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Lipschitz Modulus of the Optimal Value in Linear Programming

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Abstract

The present paper is devoted to the computation of the Lipschitz modulus of the optimal value function restricted to its domain in linear programming under different types of perturbations. In the first stage, we study separately perturbations of the right-hand side of the constraints and perturbations of the coefficients of the objective function. Secondly, we deal with canonical perturbations, i.e., right-hand side perturbations together with linear perturbations of the objective. We advance that an exact formula for the Lipschitz modulus in the context of right-hand side perturbations is provided, and lower and upper estimates for the corresponding moduli are also established in the other two perturbation frameworks. In both cases, the corresponding upper estimates are shown to provide the exact moduli when the nominal (original) optimal set is bounded. A key strategy here consists in taking advantage of the background on calmness in linear programming and providing the aimed Lipschitz modulus through the computation of a uniform calmness constant.

Keywords Lipschitz modulus · Optimal value · Linear programming · Variational analysis · Calmness

Mathematics Subject Classification 90C31 · 49J53 · 49K40 · 90C05

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1 Introduction

This paper deals with the Lipschitz continuity of the optimal value in linear programming (LP for short). Specifically, we consider the optimal value function restricted to its domain (where the value is finite), denoted by ϑ^R, and analyze its behavior around a fixed (referred to as nominal) LP problem. Along this work different type of perturbations of the nominal problem are considered and, in each of these perturbation frameworks, our goal is to compute (or at least estimate) the *Lipschitz modulus* of the corresponding optimal value (see Sects. 2.1 and 2.2 for the formal definitions). Roughly speaking, this Lipschitz modulus provides a local measure of the greatest rate of variation of the optimal value with respect to data perturbations. In this sense, the present research is focussed on a local aspect of the *sensitivity analysis* in LP, in contrast to the classical theory of parametric linear optimization (see, e.g., [1] and [2]).

First, we consider the case of right-hand-side (RHS in brief) perturbations of the constraints, where a formula for the exact Lipschitz modulus of ϑ^R at a nominal problem is obtained. Secondly, we deal with linear perturbations of the objective function (c-perturbations, for simplicity). After that, we tackle the problem of computing the Lipschitz modulus of ϑ^R in the setting of the so-called *canonical perturbations*, i.e., RHS perturbations together with c-perturbations. In the last two settings, lower and upper estimates for the aimed moduli are derived. In both cases the upper estimates turn out to be the exact moduli when the nominal optimal set is bounded.

The systematic study of stability in LP with canonical perturbations started in the 1970s. Specifically, the continuity of ϑ^R was proved through different approaches (see [3–6]). One can find a second line of research based on variational analysis like Berge's theory or Hoffman's error bounds; see [4,6–12].

The immediate antecedents of this work can be traced out from [13] and [14]. The first one, instead of ϑ^R, deals with the optimal value function, ϑ, defined on the whole space (and, so, taking values in the extended real line). As a counterpart, the local study is made around a problem which is in the interior of the domain of ϑ. This interiority condition characterizes the Lipschitz continuity of ϑ at such a problem (this fact is held in the more general setting of linear semi-infinite optimization; see [15, Lemma 10.2]) and it is equivalent to the well-known Slater constraint qualification together with the boundedness (and nonemptiness) of the nominal optimal set. Specifically, [13, Theorem 4.3] provides a formula for a particular Lipschitz constant for ϑ in terms of the so-called *distance to ill-posedness*. (See also the pioneer works [16] and [17], developed in the context of conic linear problems.) Let us point out that the new results of the current paper constitute an improvement of [13, Theorem 4.3] in different directions: first, here we do not require any interiority assumption; moreover, the Lipschitz modulus provides—roughly speaking—the more accurate Lipschitz constant; and, finally, we also tackle the case of partial perturbations (RHS or c-perturbations).

Paper [14] is focussed on the calmness of ϑ^R, which is known to be a weaker property than Lipschitz continuity. In that paper, the calmness of ϑ^R is approached through the *calmness from above* and *calmness from below*, which roughly speaking, measure the local rate of increase and decrease, respectively, with respect to the nominal problem. While calmness property compares the nominal optimal value with the optimal value of a perturbed problem, Lipschitz property involves the optimal values of two different perturbed problems around the nominal one. This fact entails notable differences between both properties and their moduli. In particular, the new contributions of the current paper are not direct consequences of the ones of [14], as we shall emphasize in the corresponding proofs. In any case, we take advantage of the background on calmness. In particular, a key strategy (inspired by [18, Section 2]) based on computing the aimed Lipschitz modulus through a uniform calmness constant is appealed to.

Finally, let us comment that both calmness and Lipschitz properties have extensions for multifunctions, closely related to metric regularity notions, which are important concepts in the field of variational analysis; see the monographs [19–22] for additional references and details. The analysis of pseudo-Lipschitz (Aubin) property for the particular case of the *argmin mapping* (resp. the *feasible set mapping*) has been addressed in [23,24] (resp. [25]).

The structure of the paper is as follows. Section 2 introduces the model we are dealing with, the main goals of this work, as well as the necessary notation and preliminary results on calmness (from [14]) which are used later on. Section 3 is devoted to the study of the Lipschitz modulus of ϑ^R under RHS perturbations. The main result of this section is Theorem 3.1. Section 4 is developed in the context of c-perturbations, and mainly consists of Theorem 4.1, where the announced lower and upper estimates (exact value when the nominal optimal set is nonempty and bounded) for the aimed modulus are provided. Section 5 deals with canonical perturbations. Theorem 5.1 provides a lower estimate of the corresponding Lipschitz modulus, while Theorem 5.2 provides an upper estimate based on a certain uniform calmness constant which is established in Lemma 5.1. The last theorem also provides the exact Lipschitz modulus under the boundedness (and nonemptiness) of the nominal optimal set. Finally, Sect. 6 gathers some conclusions.

2 Preliminaries and Main Goals

This section is devoted to formalize the main goal of the paper and to connect it with the immediate antecedents. The section is divided into three subsections: First, we introduce the parameterized optimization model and the mappings which are dealt with in the paper; secondly, we make precise the main goal of this work, which consists in computing (or estimating) the Lipschitz modulus of the optimal value function under different type of perturbations. The third subsection gathers some results about calmness of the same function traced out from [14].

2.1 The Parameterized Model

We consider a parameterized linear optimization problem, in $\mathbb{R}^n$, given in the form

$$\begin{array}{rll} \pi : & \text{minimize} & c'x \\ & \text{subject to} & \overline{a}_t'x \leq b_t,\ t \in T := \{1, 2, \ldots, m\}, \end{array} \tag{1}$$

where $x \in \mathbb{R}^n$ is the vector of decision variables, $\overline{a} \equiv (\overline{a}_t)_{t\in T} \in (\mathbb{R}^n)^T$ is fixed, $c \in \mathbb{R}^n$ and $b \equiv (b_t)_{t\in T} \in \mathbb{R}^T$. Any $z \in \mathbb{R}^n$ is considered as a column vector and z' denotes its transpose. Our problem π is identified with the pair $(c, b) \in \mathbb{R}^n \times \mathbb{R}^T$, which constitutes our parameter to be perturbed. So, as mentioned above, we are working in the setting of the so-called *canonical perturbations*.

The space of variables, $\mathbb{R}^n$, is endowed with an arbitrary norm, $\|\cdot\|$, while the parameter space $\mathbb{R}^n \times \mathbb{R}^T$ is endowed with the norm

$$\|\pi\| := \max\{\|c\|_*, \|b\|_\infty\}, \ \pi \equiv (c, b) \in \mathbb{R}^n \times \mathbb{R}^T,$$

where $\|u\|_* := \max_{\|x\|\leq 1} \left|u'x\right|$, and $\|b\|_\infty := \max_{t\in T} |b_t|$. Observe that, in relation to vector c of the objective function, we use the dual norm since it is seen as a functional.

Along the paper we deal with the following mappings: The *feasible set mapping*, $\mathcal{F} : \mathbb{R}^T \rightrightarrows \mathbb{R}^n$, defined as

$$\mathcal{F}(b) := \{x \in \mathbb{R}^n : \overline{a}_t' x \leq b_t, \ t \in T\}, \ b \in \mathbb{R}^T;$$

the *optimal value function*, $\vartheta : \mathbb{R}^n \times \mathbb{R}^T \to [-\infty, +\infty]$, given by

$$\vartheta(\pi) := \inf\{c'x : x \in \mathcal{F}(b)\},$$

(with the convention $\vartheta(\pi) := +\infty$ when $\mathcal{F}(b) = \emptyset$); and the *optimal set mapping*, $\mathcal{F}^{\text{op}} : \mathbb{R}^n \times \mathbb{R}^T \rightrightarrows \mathbb{R}^n$, which assigns to each problem $\pi \equiv (c, b)$ its optimal set

$$\mathcal{F}^{\text{op}}(\pi) := \{x \in \mathcal{F}(b) : c'x = \vartheta(\pi)\}.$$

The domain of $\mathcal{F}$, denoted by dom$\mathcal{F}$, is formed by all $b \in \mathbb{R}^T$, whose associated linear inequality systems are consistent; formally,

$$\text{dom}\mathcal{F} := \left\{b \in \mathbb{R}^T : \mathcal{F}(b) \neq \emptyset\right\}.$$

Analogously, the domain of $\mathcal{F}^{\text{op}}$, dom$\mathcal{F}^{\text{op}}$, is formed by all problems $\pi \equiv (c, b) \in \mathbb{R}^n \times \mathbb{R}^T$ having a nonempty optimal set. It is known from standard arguments in LP that dom$\mathcal{F}^{\text{op}}$ coincides with the domain of ϑ. It is also known that both dom$\mathcal{F} \subset \mathbb{R}^T$ and dom$\mathcal{F}^{\text{op}} \subset \mathbb{R}^n \times \mathbb{R}^T$ are closed and convex sets.

