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A general direct approach for decomposing profit inefficiency[☆]

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Considering any graph technical inefficiency measure, we show that the so-called standard or traditional approach for decomposing profit inefficiency relying on Fenchel-Mahler inequalities obtained from duality theory, establishes that profit inefficiency is greater than or equal to the product of technical inefficiency times a positive factor expressed in monetary units. This product is identified as the technical profit inefficiency and its difference with respect to the profit inefficiency as the allocative profit inefficiency. Dividing profit inefficiency by the mentioned positive factor one obtains the normalized (units' invariant) profit inefficiency of the firm, which is a pure number, and can be decomposed into the sum of technical inefficiency and the normalized allocative profit inefficiency, usually called allocative inefficiency. We propose a new decomposition based on equalities that starts from the input and output slacks connecting the firm with the frontier benchmark, obtained through the pre-specified technical inefficiency measure. Profit inefficiency is then decomposed into the value of the technological gap and the profit inefficiency of the frontier benchmark. Expressing the value of the technological gap as the product of the technical inefficiency times a certain normalizing factor, we deduce a new normalized profit inefficiency decomposition. Our decomposition ensures that the allocative efficiency of a firm corresponds to that of its benchmark on the frontier and therefore avoids the possibility of overestimating it. We compare the traditional and the general direct approach and show that the new decomposition is conceptually sound and more accurate, with the only exception of the family of directional distance functions, for which both decompositions are equivalent.

1. Introduction

The decomposition of profit inefficiency was introduced by Chambers et al.'s [1] based on the duality between the graph directional distance function (DDF) and the profit function. This duality enables the definition of a Fenchel-Mahler inequality where normalized profit inefficiency is greater than or equal to the graph DDF, which is interpreted as a measure of technical inefficiency. Subsequently, normalized profit inefficiency can be decomposed into two mutually exclusive terms corresponding to the technical component and an allocative residual. Since then, several authors have adapted this methodology, which we term traditional approach, to decompose profit inefficiency using alternative technical inefficiency measures.

Pastor et al. [2] offer a systematic review of the most relevant proposals besides that based on the graph DDF. Chronologically, normalized profit inefficiency can be decomposed resorting to: i) the Hölder distance functions under alternative norms, Briec and Lesourd [3]; ii) the weighted additive distance function (WADF), Aparicio, Pastor and Vidal [4]; iii) the Loss Distance Function, Aparicio, Borras, Pastor and Zofío [5]; iv) the Enhanced Russell Graph measure (ERG) (or Slack-Based Measure, SBM), Aparicio et al. [6]; v) the original graph Russell measure, Halická and Trnovská [7]; vi) the modified directional distance function; and, finally, vii) the reverse directional distance function—duality results for these last two are presented in Pastor et al. [2].

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Relying on their associated Fenchel-Mahler inequalities, each of these proposals derive a particular decomposition of profit inefficiency based on its corresponding technical efficiency measure and associated normalization factor. Profit inefficiency is normalized by these factors, giving rise to a normalized (units' invariant) decomposition expressed in pure numbers, i.e., without units of measurement. However, as we show later, for some of these decompositions, the normalized allocative component of the traditional approach can be unrelated to the profit inefficiency of its frontier benchmark, which may result in the underestimation (overestimation) of technical inefficiency (allocative inefficiency).

To solve this question, we introduce a new method, the general direct approach (GDA), that directly states an equality between overall economic efficiency and the two typical drivers (viz. technical and allocative efficiency). For any firm, and given a preferred technical inefficiency measure, the method requires just two pieces of information: the firm's technical inefficiency, and its associated frontier projection. Our approach, as the traditional one, results in an additive decomposition of profit inefficiency into two components. The first component, which we call technological profit gap or technical profit inefficiency, is the interior product of two vectors, the optimal slack vector (separating the firm under evaluation from its frontier benchmark) and the market price vector. The second component corresponds to the profit inefficiency of the benchmark at the frontier, which we call allocative profit inefficiency. This GDA decomposition of profit inefficiency is expressed in monetary units. However, following the literature, we show how to define a normalized version that allows to relate profit inefficiency with the initial technical inefficiency measure of choice based on quantities, and that is units' invariant, thereby removing the monetary units of measurement, so the decomposition satisfies the commensurability property. For this purpose, we show that the profit technological gap can be rewritten as the product of a certain normalizing factor, times the inefficiency measure. Then, by dividing the three terms of the equality by the mentioned factor, we obtain a general normalized profit inefficiency decomposition.

The general direct approach offers a unifying framework for decomposing profit inefficiency that is valid for any technical inefficiency measure. It follows similar steps to the traditional approach, but it is easier to develop and implement. After selecting a specific graph technical inefficiency measure, e.g., the already mentioned DDF, Hölder, Loss distance function, WADF, ERG, graph Russell measure, etc., we obtain the necessary information for its decomposition by solving, for each evaluated firm, the corresponding mathematical program. Contrarily, the traditional approach associated with the selected measure proves more demanding, requiring not only to solve the last mentioned program in order to know the values of the inefficiency measure and its projection, but also the *explicit* determination of the so-called Fenchel-Mahler inequality relating economic inefficiency with the selected technical inefficiency measure, from which a normalization factor can be recovered, and that can be more or less costly to obtain.[1] The GDA relies on duality *implicitly* because the normalization factor depends on the value of the technical inefficiency measure. Both proposals end up decomposing profit inefficiency, after normalizing it, into the sum of a technical and an allocative component, but the direct approach is general, easier to implement, and more reliable. It is *general* because it can be applied to any efficiency measure; direct and *easier to implement*, because it does not require formulating the normalizing factors associated to Fenchel-Mahler inequalities (although the underlying duality holds); and *more reliable* because, as mentioned above, working with equalities instead of inequalities, avoids the possibility of overestimating allocative inefficiency (the so-called essential property), something that happens with many traditional approaches as recently proven by Aparicio et al. [8].

Our general direct approach has some precursors in the literature that also use the profit levels of the projections to decompose profit inefficiency. We can highlight Cooper et al. [9] who, resorting to the (unweighted) additive model, established a decomposition of profit inefficiency into the economic value of technical inefficiency and allocative inefficiency. Following this thread, Ruiz and Sirvent [10] determine lower and upper bounds for the technical and allocative components of the profit efficiency decomposition using market prices as weights of the additive model. Portela and Thanassoulis [11] proposed a decomposition of overall economic efficiency based on the mathematical expression of the objective function of the multiplicative Geometric Distance Function (GDF) model.[2] More recently, Färe et al. [12] introduced a decomposition of profit inefficiency based on the slacks-based directional distance function (SBD). Finally, Pastor et al. [13] propose a 'reversed' decomposition of profit change that is driven by the minimization of allocative inefficiency when searching for the technically efficient projection on the frontier and using the market prices of outputs and inputs as weights of the output and inputs slacks in the additive model. However, all these previous approaches are not sufficiently general to encompass any technical efficiency measure.

