Electromagnetic Pulses: A Novel Framework for Helicity, Chirality, and Their Flows

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***Abstract*— This article delves into conserved quantities of electromagnetic pulses, which are crucial for understanding their behavior. We introduce a precise framework for computing scalar properties (energy, helicity, and chirality) and vector properties (Poynting vector, spin, and chirality flux) of electromagnetic pulses with gauge-invariant equations. Based on the far-field behavior of radiated electric field, our methodology offers accurate results akin to traditional volume integrals but at a lower computational cost. In addition, our findings enable a reinterpretation of these properties as statistical average parameters of the pulse. This innovative approach not only simplifies calculations but also enhances their accuracy, making it useful for studying the main conserved quantities for complex electromagnetic field structures, as those formed by multiple interference of pulses.**

***Index Terms*— Chirality, electromagnetic pulses, helicity, Poynting vector, spin.**

I. Introduction

IN THE vast domain of electromagnetic waves, understanding the underlying principles that govern their behavior is crucial for advancements in various scientific and technological fields. This article delves into the intricacies of electromagnetic pulses, focusing on several of the conserved quantities that define their fundamental properties. Our comprehensive analysis explores the conservation of a set of scalar properties, namely, energy, helicity, and chirality, which are of profound interest in both monochromatic and pulsed electromagnetic fields [1], [2], [3], [4], [5], [6], [7]. Moreover, the analysis extends to the vector properties associated with the flows of such scalar properties, the so-called Poynting vector (energy flow), spin (helicity flow), and chirality flow, which illuminates the directional aspects of these essential quantities, and also has been extensively studied [1], [2], [3], [4], [5], [6]. Together, these magnitudes form a comprehensive framework vital for studying light–matter interactions, offering novel insights into the intricate dynamics of electromagnetic pulses and enabling a complete characterization of their behavior in diverse contexts.

In this way, we are going to introduce an efficient and precise framework for the calculation of the main scalar (energy, helicity, and chirality) and vectorial (Poynting vector, total spin, and chirality flux) properties of electromagnetic pulses. This innovative method, based on the far-field behavior of the electric field in the frequency domain, offers results that align precisely with those obtained through volume integrals of local properties, but with significantly reduced computational cost. Furthermore, it is important to emphasize the gauge invariance of the proposed framework. This is due to the fact that the final expressions depend solely on electric and magnetic fields, enabling application to all radiated fields irrespective of gauge considerations. Gauge-related aspects will be accounted for using the standard method based on the volume integral of properties density [15]. In addition, the results presented in this article offer a profound insight into the interconnections among these properties, enabling a reinterpretation of their relationships as statistical average parameters of the pulse.

This article is organized into different sections. In Section II, we define the conserved quantities. This section lays the theoretical foundation, establishing the fundamental principles that guide our analysis. Moving forward, Section II-A delves into the deduction of expressions for scalar properties utilizing the far-field approximation. In Section II-B, our focus shifts to the deduction of expressions for vector properties, shedding light on the directional aspects of energy, helicity, and chirality flow through the examination of the Poynting vector, spin, and chirality flux. Finally, Section III presents a compelling demonstration of the validity of our methodology through an example. This practical application showcases the precision and efficacy of our approach, highlighting its potential for diverse scenarios.

II. Theory

The propagation of electromagnetic pulses in vacuum is described by the electric $\mathbf{E}$ and magnetic $\mathbf{H}$ fields, which obey the following source-free Maxwell equations [8]:

$$\nabla \cdot \mathbf{E} = 0, \quad \nabla \cdot \mathbf{H} = 0 \tag{1}$$

$$\nabla \times \mathbf{E} = -\mu_0 \dot{\mathbf{H}}, \quad \nabla \times \mathbf{H} = \epsilon_0 \dot{\mathbf{E}}. \tag{2}$$

Here, μ_0 and ϵ_0 are the magnetic permeability and the electric permittivity of vacuum, respectively, while the dot denotes the temporal derivative. In this article, the spatial and temporal dependency on magnitudes has been omitted for simplicity, while the vectors are denoted by bold letters. These fields ($\boldsymbol{E}$ and $\boldsymbol{H}$) can also be expressed as a function of

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two potential vectors $\mathbf{C}$ and $\mathbf{A}$, such that the electric field and the magnetic field can be written as follows:

$$\mathbf{E} = -(\nabla \times \mathbf{C})/\epsilon_0, \quad \mathbf{H} = (\nabla \times \mathbf{A})/\mu_0. \tag{3}$$

Thus, with the help of these four fields ($\mathbf{E}$, $\mathbf{H}$, $\mathbf{C}$, and $\mathbf{A}$), which are real quantities, a set of both scalar and vector properties that characterize the electromagnetic pulses can be defined. In this way, there are three main scalar properties, energy (U), helicity ($\mathcal{H}$), and chirality ($\mathcal{X}$), which are given by [2], [8]

$$U = \int_\Omega u\, dr^3 = \int_\Omega \frac{1}{2}(\epsilon_0|\mathbf{E}|^2 + \mu_0|\mathbf{H}|^2)\, dr^3 \tag{4}$$

$$\mathcal{H} = \int_\Omega h\, dr^3 = \int_\Omega \frac{1}{2c}(\mathbf{A}\cdot\mathbf{H} - \mathbf{C}\cdot\mathbf{E})\, dr^3 \tag{5}$$

$$\mathcal{X} = \int_\Omega \chi\, dr^3 = \int_\Omega \frac{1}{2c^2}(\mathbf{H}\cdot\dot{\mathbf{E}} - \mathbf{E}\cdot\dot{\mathbf{H}})\, dr^3. \tag{6}$$

