposition 32 (and justifying the estimates via Galerkin approximations), we obtain that there exists $M > 0$ such that the closed ball B_M in $H_0^1(\Omega) \cap L^p(\Omega)$ centered at 0 with radius M is absorbing for T_s. By Lemma 55 the set $B_1 = \overline{T_s(1, B_M)}$ is an absorbing set which is compact in $H_0^1(\Omega)$ and bounded in $L^p(\Omega)$. ■

Theorem 57 *We assume the conditions of Lemma 55 and (17). Then the semigroup T_s possesses a global attractor $\mathcal{A}_s$, which is compact in X and bounded in $L^p(\Omega)$.*

Proof. We cannot apply directly the general theory of attractors for semigroup because we do not know whether the semigroup T_s is continuous with respect to the initial datum in X.

We state that

$$\mathcal{A}_s = \omega(B_1) = \{y : \exists t_n \to +\infty,\ y_n \in T_s(t_n, B_1) \text{ such that } y_n \to y \text{ in } X\}$$

is a global compact attractor. The fact that set $\omega(B_1)$ is non-empty, compact and the minimal closed set attracting B_1 can be proved in a standard way (see for example Theorem 10.5 in [30]). Since B_1 is absorbing, $\omega(B_1)$ attracts any bounded set B. As $\omega(B_1) \subset B_1$, $\mathcal{A}_s$ is bounded in $L^p(\Omega)$.

We need to prove that it is invariant.

First, we prove that it is negatively invariant. Let $y \in \mathcal{A}_s$ and $t > 0$ be arbitrary. We take a sequence $y_n \in T_s(t_n, B_1)$ such that $y_n \to y$, $t_n \to +\infty$. Since $T_s(t_n, B_1) = T_s(t, T_s(t_n - t, B_1))$, there are $x_n \in T_s(t_n - t, B_1)$ such that $y_n = T_s(t, x_n)$. As for n large $T_s(t_n - t, B_1) \subset B_1$, the sequence $\{x_n\}$ is bounded in $L^p(\Omega)$ and relatively compact in $H_0^1(\Omega)$. Hence, up to a subsequence $x_n \to x \in \mathcal{A}_s$ weakly in $L^p(\Omega)$ and strongly in $H_0^1(\Omega)$. We deduce by Lemma 55 that $T_s(t, x_n) \to T_s(t,x)$ weakly in $L^p(\Omega)$ and strongly in $H_0^1(\Omega)$. Thus, $y = T_s(t,x) \subset T_s(t, \mathcal{A}_s)$.

Second, we prove that it is positively invariant. As $\mathcal{A}_s = T_s(\tau, \mathcal{A}_s)$ for any $\tau \geq 0$, this follows from

$$dist_X(T_s(t, \mathcal{A}_s), \mathcal{A}_s) = dist_X(T_s(t, T_s(\tau, \mathcal{A}_s)), \mathcal{A}_s) = dist_X(T_s(t+\tau, \mathcal{A}_s), \mathcal{A}_s) \underset{\tau \to +\infty}{\to} 0.$$

■

Lemma 58 *Under the conditions of Theorem 57, $\mathcal{A}_s = \mathcal{A}_r$, where $\mathcal{A}_r$ is the attractor of Theorem 17.*

Proof. Since $T_r(t, u_0) = T_s(t, u_0)$ for any $u_0 \in X$, we have

$$dist_{L^2}(\mathcal{A}_s, \mathcal{A}_r) = dist_{L^2}(T_s(t, \mathcal{A}_s), \mathcal{A}_r) = dist_{L^2}(T_r(t, \mathcal{A}_s), \mathcal{A}_r) \underset{t \to +\infty}{\to} 0,$$

so $\mathcal{A}_s \subset \mathcal{A}_r$. In the same way,

$$dist_X(\mathcal{A}_r, \mathcal{A}_s) = dist_X(T_r(t, \mathcal{A}_r), \mathcal{A}_s) = dist_X(T_s(t, \mathcal{A}_r), \mathcal{A}_s) \underset{t \to +\infty}{\to} 0,$$

and then $\mathcal{A}_r \subset \mathcal{A}_s$. ■

The following two theorems are proved in the same way as Theorem 18 and Proposition 40

Theorem 59 *Assume the conditions of Theorem 57. Then the global attractor $\mathcal{A}_s$ is bounded in $L^\infty(\Omega)$ provided that $h \in L^\infty(\Omega)$.*

As before, we denote by $\mathfrak{R}$ the set of fixed points of T_s. Also, the global attractor is the union of all bounded complete trajectories

$$\mathcal{A}_s = \{\phi(0) : \phi \text{ is a bounded complete trajectory of } T_s\}.$$

Theorem 60 *We assume the conditions of Theorem 57 and one of the following assumptions:*

1. *$h \in L^\infty(\Omega)$;*
2. *$p \leq \frac{2n}{n-2}$ if $n \geq 3$.*

Then the global attractor $\mathcal{A}_s$ can be characterized as follows

$$\mathcal{A}_s = M_s^u(\mathfrak{R}) = M_s^s(\mathfrak{R}),$$

where the sets $M_s^u(\mathfrak{R})$, $M_s^s(\mathfrak{R})$ are defined in (101)-(102).

We obtain additionally that the attractor is bounded in $H^2(\Omega)$.

Proposition 61 *Assume the conditions of Theorem 57 and also that (15) is satisfied. Then $\mathcal{A}_s$ is bounded in $H^2(\Omega)$.*

Proof. The proof follows the same lines as in Proposition 19, so we omit it. ■

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