Other relevant contributions related to Data Envelopment Analysis and economic efficiency are the following: Juo et al. [14] proposed a non-oriented slacks-based measure (SBM) model to decompose the change in the operating profit into various meaningful components (quantity and price effects); Fu et al. [15] assumed that Taiwanese and Chinese banks operate under a common global frontier and measured profit inefficiency; Jradi et al. [16] measured and decomposed revenue inefficiency over time through the Luenberger indicator; Yu [17] proposed a decomposition of profit inefficiency for any directional distance function under non-perfect competition into three components: price, technical and allocative inefficiency; and Aparicio and Zofio [18] combined cross-efficiency and profit inefficiency and decomposed the corresponding score into technical and allocative inefficiency.

The paper develops as follows. In Section 2 we propose the new General Direct Approach for decomposing profit inefficiency. Here, we compare the traditional approach based on the Fenchel-Mahler inequality with the GDA and show that the traditional technical profit inefficiency can underestimate its GDA counterpart, proving this result for the ERG graph measure.[3] We further prove for the ERG under which conditions, certainly demanding, both decompositions are equal. We also show that for the family of directional distance functions the traditional and GDA decomposition are equal. We finalize this section presenting the normalized version of both decompositions and defining the necessary GDA normalization factors. Section 3 illustrates the new model with an empirical application to Taiwanese banks, discussing the differences between the traditional and GDA decompositions. Section 4 concludes.

[1] For what is probably the most complicated case, see the results by Halická and Trnovská [7] for the non-linear graph Russell measure proposed by Färe and Lovell [34].

[2] The GDF minimizes the ratio of the geometric mean of proportional contraction rates of inputs to the geometric mean of proportional expansion rates of outputs.

[3] The enhanced Russell graph measure introduced by Pastor et al. [22] was formulated in terms of individual input and output proportional reductions, consistent with the definition of the Russell graph measure [35]. These authors showed that it could reformulated in terms of input and output slacks obtaining a specification that coincided with that proposed by Tone [36], who called it Slack Based Measure (SBM). Therefore, it can be referred indistinctly as ERG or SBM. A search in the ISI WoK at the time of writing shows that both articles received about 2,650 citations, followed by the additive model of Charnes et al. [23] with 950 citations—although this number does not include cites to all its weighted variants (MIP, RAM, BAM, etc.). Both measures are followed by the directional distance function proposed by Chambers et al. [37], receiving around 750 citations.

2. The general direct approach (GDA) for decomposing profit inefficiency

Let $x \in \mathbb{R}_+^M$ denote the vector of inputs and $y \in \mathbb{R}_+^N$ the vector of outputs; the technology is given by

$$T = \left\{(x, y) : x \in \mathbb{R}_+^M, y \in \mathbb{R}_+^N, \ x \text{ can produce } y\right\} \quad (1)$$

We assume that T is a subset of $\mathbb{R}_+^{M+N}$ that satisfies the usual axioms (see, for example [19]). We also identify two reference sets of the technology, namely the weakly and strongly efficient production possibility sets: $\partial^W(T) = \{(x, y) \in T : (x', -y') < (x, -y) \Rightarrow (x', y') \notin T\}$ and $\partial^S(T) = \{(x, y) \in T : (x', -y') \leq (x, -y), \ (x', y') \neq (x, y) \Rightarrow (x', y') \notin T\}$. Intuitively the strongly efficient subset includes all feasible production plans that are not dominated and, therefore, are technically efficiency in the sense of Koopmans [[20], p. 60], who provides a practical definition of efficiency based on the notion of Pareto optimality. Contrarily, the weakly efficient subset of the technology comprises production plans that may be dominated in some input and output dimension. Therefore $\partial^S(T) \subset \partial^W(T)$. Regarding returns to scale, since we are interested in profit inefficiency, we assume that the technology is characterized by variable returns to scale, VRS, guaranteeing that at least one profit maximizing input-output bundle exists and belongs to $\partial^S(T)$.

Let us also assume that common positive market prices for each input and each output are known, $(w, p) \in \mathbb{R}_{++}^{M+N}$. Then the profit function is defined as:

$$\Pi(w, p) = \max_{x,y} \{p \cdot y - w \cdot x : (x, y) \in T\} \quad (2)$$

Profit inefficiency of firm (x_j, y_j), $j = 1,\ldots,J$, denoted by $\Pi I(x_j, y_j, w, p)$, measures the difference between maximum profit, $\Pi(w, p)$, and observed profit: $\Pi_j = p \cdot y_j - w \cdot x_j$. That is:

$$\Pi I\left(x_j, y_j, w, p\right) = \Pi(w, p) - \left(p \cdot y_j - w \cdot x_j\right), j = 1, \ldots, J \quad (3)$$

For any firm j, $\Pi I(x_j, y_j, w, p) \geq 0_{\$}$;, where \$ denotes the monetary unit (currency) in which profit is denominated–if (x_j, y_j) is a profit maximizing firm, then $\Pi I(x_j, y_j, w, p) = 0_{\$}$.

Let us now assume that we are considering a graph inefficiency measure G that allows calculating the technical inefficiency of firm j, $TI_G(x_j, y_j)$; i.e., DDF, Hölder, ERG (or SBM), WADF, etc. For each firm (x_j, y_j) these measures yield the value of technical inefficiency, which is a dimensionless number, and identifies a projection, $(\hat{x}_j^G, \hat{y}_j^G)$ that belongs to either $\partial^W(T)$ or $\partial^S(T)$ of the production possibility set.

The profit inefficiency of $(\hat{x}_j^G, \hat{y}_j^G)$ is given by

$$\begin{aligned}\Pi I\left(\hat{x}_j^G, \hat{y}_j^G, w, p\right) &= \Pi(w, p) - \left(p \cdot \hat{y}_j^G - w \cdot \hat{x}_j^G\right) = \Pi I\left(x_j, y_j, w, p\right) \\ &\quad -\left(p \cdot \hat{s}_j^{+G} + w \cdot \hat{s}_j^{-G}\right). \quad (4)\end{aligned}$$

where $(\hat{s}_j^{-G}, \hat{s}_j^{+G})$ is the vector of input and outputs slacks, corresponding to the L_1-path connecting the firm under evaluation and its projection: $\hat{x}_j^G = x_j - \hat{s}_j^{-G}, \hat{y}_j^G = y_j + \hat{s}_j^{+G}$, $j = 1, \ldots, J$.[4] Finally, substituting (4) into (3) and simplifying delivers[5]

$$\Pi I\left(x_j, y_j, w, p\right) = \left(w \cdot \hat{s}_j^{-G} + p \cdot \hat{s}_j^{+G}\right) + \Pi I\left(\hat{x}_j^G, \hat{y}_j^G, w, p\right) \quad (5)$$

This expression represents the general direct approach, GDA, showing that profit inefficiency decomposes into the sum of two terms expressed in the same monetary values: the price value of the slacks $(w \cdot \hat{s}_j^{-G} + p \cdot \hat{s}_j^{+G})$ and the profit inefficiency of the projection, $\Pi I(\hat{x}_j^G, \hat{y}_j^G, w, p)$. The above decomposition of profit inefficiency can be interpreted in terms of the usual technical and allocative components. The profit value of the slacks provides a natural measure of technical inefficiency, i.e., $(w \cdot \hat{s}_j^{-G} + p \cdot \hat{s}_j^{+G}) \equiv T\Pi I_G^{GDA}(x_j, y_j, w, p)$, while the profit inefficiency of the projection corresponds to the profit loss due to allocative inefficiency, i.e., $\Pi I(\hat{x}_j^G, \hat{y}_j^G, w, p) \equiv A\Pi I_G^{GDA}(x_j, y_j, w, p)$.