Here, u, χ, and h are the energy, chirality, and helicity densities, respectively, while Ω denotes that the integration must be made over the entire volume. In the same way, there are also three main vectorial properties, associated with the fluxes of energy (Poynting vector $\mathbf{P}$), helicity (total spin $\mathbf{S}$), and chirality ($\mathbf{Y}$), which can be expressed as follows [2], [6], [8], [9]:

$$\mathbf{P} = \int_\Omega \mathbf{p}\, dr^3 = \int_\Omega \mathbf{E}\times\mathbf{H}\, dr^3 \tag{7}$$

$$\mathbf{S} = \int_\Omega \frac{\mathbf{\Phi}_h}{c} dr^3 = \frac{1}{2}\int_\Omega (\epsilon_0\dot{\mathbf{A}}\times\mathbf{A} + \mu_0\dot{\mathbf{C}}\times\mathbf{C}) dr^3 \tag{8}$$

$$\mathbf{Y} = \int_\Omega \mathbf{\Phi}_{\mathcal{X}} dr^3 = \frac{1}{2}\int_\Omega (\epsilon_0\mathbf{E}\times\dot{\mathbf{E}} + \mu_0\mathbf{H}\times\dot{\mathbf{H}}) dr^3 \tag{9}$$

where $\mathbf{p}$, $\mathbf{\Phi}_h$, and $\mathbf{\Phi}_{\mathcal{X}}$ denote the respective fluxes of energy, helicity, and chirality densities. Thus, the calculation of the scalar and vector properties of electromagnetic pulses involves the knowledge of the fields over all of space and time, and the integration of the corresponding densities over the entire volume. In this sense, in many cases, it will be difficult to find analytical solutions for the properties (especially in systems with pulse superposition), and even the numerical calculation will be very extensive. We propose an alternative for the calculation of all the properties that only involves knowing the electric field in the far-field. For this, we have to take into account that the main scalar (energy, helicity, and chirality) and vectorial (energy flux, helicity flux, and chirality flux) properties of an electromagnetic field are subject to the following local conservation theorem [2], [6], [10], [11], [12]:

$$\dot{\alpha} + \nabla\cdot\mathbf{\Phi}_\alpha = 0 \tag{10}$$

where α denotes a scalar property density or the different components of vector properties, while $\mathbf{\Phi}_\alpha$ is the flux of the property. Therefore, integration of (10) over all the space Ω (which denotes the total volume, covering the interval [0,∞] for radial distance in spherical coordinates ρ) yields for all the scalar and vector properties analyzed to the conservation relation dΛ/dt = 0 [2], [6], since the flux is null to infinity,

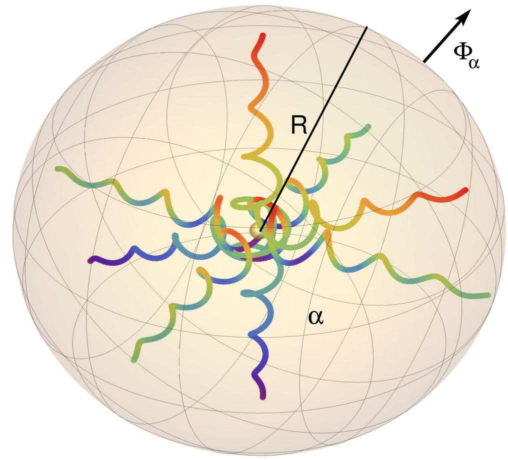

Fig. 1. Schematic of the large sphere considered for the calculation of electromagnetic properties of light pulses, where α denotes the property density defined in all the points of the volume and Φ is the flux of the property that crosses the surface of the sphere.

being Λ the different properties defined as the volume integral of the respective densities as follows:

$$\Lambda = \int_\Omega \alpha dr^3. \tag{11}$$

Let us now consider a large sphere with radius R as that shown in Fig. 1, which encompasses a very large but finite volume Ω' (where ρ pertains to the interval [0, R]) in which there are defined the property density α and the flux of the property Φ_α that crosses the surface Σ of the sphere (being the surface differential $d\mathbf{\Sigma} = d\Sigma_\rho \boldsymbol{u}_\rho = \rho^2\sin(\theta)d\theta d\phi \boldsymbol{u}_\rho$). Therefore, the integration of (10) over all the time and the volume Ω' gives the following equation using the divergence theorem:

$$\Big[\Lambda_{\Omega'}\Big]_{t=-\infty}^{t=\infty} = \left[\int_{\Omega'} \alpha dr^3\right]_{t=-\infty}^{t=\infty} = -\int_{-\infty}^{\infty}\left(\int_\Sigma \mathbf{\Phi}_\alpha \cdot d\mathbf{\Sigma}\right)dt \tag{12}$$