This paper mainly deals with the *optimal value function restricted to its domain*, $\vartheta^R : \text{dom}\mathcal{F}^{\text{op}} \to]-\infty, +\infty[$, i.e.,

$$\vartheta^R := \vartheta|_{\text{dom}\mathcal{F}^{\text{op}}},$$

and two other functions coming from considering perturbations of b and c independently. Specifically, given a nominal (fixed) $\overline{\pi} \equiv \left(\overline{c}, \overline{b}\right) \in \text{dom}\mathcal{F}^{\text{op}}$ we define

$$\vartheta^R_{\overline{c}} : \text{dom}\mathcal{F} \to]-\infty, +\infty[\text{ and } \vartheta^R_{\overline{b}} : C \to]-\infty, +\infty[,$$

with

$$C = -\text{cone}\{\overline{a}_t, t \in T\}, \tag{2}$$

(where 'cone' means conical convex hull) given, respectively, by

$$\vartheta^R_{\overline{c}}(b) = \vartheta^R(\overline{c}, b) \text{ and } \vartheta^R_{\overline{b}}(c) = \vartheta^R(c, \overline{b}).$$

Observe that the previous two functions are finite valued, since we are not perturbing $\overline{a}$, which entails that $\{\overline{c}\} \times \text{dom}\mathcal{F}$ and $C \times \{\overline{b}\}$ are both included in $\text{dom}\mathcal{F}^{\text{op}}$ (recall that, in LP, optimality is equivalent to primal and dual feasibility).

One can find different proofs (from different approaches) for the next theorem; see, e.g., [4, Theorem 4.5.2], [5, Theorem 2.7] and [6, Theorem 14]; see also [11, p. 214] for a stronger version (ϑ^R is Lipschitz on bounded subsets of $\text{dom}\mathcal{F}^{\text{op}}$) in the more general context of canonically perturbed convex quadratic problems; see also [6, p. 25] and [9] for (generally non-convex) quadratic programs.

Theorem 2.1 *ϑ^R is continuous on* $\text{dom}\mathcal{F}^{\text{op}}$.

Finally, the following theorem is a well-known result of stability theory in LP (see, e.g., [26, p. 232] or [8, Chapter IX (Section 7)]). In it, we appeal to the Painlevé-Kuratowski convergence of sequences of sets. More in detail, given $X_r \subset \mathbb{R}^n$, $r \in \mathbb{N}$, $\text{Lim inf}_r X_r$ consists of all points which may be written as $\lim_r x^r$ with $x^r \in X_r$ for r large enough; whereas elements of $\text{Lim sup}_r X_r$ are those of the form $\lim_k x^k$ with $x^k \in X_{r_k}$ for some subsequence $r_1 < r_2 < ...$ Obviously, $\text{Lim inf}_r X_r \subset \text{Lim sup}_r X_r$, and when both coincide, we just write $\text{Lim}_r X_r$.

Theorem 2.2 *Let $\overline{c} \in C$. For any $\{b^r\}_{r\in\mathbb{N}} \subset \text{dom}\mathcal{F}$ converging to $\overline{b}$, we have*

$$\mathcal{F}^{\text{op}}(\overline{\pi}) = \text{Lim}_r \mathcal{F}^{\text{op}}\left(\overline{c}, b^r\right).$$

Remark 2.1 In general, the boundedness of a Painlevé–Kuratowski limit of sets does not imply the boundedness of those sets. For instance, $\text{Lim}_r\{1\} \cup [r, +\infty[= \{1\}$. Nevertheless, in the previous theorem the boundedness of $\mathcal{F}^{\text{op}}(\overline{\pi})$ does imply, indeed, the uniform boundedness of $\{\mathcal{F}^{\text{op}}(\overline{c}, b^r)\}_{r\in\mathbb{N}}$. This follows from the convexity of each $\mathcal{F}^{\text{op}}(\overline{c}, b^r)$ or, alternatively, from [15, Corollary 6.2.1] together with Theorem 2.1.

2.2 Main Goals

This subsection is devoted to formalize the main goals of the current work and to integrate them in the existing literature. At this moment, we advance that our aim is focussed on the Lipschitzian behavior of the optimal value function in different frameworks of perturbations; specifically, on the Lipschitzian behavior of ϑ, ϑ^R, $\vartheta^R_{\overline{c}}$, and $\vartheta^R_{\overline{b}}$.

Recall that a function $f : A \subset \mathbb{R}^p \to [-\infty, +\infty]$, $p \in \mathbb{N}$, is said to be *Lipschitz continuous* at $\overline{z} \in A$, with $f(\overline{z})$ finite, if there exist a constant $\kappa \geq 0$ (called *Lipschitz constant*) and a neighborhood U of $\overline{z}$ such that

$$|f(z) - f(\widetilde{z})| \leq \kappa \|z - \widetilde{z}\|, \text{ for all } z, \widetilde{z} \in U \cap A. \tag{3}$$

The infimum of constants κ for which (3) holds, for some associated neighborhood, is the *Lipschitz modulus* of f at $\overline{z}$, denoted by $\operatorname{lip} f(\overline{z})$. Observe that the Lipschitz modulus can be expressed as

$$\operatorname{lip} f(\overline{z}) = \limsup_{\substack{z,\widetilde{z}\to\overline{z} \\ z,\widetilde{z}\in A}} \frac{|f(z) - f(\widetilde{z})|}{\|z - \widetilde{z}\|}. \tag{4}$$

(In the previous expressions, we do not exclude coincidences among z, $\widetilde{z}$, and $\overline{z}$, under the convention $\frac{0}{0} := 0$ and $\infty - \infty := 0$.)

In relation to the optimal value function, it is well known that ϑ is Lipschitz continuous at $\overline{\pi} \equiv \left(\overline{c}, \overline{b}\right)$ if and only if $\overline{\pi} \in \operatorname{int} \operatorname{dom}\mathcal{F}^{\mathrm{op}}$ (the interior of $\operatorname{dom}\mathcal{F}^{\mathrm{op}}$). In fact, as commented above, this characterization is held in the more general framework of linear semi-infinite problems (with—possibly—infinitely many constraints); see, [15, Lemma 10.2]. Moreover, it is also known (see, e.g., [15, Theorem 6.1 and Lemma 10.2]) that condition '$\overline{\pi} \in \operatorname{int} \operatorname{dom}\mathcal{F}^{\mathrm{op}}$' is equivalent to the simultaneous fulfillment of two conditions: $\mathcal{F}^{\mathrm{op}}(\overline{\pi})$ is nonempty and bounded, and the *Slater constraint qualification* (SCQ, in brief) is satisfied at $\overline{b}$. Recall that SCQ is satisfied at $\overline{b}$ if there exists $\widehat{x} \in \mathbb{R}^n$, called a *Slater point* , such that $\overline{a}_t'\widehat{x} < \overline{b}_t$ for all $t \in T$.

Remark 2.2 Observe that, in the case when $\overline{\pi} \in \operatorname{int} \operatorname{dom}\mathcal{F}^{\mathrm{op}}$ one clearly has $\operatorname{lip}\vartheta(\overline{\pi}) = \operatorname{lip}\vartheta^R(\overline{\pi})$. On the other hand, if $\overline{\pi} \in \operatorname{bd} \operatorname{dom}\mathcal{F}^{\mathrm{op}}$ (bd standing for boundary), one has $\operatorname{lip}\vartheta(\overline{\pi}) = +\infty$, whereas $\operatorname{lip}\vartheta^R(\overline{\pi})$ is still finite, as it is shown in the current work (as a consequence of Theorem 5.2).

The previous remark motivates that we focus this paper on computing (or at least estimating) $\operatorname{lip}\vartheta^R(\overline{\pi})$. For solvable problems, $\operatorname{lip}\vartheta^R(\overline{\pi})$ is always finite and provides a quantitative measure of the stability of the optimal value under data perturbations (provided that they yield solvable problems).

We advance that $\operatorname{lip}\vartheta^R_{\overline{c}}(\overline{b})$ is completely determined through a point-based formula (depending only on the nominal data) without any assumption (see Theorem 3.1), while $\operatorname{lip}\vartheta^R_{\overline{b}}(\overline{c})$ and $\operatorname{lip}\vartheta^R(\overline{\pi})$ are upper and lower estimated in general (see Theorems 4.1, 5.1, and 5.2). It is also shown that under the boundedness of $\mathcal{F}^{\mathrm{op}}(\overline{\pi})$, both $\operatorname{lip}\vartheta^R_{\overline{b}}(\overline{c})$ and $\operatorname{lip}\vartheta^R(\overline{\pi})$ are also completely determined. All the mentioned estimates (or exact values) are given exclusively in terms of $\overline{\pi} \equiv \left(\overline{c}, \overline{b}\right)$.

2.3 Antecedents on Calmness

This subsection mainly gathers some results about the calmness of ϑ^R, traced out from [14], which are used in the remaining sections.

Recall that the calmness property is weaker than the Lipschitz one, as far as it comes from fixing $\widetilde{z} = \overline{z}$ in (3). With the notation before (3), the *calmness modulus* of f at $\overline{z}$ is given by

$$\operatorname{clm} f(\overline{z}) = \limsup_{z\to\overline{z},\ z\in A} \frac{|f(z) - f(\overline{z})|}{\|z - \overline{z}\|}.$$

Obviously, $\operatorname{clm} f(\overline{z}) \leq \operatorname{lip} f(\overline{z})$.