Consequently, expression (5) can be equivalently rewritten as

$$\Pi I\left(x_j, y_j, w, p\right) = T\Pi I_G^{GDA}\left(x_j, y_j, w, p\right) + A\Pi I_G^{GDA}\left(x_j, y_j, w, p\right) \quad (6)$$

2.1. Comparing the general direct and traditional decompositions of profit inefficiency

It is possible to derive an equality like (6) for the traditional profit inefficiency decompositions based on any of the technical inefficiency measures, G, surveyed in the introduction. The traditional approach relies on duality theory to derive a Fenchel-Mahler inequality according to which *normalized* profit inefficiency is greater or equal than the specific technical inefficiency measure G. In the traditional decompositions the second allocative profit term is retrieved as a residual and, generally speaking, it is unrelated to its projection on the frontier with only one remarkable exception, the DDF, that we discuss later on.

Given a graph technical inefficiency measure $TI_G(x_j, y_j)$, the traditional approach generates the following Fenchel-Mahler inequality for firm (x_j, y_j):

$$\Pi I\left(x_j, y_j, w, p\right) \geq TI_G\left(x_j, y_j\right) NF_G^{TRA}\left(x_j, y_j, p, w\right) \quad (7)$$

where the normalizing factor appears multiplying technical inefficiency in the right-hand side instead of dividing profit inefficiency in the left-hand side, as usually presented. In expression (7) profit inefficiency is once again expressed in monetary units, and the right-hand side term is the product of the technical inefficiency $TI_G(x_j, y_j)$, which is a nonnegative number, and the corresponding traditional normalization factor $NF_G^{TRA}(x_j, y_j, w, p)$, which is positive and expressed in the same monetary units as profit inefficiency. Their product is defined as the traditional technical profit inefficiency $T\Pi I_G^{TRA}(x_j, y_j, w, p)$, which is the counterpart of the profit technical inefficiency term of the GDA decomposition (6). Expression (7) can be transformed into an equality that defines the traditional profit inefficiency decomposition by retrieving as residual its second component, corresponding to the traditional allocative profit inefficiency; i.e., $A\Pi I_G^{TRA}(x_j, y_j, w, p) = \Pi I(x_j, y_j, w, p) - TI_G(x_j, y_j) NF_G^{TRA}(x_j, y_j, p, w)$. Hence, the traditional decomposition of profit inefficiency corresponds to:

$$\Pi I\left(x_j, y_j, w, p\right) = T\Pi I_G^{TRA}\left(x_j, y_j, w, p\right) + A\Pi I_G^{TRA}\left(x_j, y_j, w, p\right) \quad (8)$$

Formally, the structures of the general and traditional approaches, (6) and (8), are equal, although the two terms of both right-hand sides are frequently different for the subset of inefficient firms. This mismatch emerges because expression (7), which is defined as an inequality, makes the traditional normalization factor liable to *undervaluation*, in which case the same happens to $T\Pi I_G^{TRA}(x_j, y_j, w, p)$. This does not happen in the GDA approach based on an equality, where its technical component is as large as possible, since it is based on the slacks that connect each firm with its frontier projection.

2.2. Normalized decompositions of profit inefficiency

Based on the previous relationships, it is possible to consider the traditional decomposition of *normalized* profit ineffi-

[4] This L_1-path can alaways be calculated for any type of measure, multiplicative or additive.

[5] The rearrangement of the two terms of the GDA profit inefficiency decomposition in (5) facilitates its future comparison with the equality associated with the traditional profit inefficiency decomposition.

ciency. Dividing profit inefficiency by the above normalization factor $NF_G^{TRA}(x_j, y_j, w, p)$ one can define a Fenchel-Mahler inequality by which normalized profit inefficiency is greater than or equal to technical inefficiency. The advantage of the normalized decomposition over its unnormalized (monetary) counterpart is that it is units' invariant, satisfying the so-called *commensurability* property. Reformulating (7) and adding a second term to the technical inefficiency on the right-hand side by closing the inequality—namely the residual normalized allocative inefficiency term $AI_G^{TRA}(x_j, y_j, \tilde{w}, \tilde{p})$, we obtain the traditional decomposition of normalized profit inefficiency:

$$N\Pi I_G^{TRA}\left(x_j, y_j, \tilde{w}, \tilde{p}\right) = TI_G\left(x_j, y_j\right) + \frac{A\Pi I_G^{TRA}\left(x_j, y_j, w, p\right)}{NF_G^{TRA}\left(x_j, y_j, w, p\right)} = TI_G\left(x_j, y_j\right) + AI_G^{TRA}\left(x_j, y_j, \tilde{w}, \tilde{p}\right). \tag{9}$$

In the same way, we propose a normalized version of the GDA decomposition, which requires the definition of its associated normalizing factor:

$$NF_G^{GDA}\left(X_j, y_j, w, p\right) \left\{ \begin{array}{ll} \left(\frac{w\cdot \hat{s}_j^{-G} + p\cdot \hat{s}_j^{+G}}{TI_G(x_j, y_j)}\right) & , \text{ when } TI_G\left(x_j, y_j\right) > 0 \\ k_j > 0_{\$}; & , \text{ when } TI_G\left(x_j, y_j\right) = 0 \end{array} \right\} \tag{10}$$

This expression assigns a positive normalization factor to the technically efficient firms, which is expressed in the same monetary units as profit inefficiency.

Dividing expression (5) by $NF_G^{GDA}(x_j, y_j, w, p)$, we define the normalized GDA profit inefficiency decomposition as follows:

$$N\Pi I_G^{GDA}\left(x_j, y_j, \tilde{w}, \tilde{p}\right) = \left(\frac{\Pi I\left(x_j, y_j, w, p\right)}{NF_G^{GDA}\left(x_j, y_j, w, p\right)}\right) = TI_G\left(x_j, y_j\right) + \frac{\Pi I\left(\hat{x}_j^G, \hat{y}_j^G, w, p\right)}{NF_G^{GDA}\left(x_j, y_j, w, p\right)}, \tag{11}$$

where the last term is identified as allocative inefficiency $AI_G^{GDA}(x_j, y_j, \tilde{w}, \tilde{p})$.

Previous papers in the literature have already dealt with traditional profit decompositions by using various normalization factors (see, for example [1,4,12,21]). In our case, the normalization factor is general and can encompass any graph technical inefficiency measure G.