where $\Lambda_{\Omega'}$ denotes the integral of the density α over the volume Ω'. Let us now analyze the physical meaning of $\Lambda_{\Omega'}(t = \infty)$ and $\Lambda_{\Omega'}(t = -\infty)$ for the problem considered. Thus, at the beginning of time (t $= -\infty$ in this case, but similar discussion can be made for t = 0), the electromagnetic pulse will be localized in a limited region of space, so integrating the density of the different properties over a volume Ω' with a sufficiently large R is equal to the integration over the entire volume, i.e., all the fields vanish outside the large sphere at the initial time, and therefore, $\Lambda = \Lambda_{\Omega'}(t = -\infty)$. On the other hand, at the end of the time (t $= \infty$), we can consider that all the fields have propagated outside the finite sphere, so that the fields inside the volume Ω' vanish, and as a consequence the density of the different properties within it is zero, and therefore, $\Lambda_{\Omega'}(t = \infty) = 0$. This assumption is in line with the Sommerfeld radiation condition [13], which is used to ensure that the radiated fields from localized sources propagate outward in a physically reasonable manner. Therefore, the electromagnetic properties of the light pulses can also be defined as follows:

$$\Lambda = \Lambda_{\Omega'}(t=-\infty) = \int_{-\infty}^{\infty}\left(\int_{\Sigma}\mathbf{\Phi}_{\alpha}\cdot d\mathbf{\Sigma}\right)dt. \tag{13}$$

Here, Σ denotes a surface that encloses the volume of integration of the electromagnetic properties densities. In Sections II-A and II-B, we are going to demonstrate that all the electromagnetic properties of the pulse can be expressed as a function of statistical parameters of the spectrum distribution of the beam using the last term of (13), which only implies the knowledge of the electric field and magnetic fields in the far-field.

A. Energy, Helicity, and Chirality

First, we are going to analyze the three main scalar properties of electromagnetic pulses: energy, helicity, and chirality. The flux of these properties takes the same form, given by the cross product of two vector fields

$$\mathbf{\Phi}_{\alpha} = \mathbf{M}\times\mathbf{N} \tag{14}$$

where, for each property, we have to take the appropriate vector fields (for example, $\mathbf{M} = \mathbf{E}$ and $\mathbf{N} = \mathbf{H}$ for the energy [8]). Therefore, introducing this equation into (13) and switching the order of integration, we obtain

$$\Lambda = \int_{\Sigma}\left(\int_{-\infty}^{\infty}(\mathbf{M}\times\mathbf{N})\cdot\boldsymbol{u}_{\rho}dt\right)d\Sigma_{\rho}. \tag{15}$$

Now, we apply the Phancherel's theorem to the temporal integral given by

$$\int_{-\infty}^{\infty}(\mathbf{M}\times\mathbf{N})\cdot\boldsymbol{u}_{\rho}dt = \frac{1}{\pi}\int_{0}^{\infty}(\widetilde{\mathbf{M}}\times\widetilde{\mathbf{N}}^{*})\cdot\boldsymbol{u}_{\rho}d\omega \tag{16}$$

where $\widetilde{\mathbf{M}}$ and $\widetilde{\mathbf{N}}^{*}$ denote the Fourier transforms of the corresponding vector fields $\mathbf{M}$ and $\mathbf{N}$, while the asterisk implies the complex conjugate. Using (16) in (15) and switching the order of integration, we have

$$\Lambda = \int_{0}^{\infty}V_{\omega}^{\Lambda}d\omega = \frac{1}{\pi}\int_{0}^{\infty}\left(\int_{\Sigma}\left(\widetilde{\mathbf{M}}\times\widetilde{\mathbf{N}}^{*}\right)\cdot\boldsymbol{u}_{\rho}d\Sigma_{\rho}\right)d\omega \tag{17}$$

where V_{ω}^{Λ} denotes the spectral density of the property Λ.

Let us now analyze the particular cases of the different scalar magnitudes. Regarding the energy, its flux is given by Poynting vector $\mathbf{\Phi}_{U} = \mathbf{E}\times\mathbf{H}$, so it is easy to see that the energy spectral density can be written as follows:

$$V_{\omega}^{U} = \frac{1}{\pi}\int_{\Sigma}\left(\widetilde{\mathbf{E}}\times\widetilde{\mathbf{H}}^{*}\right)\cdot\boldsymbol{u}_{\rho}d\Sigma_{\rho}. \tag{18}$$

In the case of the helicity, the flux is $\mathbf{\Phi}_{\mathcal{H}} = 1/2(\epsilon_0 c\mathbf{A}\times\dot{\mathbf{A}} + 1/(\epsilon_0 c)\mathbf{C}\times\dot{\mathbf{C}})$, so the spectral density has two contributions, and taking into account the relations $\mathbf{E} = -\dot{\mathbf{A}}$ and $\mathbf{H} = -\dot{\mathbf{C}}$ (in the Coulomb gauge), which implies that $\widetilde{\mathbf{A}} = \mathrm{j}\widetilde{\mathbf{E}}/(\omega)$ and $\widetilde{\mathbf{C}} = j\widetilde{\mathbf{H}}/(\omega)$, and therefore, the spectral density can be written as follows:

$$V_{\omega}^{\mathcal{H}} = \frac{1}{2\pi\omega}\int_{\Sigma}\Im\left[\left(\epsilon_0 c\widetilde{\mathbf{E}}\times\widetilde{\mathbf{E}}^{*} + \frac{1}{\epsilon_0 c}\widetilde{\mathbf{H}}\times\widetilde{\mathbf{H}}^{*}\right)\right]\cdot\boldsymbol{u}_{\rho}d\Sigma_{\rho} \tag{19}$$