At this moment, we introduce some necessary notation used along the paper. To start with, given $X \subset \mathbb{R}^p$, $p \in \mathbb{N}$, we denote by $\text{conv}X$, $\text{span}X$, and $\text{extr}X$ the *convex hull*, the *linear hull* of X, and the set of extreme points of X, respectively. Recall that $\text{cone}X$ stands for the *conical convex hull* of X.

For $b \in \text{dom}\mathcal{F}$ and $x \in \mathcal{F}(b)$, we denote by $T_b(x)$ the *set of active indices* at x; i.e.,

$$T_b(x) := \left\{ t \in T : \overline{a}_t' x = b_t \right\}.$$

Associated with $\pi \equiv (c, b) \in \text{dom}\mathcal{F}^{\text{op}}$, we consider the following *family of minimal Karush–Kuhn–Tucker (KKT) subsets of indices*

$$\mathcal{M}_\pi := \left\{ D \subset T_b(x) : \begin{array}{c} -c \in \text{cone}\{\overline{a}_t,\ t \in D\}, \\ D \text{ is minimal for the inclusion order} \end{array} \right\}, \tag{5}$$

for some $x \in \text{dom}\mathcal{F}^{\text{op}}$. Observe that $\mathcal{M}_\pi$ is correctly defined since the right member of (5) indeed does not depend on x (this comes from a standard fact in LP; see [14, Remark 2]). It is also standard that $\{\overline{a}_t, t \in D\}$ is linearly independent for any $D \in \mathcal{M}_\pi$, and this fact justifies the well definedness of the following elements associated with our nominal problem $\overline{\pi} \equiv \left(\overline{c}, \overline{b}\right) \in \text{dom}\mathcal{F}^{\text{op}}$, which were already introduced in [14]:

$$k^- := \min_{D \in \mathcal{M}_{\overline{\pi}}} \left\| \lambda^D \right\|_1 \text{ and } k^+ := \max_{D \in \mathcal{M}_{\overline{\pi}}} \left\| \lambda^D \right\|_1, \tag{6}$$

where, for $D \in \mathcal{M}_{\overline{\pi}}$, $\lambda^D = \left(\lambda_t^D\right)_{t \in T} \in \mathbb{R}_+^T$ is the unique element such that $-\overline{c} = \sum_{t \in D} \lambda_t^D \overline{a}_t$ and $\lambda_t^D = 0$ for all $t \in T \backslash D$, and $\left\| \lambda^D \right\|_1 := \sum_{t \in T} \lambda_t^D$.

Paper [14] analyzes the calmness modulus of the optimal value function under right-hand-side perturbations, $\text{clm}\vartheta_{\overline{c}}^R(\overline{b})$, as well as the calmness modulus under canonical perturbations, $\text{clm}\vartheta^R(\overline{\pi})$. In that paper, each of the moduli is studied by splitting it into the so-called calmness from above and calmness from below moduli. The reader is addressed to [14] for details, since these concepts do not have their counterpart for the Lipschitz modulus. Nevertheless, we need some tools from that paper.

Recall (see, e.g., [27, p. 65]) that any non-empty convex set F can be decomposed as the direct sum

$$F = L_F + \left(F \cap L_F^\perp \right),$$

where L_F is the lineality space of F and $L_F^\perp$ is the orthogonal complement of L_F. In our case, when either $F = \mathcal{F}(b)$ for $b \in \text{dom}\mathcal{F}$ or $F = \mathcal{F}^{\text{op}}(\pi)$ for $\pi \in \text{dom}\mathcal{F}^{\text{op}}$, one has that $L_F^\perp = \text{span}\{\overline{a}_t, t \in T\}$. In [14, Section 2.2] we appeal to the following set of extreme points:

$$\mathcal{E}^{\text{op}}(\pi) := \text{extr}\left(\mathcal{F}^{\text{op}}(\pi) \cap \text{span}\{\overline{a}_t,\ t \in T\} \right),\ \pi \in \text{dom}\mathcal{F}^{\text{op}}, \tag{7}$$

which is clearly nonempty and finite.

The following lemmas will be used later on. The first one comes from [14, Lemma 2] together with a standard argument of LP. Specifically, the uniform boundedness of the sequence $\{\mathcal{E}^{\text{op}}(\pi^r)\}_{r \in \mathbb{N}}$ comes from the fact that any point of $\mathcal{E}^{\text{op}}(\pi^r)$, $r \in \mathbb{N}$, is the unique solution of a Cramer's system. The proof of the second one can be directly extracted from the proof of [14, Theorem 5] (see equation (20) therein). The third comes from [14, Lemma 4].

Lemma 2.1 *Let* $\overline{\pi} \in \mathrm{dom}\mathcal{F}^{\mathrm{op}}$. *For any* $\{\pi^r\}_{r\in\mathbb{N}} \subset \mathrm{dom}\mathcal{F}^{\mathrm{op}}$ *converging to* $\overline{\pi}$, *we have that* $\{\mathcal{E}^{\mathrm{op}}(\pi^r)\}_{r\in\mathbb{N}}$ *is uniformly bounded and*

$$\emptyset \neq \mathrm{Lim\,sup}_r\, \mathcal{E}^{\mathrm{op}}\left(\pi^r\right) \subset \mathcal{E}^{\mathrm{op}}\left(\overline{\pi}\right).$$

Lemma 2.2 *Let* $\overline{\pi} \equiv (\overline{c}, \overline{b}) \in \mathrm{dom}\mathcal{F}^{\mathrm{op}}$ *and* $\{\pi^r \equiv (c^r, b^r)\}_{r\in\mathbb{N}}$ *be a sequence converging to* $\overline{\pi}$, *with* $b^r \in \mathrm{dom}\mathcal{F}$ *for all* $r \in \mathbb{N}$. *If* $x \mapsto (c^r)'x$ *is bounded from below on* $\mathcal{F}^{\mathrm{op}}(\overline{c}, b^r)$ *for all* r, *then*

$$\pi^r \in \mathrm{dom}\mathcal{F}^{\mathrm{op}}$$

for r *large enough.*

Lemma 2.3 *Let* $\overline{\pi} \equiv (\overline{c}, \overline{b}) \in \mathrm{dom}\mathcal{F}^{\mathrm{op}}$ *and* $\{\pi^r \equiv (c^r, b^r)\}_{r\in\mathbb{N}} \subset \mathrm{dom}\mathcal{F}^{\mathrm{op}}$ *be a sequence converging to* $\overline{\pi}$. *Then*

$$\mathcal{F}^{\mathrm{op}}\left(\pi^r\right) \subset \mathcal{F}^{\mathrm{op}}\left(\overline{c}, b^r\right)$$

for r *large enough.*

From now on $e\left(\mathcal{E}^{\mathrm{op}}(\overline{\pi}), 0_n\right)$ denotes the Hausdorff excess of $\mathcal{E}^{\mathrm{op}}(\overline{\pi})$ over $\{0_n\}$, which may be written alternatively as $\max_{x\in\mathcal{E}^{\mathrm{op}}(\overline{\pi})} \|x\|$. On the other hand, $d\left(0_n, \mathcal{F}^{\mathrm{op}}\left(\overline{\pi}\right)\right)$ represents the distance from the origin to the set $\mathcal{F}^{\mathrm{op}}\left(\overline{\pi}\right)$; i.e., $d\left(0_n, \mathcal{F}^{\mathrm{op}}\left(\overline{\pi}\right)\right) = \min_{x\in\mathcal{F}^{\mathrm{op}}(\overline{\pi})} \|x\|$.

Theorem 2.3 [14, Theorem 4, Corollary 3, and Section 5] *Let* $\overline{\pi} \equiv (\overline{c}, \overline{b}) \in \mathrm{dom}\mathcal{F}^{\mathrm{op}}$. *Then*

(*i*) $\mathrm{clm}\vartheta_{\overline{c}}^{R}(\overline{b}) \leq k^+$, *and equality holds when SCQ is satisfied at* $\overline{b}$.

(*ii*) $\mathrm{clm}\vartheta^{R}(\overline{\pi}) \leq \max\{k^- + e\left(\mathcal{E}^{\mathrm{op}}\left(\overline{\pi}\right), 0_n\right), k^+ + d\left(0_n, \mathcal{F}^{\mathrm{op}}\left(\overline{\pi}\right)\right)\}$, *and equality holds when* $\overline{\pi} \in \mathrm{int\,dom}\mathcal{F}^{\mathrm{op}}$.

3 Lipschitz Modulus Under RHS Perturbations

This section is devoted to compute the Lipschitz modulus of the optimal value under perturbations of b (RHS perturbations); i.e., to compute $\mathrm{lip}\vartheta_{\overline{c}}^{R}(\overline{b})$. First, we recall a useful result which provides an explicit expression (as the maximum of a finite amount of linear functions) for the optimal value function in the current perturbation setting. Recall that we are considering a nominal problem $\overline{\pi} \equiv \left(\overline{c}, \overline{b}\right)$.

Lemma 3.1 [14, Lemma 3 and Corollary 1] *Let* $\overline{\pi} \in \mathrm{dom}\mathcal{F}^{\mathrm{op}}$. *There exists a neighborhood* $U_{\overline{b}} \subset \mathbb{R}^T$ *of* $\overline{b}$ *such that*

$$\vartheta\left(\overline{c}, b\right) = \max_{D\in\mathcal{M}_{\overline{\pi}}} -b'\lambda^D, \text{ for all } b \in \mathrm{dom}\mathcal{F} \cap U_{\overline{b}}.$$

Observe that, by the KKT conditions, with respect to the nominal problem, we have

$$\vartheta(\overline{\pi}) = -\overline{b}'\lambda^D, \text{ for all } D \in \mathcal{M}_{\overline{\pi}}. \tag{8}$$

The next proposition follows an analogous argument to the one used for establishing [14, Corollary 2]. Nevertheless, due to its simplicity, and for completeness purposes, we include its proof. Along this section, we use indistinctly $\vartheta_{\overline{c}}^R(b)$ or $\vartheta(\overline{c}, b)$, provided that $b \in \text{dom}\mathcal{F}$. Indeed, for the sake of simplicity in the notation, we usually write $\vartheta_{\overline{c}}^R$ when referring to the function itself and $\vartheta(\overline{c}, b)$, $b \in \text{dom}\mathcal{F}$, for its images.