2.3. Implementing the general direct approach

2.3.1. The case of the enhanced Russell graph measure, ERG

To implement the general direct approach, one needs to choose first a technical inefficiency measure. We decide on the additive ERG measure, which is defined as:

$$TE_{ERG}\left(x_j, y_j\right) = \min_{s^-, s^+} \left\{ \frac{1 - \frac{1}{M}\sum_{m=1}^{M} \frac{s_m^-}{x_{jm}}}{1 + \frac{1}{N}\sum_{n=1}^{N} \frac{s_n^+}{y_{jn}}} \middle| \left(x_j - s_j^-, y_j + s_n^+\right) \in T \right\} \tag{12}$$

An appealing feature of the ERG is that it projects firms to the strongly efficient subset of the of technology $\partial^S(T)$ and therefore their benchmarks comply with the notion of Pareto—Koopmans efficiency. This is not the case for measures projecting firms to $\partial^W(T)$, where individual input reductions and output expansions may still be feasible—as the DDF discussed below.

The traditional profit inefficiency decomposition for the ERG measure was recently proposed by Aparicio et al. [6], who formulated the corresponding Fenchel-Mahler inequality, showing that its specific normalizing factor is:

$$NF_{ERG}^{TRA}\left(x_j, y_j, p, w\right) = \left(1 + \frac{1}{N}\sum_{n=1}^{N} \frac{\hat{s}_{jn}^{+ERG}}{y_{jn}}\right) \cdot \min\left\{Mw_1x_{j1}, ..., Mw_Mx_{jM}, Np_1y_{j1}, ..., Np_Ny_{jN}\right\}. \tag{13}$$

Based on the formulation of the ERG technical efficiency (12), $TE_{ERG}(x_j, y_j) \in [0,1[$, see Pastor et al. [22], so its associated technical inefficiency, with range]0,1], is defined as follows:

$$TI_{ERG}\left(x_j, y_j\right) = 1 - TE_{ERG}\left(x_j, y_j\right) = \left(\frac{\frac{1}{M}\sum_{m=1}^{M} \frac{\hat{s}_{jm}^{-ERG}}{x_{jm}} + \frac{1}{N}\sum_{n=1}^{N} \frac{\hat{s}_{jn}^{+ERG}}{y_{jn}}}{1 + \frac{1}{N}\sum_{n=1}^{N} \frac{\hat{s}_{jn}^{+ERG}}{y_{jn}}}\right). \tag{14}$$

Consequently, the right-hand side expression of the specific Fenchel-Mahler inequality (7), which corresponds to $T\Pi I_{ERG}^{TRA}(x_j, y_j, w, p) = TI_{ERG}(x_j, y_j) NF_{ERG}^{TRA}(x_j, y_j, p, w)$, is equal to

$$\left(\frac{\frac{1}{M}\sum_{m=1}^{M} \frac{\hat{s}_{jm}^{-ERG}}{x_{jm}} + \frac{1}{N}\sum_{n=1}^{N} \frac{\hat{s}_{jn}^{+ERG}}{y_{jn}}}{1 + \frac{1}{N}\sum_{n=1}^{N} \frac{\hat{s}_{jn}^{+ERG}}{y_{jn}}}\right)\left(\left(1 + \frac{1}{N}\sum_{n=1}^{N} \frac{\hat{s}_{jn}^{+ERG}}{y_{jn}}\right) \cdot \min\left\{Mw_1x_{j1}, ..., Mw_Mx_{jM}, Np_1y_{j1}, ..., Np_Ny_{jN}\right\}\right) = \left(\frac{1}{M}\sum_{n=1}^{N} \frac{s_{jm}^{-*ERG}}{x_{jm}} + \frac{1}{N}\sum_{n=1}^{N} \frac{s_{jn}^{+*ERG}}{y_{jn}}\right) \cdot \min\left\{Mw_1x_{j1}, ..., Mw_Mx_{jM}, Np_1y_{j1}, ..., Np_Ny_{jN}\right\}. \tag{15}$$

We enunciate next proposition related to the ERG measure, establishing three relevant results:

Proposition 1.

1.1 *For any firm (x_j, y_j) of the production possibility set the following inequality holds, $T\Pi I_{ERG}^{TRA}(x_j, y_j, w, p) \leq T\Pi I_{ERG}^{GDA}(x_j, y_j, w, p)$, or, equivalently, $A\Pi I_{ERG}^{TRA}(x_j, y_j, w, p) \geq A\Pi I_{ERG}^{GDA}(x_j, y_j, w, p)$.*

1.2 *In particular, any efficient firm satisfies $T\Pi I_{ERG}^{TRA}(x_j, y_j, w, p) = T\Pi I_{ERG}^{GDA}(x_j, y_j, w, p)$, or, equivalently, $A\Pi I_{ERG}^{TRA}(x_j, y_j, w, p) = A\Pi I_{ERG}^{GDA}(x_j, y_j, w, p)$.*

1.3 *Moreover, for any inefficient firm (x_j, y_j), the equality $T\Pi I_{ERG}^{TRA}(x_j, y_j, w, p) = T\Pi I_{ERG}^{GDA}(x_j, y_j, w, p)$ holds if, and only if, the next chain of equalities holds:$Mw_1x_{j1} ... = Mw_Mx_{jM} = Np_1y_{j1} ... = Np_Ny_{jN} = \min\{Mw_1x_{j1}, ..., Mw_Mx_{jM}, Np_1y_{j1}, ..., Np_Ny_{jN}\}$.*

If all inefficient firms have a common subset of zero-valued input or output slacks, the corresponding inputs and outputs can be dropped from the last chain of equalities.

Proof. See Appendix 1

The chain of equalities required for the latter result *1.3* are highly demanding, making it unlikely that inefficient firms satisfy them. This explains why the traditional approach to decompose profit inefficiency normally underestimates the technical profit inefficiency value of the subset of inefficient firms. Furthermore, since similar requirements can be established for other additive technical efficiency measures, e.g., the additive model by Charnes et al. [23] and its weighted variants, this is a pervasive result plaguing the traditional approach to decompose profit inefficiency.

From Proposition 1, for any firm, the technical profit inefficiency of the traditional approach is always smaller than or equal to the technical profit inefficiency of the GDA approach, which implies that the traditional normalization factor is always smaller than or equal to the GDA normalization factor: $NF_{ERG}^{TRA}(x_j, y_j, w, p) \leq NF_{ERG}^{GDA}(x_j, y_j, w, p)$.

Table 1
Descriptive statistics. Input and output quantities, Taiwanese banks, 2010.

	Inputs			Outputs	
	x_1	x_2	x_3	y_1	y_2
Average	795,536	3826	13,393	196,808	609,489
Median	428,995	3146	8721	157,870	328,574
Max.	3171,493	9538	76,576	904,580	2091,100
Min.	25,019	202	505	1681	66,947
Stand. Dev.	768,008	2729	15,185	215,063	582,854

Source: Juo et al. [24].