where $\Im[]$ denotes that the imaginary part must be taken. Finally, the chirality flux is $\mathbf{\Phi}_{\mathcal{X}} = 1/2(\epsilon_0\mathbf{E}\times\dot{\mathbf{E}} + \mu_0\mathbf{H}\times\dot{\mathbf{H}})$, and therefore, its corresponding spectral density is given by

$$V_{\omega}^{\mathcal{X}} = \frac{1}{2\pi}\frac{\omega}{c}\int_{\Sigma}\Im\left[\left(\epsilon_0 c\widetilde{\mathbf{E}}\times\widetilde{\mathbf{E}}^{*} + \frac{1}{\epsilon_0 c}\widetilde{\mathbf{H}}\times\widetilde{\mathbf{H}}^{*}\right)\right]\cdot\boldsymbol{u}_{\rho}d\Sigma_{\rho}. \tag{20}$$

Equations (18)–(20) provide the spectral density of the three scalar properties for any type of electromagnetic pulse as a function of the asymptotic behavior of the electric field and the magnetic field, in such a way that the property can be obtained by means of the integral over all frequencies of the respective spectral density according to (17). It is noteworthy that the sole distinction between the spectral density of helicity and chirality lies in a scaling factor applied to the power of ω and a constant c.

Polarization State Effect on Scalar Properties: While (18)–(20) remain applicable to any electromagnetic field when the asymptotic behavior of both the electric and magnetic fields is known, delving into the polarization state of the field reveals intriguing connections among various properties. To elaborate, we will consider the decomposition of any field into a superposition of a left-handed circularly polarized field and its right-handed counterpart. This analysis yields a representation, wherein the electric and magnetic fields in the far-field are expressed as follows:

$$\widetilde{\mathbf{E}} = Exp\left[-j\frac{\omega}{c}\rho\right]\frac{F(\theta,\phi,\omega)}{\rho}\left(A_d\boldsymbol{u}_d + A_l Exp[j\delta]\boldsymbol{u}_l\right) \tag{21}$$

$$\widetilde{\mathbf{H}} = Exp\left[-j\frac{\omega}{c}\rho\right]\frac{jF(\theta,\phi,\omega)}{c\mu_0\rho}\left(-A_d\boldsymbol{u}_d + A_l Exp[j\delta]\boldsymbol{u}_l\right). \tag{22}$$

Here, we define $\boldsymbol{u}_d$ as $(\boldsymbol{u}_\theta - j\boldsymbol{u}_\phi)/\sqrt{2}$ and $\boldsymbol{u}_l$ as $(\boldsymbol{u}_\theta + j\boldsymbol{u}_\phi)/\sqrt{2}$, representing the unitary vectors of right-handed and left-handed fields, respectively. A_D and A_L correspond to the fractions of right-handed and left-handed photons, subject to the condition $A_d^2 + A_l^2 = 1$. In addition, δ introduces a phase shift to account for all possible polarization states. This behavior conforms to the standard characteristics of radiated fields in the far-field, where there is an absence of a radial component, the field amplitude follows a decay proportional to $1/\rho$, and $F(\theta,\phi,\omega)$ denotes a function describing the directionality and spectral distribution of the field. Thus, substituting (21) and (22) into expressions (18)–(20), we can obtain the following spectral densities for the different scalar properties:

$$V_{\omega}^{U} = \left(A_d^2 + A_l^2\right)\frac{\epsilon_0 c}{\pi}\int_{\Sigma}\frac{|F|^2}{\rho^2}d\Sigma_{\rho} = \frac{\epsilon_0 c}{\pi}\int_{\Sigma}|\widetilde{\mathbf{E}}|^2 d\Sigma_{\rho} \tag{23}$$

$$V_{\omega}^{\mathcal{H}} = \frac{\Delta_d^l}{\omega}\frac{\epsilon_0 c}{\pi}\int_{\Sigma}\frac{|F|^2}{\rho^2}d\Sigma_{\rho} = \frac{\Delta_d^l}{\omega}\frac{\epsilon_0 c}{\pi}\int_{\Sigma}|\widetilde{\mathbf{E}}|^2 d\Sigma_{\rho} \tag{24}$$

$$V_{\omega}^{\mathcal{X}} = \Delta_d^l\frac{\epsilon_0}{\pi}\omega\int_{\Sigma}\frac{|F|^2}{\rho^2}d\Sigma_{\rho} = \Delta_d^l\frac{\epsilon_0}{\pi}\omega\int_{\Sigma}|\widetilde{\mathbf{E}}|^2 d\Sigma_{\rho} \tag{25}$$

where we have taken into account that $|\widetilde{\mathbf{E}}| = |F|/\rho$ and $\Delta_d^l = (A_d^2 - A_l^2)$. Examining (23)–(25), it becomes apparent that a direct relationship exists among the three spectral densities. Consequently, the estimation of the three scalar properties can be accomplished through the energy spectral density, representing the expected value of various powers of the frequency, expressed as follows:

$$U = \int_0^\infty V_\omega^U d\omega \tag{26}$$

$$\mathcal{H} = \Delta_d^l \int_0^\infty \frac{1}{\omega} V_\omega^U d\omega \tag{27}$$

$$\mathcal{X} = \frac{\Delta_d^l}{c} \int_0^\infty \omega V_\omega^U d\omega. \tag{28}$$

Equation (26) is well known and was previously used by Hellwarth and Nouchi [14] to calculate the energy of electromagnetic pulses; however, (27) and (28) show that the energy spectral density can be used to calculate helicity and chirality of an electromagnetic pulse, taking into account the fraction of right-handed and left-handed photons.