Proposition 3.1 *Let $\overline{\pi} \in \text{dom}\mathcal{F}^{\text{op}}$ and let $U_{\overline{b}}$ be as in the previous lemma. Then,*

$$\left|\vartheta(\overline{c}, b) - \vartheta(\overline{c}, \widetilde{b})\right| \le k^+\|b - \widetilde{b}\|_\infty \textit{ for all } b, \widetilde{b} \in \text{dom}\mathcal{F} \cap U_{\overline{b}}.$$

Consequently

$$\text{lip}\vartheta_{\overline{c}}^R(\overline{b}) \le k^+.$$

Proof Take $b, \widetilde{b} \in \text{dom}\mathcal{F} \cap U_{\overline{b}}$. Applying the previous lemma, we have

$$\vartheta(\overline{c}, b) - \vartheta(\overline{c}, \widetilde{b}) = \max_{D \in \mathcal{M}_{\overline{\pi}}} \left(-b'\lambda^D\right) - \max_{D \in \mathcal{M}_{\overline{\pi}}} \left(-\widetilde{b}'\lambda^D\right),$$

and let us assume the first maximum is reached at $\widehat{D} \in \mathcal{M}_{\overline{\pi}}$, then

$$\begin{aligned}\vartheta(\overline{c}, b) - \vartheta(\overline{c}, \widetilde{b}) &= -b'\lambda^{\widehat{D}} + \min_{D \in \mathcal{M}_{\overline{\pi}}} \widetilde{b}'\lambda^D \le -b'\lambda^{\widehat{D}} + \widetilde{b}'\lambda^{\widehat{D}} \\ &= (\widetilde{b} - b)'\lambda^{\widehat{D}} \le k^+\|b - \widetilde{b}\|_\infty.\end{aligned}$$

Since b and $\widetilde{b}$ have been arbitrarily chosen, switching them in the preceding argument we obtain the aimed inequality. □

Theorem 3.1 *Let $\overline{\pi} \equiv \left(\overline{c}, \overline{b}\right) \in \text{dom}\mathcal{F}^{\text{op}}$. Then, $\vartheta_{\overline{c}}^R$ is Lipschitz continuous at $\overline{b}$ and*

$$\text{lip}\vartheta_{\overline{c}}^R(\overline{b}) = k^+. \tag{9}$$

Proof According to the previous proposition, it remains to prove $\text{lip}\vartheta_{\overline{c}}^R(\overline{b}) \ge k^+$. To do that take any $\overline{D} \in \mathcal{M}_{\overline{\pi}}$ such that $\left\|\lambda^{\overline{D}}\right\|_1 = k^+$ and let us construct two sequences $\{b^r\}, \{\widetilde{b}^r\} \subset \text{dom}\mathcal{F}$ converging to $\overline{b}$ such that

$$\limsup_r \frac{\left|\vartheta\left(\overline{c}, b^r\right) - \vartheta\left(\overline{c}, \widetilde{b}^r\right)\right|}{\left\|b^r - \widetilde{b}^r\right\|_\infty} = \left\|\lambda^{\overline{D}}\right\|_1, \tag{10}$$

which will establish our aimed inequality.

Let $\overline{x} \in \mathcal{F}^{\text{op}}(\overline{\pi})$. Fix an arbitrary $r \in \mathbb{N}$. Observe that

$$W_r := \left\{x \in \mathbb{R}^n : \overline{a}_t'x < \overline{b}_t + \frac{1}{r},\ t \in T \backslash \overline{D}\right\}$$

is a neighborhood of $\overline{x}$. Now, since $\overline{a}_t'\overline{x} = \overline{b}_t$, $t \in \overline{D}$, and $\left\{\overline{a}_t,\ t \in \overline{D}\right\}$ is linearly independent, a standard argument in LP yields the existence of $0 < \delta_r < \frac{1}{r}$ such that the systems of linear equations

$$\left\{\overline{a}_t'x = \overline{b}_t - \delta_r,\ t \in \overline{D}\right\} \text{ and } \left\{\overline{a}_t'x = \overline{b}_t + \delta_r,\ t \in \overline{D}\right\} \tag{11}$$

have solutions inside W_r; pick x^r and $\widetilde{x}^r$ as solutions of the respective systems in (11) and such that $x^r, \widetilde{x}^r \in W_r$.

Now, let us define $b^r = \left(b_t^r\right)_{t\in T}$ and $\widetilde{b}^r = \left(\widetilde{b}_t^r\right)_{t\in T}$ as follows

$$b_t^r := \begin{cases} \overline{b}_t - \delta_r, & \text{if } t \in \overline{D}, \\ \overline{b}_t + \frac{1}{r}, & \text{if } t \in T\backslash\overline{D}, \end{cases} \quad \text{and } \widetilde{b}_t^r := \begin{cases} \overline{b}_t + \delta_r, & \text{if } t \in \overline{D}, \\ \overline{b}_t + \frac{1}{r}, & \text{if } t \in T\backslash\overline{D}. \end{cases}$$

In particular, $x^r \in \mathcal{F}(b^r)$ and $\widetilde{x}^r \in \mathcal{F}\left(\widetilde{b}^r\right)$; in fact, $x^r \in \mathcal{F}^{\mathrm{op}}(\overline{c}, b^r)$ and $\widetilde{x}^r \in \mathcal{F}^{\mathrm{op}}\left(\overline{c}, \widetilde{b}^r\right)$, since $\overline{D} \subset T_{b^r}(x^r) \cap T_{\widetilde{b}^r}(\widetilde{x}^r)$. Moreover, according to the KKT conditions and taking into account that $\lambda^{\overline{D}}$ is a vector of KKT multipliers associated with both problems $(\overline{c}, b^r)$ and $\left(\overline{c}, \widetilde{b}^r\right)$, by duality in LP we have that

$$\vartheta\left(\overline{c}, b^r\right) = -\left(b^r\right)'\lambda^{\overline{D}} \text{ and } \vartheta\left(\overline{c}, \widetilde{b}^r\right) = -\left(\widetilde{b}^r\right)'\lambda^{\overline{D}}. \tag{12}$$

In this way, and since clearly both sequences $\{b^r\}_{r\in\mathbb{N}}$ and $\{\widetilde{b}^r\}_{r\in\mathbb{N}}$ converge to $\overline{b}$, by applying (12) and, recalling that $\lambda_t^{\overline{D}} = 0$ for $t \in T\backslash\overline{D}$, we have

$$\begin{aligned} \limsup_r \frac{\left|\vartheta\left(\overline{c}, b^r\right) - \vartheta\left(\overline{c}, \widetilde{b}^r\right)\right|}{\left\|b^r - \widetilde{b}^r\right\|_\infty} &= \limsup_r \frac{\left|-\left(b^r - \widetilde{b}^r\right)\lambda^{\overline{D}}\right|}{2\delta_r} \\ &= \limsup_r \frac{\left|-\sum_{t\in\overline{D}}\left(-2\delta_r\lambda_t^{\overline{D}}\right)\right|}{2\delta_r} \\ &= \left\|\lambda^{\overline{D}}\right\|_1, \end{aligned}$$

which finishes the proof. □

The following corollary is a direct consequence of the previous theorem, together with Theorem 2.3(i).

Corollary 3.1 *Let $\overline{\pi} \in \mathrm{dom}\mathcal{F}^{\mathrm{op}}$ and assume that SCQ holds at $\overline{b}$. Then we have*

$$\mathrm{lip}\vartheta_{\overline{c}}^R(\overline{b}) = \mathrm{clm}\vartheta_{\overline{c}}^R(\overline{b}) = k^+.$$

The next example, inspired in [14, Example 1], shows that $\mathrm{clm}\vartheta_{\overline{c}}^R(\overline{b})$ can be strictly less than $\mathrm{lip}\vartheta_{\overline{c}}^R(\overline{b})$ when SCQ fails. Observe that in this example $\mathrm{lip}\vartheta_{\overline{c}}(\overline{b}) = +\infty$, since $\overline{b} \in \mathrm{bd}\ \mathrm{dom}\mathcal{F}$, while $\mathrm{lip}\vartheta_{\overline{c}}^R(\overline{b})$ is finite.

Example 3.1 Consider the problem in $\mathbb{R}$ given by

$$\begin{array}{lll}\overline{\pi}: & \text{minimize} & x \\ & \text{subject to} & -x \leq 0,\ t=1, \\ & & -2x \leq 0,\ t=2, \\ & & 2x \leq 0,\ t=3.\end{array}$$

Observe that $\overline{c} = 1$ and $\bar{b} = 0_3$. Obviously, $\vartheta(\overline{\pi}) = 0$, $\mathcal{M}_{\overline{\pi}} = \{\{1\},\{2\}\}$, $\lambda^{\{1\}} = 1$, $\lambda^{\{2\}} = \frac{1}{2}$, and so $k^+ = 1 = \operatorname{lip}\vartheta^R_{\overline{c}}(\overline{b})$. Let us check that $\operatorname{clm}\vartheta^R_{\overline{c}}(\overline{b}) = \frac{1}{2}$.