2.3.2. The case of the directional distance function, DDF

However, it is worth mentioning that the additive directional distance function is the only known technical inefficiency measure that gives rise to the same profit decomposition regardless the generalized direct or traditional approach, as next proposition establishes:

Proposition 2. *The directional distance function, DDF, always satisfies that its traditional and GDA profit inefficiency decompositions are equal, which is equivalent to saying that the equality* $T\Pi I^{TRA}_{DDF}(x_j, y_j, w, p) = T\Pi I^{GDA}_{DDF}(x_j, y_j, w, p)$ *holds for any firm* (x_j, y_j) *of the production possibility set. Moreover, any graph measure G that satisfies last conditions is necessarily a DDF.*

Proof. See Appendix 2

Nevertheless, the choice of the directional distance function implies that firms may be projected to the weakly efficient subset of the technology $\partial^W(T)$, resulting in the underestimation of technical inefficiency and, equivalently, overestimation of allocative inefficiency. Therefore, while the use of the DDF ensures that profit technical inefficiency is not underestimated in the traditional approach due to the particular specification of the normalization factor, it may still be underestimated if the projected benchmark does not belong to the strongly efficient frontier, $\partial^W(T)$.

Regarding the normalized versions of the general direct and traditional approaches, we establish an interesting result between their corresponding normalization factors.

Proposition 3. *For any graph directional distance function, DDF, its GDA and traditional normalization factors are equal. Moreover, any graph measure G that satisfies the mentioned condition is necessarily a DDF.*

Proof. It is derived from the proof of Proposition 2.

3. An application: Taiwanese banking industry

We illustrate the monetary and normalized GDA and traditional models resorting to a set of 31 Taiwanese banks observed in 2010, previously studied by Juo et al. [24]. A complete discussion of the statistical sources and variables specification can be found there.[6] Table 1 presents descriptive statistics for these variables. Regarding the technology and interrelations between inputs and outputs, the variables reflect the so-called intermediation approach suggested by Sealey and Lindley [25], whereby financial institutions, through labor and capital, collect deposits from savers to produce loans and other earning assets for borrowers. Inputs are financial funds (x_1), number of employees (x_2), and physical capital (x_3). The output vector includes financial investments (y_1) and loans (y_2). Monetary variables are measured in millions of New Taiwan Dollar (TWD). The unit prices of inputs include average interest paid per TWD of financial funds (w_1), the ratio of personnel expenses to the number of employees (w_2), and the non-labor operational cost (operational expenses net of personnel expenses) per TWD of fixed assets (w_3). The unit prices of outputs correspond to average interest earned per TWD of investment (p_1), and average interest earned per TWD of loan (p_2). Common input and output prices are calculated as unit values; that is, individual costs and revenues divided by quantities. Inputs prices are w_1=0.0064, w_2=1.2586, w_3=0.3171, while output prices are p_1=0.0349 and p_2=0.0211.

We first calculate and decompose the monetary GDA and traditional profit inefficiency models based on the ERG (or SBM). The ERG measure is calculated using Data Envelopment Analysis (DEA)[7] techniques as shown in program (4) in Pastor et al. [22]. Table 2 presents both decompositions, where profit inefficiency, expressed in million TWD, is common to the GDA and traditional approach. The GDA decomposition−expressions (5) and (6), is reported in columns 2 thru 4. As for the traditional approach based on Fenchel-Mahler inequalities, its decomposition−expression (8)−is reported in columns 5 thru 8. The Bank of Taiwan (#2) maximized profit efficiency, i.e., its profit inefficiency value is null, and therefore it is technically and allocatively efficient, constituting the economic benchmark for the remaining banks. A total of 11 banks were technically efficient (out of 31), while the rest were technically and allocatively inefficient. It is easy to observe that, as stated in Proposition 1.1, the value of the technical profit inefficiency corresponding to the traditional approach is smaller than or equal to that of the generalized approach. As anticipated in Section 2.3.1 equality is verified for the previous 11 technically efficient firms, while a strict inequality is observed for the remaining 20 banks, implying that the condition stated in Proposition 1.3 is not met. Consequently, the traditional approach underestimates the technical profit inefficiency, preventing banks from exploiting the potential production frontier, while overstating, in the exact same amount, the allocative profit inefficiency, which is associated to suboptimal mixes of input demands and output supplies given their market prices. Managers of these banks could had been misled by the source of profit inefficiency, which may had resulted in wrong efficiency enhancing strategies. For example, for bank #11, technical profit inefficiency is underestimated in \$4964.9 million TWD (\$15,021.2−\$10,056.4), which represent 23.7% of the overall profit inefficiency (\$4964.9/\$20,955.5 × 100) and as much as 33.1% of the GDA technical profit inefficiency value (\$4964.9/\$15,021.2 × 100). As many as six banks present two-digit deviations in percentage points. Therefore, we observe that the discrepancy can be substantial. Generally, the average disparity in the measurement of technical profit inefficiency in the whole industry amounts 6.4% of overall profit inefficiency.

We observe considerable inefficiency dispersion across banks and different patterns of technical and allocative efficiency. To provide some insights about individual performance we discuss the best five and worst five performing banks in terms of profit efficiency and its components. The profit efficient Bank of Taiwan (#2) is followed by Cooperative Bank (#6), First Bank (#7), Land Bank (#5), and Hua Nan Bank (#8). All these banks, except the last one, were technically efficient, showing that the main cause of profit loss for the top performing banks is allocative inefficiency. Furthermore, all these banks were publicly owned, suggesting that some of the weaknesses associated to profit maximization in public enterprises were not at play in the Taiwanese banking sector; e.g., multiplicity of objectives−like securing employment levels, higher pressure from political parties and lobbies, weaker budget constraints since central banks acts as lenders of last resort, or dis-

[6] We are grateful to these authors for sharing the data. The same data set has been used to illustrate symmetric decompositions of cost variation by Balk and Zofío [38], and the decompositions of total factor productivity change using quantities-only and price-based indices by Balk [39].

[7] Cook et al. [40] addressed several problems in the context of DEA such as model orientation, input and output selection/definition, the use of mixed and raw data, and the number of variables to use versus the number of DMUs.

Table 2
Monetary General Direct Approach and Traditional decompositions of profit inefficiency based on the ERG measure.