Otherwise, from energy spectral density, we can define the continuous probability density function (pdf) $\hat{V}_\omega$ of polychromatic beam as $\hat{V}_\omega = V_\omega^U/U$, so the expected value for different powers of the frequency are given by

$$\langle \omega^n \rangle = \int_0^\infty \omega^n \hat{V}_\omega d\omega = \frac{\int_0^\infty \omega^n V_\omega d\omega}{\int_0^\infty V_\omega d\omega}. \tag{29}$$

This allow us to analyze the connection between helicity and chirality for electromagnetic pulses, which has been extensively analyzed for circularly polarized monochromatic fields [1], [2], [5]. In the case of such fields, the helicity and chirality densities have the following relations with the energy density $\chi/u = \pm\omega/c$ and $h/u = \pm 1/\omega$, so the relation between both magnitudes is $\chi = \omega^2 h/c$ [1], [2], but this proportionality only holds for the special case of monochromatic fields [1], [2]. Thus, from (26)–(29), we can express the ratios between helicity and chirality with the energy for circularly polarized polychromatic beams as $\mathcal{H}/U = \pm\langle\omega^{-1}\rangle$ and $\mathcal{X}/U = \pm\langle\omega\rangle/c$ (the sign depends on the circularly polarization considered), where $\langle\omega\rangle = \bar{\omega}$ represents the mean frequency of the distribution and $\langle\omega^{-1}\rangle = 1/\omega_2$ is related to the number of photons N by $N = U\langle\omega^{-1}\rangle/\hbar = U/(\hbar\omega_2)$ ($\hbar$ denotes the Planck constant divided by 2π). Therefore, for electromagnetic pulses, the relation between chirality and helicity is given by

$$\mathcal{X} = \frac{1}{c}\frac{\langle\omega\rangle}{\langle\omega^{-1}\rangle}\mathcal{H} = \frac{1}{c}\bar{\omega}\omega_2\mathcal{H}. \tag{30}$$

This result implies that for polychromatic beams, the proportionality between chirality and helicity depends on two different frequencies, one related to the average energy per photon ($\hbar\bar{\omega}$) and the other to the energy per photon ($\hbar\omega_2 = U/N$).

B. Energy Flux, Helicity Flux, and Chirality Flux

In the same way as for the scalar properties analyzed in Section II-A, the components of vector properties associated with the energy flux (Poynting vector **P**), helicity flux (total spin **S**), and chirality flux also obey the local conservation equation given by (10), where the corresponding flux for i component takes the general form [2], [6], [8]

$$\mathbf{\Phi}_{\boldsymbol{\alpha}_i} = \frac{1}{2}\Big(- \delta_{ij}\mathbf{M}\cdot\mathbf{N} + M_j N_i + M_i N_j\Big). \tag{31}$$

Therefore, the scalar product of the flux (31) and the surface differential of the sphere is given by

$$\mathbf{\Phi}_{\alpha_i}\cdot d\mathbf{\Sigma} = \frac{1}{2}\big(\mathbf{M}\cdot\mathbf{N}\big)u_i d\Sigma_\rho \tag{32}$$

where the values of u_i are the components of the unitary vectors in the propagation direction of the flux, for example, for Cartesian components $u_x = \cos(\phi)\sin(\theta)$, $u_y = \sin(\phi)\sin(\theta)$, and $u_z = \cos(\theta)$, while for the radial component of spherical coordinates, $u_\rho = 1$. In this point, following similar procedure as explained for scalar properties with the help of the Plancherel's theorem, the components of vector properties can be expressed as follows:

$$\Lambda_i = \int_0^\infty V_\omega^{\Lambda_i} d\omega = \frac{1}{2\pi}\int_0^\infty \Big(\int_\Sigma \big(\widetilde{\mathbf{M}}\cdot\widetilde{\mathbf{N}}^*\big)u_i d\Sigma_\rho\Big)d\omega \tag{33}$$

where $V_\omega^{\Lambda_i}$ denotes the spectral density for the ith component of vector property Λ. Therefore, similar to the previous section, the spectral densities of the different components of the vector properties can be obtained. Let us start with the energy flux, so taking into account that the flux of Poynting vector is given by the Maxwell stress tensor [6]

$$\mathbf{\Phi}_{\mathbf{P}_i} = \frac{1}{2\mu_0}\Big(- \delta_{ij}\mathbf{E}\cdot\mathbf{E} + 2E_j E_i\Big) + \frac{1}{2\epsilon_0}\Big(- \delta_{ij}\mathbf{H}\cdot\mathbf{H} + 2H_j H_i\Big). \tag{34}$$