According to Lemma 3.1 we have

$$\begin{aligned}\operatorname{clm}\vartheta^R_{\overline{c}}(\overline{b}) &= \limsup_{b\to\overline{b},\ b\in\operatorname{dom}\mathcal{F}} \frac{|\vartheta(\overline{c}, b) - \vartheta(\overline{\pi})|}{\left\|b-\overline{b}\right\|_\infty} \\ &= \limsup_{b\to\overline{b},\ b\in\operatorname{dom}\mathcal{F}} \frac{\left|\max\{-b_1, -\frac{1}{2}b_2\}\right|}{\|b\|_\infty} \leq \frac{1}{2},\end{aligned}$$

where we have appealed to the fact that $b \in \operatorname{dom}\mathcal{F}$ implies $-b_1 \leq \frac{1}{2}b_3$. We may attain $\frac{1}{2}$ by considering $(b^r) = \left(\frac{1}{r}, \frac{1}{r}, \frac{1}{r}\right)'$, $r \in \mathbb{N}$.

4 Lipschitz Modulus Under *c*-Perturbations

This section is devoted to study $\operatorname{lip}\vartheta^R_{\overline{b}}(\overline{c})$, where $\overline{b} \in \operatorname{dom}\mathcal{F}$ is fixed. Recall the notation $C = -\operatorname{cone}\{\overline{a}_t, t \in T\}$, and the standard fact (in LP) that $c \in C$ if and only if $(c, \overline{b}) \in \operatorname{dom}\mathcal{F}^{\mathrm{op}}$. Recall also that $\vartheta^R_{\overline{b}}(c) := \vartheta\left(c, \overline{b}\right)$, for any $c \in C$.

The next proposition is intended to clarify the role played by the set of extreme points (7) when we deal with perturbations of parameter c.

Proposition 4.1 *Let* $\overline{\pi} \in \operatorname{dom}\mathcal{F}^{\mathrm{op}}$. *Then, there exists a neighborhood* $U_{\overline{c}}$ *of* $\overline{c}$ *such that*

$$\vartheta(c, \overline{b}) = \min_{x\in\mathcal{E}^{\mathrm{op}}(\overline{\pi})} c'x \quad \textit{for } (c, \overline{b}) \in \operatorname{dom}\mathcal{F}^{\mathrm{op}} \textit{ and } c \in U_{\overline{c}}.$$

Proof Clearly, for $c \in C$, $\mathcal{E}^{\mathrm{op}}(c, \overline{b}) \subset \operatorname{extr}\left(\mathcal{F}(\overline{b}) \cap \operatorname{span}\{\overline{a}_t,\ t \in T\}\right)$, which is a fixed finite set. This fact, combined with Lemma 2.1, yields $\mathcal{E}^{\mathrm{op}}(c, \overline{b}) \subset \mathcal{E}^{\mathrm{op}}(\overline{\pi})$ for c in some neighborhood $U_{\overline{c}}$ of $\overline{c}$. Since $\mathcal{E}^{\mathrm{op}}(\overline{\pi}) \subset \mathcal{F}(\overline{b})$, the thesis of the proposition holds. □

The following theorem provides a lower and an upper estimate for the aimed Lipschitz modulus. Moreover, it shows that the upper estimate becomes the exact modulus when $\mathcal{F}^{\mathrm{op}}(\overline{\pi})$ is bounded.

Theorem 4.1 *Let* $\overline{\pi} \in \operatorname{dom}\mathcal{F}^{\mathrm{op}}$. *Then,*

$$d(0_n, \mathcal{F}^{\mathrm{op}}(\overline{\pi})) \leq \operatorname{clm}\vartheta^R_{\overline{b}}(\overline{c}) \leq \operatorname{lip}\vartheta^R_{\overline{b}}(\overline{c}) \leq e\left(\mathcal{E}^{\mathrm{op}}(\overline{\pi}), 0_n\right).$$

Moreover, if we assume that $\mathcal{F}^{\text{op}}(\overline{\pi})$ is bounded, then

$$\text{lip}\vartheta_{\overline{b}}^{R}(\overline{c}) = \text{clm}\vartheta_{\overline{b}}^{R}(\overline{c}) = e\left(\mathcal{F}^{\text{op}}(\overline{\pi}), 0_n\right).$$

Proof First, let us see $\text{clm}\vartheta_{\overline{b}}^{R}(\overline{c}) \geq d(0_n, \mathcal{F}^{\text{op}}(\overline{\pi}))$ in the nontrivial case $d(0_n, \mathcal{F}^{\text{op}}(\overline{\pi})) > 0$. Let $\overline{x} \in \mathcal{F}^{\text{op}}(\overline{\pi})$ with $\|\overline{x}\| = d\left(0_n, \mathcal{F}^{\text{op}}(\overline{\pi})\right)$. According to [28, Lemma 9], there exists $u \in \mathbb{R}^n$ with $\|u\|_* = 1$ such that $u'x \geq u'\overline{x} = \|\overline{x}\|$ for all $x \in \mathcal{F}^{\text{op}}(\overline{\pi})$. Define

$$c^r := \overline{c} + \frac{1}{r}u, \text{ for each } r \in \mathbb{N}.$$

For all $x \in \mathcal{F}^{\text{op}}(\overline{\pi})$ we have

$$\left(c^r\right)' x = \overline{c}'x + \frac{1}{r}u'x \geq \overline{c}'\overline{x} + \frac{1}{r}u'\overline{x} = \left(c^r\right)' \overline{x}. \tag{13}$$

This implies that $x \mapsto (c^r)' x$ is bounded from below on $\mathcal{F}^{\text{op}}(\overline{\pi})$ and, by Lemma 2.2, $(c^r, \overline{b}) \in \text{dom}\mathcal{F}^{\text{op}}$ for r large enough (say for all r). Lemma 2.3 entails $\mathcal{F}^{\text{op}}(c^r, \overline{b}) \subset \mathcal{F}^{\text{op}}(\overline{\pi})$, for r large enough, and indeed (13) yields $\overline{x} \in \mathcal{F}^{\text{op}}(c^r, \overline{b})$. Then, we have

$$\begin{aligned}\text{clm}\vartheta_{\overline{b}}^{R}(\overline{c}) &\geq \limsup_r \frac{\vartheta(c^r, \overline{b}) - \vartheta(\overline{\pi})}{\|c^r - \overline{c}\|_*} \\ &= \limsup_r \frac{(c^r - \overline{c})' \overline{x}}{\frac{1}{r}\|u\|_*} = u'\overline{x} = \|\overline{x}\| = d\left(0_n, \mathcal{F}^{\text{op}}(\overline{\pi})\right).\end{aligned}$$

Recall that $\text{clm}\vartheta_{\overline{b}}^{R}(\overline{c}) \leq \text{lip}\vartheta_{\overline{b}}^{R}(\overline{c})$ is always true. Now let us check $\text{lip}\vartheta_{\overline{b}}^{R}(\overline{c}) \leq e\left(\mathcal{E}^{\text{op}}(\overline{\pi}), 0_n\right)$. Write

$$\text{lip}\vartheta_{\overline{b}}^{R}(\overline{c}) = \limsup_r \frac{\left|\vartheta(c^r, \overline{b}) - \vartheta(\widetilde{c}^r, \overline{b})\right|}{\|c^r - \widetilde{c}^r\|_*}, \tag{14}$$

for appropriate sequences $\{c^r\}_r, \{\widetilde{c}^r\}_r \subset C$ converging to $\overline{c}$. Because of the symmetry of the quotients in (14), it is not restrictive to assume $\vartheta(c^r, \overline{b}) - \vartheta(\widetilde{c}^r, \overline{b}) \geq 0$ for all r.

According to Lemma 2.1, there exist a certain $\overline{x} \in \text{Lim sup}_r \, \mathcal{E}^{\text{op}}(\widetilde{c}^r, \overline{b})$ and associated $x^k \in \mathcal{E}^{\text{op}}(\widetilde{c}^{r_k}, \overline{b}) \subset \mathcal{F}^{\text{op}}(\widetilde{c}^{r_k}, \overline{b})$, for $r_1 < r_2 < \ldots < r_k < \ldots$, such that $x^k \to \overline{x} \in \mathcal{E}^{\text{op}}(\overline{\pi})$. Then, for all $k \in \mathbb{N}$ we have

$$0 \leq \vartheta(c^{r_k}, \overline{b}) - \vartheta(\widetilde{c}^{r_k}, \overline{b}) \leq \left(c^{r_k}\right)' x^k - \left(\widetilde{c}^{r_k}\right)' x^k \leq \|c^{r_k} - \widetilde{c}^{r_k}\|_* \|x^k\|,$$

which implies

$$\begin{aligned}\text{lip}\vartheta_{\overline{b}}^{R}(\overline{c}) &= \limsup_k \frac{\vartheta(c^{r_k}, \overline{b}) - \vartheta(\widetilde{c}^{r_k}, \overline{b})}{\|c^{r_k} - \widetilde{c}^{r_k}\|_*} \\ &\leq \limsup_k \|x^k\| = \|\overline{x}\| \leq e\left(\mathcal{E}^{\text{op}}(\overline{\pi}), 0_n\right).\end{aligned}$$