	Economic inefficiency (GDA)			Economic inefficiency (TRA)		
Bank	Profit Ineff. $\Pi I(x_j, y_j, w, p)$	Tech. Profit Ineff. $T\Pi I_{ERG}^{GDA}(x_j, y_j, w, p)$	Alloc. Prof. Ineff. $A\Pi I_{ERG}^{GDA}(x_j, y_j, w, p)$	Profit Ineff. $\Pi I(x_j, y_j, w, p)$	Tech. Prof. Ineff. $T\Pi I_{ERG}^{TRA}(x_j, y_j, w, p)$	Allocative Ineff. $A\Pi I_{ERG}^{TRA}(x_j, y_j, w, p)$
Export-Import Bank	19,908.3	0.0	19,908.3	19,908.3	0.0	19,908.3
Bank of Taiwan	0.0	0.0	0.0	0.0	0.0	0.0
Taipei Fubon Bank	8922.1	0.0	8922.1	8922.1	0.0	8922.1
Bank of Kaohsiung	20,216.3	4149.9	16,066.5	20,216.3	4124.1	16,092.2
Land Bank	3396.7	0.0	3396.7	3396.7	0.0	3396.7
Cooperative Bank	1603.6	0.0	1603.6	1603.6	0.0	1603.6
First Bank	3202.5	0.0	3202.5	3202.5	0.0	3202.5
Hua Nan Bank	7981.6	5248.7	2732.9	7981.6	4919.3	3062.3
Chang Hwa Bank	13,550.2	8864.5	4685.7	13,550.2	7367.4	6182.9
Mega Bank	1791.2	0.0	1791.2	1791.2	0.0	1791.2
Cathay United Bank	20,955.5	15,021.2	5934.3	20,955.5	10,056.4	10,899.2
The Shanghai Bank	14,638.4	2773.3	11,865.1	14,638.4	2254.1	12,384.4
Union Bank	24,839.3	11,277.2	13,562.1	24,839.3	8262.0	16,577.3
Far Eastern Bank	12,052.0	0.0	12,052.0	12,052.0	0.0	12,052.0
E. Sun Bank	12,220.6	3361.8	8858.8	12,220.6	2933.9	9286.6
Cosmos Bank	24,409.6	7164.9	17,244.7	24,409.6	4008.8	20,400.8
Taishin Bank	20,760.0	11,521.3	9238.8	20,760.0	9455.7	11,304.4
Ta Chong Bank	18,916.4	4897.4	14,018.9	18,916.4	1768.1	17,148.2
Jih Sun Bank	21,933.0	6472.0	15,461.0	21,933.0	5724.1	16,209.0
Entie Bank	20,925.3	5875.9	15,049.4	20,925.3	4835.9	16,089.4
China Trust Bank	14,248.8	0.0	14,248.8	14,248.8	0.0	14,248.8
Sunny Bank	23,815.2	8567.9	15,247.3	23,815.2	5915.1	17,900.1
Bank of Panhsin	23,693.5	7535.6	16,157.9	23,693.5	5265.8	18,427.7
Taiwan Business Bank	13,339.1	4494.1	8845.0	13,339.1	3708.7	9630.4
Taichung Bank	20,773.8	6959.4	13,814.4	20,773.8	6856.9	13,916.9
China Development	17,244.7	0.0	17,244.7	17,244.7	0.0	17,244.7
Hwatai Bank	21,469.5	4739.0	16,730.6	21,469.5	4520.8	16,948.7
Cota Bank	21,642.4	3945.5	17,696.9	21,642.4	3183.0	18,459.3
Industrial Bank of Taiwan	20,302.6	0.0	20,302.6	20,302.6	0.0	20,302.6
Bank SinoPac	11,687.0	3960.5	7726.4	11,687.0	2044.6	9642.4
Shin Kong Bank	22,591.6	10,790.1	11,801.5	22,591.6	9268.2	13,323.4
Average	15,581.6	4439.4	11,142.3	15,581.6	3434.6	12,147.0
Median	18,916.4	4149.9	12,052.0	18,916.4	3183.0	13,323.4
Maximum	24,839.3	15,021.2	20,302.6	24,839.3	10,056.4	20,400.8
Minimum	0.0	0.0	0.0	0.0	0.0	0.0
Std. Dev.	7590.8	4233.4	5882.6	7590.8	3275.0	6141.8

parate incentives structures away from financial performance (public banks are indicated in italics in Tables 2 and 3). This result is in accordance with the findings of Juo et al. [24], who further discuss the good performance of these public banks in terms of key financial indicators such as 'non-performing loan (NPL) ratio', 'net income before taxes to asset (NIBT/A) ratio', and 'NBIT/employee (NIBT/E) ratio'. The last two ratios are highly related to the outputs and inputs used in the study. We also refer the reader to their article to gain insights about the individual course of some of these banks that may had favored their good performance, e.g., mergers and acquisitions favoring the realization of economics of scale and scope.

Regarding the worst five performing banks, we find Union Bank (#13), Cosmos Bank (#16), Sunny Bank (#22), Bank of Panhsin (#23) and Shin Kong Bank (#33). As opposed to their best performing counterparts, these banks exhibit both technical and allocative inefficiency. Besides being privately owned and enduring a small production scale, most of these banks were either newly established or transformed from credit cooperatives into commercial banks. In this regard the adjustments required to start the enterprise or complete the necessary transition to a commercial setting seem to have hampered their financial performance. This is also confirmed from the financial perspective of the above key performance indicators which were well above the industry average for 'non-performing loans' and below for the 'net income before taxes ratio', either to assets or employees.

The monetary decompositions of profit inefficiency do not comply with the commensurability property. Relying on the normalized versions of the GDA and traditional profit inefficiency we ensure that our results are units' invariant. Normalized profit inefficiency along with its technical and allocative inefficiency terms are reported in columns 2 thru 4 in Table 3. As for the traditional decomposition, we calculate for each firm its specific ERG normalizing factor, see (13), and report the corresponding profit, technical and allocative inefficiency values in columns 5 thru 8. We observe that technical inefficiency is common to both approaches as they share the same measure, while the values of normalized profit and allocative inefficiencies differ for technically inefficient firms due to different normalization factors: $NF_{ERG}^{TRA}(x_j, y_j, w, p) < NF_{ERG}^{GDA}(x_j, y_j, w, p)$. Regarding the normalized profit inefficiency values, these are shown in columns 2 and 5. Consistent with the monetary decomposition, the second bank maximized profit efficiency, so its generalized and traditional profit inefficiencies are null. Again, a total of eleven banks were technically efficient. For the technically inefficient firms, and given the direction of the inequality in the respective normalization factors above, we observe that normalized profit inefficiency is greater in the traditional approach than in the generalized approach, with the amount in excess being attributed to allocative inefficiency. Profit and allocative inefficiency are overvalued for all 20 inefficient banks, representing just 8.4% of average profit inefficiency in the GDA approach (=0.435/5.605 × 100). However, in some cases, the disparity in the

Table 3
Normalized General Direct Approach and Traditional decompositions of profit inefficiency based on the ERG measure.