The spectral density of the components of Poynting vector is given by the following expression:

$$V_\omega^{P_i} = \frac{1}{2\pi}\int_\Sigma \Big(\frac{1}{\mu_0}|\widetilde{\mathbf{E}}|^2 + \frac{1}{\epsilon_0}|\widetilde{\mathbf{H}}|^2\Big)u_i d\Sigma_\rho. \tag{35}$$

In the case of the other two vector properties studied, the corresponding fluxes can be written as follows [2]:

$$\mathbf{\Phi}_{\mathbf{S}_i} = \frac{c}{2}\Big(\delta_{ij}\mathbf{A}\cdot\mathbf{H} - A_j H_i - A_i H_j - \delta_{ij}\mathbf{C}\cdot\mathbf{E} + C_j E_i + C_i E_j\Big) \tag{36}$$

$$\mathbf{\Phi}_{\mathbf{Y}_i} = \frac{1}{2}\Big(\delta_{ij}\mathbf{H}\cdot\dot{\mathbf{E}} - H_j\dot{E}_i - H_i\dot{E}_j - \delta_{ij}\mathbf{E}\cdot\dot{\mathbf{H}} + E_j\dot{H}_i + E_i\dot{H}_j\Big). \tag{37}$$

With these definitions and taking into account (33), the spectral densities for the components of the spin $\mathbf{S}_i$ and chirality flux ($\mathbf{Y}_i$) can be written as follows:

$$V_\omega^{S_i} = \frac{1}{\pi}\frac{c}{\omega}\int_\Sigma \Im\Big[\widetilde{\mathbf{E}}\cdot\widetilde{\mathbf{H}}^*\Big]u_i d\Sigma_\rho \tag{38}$$

$$V_\omega^{Y_i} = \frac{1}{\pi}\omega\int_\Sigma \Im\Big[\widetilde{\mathbf{E}}\cdot\widetilde{\mathbf{H}}^*\Big]u_i d\Sigma_\rho \tag{39}$$

where we have used the relations between the electric and magnetic fields Fourier transforms and the potentials $\widetilde{\mathbf{A}}$ and $\widetilde{\mathbf{C}}$ exposed in the previous section and the properties of the Fourier transform of temporal derivatives. Therefore, (35), (38), and (39) offer the spectral density for the different components of the vector properties applicable to any kind of electromagnetic pulse, where we only need to know the far-field behavior of the Fourier transform of electric and magnetic fields ($\widetilde{\mathbf{E}}$ and $\widetilde{\mathbf{H}}$). Recently, the gauge invariance of helicity continuity equation has been studied [15]. In this way, it is important to note that (35), (38), and (39), together with (18)–(20), are gauge-invariant, since they solely depend on magnitudes that are gauge-invariant, despite of that in the deduction procedure the Coulomb gauge has been used.

Polarization State Effect on Vector Properties: In the same way of the scalar properties, interesting relationships between vector properties can be obtained if we express the electric and magnetic fields in the far-field as the superposition of right-handed and left-handed fields given by (21) and (22). Therefore, substituting such equations into (35), (38), and (39) gives the following expressions for the spectral densities of vector components:

$$V_\omega^{P_i} = \frac{\epsilon_0 c^2}{\pi} \int_\Sigma |\widetilde{\mathbf{E}}|^2 u_i \, d\Sigma_\rho \tag{40}$$

$$V_\omega^{S_i} = \Delta_d^l \frac{1}{\omega} \frac{\epsilon_0 c^2}{\pi} \int_\Sigma |\widetilde{\mathbf{E}}|^2 u_i \, d\Sigma_\rho \tag{41}$$

$$V_\omega^{Y_i} = \Delta_d^l \frac{\epsilon_0 c}{\pi} \omega \int_\Sigma |\widetilde{\mathbf{E}}|^2 u_i \, d\Sigma_\rho . \tag{42}$$

Now, as in the case of the scalar properties, it is easy to see that there is a direct relationship between the spectral densities of the three vector properties, so $V_\omega^{S_i} = \Delta_d^l V_\omega^{P_i}/\omega$ and $V_\omega^{Y_i} = \Delta_d^l V_\omega^{P_i}\omega/c$, and therefore, the different components of these vector properties can be obtained from the following expressions:

$$\mathbf{P}_i = \int_0^\infty V_\omega^{P_i} d\omega \tag{43}$$

$$\mathbf{S}_i = \Delta_d^l \int_0^\infty \frac{1}{\omega} V_\omega^{P_i} d\omega \tag{44}$$

$$\mathbf{Y}_i = \frac{\Delta_d^l}{c} \int_0^\infty \omega V_\omega^{P_i} d\omega . \tag{45}$$

In the case of the spherical coordinates, $u_\rho = 1$, so $V_\omega^{P_\rho} = cV_\omega^U$, and therefore, $\mathbf{P}_\rho = cU$, $\mathbf{S}_\rho = c\mathcal{H}$, and $\mathbf{Y}_\rho = c\mathcal{X}$, which means that the radial component of the fluxes of each scalar property is basically the property multiplied by the speed of the light. In the case of Cartesian components, there is not a direct relationship between the spectral densities of scalar and vector properties, so, for example, if we are interested in the flux in z-direction, we need to calculate the $V_\omega^{P_i}$ for $u_z = \cos(\theta)$, given by

$$V_\omega^{P_z} = \frac{\epsilon_0 c^2}{\pi} \int_\Sigma |\widetilde{\mathbf{E}}|^2 cos(\theta) \, d\Sigma_\rho . \tag{46}$$

From such definition, and taking into account the results for radial component of Poynting vector, it can be deduced from $V_\omega^{P_z} < V_\omega^{P_\rho}$ that the z component of Poynting vector is bounded by $\mathbf{P}_z < cU$, as has been pointed out by Lekner [6] as a consequence of the Maxwell equations.