Finally, let us assume that $\mathcal{F}^{\mathrm{op}}(\overline{\pi})$ is bounded, which entails $\operatorname{span}\{\overline{a}_t, t \in T\} = \mathbb{R}^n$, hence $\mathcal{E}^{\mathrm{op}}(\overline{\pi}) = \operatorname{extr}\mathcal{F}^{\mathrm{op}}(\overline{\pi})$ and $e(\operatorname{extr}\mathcal{F}^{\mathrm{op}}(\overline{\pi}), 0_n) = e(\mathcal{F}^{\mathrm{op}}(\overline{\pi}), 0_n)$ (this last follows a standard argument by using the convexity of the norm). Observe that we only have to prove $\operatorname{clm}\vartheta_{\overline{b}}^{R}(\overline{c}) \geq e(\mathcal{F}^{\mathrm{op}}(\overline{\pi}), 0_n)$. Let $\overline{x} \in \mathcal{F}^{\mathrm{op}}(\overline{\pi})$ with $\|\overline{x}\| = e(\mathcal{F}^{\mathrm{op}}(\overline{\pi}), 0_n)$. Take $u \in \mathbb{R}^n$ with $\|u\|_* = 1$ be such that $u'\overline{x} = \|\overline{x}\|$. Define the perturbation $c^r := \overline{c} - \frac{1}{r}u$ for all r. Since $\mathcal{F}^{\mathrm{op}}(\overline{\pi})$ is bounded, from [15, Lemma 10.2] $(c^r, \overline{b}) \in \operatorname{dom}\mathcal{F}^{\mathrm{op}}$ for r large enough. Then, since both problems $(c^r, \overline{b})$ and $\overline{\pi}$ have the same feasible set, we have

$$\vartheta(c^r, \overline{b}) \leq (c^r)'\overline{x} = \overline{c}'\overline{x} - \frac{1}{r}u'\overline{x} = \vartheta(\overline{\pi}) - \|c^r - \overline{c}\|_* \|\overline{x}\|.$$

Therefore,

$$\operatorname{clm}\vartheta_{\overline{b}}^{R}(\overline{c}) \geq \limsup_{r} \frac{\vartheta(\overline{\pi}) - \vartheta(c^r, \overline{b})}{\|\overline{c} - c^r\|_*} \geq \|\overline{x}\| = e\left(\mathcal{F}^{\mathrm{op}}(\overline{\pi}), 0_n\right).$$

Finally, since

$$\operatorname{lip}\vartheta_{\overline{b}}^{R}(\overline{c}) \geq \operatorname{clm}\vartheta_{\overline{b}}^{R}(\overline{c}) \geq e\left(\mathcal{F}^{\mathrm{op}}(\overline{\pi}), 0_n\right) \geq \operatorname{lip}\vartheta_{\overline{b}}^{R}(\overline{c}),$$

we get the aimed equality. □

The next two examples are intended to show that all the inequalities in the statement of Theorem 4.1 may be strict. The first example is concerned with the two first inequalities.

Example 4.1 Consider the nominal problem, in $\mathbb{R}^3$ with the Euclidean norm,

$$\overline{\pi} : \text{minimize } x_3 \text{ s.t. } x_1 \leq -1, \quad -x_2 \leq 2, \quad -x_3 \leq 0.$$

Clearly, $d(0_3, \mathcal{F}^{\mathrm{op}}(\overline{\pi})) = 1$, $\mathcal{E}^{\mathrm{op}}(\overline{\pi}) = \left\{(-1, -2, 0)'\right\}$, and hence $e(\mathcal{E}^{\mathrm{op}}(\overline{\pi}), 0_3) = \sqrt{5}$. Let us prove that $\operatorname{clm}\vartheta_{\overline{b}}^{R}(\overline{c}) = 2$ and $\operatorname{lip}\vartheta_{\overline{b}}^{R}(\overline{c}) = \sqrt{5}$. Consider any $0 < \varepsilon < 1$ and any $c \in \mathbb{R}^3$ with $\|c - \overline{c}\|_* = \varepsilon$, which may be written as $c = (\varepsilon_1, \varepsilon_2, 1 + \varepsilon_3)'$ with $\varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2 = \varepsilon^2$. Then $\left(c, \overline{b}\right) \in \operatorname{dom}\mathcal{F}^{\mathrm{op}}$ if and only if $\varepsilon_1 \leq 0$ and $\varepsilon_2 \geq 0$, in which case $\vartheta\left(c, \overline{b}\right) = c'(-1, -2, 0)' = -\varepsilon_1 - 2\varepsilon_2$. Accordingly,

$$\min_{\substack{\|c-\overline{c}\|_*=\varepsilon \\ (c,\overline{b})\in\operatorname{dom}\mathcal{F}^{\mathrm{op}}}} \vartheta\left(c, \overline{b}\right) = \min_{\substack{\varepsilon_1^2+\varepsilon_2^2=\varepsilon^2 \\ \varepsilon_1\leq 0,\ \varepsilon_2\geq 0}} -\varepsilon_1 - 2\varepsilon_2 = -2\varepsilon, \tag{15}$$

attained at $c = (0, \varepsilon, 1)'$. The corresponding maximum equals ε and is attained at $c = (-\varepsilon, 0, 1)'$. Consequently, for any $0 < \varepsilon < 1$,

$$\max_{\substack{\|c-\overline{c}\|_*=\varepsilon \\ (c,\overline{b})\in\operatorname{dom}\mathcal{F}^{\mathrm{op}}}} \left|\vartheta\left(c, \overline{b}\right) - \vartheta(\overline{\pi})\right| = 2\varepsilon,$$

which, clearly entails $\mathrm{clm}\vartheta_{\overline{b}}^{R}(\overline{c}) = 2$. Now let us compute the Lipschitz modulus of $\vartheta_{\overline{b}}^{R}$ at $\overline{c}$. As a motivation of such computation note that

$$\max_{\varepsilon_1^2+\varepsilon_2^2=\varepsilon^2} -\varepsilon_1 - 2\varepsilon_2 = \sqrt{5}\varepsilon,$$

and this maximum is attained at $(\varepsilon_1, \varepsilon_2) = \left(-\varepsilon/\sqrt{5}, -2\varepsilon/\sqrt{5}\right)$. Let us consider $c := \left(-\varepsilon/\sqrt{5}, 0, 1\right)'$ and $\widetilde{c} := \left(0, 2\varepsilon/\sqrt{5}, 1\right)'$. Then

$$\frac{\left|\vartheta\left(c, \overline{b}\right) - \vartheta\left(\widetilde{c}, \overline{b}\right)\right|}{\|c - \widetilde{c}\|_*} = \frac{\varepsilon/\sqrt{5} - \left(-4\varepsilon/\sqrt{5}\right)}{\varepsilon} = \sqrt{5}.$$

Since this happens for all $0 < \varepsilon < 1$, we conclude $\mathrm{lip}\vartheta_{\overline{b}}^{R}(\overline{c}) \geq \sqrt{5}$. The converse inequality comes from Theorem 4.1.

Next we provide an example where $\mathrm{lip}\vartheta_{\overline{b}}^{R}(\overline{c}) \leq e\left(\mathcal{E}^{\mathrm{op}}(\overline{\pi}), 0_n\right)$ holds strictly.

Example 4.2 Consider $\mathbb{R}^2$ endowed with the norm given by

$$\|x\| := \max\left\{|2x_1 + x_2|, |2x_1 + 3x_2|\right\},$$

whose dual norm $\|\cdot\|_*$ has as its closed unit ball the set

$$B_* := \mathrm{conv}\{\pm(2, 1)', \pm(2, 3)'\}.$$

Alternatively, we may start by considering B_* and define $\|\cdot\|$ as $(\|\cdot\|_*)_*$. Consider the nominal problem in $\mathbb{R}^2$

$$\begin{aligned} \overline{\pi}: \ & \text{minimize } x_1 \\ & \text{subject to } -x_1 \leq -2, \end{aligned}$$

i.e., $\overline{c} = (1, 0)'$ and $\overline{b} = -2$. Then $\mathcal{E}^{\mathrm{op}}(\overline{\pi}) = \left\{(2, 0)'\right\}$ and, accordingly, $e\left(\mathcal{E}^{\mathrm{op}}(\overline{\pi}), 0_2\right) = \left\|(2, 0)'\right\| = 4$. On the other hand, $\left(c, \overline{b}\right) \in \mathrm{dom}\mathcal{F}^{\mathrm{op}}$ if and only if $c = (1 + \alpha, 0)'$ for some $\alpha \geq -1$, in which case $\vartheta(c, \overline{b}) = c'(2, 0)' = 2\alpha + 2$. Therefore, recalling our convention $\frac{0}{0} := 0$,

$$\mathrm{lip}\vartheta_{\overline{b}}^{R}(\overline{c}) = \lim_{\alpha_1,\alpha_2 \to 0} \frac{|(2\alpha_1 + 2) - (2\alpha_2 + 2)|}{\left\|(1 + \alpha_1, 0)' - (1 + \alpha_2, 0)'\right\|_*} = 2,$$

since $(1, 0)'$ is in the boundary of B_*.

Remark 4.1 Note that the norm under consideration plays a key role in the previous example. If $\mathbb{R}^2$ were endowed with the Euclidean norm, we would have $\mathrm{lip}\vartheta_{\overline{b}}^{R}(\overline{c}) = 2 = e\left(\mathcal{E}^{\mathrm{op}}(\overline{\pi}), 0_2\right)$ (for the same $\overline{\pi}$). See [27, Theorem 15.2] for a characterization of all possible norms in $\mathbb{R}^n$ in terms of their closed unit balls.

5 Lipschitz Modulus Under Canonical Perturbations

The objective of this section is to compute (or at least estimate) the Lipschitz modulus of the optimal value function, restricted to dom$\mathcal{F}^{\text{op}}$, at a nominal parameter $\overline{\pi} \in$ dom$\mathcal{F}^{\text{op}}$ under canonical perturbations, i.e., when the RHS of the constraints and the coefficients of the objective function can be simultaneously perturbed.

The following theorem provides a lower bound of the Lipschitz modulus lip$\vartheta^R(\overline{\pi})$.