Bank	Economic inefficiency (GDA)			Economic inefficiency (TRA)		
	Profit Ineff. $N\Pi I_{ERG}^{GDA}(x_j, y_j, \tilde{w}, \tilde{p})$	Technical Ineff. $TI_{ERG}(x_j, y_j)$	Allocative Ineff. $AI_{ERG}^{GDA}(x_j, y_j, \tilde{w}, \tilde{p})$	Profit Ineff. $N\Pi I_{ERG}^{TRA}(x_o, y_o, \tilde{w}, \tilde{p})$	Technical Ineff. $TI_{ERG}(x_j, y_j)$	Allocative Ineff. $AI_{ERG}^{TRA}(x_j, y_j, \tilde{w}, \tilde{p})$
Export-Import Bank	91.207	0.000	91.207	91.207	0.000	91.207
Bank of Taiwan	0.000	0.000	0.000	0.000	0.000	0.000
Taipei Fubon Bank	0.776	0.000	0.776	0.776	0.000	0.776
Bank of Kaohsiung	3.814	0.783	3.031	3.838	0.783	3.055
Land Bank	0.214	0.000	0.214	0.214	0.000	0.214
Cooperative Bank	0.050	0.000	0.050	0.050	0.000	0.050
First Bank	0.147	0.000	0.147	0.147	0.000	0.147
Hua Nan Bank	0.316	0.208	0.108	0.337	0.208	0.129
Chang Hwa Bank	0.556	0.364	0.192	0.669	0.364	0.305
Mega Bank	0.139	0.000	0.139	0.139	0.000	0.139
Cathay United Bank	0.827	0.593	0.234	1.236	0.593	0.643
The Shanghai Bank	1.359	0.257	1.101	1.672	0.257	1.414
Union Bank	2.056	0.934	1.123	2.807	0.934	1.873
Far Eastern Bank	4.437	0.000	4.437	4.437	0.000	4.437
E. Sun Bank	0.790	0.217	0.573	0.905	0.217	0.688
Cosmos Bank	3.366	0.988	2.378	6.016	0.988	5.028
Taishin Bank	0.959	0.532	0.427	1.168	0.532	0.636
Ta Chong Bank	1.607	0.416	1.191	4.450	0.416	4.034
Jih Sun Bank	2.754	0.813	1.941	3.113	0.813	2.301
Entie Bank	2.619	0.735	1.883	3.182	0.735	2.446
China Trust Bank	0.556	0.000	0.556	0.556	0.000	0.556
Sunny Bank	2.665	0.959	1.706	3.860	0.959	2.902
Bank of Panhsin	3.086	0.981	2.104	4.416	0.981	3.435
Taiwan Business Bank	0.829	0.279	0.549	1.004	0.279	0.725
Taichung Bank	2.617	0.877	1.740	2.656	0.877	1.779
China Development	14.870	0.000	14.870	14.870	0.000	14.870
Hwatai Bank	4.343	0.959	3.385	4.553	0.959	3.594
Cota Bank	5.169	0.942	4.227	6.407	0.942	5.465
Industrial Bank of Taiwan	19.067	0.000	19.067	19.067	0.000	19.067
Bank SinoPac	0.669	0.227	0.442	1.296	0.227	1.069
Shin Kong Bank	1.898	0.906	0.991	2.210	0.906	1.303
Average	5.605	0.418	5.187	6.041	0.418	5.622
Median	1.607	0.279	1.101	2.210	0.279	1.414
Maximum	91.207	0.988	91.207	91.207	0.988	91.207
Minimum	0.000	0.000	0.000	0.000	0.000	0.000
Std. Dev.	16.407	0.398	16.490	16.339	0.398	16.414

allocative inefficiency can reach three orders of magnitude, as for bank #18, whose profit or allocative inefficiency in excess, 2.843 (4.034−1.191) represents 238.8% of the allocative inefficiency under the GDA (2.843/1.191 × 100), and 177.0% of overall profit inefficiency (2.843/1.607 × 100).

These discrepancies confirm the concerns raised about the characterization of profit inefficiency and its decomposition using the traditional approach. We further study the (dis)similarity between the normalized profit (or allocative) inefficiencies under the traditional and general approaches. We look first at their *ranking compatibility* by means of Kendall's tau-b correlation (handling ties) and second, relying on kernel density estimations, determine whether the two profit inefficiency *distributions are equal or not* according to the test proposed by Simar and Zelenyuk [26]—which is an extension of the nonparametric test for the equality of two densities of Li [27]. Their Kendall correlation is 0.891, which is significant at the 5% level. Although this value suggests that the ranking compatibility is relatively high, the value of the Li test also confirms that there is a significant statistical difference between the two models at the same significance level.

4. Conclusions

We introduce a new method to decompose profit inefficiency, identified as the General Direct Approach (GDA) that, based on equalities, identifies a technical profit inefficiency term and an allocative profit inefficiency term. The advantage of the new approach is that its decomposition in monetary terms with respect to the technical profit inefficiency does not depend on the normalization factor as the traditional approach does. Indeed, as we prove theoretically and illustrate in the empirical application—considering the enhanced Russell graph measure as underlying technical efficiency, when both approaches differ for technical inefficient firms, the traditional approach undervalues (overvalues) the value of profit technical (allocative) inefficiency. Last sentence can be reformulated in terms of the normalizing factors, which may be underestimated by the traditional approach for technically inefficient firms, giving rise to different general and traditional decompositions. Moreover, the GDA satisfies by construction the essential property which requires that the allocative inefficiency of the firm under evaluation is null when projected to a profit maximizing benchmark—a property that traditional decompositions based on relevant technical inefficiency measures fail to satisfy [8]. As a result, we advocate for the use of the general direct approach to prevent the discrepancies detected in the traditional approach and perform reliable comparisons of profit inefficiency and its sources across firms. Besides, it can be adapted for any (in)efficiency measure and eases the calculation and decomposition of profit inefficiency in light of already existing or newly proposed measures, whose duality results, necessary to obtain the Fenchel-Mahler inequalities and their associated normalizations factors, are more complex and need to be developed explicitly, whereas our approach, although requiring duality criteria implicitly, offers an easier approach to obtain the normalization factor. Duality refers to the methods allowing to relate price measures of economic efficiency with quantity measures of technical efficiency, and therefore the GDA approach falls within this theoretical framework. We have also established that both approaches are equivalent when the directional distance function is used as measure of technical inefficiency. Therefore, despite this measure projecting firms to the weakly efficient subset of the technology, it is superior to other alternative measures when resorting to the traditional approach to decompose profit inefficiency. From a managerial perspective, the unnormalized version of the general direct approach would be recommended since both the technical and allocative terms are expressed in monetary units and therefore easier to interpret. Nevertheless, being dependent on the units of measurement, the normalized version is also available for economic efficiency studies involving, for example, firms belonging to countries with different currencies.