III. Results and Discussion

Finally, we will verify that the main results of this work, described in (26), (27), (28), (43), (44), and (45), exactly estimate the electromagnetic properties of the pulses, so that they provide the same results as through the volume integrals of the corresponding densities [see (4)–(9)]. To do this, we are going to use the electromagnetic pulses with finite energy and momentum generated through the potential proposed by Ziolkowsky et al. [16] with the parameters analyzed by Hellwarth and Nouchi [14], which have analytical solutions for the volume integrals [14], [17]. The starting point is the following scalar potential, which is a solution to the wave equation:

$$\psi = \frac{f_0}{(s+q_2)(j\tau + q_1)}, \quad s = \frac{r^2}{j\tau + q_1} - j(ct+z) \tag{47}$$

where r and z are the radial and axial coordinates in the cylindrical system; q_1 and q_2 are parameters that characterize the pulse, which are defined strictly positive; f_0 is a constant; and τ = z-ct. Since ψ is a complex function, we can define its real and imaginary parts as $\psi_R = \Re(\psi)$ and $\psi_I = \Im(\psi)$, which allows us to construct the following Hertz potentials in cylindrical coordinates:

$$\mathbf{\Pi}_e = \psi_R \boldsymbol{u}_z \tag{48}$$

$$\mathbf{\Pi}_m = -\frac{\psi_I}{\mu_0 c} \boldsymbol{u}_z \tag{49}$$

where $\mathbf{\Pi_e}$ and $\mathbf{\Pi_m}$ denote the electric and magnetic Hertz potential, respectively. These Hertz potentials are a solution to the vector wave equation and generate a circularly polarized electromagnetic field in the far-field. The electric field, magnetic field, and their corresponding potential vectors ($\mathbf{C}$ and $\mathbf{A}$) can be obtained by the expressions

$$\mathbf{E} = \nabla \times \nabla \times \mathbf{\Pi}_e - \mu_0 \nabla \times \dot{\mathbf{\Pi}}_m \tag{50}$$

$$\mathbf{H} = \nabla \times \nabla \times \mathbf{\Pi}_m + \epsilon_0 \nabla \times \dot{\mathbf{\Pi}}_e \tag{51}$$

$$\mathbf{A} = \frac{1}{c^2} \dot{\mathbf{\Pi}}_e + \mu_0 \nabla \times \mathbf{\Pi_m} \tag{52}$$

$$\mathbf{C} = \frac{1}{c^2} \dot{\mathbf{\Pi}}_m - \epsilon_0 \nabla \times \mathbf{\Pi_e} . \tag{53}$$

Therefore, these fields can be estimated for the defined potentials [see (47)–(49)] and later introduced in (4)–(6) to calculate the scalar properties. The results obtained after the relatively high time-consuming calculation are the following:

$$U = \beta \frac{q_1 + q_2}{q_1^3 q_2^3} \tag{54}$$

$$\mathcal{H} = \beta \frac{1}{c q_1^2 q_2^2} \tag{55}$$

$$\mathcal{X} = \beta \frac{1}{2} \left(3q_1^2 + 4q_1 q_2 + 3q_2^2 \right) \tag{56}$$

where β is a constant defined as $\beta = f_0 \epsilon_0 \pi^2/2$. In the same way as the scalar properties, the vectorial properties can be obtained from the volume integrals of (7)–(9). For these beams, the fluxes of the energy, helicity, and chirality only have z component, which can be written as follows:

$$\mathbf{P}_z = \beta \frac{q_2 - q_1}{q_1^3 q_2^3} c \tag{57}$$

$$\mathbf{S}_z = \beta \, {}_2F_1\left[\frac{1}{2}; 1; \frac{7}{2}; \frac{(q_2 - q_1)^2}{(q_2 + q_1)^2}\right] \frac{4(q_2 - q_1)}{5(q_2 + q_1) q_1^2 q_2^2} \tag{58}$$

$$\mathbf{Y}_z = \beta \frac{3}{2} \frac{\left(q_2^2 - q_1^2\right)}{q_1^4 q_2^4} c \tag{59}$$

where ${}_2F_1$ indicates the hypergeometric function. On the other hand, these properties can also be estimated with the respective spectral densities for scalar [see (23)] and vectorial [see (40)] properties, which are generated from the Fourier transform of the electric field in the far-field approximation. Thus, the starting point to calculate such spectral densities is the far-field approximation for the potential (47) in spherical coordinates (ρ, θ, and ϕ). This far-field expansion involves various steps. First, the time variable must be defined as t = t′-ρ/c, since we must take into account the retarded time that the pulse needs to arrive to the far-field. Second, the radial coordinate must be replaced by $\rho = 1/\varrho$. The last step involves the series expansion around $\varrho = 0$ truncated in the first term, and finally, undoing the variable changes, so the final expression is given by

$$\psi_a = \frac{f_0}{2\rho(\rho + ct + jQ)}, \quad Q = \frac{1}{2}(q_1 + q_2 + (q_2 - q_1) cos\theta). \tag{60}$$