Theorem 5.1 *Let* $\overline{\pi} \in$ dom$\mathcal{F}^{\text{op}}$. *Then*

$$\text{lip}\vartheta^R(\overline{\pi}) \geq k^+ + d\left(0_n, \mathcal{F}^{\text{op}}(\overline{\pi})\right).$$

Proof The case $0_n \in \mathcal{F}^{\text{op}}(\overline{\pi})$ is trivial due to the fact that $\text{lip}\vartheta^R(\overline{\pi}) \geq \text{lip}\vartheta^R_{\overline{c}}(\overline{b})$. So, let us assume $0_n \notin \mathcal{F}^{\text{op}}(\overline{\pi})$. Take $\overline{x} \in \mathcal{F}^{\text{op}}(\overline{\pi})$ with $\|\overline{x}\| = d\left(0_n, \mathcal{F}^{\text{op}}(\overline{\pi})\right)$. Let us consider sequences $\{b^r\}_r, \{\widetilde{b^r}\}_r \subset \text{dom}\mathcal{F}$ such that

$$k^+ = \text{lip}\vartheta^R_{\overline{c}}(\overline{b}) = \lim_r \frac{\vartheta(\overline{c}, b^r) - \vartheta(\overline{c}, \widetilde{b^r})}{\|b^r - \widetilde{b^r}\|_\infty}.$$

The next step is analogous to its counterpart for calmness in the proof of [14, Theorem 5], so that we will focus on the differences. As in formula (18) in the referred proof, there exist sequences $\{x^r\}_r$ and $\{u^r\}_r$ in $\mathbb{R}^n$, with $\|x^r\| \to \|\overline{x}\|$, such that, for each r, $x^r \in \mathcal{F}^{\text{op}}(\overline{c}, b^r)$, $\|u^r\|_* = 1$ and

$$(u^r)'x \geq (u^r)'x^r = \|x^r\| = d\left(0_n, \mathcal{F}^{\text{op}}(\overline{c}, b^r)\right), \text{ whenever } x \in \mathcal{F}^{\text{op}}(\overline{c}, b^r).$$

Now, we define $c^r := \overline{c} + \|b^r - \widetilde{b^r}\|_\infty u^r$. For $x \in \mathcal{F}^{\text{op}}(\overline{c}, b^r)$ one has

$$(c^r)'x = \overline{c}'x + \left\|b^r - \widetilde{b^r}\right\|_\infty (u^r)'x \geq \vartheta(\overline{c}, b^r) + \left\|b^r - \widetilde{b^r}\right\|_\infty \left\|x^r\right\|, \tag{16}$$

so $x \mapsto (c^r)'x$ is bounded from below on $\mathcal{F}^{\text{op}}(\overline{c}, b^r)$. Because of Lemma 2.2, there exists $r_0 \in \mathbb{N}$ such that $\pi^r \equiv (c^r, b^r) \in \text{dom}\mathcal{F}^{\text{op}}$ for $r \geq r_0$. Then Lemma 2.3 yields $\mathcal{F}^{\text{op}}(\pi^r) \subset \mathcal{F}^{\text{op}}(\overline{c}, b^r)$ for $r \geq r_0$ large enough. Accordingly, by the restriction of (16) to points $x \in \mathcal{F}^{\text{op}}(\pi^r)$, we get

$$\vartheta(\pi^r) = (c^r)'x \geq \vartheta(\overline{c}, b^r) + \left\|b^r - \widetilde{b^r}\right\|_\infty \left\|x^r\right\|.$$

Let us define $\widetilde{\pi}^r :\equiv (\overline{c}, \widetilde{b^r})$ which belongs to dom$\mathcal{F}^{\text{op}}$ (because $\{\widetilde{b^r}\}_r \subset \text{dom}\mathcal{F}$ and $\overline{c} \in C$). Note that $\|\pi^r - \widetilde{\pi}^r\| = \left\|b^r - \widetilde{b^r}\right\|_\infty$.

Then we have

$$\begin{aligned}
\text{lip}\vartheta^R(\overline{\pi}) &\geq \limsup_r \frac{|\vartheta(\pi^r) - \vartheta(\widetilde{\pi}^r)|}{\|\pi^r - \widetilde{\pi}^r\|} \\
&\geq \limsup_r \frac{\vartheta(\pi^r) - \vartheta(\overline{c}, b^r) + \vartheta(\overline{c}, b^r) - \vartheta(\overline{c}, \widetilde{b^r})}{\|b^r - \widetilde{b^r}\|_\infty}
\end{aligned}$$

$$= \lim_r \frac{\vartheta(\overline{c}, b^r) - \vartheta(\overline{c}, \widetilde{b}^r)}{\|b^r - \widetilde{b}^r\|_\infty} + \limsup_r \frac{\vartheta(\pi^r) - \vartheta(\overline{c}, b^r)}{\|b^r - \widetilde{b}^r\|_\infty}$$
$$\geq \operatorname{lip}\vartheta_{\overline{c}}^R(\overline{b}) + \lim_r \|x^r\| = k^+ + \|\overline{x}\|,$$

which completes the proof. □

In order to establish an upper bound for the Lipschitz modulus of ϑ^R at $\overline{\pi}$, we appeal to the technique developed in [18, Section 2]. Specifically, Wu Li proved that if a set-valued mapping is Hausdorff lower semicontinuous, a uniform upper Lipschitz constant for that mapping in a convex neighborhood of the nominal parameter becomes a Lipschitz constant in such a neighborhood (see [18, Theorem 2.1] for details). Translating it into our context, a uniform calmness constant for ϑ^R in a neighborhood (relative to $\operatorname{dom}\mathcal{F}^{\text{op}}$) of $\overline{\pi}$ becomes a Lipschitz constant at $\overline{\pi}$. This technique was already applied in [24] for obtaining the so-called sharp Lipschitz constant for $\mathcal{F}^{\text{op}}$ under suitable hypotheses.

Lemma 5.1 *Let* $\overline{\pi} \in \operatorname{dom}\mathcal{F}^{\text{op}}$. *For all* $\varepsilon > 0$, *there exists* $\delta > 0$ *such that*

$$k^+ + e\left(\mathcal{E}^{\text{op}}(\overline{\pi}), 0_n\right) + \varepsilon$$

is a calmness constant of ϑ^R *at any* $\pi \in (\operatorname{dom}\mathcal{F}^{\text{op}}) \cap B(\overline{\pi}, \delta)$ *(the closed ball centered at* $\overline{\pi}$ *of radius* δ*).*

Proof We start by observing that, from Lemma 2.1, $\mathcal{E}^{\text{op}} : \operatorname{dom}\mathcal{F}^{\text{op}} \rightrightarrows \mathbb{R}^n$ is Hausdorff-upper semicontinuous at $\overline{\pi}$; i.e., $\lim_{\pi \to \overline{\pi}} e\left(\mathcal{E}^{\text{op}}(\pi), \mathcal{E}^{\text{op}}(\overline{\pi})\right) = 0$.

Now, let us abuse the notation and identify also constant k^+ as a function $k^+ : \operatorname{dom}\mathcal{F}^{\text{op}} \longrightarrow \mathbb{R}_+$ defined as $k^+(\pi) = \max_{D \in \mathcal{M}_\pi} \left\|\lambda^D\right\|_1$, where $k^+(\overline{\pi})$ is our original k^+ as defined in (6). We need to prove that function k^+ is also upper semicontinuous at $\overline{\pi}$, that is, for all $\varepsilon > 0$ there exists $\delta > 0$ such that if $\|\pi - \overline{\pi}\| < \delta$, for $\pi \in \operatorname{dom}\mathcal{F}^{\text{op}}$, then $k^+(\pi) \leq k^+(\overline{\pi}) + \varepsilon$. Reasoning by contradiction, suppose that there exists a sequence $\{\pi^r\}_r \subset \operatorname{dom}\mathcal{F}^{\text{op}}$ converging to $\overline{\pi}$ such that $k^+(\pi^r) \geq k^+(\overline{\pi}) + \varepsilon_0$ for a certain $\varepsilon_0 > 0$. Suppose that the maximum defining $k^+(\pi^r)$ is attained at a certain $D^r \in \mathcal{M}_{\pi^r}$. Since T is finite, we can assume the existence of a constant subsequence, say $D^r = D$ for all r. The fact that $-c^r \in \operatorname{cone}\{\overline{a}_t, t \in D\}$ entails $-\overline{c} \in \operatorname{cone}\{\overline{a}_t, t \in D\}$, although we cannot ensure the minimality of D for $\overline{\pi}$. Recall that $\{\overline{a}_t', t \in D\}$ is linearly independent. Write

$$-c^r = \sum_{t \in D} \lambda_t^r \overline{a}_t \text{ for all } r, \text{ and } -\overline{c} = \sum_{t \in D} \lambda_t^D \overline{a}_t \ .$$

Using a standard argument it is easy to see that $\left\{\sum_{t \in D} \lambda_t^r\right\}_r$ is bounded so, taking a subsequence, if necessary, it may be assumed to converge to $\sum_{t \in D} \lambda_t^D$. Although, we cannot assume $D \in \mathcal{M}_{\overline{\pi}}$, we know that D contains at least a minimal element for $\overline{\pi}$, so let $\widetilde{D} \in \mathcal{M}_{\overline{\pi}}$ with $\widetilde{D} \subset D$ and $\lambda_t^D = 0$ for all $t \notin \widetilde{D}$. Therefore, we have

$$k^+(\pi^r) = \sum_{t \in D} \lambda_t^r \longrightarrow \sum_{t \in D} \lambda_t^D = \sum_{t \in \widetilde{D}} \lambda_t^D \leq k^+(\overline{\pi}),$$

hence we attain a contradiction.