We finish with a relevant remark and further venues of research. First, we focus on the additive decomposition of profit inefficiency defined in difference form, i.e., as maximum profit less observed profit. Therefore, we leave out possible decomposition based on multiplicative measures such as the generalized distance function proposed by Chavas and Cox [28] or the geometric distance function by Portela and Thanassoulis [11], which fall outside the objectives of this paper, since they cannot be related by duality to the profit function. However, a similar study analyzing the functioning of multiplicative measures should be welcome. Also, one interesting extension of our new model would be to consider the case of different prices across firms. In the literature reviewed in the introduction, as well as in our proposed general direct approach, it is customarily assumed that firms face common input and output prices that are exogenously determined in perfectly competitive markets, i.e., firms are price takers. However, under imperfectly competitive markets firms may exhibit different prices. In this case one can think of price changes as yet another strategy available to the firms to improve economic efficiency. In a cost efficiency framework Camanho and Dyson [29] suggest different models allowing firms to reduce their observed input prices by mirroring those of their peers, while Portela and Thanassoulis [30] propose a more general model that simultaneously reduce input quantities (resulting in technical savings) and input prices (price savings). Very recently, Boussemart et al. [31] introduced a new model, assuming different prices across firms, decomposing first cost efficiency as the product of a price effect and a quantity effect, and ending with a similar decomposition for profitability efficiency (which is defined as revenue to cost). From the perspective of either the general direct approach or the traditional approach the existence of different prices represents yet another source of variation resulting in firm-specific normalization factors, hampering the comparability of profit inefficiency and its components between firms.

Other possible extensions could be the application of our approach and decomposition of profit inefficiency in the context of Network DEA ([32], and [33]).

Declaration of Competing Interest

The authors declare no conflict of interest.

CRediT authorship contribution statement

Jesus T. Pastor: Conceptualization, Methodology. **José Luis Zofío:** Methodology, Writing – original draft, Writing – review & editing. **Juan Aparicio:** Methodology. **D. Pastor:** Software, Validation.

Data availability

Data will be made available on request.

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Appendix 1. Proof of Proposition 1. We only prove the relationships between technical profit inefficiencies since the association between allocative profit inefficiencies can be trivially derived from them

1.1 $T\Pi I_{ERG}^{TRA}(x_j,y_j,w,p) = TI_{ERG}(x_j,y_j)NF_{ERG}^{TRA}(x_j,y_j,p,w)$ with $TI_{ERG}(x_j,y_j) = \frac{1}{M}\sum_{m=1}^{M}\frac{\hat{s}_{jm}^{-ERG}}{x_{jm}} + \frac{1}{N}\sum_{n=1}^{N}\frac{\hat{s}_{jn}^{+ERG}}{y_{jn}}$ and $NF_{ERG}^{TRA}(x_j,y_j,p,w) = \min\{Mw_1x_{j1},\dots,Mw_Mx_{jM},\ Np_1y_{j1},\dots,Np_Ny_{jN}\}$. Then, $NF_{ERG}^{TRA}(x_j,y_j,p,w) \le Mw_1x_{jm}$, $\forall m$, and $NF_{ERG}^{TRA}(x_j,y_j,p,w) \le Np_1y_{j1}$, $\forall n$. This implies that $T\Pi I_{ERG}^{TRA}(x_j,y_j,w,p) = \frac{1}{M}\sum_{m=1}^{M}\frac{NF_{ERG}^{TRA}(x_j,y_j,p,w)\hat{s}_{jm}^{-ERG}}{x_{jm}} + \frac{1}{N}\sum_{n=1}^{N}\frac{NF_{ERG}^{TRA}(x_j,y_j,p,w)\hat{s}_{jn}^{+ERG}}{y_{jn}} \le \frac{1}{M}\sum_{m=1}^{M}\frac{(Mw_mx_{jm})\hat{s}_{jm}^{-ERG}}{x_{jm}} + \sum_{n=1}^{N}\frac{(Np_ny_{jn})\hat{s}_{jn}^{+ERG}}{y_{jn}} = \sum_{m=1}^{M}w_m\hat{s}_{jm}^{-ERG} + \sum_{n=1}^{N}p_n\hat{s}_{jn}^{+ERG} = T\Pi I_{ERG}^{GDA}(x_j,y_j,w,p)$.

1.2 If (x_j,y_j) is technically efficient under the ERG, then $\hat{s}_{jm}^{-ERG} = \hat{s}_{jn}^{+ERG} = 0, \forall m,n$ and $TI_{ERG}(x_j,y_j) = 0$. This implies that $T\Pi I_{ERG}^{GDA}(x_j,y_j,w,p) = 0$ and $T\Pi I_{ERG}^{TRA}(x_j,y_j,w,p) = 0$.

1.3 From the proof of 1.1, for a technically inefficient firm (x_j,y_j), i.e., a firm such that $\exists m$ such that $\hat{s}_{jm}^{-ERG} > 0$ and/or $\exists n$ such that $\hat{s}_{jn}^{+ERG} > 0$, the only way of having the equality $T\Pi I_{ERG}^{TRA}(x_j,y_j,w,p) = T\Pi I_{ERG}^{GDA}(x_j,y_j,w,p)$ is that $Mw_1x_{j1} = \dots = Mw_Mx_{jM} = Np_1y_{j1} = \dots = Np_Ny_{jN}.= \min\{Mw_1x_{j1},\dots,Mw_Mx_{jM},Np_1y_{j1},\dots,Np_Ny_{jN}\}$

Appendix 2. Proof of Proposition 2

Regarding the first result, by Chambers et al. [1], $T\Pi I_{DDF}^{TRA}(x_j,y_j,w,p) = TI_{DDF}(x_j,y_j;g_j^-,g_j^+)(w\cdot g_j^- + p\cdot g_j^+)$. In the case of the DDF, we have the relationship $\hat{s}_j^{-DDF} = TI_{DDF}(x_j,y_j;g_j^-,g_j^+)g_j^-$ and $\hat{s}_j^{+DDF} = TI_{DDF}(x_j,y_j;g_j^-,g_j^+)g_j^+$. Therefore, $T\Pi I_{DDF}^{TRA}(x_j,y_j,w,p) = w\cdot s_j^{-DDF} + p\cdot s_j^{+DDF}$, which is, by definition, equal to $T\Pi I_{DDF}^{GDA}(x_j,y_j,w,p)$. As for the second result, let G be any graph measure satisfying $T\Pi I_G^{TRA}(x_j,y_j,w,p) = T\Pi I_G^{GDA}(x_j,y_j,w,p)$. Given that, by definition, $T\Pi I_G^{GDA}(x_j,y_j,w,p) = w\cdot\hat{s}_j^{-G} + p\cdot\hat{s}_j^{+G}$ and $T\Pi I_G^{TRA}(x_j,y_j,w,p) = TI_G(x_j,y_j)NF_G^{TRA}(x_j,y_j,w,p)$, we conclude that $NF_G^{TRA}(x_j,y_j,w,p) = \frac{T\Pi I_G^{GDA}(x_j,y_j,w,p)}{TI_G(x_j,y_j)} = w\cdot\frac{\hat{s}_j^{-G}}{TI_G(x_j,y_j)} + p\cdot\frac{\hat{s}_j^{+G}}{TI_G(x_j,y_j)}$, which corresponds to the normalization factor of a *DDF* whose directional vector is $(g_j^-,g_j^+) = \left(\frac{\hat{s}_j^{-G}}{TI_G(x_j,y_j)},\frac{\hat{s}_j^{+G}}{TI_G(x_j,y_j)}\right)$, i.e., $TI_G(x_j,y_j) = TI_{DDF}(x_j,y_j;g_j^-,g_j^+)$.

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