Fourier transforms of the real and imaginary parts of this far-field approximation are given by

$$\widetilde{\psi}_{a,R} = \frac{j}{c}\frac{\pi}{2} Exp\left[-\frac{\omega}{c} Q\right] Exp\left[-j\frac{\omega}{c}\rho\right], \quad \widetilde{\psi}_{a,I} = j\widetilde{\psi}_{a,R}. \tag{61}$$

Therefore, the far-field approximation of the Fourier transform for Hertz potentials (48) and (49) is $\widetilde{\mathbf{\Pi}}_e = \widetilde{\psi}_{a,R}(cos\theta \boldsymbol{u}_\rho - sin\theta \boldsymbol{u}_\theta)$ and $\widetilde{\mathbf{\Pi}}_m = -\widetilde{\psi}_{a,I}/(\mu_0 c)(cos\theta \boldsymbol{u}_\rho - sin\theta \boldsymbol{u}_\theta)$ in spherical coordinates, so taking into account that Fourier transform of (50) is $\widetilde{\mathbf{E}} = \nabla \times \nabla \times \widetilde{\mathbf{\Pi}}_e - j\mu_0 \omega \nabla \times \widetilde{\mathbf{\Pi}}_m$, the corresponding electric field in the frequency domain for the far-field is given by

$$\widetilde{\mathbf{E}}_a = f_0 \frac{\pi}{2} \frac{\omega^2}{c^3} \frac{sin\theta}{\rho} Exp\left[-\frac{\omega}{c} Q\right] Exp\left[-j\frac{\omega}{c}\rho\right] (\boldsymbol{u}_\theta + j\boldsymbol{u}_\phi). \tag{62}$$

In this case, electric field is purely right-handed circularly polarized, so $A_d = 1$ and $A_l = 0$. Finally, the respective spectral densities are calculated by introducing expression (62) into (23) and (40), can be written as follows:

$$V_\omega^U = \frac{8\beta}{c^5} Exp[\sigma] \frac{\omega^4}{\kappa^3} \Big(\kappa \, cosh(\kappa) - sinh(\kappa)\Big) \tag{63}$$

$$V_\omega^{P_z} = \frac{8\beta}{c^4} Exp[\sigma] \frac{\omega^4}{\kappa^4} \Big(-3\kappa \, cosh(\kappa) + (3 + \kappa^2) sinh(\kappa)\Big) \tag{64}$$

where $\sigma = -(q_1 + q_2)\omega/c$ and $\kappa = (q_1 - q_2)\omega/c$. Thus, using this expression of $V\omega^U$ in (26)–(28), and integrating over all the frequencies, the values of energy, helicity, and chirality are calculated, which are exactly the same as those provided by (54)–(56) obtained by the volume integrals of the properties densities. In the same way, using this expression of $V_\omega^{P_i}$ in (43)–(45), we obtain the corresponding z components of the energy, helicity, and chirality fluxes, which again exactly correspond to those obtained through the volume integrals, the results of which are shown in (57)–(59). We must note that, using this second procedure, the calculation time is much less expensive than the classical one, since only implies the integration over the spatial angular distribution of the electric field modulus in the far-field, which has a smooth behavior in the form of trigonometric functions that are usually easily integrable. Specifically, the calculation of the volume integral at t = 0 for energy, helicity, and chirality takes 52.78, 27.98, and 16.22 s, respectively, while the calculation times using the proposed method are 3.58, 2.51, and 2.34 s, an order of magnitude lower (Mathematica program was used to perform all the calculations). Therefore, this methodology allows the calculation of the scalar and vector properties of electromagnetic pulses in a simple and fast way through expressions that are gauge-invariant, even in pulses in which their calculation through volume integrals will be very complicated, such as the fields generated by the interference of multiple pulses.

IV. Conclusion

In conclusion, we have established a novel framework for calculating the main scalar (energy, helicity, and chirality) and vectorial (Poynting vector, total spin, and chirality flux) properties of electromagnetic pulses. This achievement is realized through gauge-invariant expressions derived from the far-field behavior of electric and magnetic fields, independent of polarization state, ensuring robustness, and consistency. Furthermore, by decomposing the fields into a superposition of a left-handed circularly polarized part and its right-handed counterpart, we have demonstrated that all properties can be obtained through the spectral density of the energy and the spectral densities of the Poynting vector components. These quantities can be calculated by considering only the modulus of the electric field in the frequency domain for the far-field. We have validated our methodology with a simple pulse, showing that the results align exactly with those obtained through corresponding volume integrals of local properties in all cases, and at a significantly lower computational cost. The simplicity, precision, and computational efficiency make this method particularly advantageous for studying the electromagnetic properties of complex field structures, including interference from multiple pulses or scattered fields due to interactions with matter. In addition, as a consequence of our developed method, we have established the proportionality between helicity and chirality for electromagnetic pulses, dependent on statistical average parameters, such as the mean frequency of the pulse and the energy per photon.

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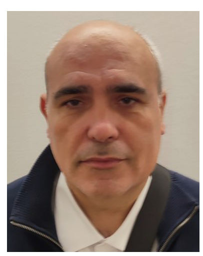

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