Applying the upper semicontinuity of both, $\mathcal{E}^{\mathrm{op}}$ and k^{+}, for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$e\left(\mathcal{E}^{\mathrm{op}}(\pi), 0_n\right) \leq e\left(\mathcal{E}^{\mathrm{op}}(\overline{\pi}), 0_n\right) + \varepsilon/2$$
$$\text{and } k^{+}(\pi) \leq k^{+}(\overline{\pi}) + \varepsilon/2,$$

for all $\pi \in \mathrm{dom}\mathcal{F}^{\mathrm{op}}$ with $\|\pi - \overline{\pi}\| < \delta$, and therefore

$$\mathrm{clm}\vartheta^{R}(\pi) \leq k^{+}(\pi) + e\left(\mathcal{E}^{\mathrm{op}}(\pi), 0_n\right) \leq k^{+}(\overline{\pi}) + e\left(\mathcal{E}^{\mathrm{op}}(\overline{\pi}), 0_n\right) + \varepsilon\ ,$$

taking Theorem 2.3(ii) into account. □

Theorem 5.2 *Let* $\overline{\pi} \in \mathrm{dom}\mathcal{F}^{\mathrm{op}}$. *Then*

$$\mathrm{lip}\vartheta^{R}(\overline{\pi}) \leq k^{+} + e\left(\mathcal{E}^{\mathrm{op}}(\overline{\pi}), 0_n\right). \tag{17}$$

If, additionally, $\mathcal{F}^{\mathrm{op}}(\overline{\pi})$ *is bounded, then equality hods in (17), which reads as*

$$\mathrm{lip}\vartheta^{R}(\overline{\pi}) = k^{+} + e\left(\mathcal{F}^{\mathrm{op}}(\overline{\pi}), 0_n\right).$$

Proof Recall that $\mathrm{dom}\mathcal{F}^{\mathrm{op}}$ is convex in $\mathbb{R}^n \times \mathbb{R}^T$ and Theorem 2.1 establishes the continuity of ϑ^{R} on $\mathrm{dom}\mathcal{F}^{\mathrm{op}}$. Then, the previous lemma and its preceding comments ensure that $k^{+} + e\left(\mathcal{E}(\overline{\pi}), 0_n\right) + \varepsilon$ is a Lipschitz constant of ϑ^{R} at $\overline{\pi}$ for each $\varepsilon > 0$. Letting $\varepsilon \downarrow 0$ we obtain (17).

Now, assume that $\mathcal{F}^{\mathrm{op}}(\overline{\pi})$ is bounded. In order to establish the converse inequality, consider sequences $\{b^r\}_r, \{\widetilde{b^r}\}_r \subset \mathrm{dom}\mathcal{F}$ such that

$$k^{+} = \mathrm{lip}\vartheta^{R}_{\overline{c}}(\overline{b}) = \lim_r \frac{\vartheta(\overline{c}, \widetilde{b^r}) - \vartheta(\overline{c}, b^r)}{\|\widetilde{b^r} - b^r\|_{\infty}}.$$

Apply Theorem 2.2 and Remark 2.1 to conclude that $\mathcal{F}^{\mathrm{op}}(\overline{c}, b^r)$ is nonempty and bounded for r large enough (say for all r). For each $r \in \mathbb{N}$ take $x^r \in \mathcal{F}^{\mathrm{op}}(\overline{c}, b^r)$ such that $\|x^r\| = e\left(\mathcal{F}^{\mathrm{op}}(\overline{c}, b^r), 0_n\right)$ and let $u^r \in \mathbb{R}^n$ be such that $\|u^r\|_* = 1$ and $(u^r)' x^r = \|x^r\|$.

The sequence $\{x^r\}_{r\in\mathbb{N}}$ may not converge, although it has for sure a convergent subsequence, but we can ensure, again by Theorem 2.2, that $\|x^r\| \to e\left(\mathcal{F}^{\mathrm{op}}(\overline{\pi}), 0_n\right)$.

For each r let us define $c^r := \overline{c} - \|\widetilde{b^r} - b^r\|_{\infty} u^r$. Obviously $x \mapsto (c^r)' x$ is bounded from below on $\mathcal{F}^{\mathrm{op}}(\overline{c}, b^r)$, because this set is compact; so that, Lemma 2.2 yields $(c^r, b^r) \in \mathrm{dom}\mathcal{F}^{\mathrm{op}}$ for r large enough, and then

$$\vartheta(\overline{c}, b^r) - \vartheta(c^r, b^r) \geq \left(\overline{c} - c^r\right)' x^r = \|\widetilde{b^r} - b^r\|_{\infty} \left\|x^r\right\|.$$

Therefore

$$\mathrm{lip}\vartheta^{R}(\overline{\pi}) \geq \limsup_r \frac{\vartheta(\overline{c}, \widetilde{b^r}) - \vartheta(c^r, b^r)}{\|(\overline{c}, \widetilde{b^r}) - (c^r, b^r)\|}$$

$$= \limsup_r \frac{\vartheta(\overline{c}, \widetilde{b^r}) - \vartheta(\overline{c}, b^r) + \vartheta(\overline{c}, b^r) - \vartheta(c^r, b^r)}{\|\widetilde{b^r} - b^r\|_\infty}$$

$$= \text{lip}\vartheta^R_{\overline{c}}(\overline{b}) + \limsup_r \frac{\vartheta(\overline{c}, b^r) - \vartheta(c^r, b^r)}{\|\widetilde{b^r} - b^r\|_\infty}$$

$$\geq k^+ + \lim_r \|x^r\| = k^+ + e\left(\mathcal{F}^{\text{op}}(\overline{\pi}), 0_n\right) ,$$

which yields the asserted formula. □

Corollary 5.1 *Let* $\overline{\pi} \in \text{dom}\mathcal{F}^{\text{op}}$*, with* $\mathcal{F}^{\text{op}}(\overline{\pi})$ *bounded. Then*

$$\text{lip}\vartheta^R(\overline{\pi}) = \text{lip}\vartheta^R_{\overline{c}}(\overline{b}) + \text{lip}\vartheta^R_{\overline{b}}(\overline{c}).$$

Proof It comes from Theorems 3.1, 4.1 and 5.2. □

6 Conclusions

The main original contributions of the present paper are focussed on the Lipschitz moduli of the optimal value functions restricted to their domains in different parametric contexts ($\vartheta^R_{\overline{c}}$, in the context of RHS perturbations, $\vartheta^R_{\overline{b}}$, in the one of c-perturbations, and ϑ^R, for canonical perturbations; see Sect. 2.1 for the definitions). The analysis is developed around a nominal LP problem $\overline{\pi}$, which is identified with the pair formed by a nominal vector of the objective function, $\overline{c}$, and a nominal RHS, $\overline{b}$. As a brief discussion about the convenience of dealing with such functions, restricted to their domains, we underline the fact that it allows us to avoid a typical interiority assumption under which some preliminary results are stated (see [15, Lemma 10.2] and [13]). Specifically, the nominal elements $\overline{b}$, $\overline{c}$, and $\overline{\pi}$ are not required to be in the interior of the respective domains of $\vartheta^R_{\overline{c}}$, $\vartheta^R_{\overline{b}}$, and ϑ^R.

In contrast, [15, Lemma 10.2] and [13] deal with the optimal function ϑ defined on the whole space, and in this case the condition '$\overline{\pi}$ is in the interior of the domain of ϑ' is not avoidable as far as it characterizes the Lipschitz continuity of ϑ at $\overline{\pi}$ (so, the Lipschitz modulus of ϑ is infinite when the interiority condition does not hold). It is known that this interiority condition is equivalent to the simultaneous fulfillment of the Slater CQ and the boundedness (and nonemptiness) of the nominal optimal set. In the next paragraphs, we comment the most important contributions of this work and, at the same time, we try to clarify the role played by the two assumptions, Slater CQ and boundedness, separately, in relation to the computation/estimation of our Lipschitz moduli. The boundedness of the optimal set does play an important role:

– When $\overline{\pi}$ is a solvable problem (without any extra assumption), the Lipschitz modulus of $\vartheta^R_{\overline{c}}$ is completely determined (Theorem 3.1), and the corresponding moduli for $\vartheta^R_{\overline{b}}$, and ϑ^R are lower and upper estimated (Theorems 4.1, 5.1, and 5.2). In particular, all these functions are always Lipschitz continuous at $\overline{\pi}$.
– When $\overline{\pi}$ is solvable and Slater CQ holds, we additionally have that the Lipschitz modulus of $\vartheta^R_{\overline{c}}$ does coincide with its calmness modulus (Corollary 3.1).

– When $\overline{\pi}$ is solvable and the nominal optimal set is bounded, the upper estimates of ϑ_{b}^{R} and ϑ^{R} turn out to be the exact moduli. Moreover, in this case, the Lipschitz modulus of ϑ^{R} coincides with the sum of the corresponding moduli of ϑ_{c}^{R} and ϑ_{b}^{R} (Corollary 5.1).
– When $\overline{\pi}$ is in the interior of solvable problems (Slater CQ together with boundedness of the nominal optimal set), then, in addition to the previous statements, the Lipschitz modulus of ϑ does coincide with the one of ϑ^{R}. Moreover, the reader can easily check that the calmness modulus of ϑ may be strictly less than the Lipschitz one from the exact expressions of both moduli (Theorems 2.3 and 5.2).

Finally, let us comment that all formulas obtained in this work for computing or estimating our aimed moduli are point-based, in the sense that all ingredients used in them only involve the nominal elements (the nominal point and problem's data), not appealing to parameters or points in a neighborhood. In this way they are implementable in practice.

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