# Characterization of the attractor for nonautonomous reaction-diffusion equations with discontinuous nonlinearity

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**Abstract**

In this paper, we study the asymptotic behavior of the solutions of a nonautonomous differential inclusion modeling a reaction-diffusion equation with a discontinuous nonlinearity.

We obtain first several properties concerning the uniqueness and regularity of non-negative solutions. Then we study the structure of the pullback attractor in the positive cone, showing that it consists of the zero solution, the unique positive nonautonomous equilibrium and the heteroclinic connections between them, which can be expressed in terms of the solutions of an associated linear problem.

Finally, we analyze the relationship of the pullback attractor with the uniform, the cocycle and the skew product semiflow attractors.

**Keywords:** differential inclusions, reaction-diffusion equations, pullback attractors, nonautonomous dynamical systems, multivalued dynamical systems, structure of the attractor

**AMS Subject Classification (2010):** 35B40, 35B41, 35B51, 35K55, 35K57

## 1 Introduction

The study of the structure of the global attractor for dynamical systems is an important question as it leads to a deep understanding of the dynamics of the solutions of differential equations. In the single-valued autonomous case, there are good examples in which the dynamics inside the attractor has been fully described (see for example [9], [10], [20], [24], [25], [34], [35]). In the multivalued autonomous case, it is more difficult to carry out the study of the structure of attractors. Nevertheless, several results have been published in this direction (see [2], [12], [27], [28]). In all these papers, the dynamical system is of gradient type, so the attractor is described by means of the set of stationary points and its unstable manifold. Moreover, in some of them, the dynamics is described in detail in terms of the stationary points and their heteroclinic connections. The Chaffee-Infante equation [25] is a paradigmatic example in which the dynamics has been completely understood.

In the nonautonomous case, the problem is more complex as stationary solutions do not exist in general in a classical sense. Instead, we need to replace them by a special type of bounded complete trajectories, which play the role of "nonautonomous equilibria". In this direction, the existence of a complete bounded positive non-degenerate trajectory is a key fact which has been proved successfully for parabolic equations of certain types (see [15], [29], [30], [36], [37]). Moreover, in the special case of the nonautonomous Chaffee-Infante equation it is proved that there exist a finite number of nonautonomous equilibria, which are analogous to the equilibria in the autonomous situation, and that the omega and alpha limits of any bounded complete trajectory belong to the set of these equilibria when we consider all the equations generated by the translations of the nonautonomous term [8]. Therefore, the pullback attractor is characterized in terms of the nonautonomus equilibria and their heteroclinic connections, leading to a gradient structure of the attractor of the associated skew product semiflow.

In this paper, we study the dynamical system generated by the differential inclusion

$$\begin{cases} \dfrac{\partial u}{\partial t} - \dfrac{\partial^2 u}{\partial x^2} \in b(t)H_0(u) + \omega(t)u, \text{ on } (\tau, +\infty) \times (0,1), \\ u(t,0) = u(t,1) = 0, \\ u(\tau, x) = u_\tau(x), \ x \in (0,1), \end{cases} \tag{1}$$

where

$$H_0(u) = \begin{cases} -1, & \text{if } u < 0, \\ [-1,1], & \text{if } u = 0, \\ 1, & \text{if } u > 0, \end{cases} \tag{2}$$

is the Heaviside function, so we have in fact a differential equation with a discontinuous nonlinearity. This dynamical system is both nonautonomous and multivalued. Well known models like combustion in porous media [19], the conduction of electrical impulses in nerve axons (see [40], [41]) or the surface temperature on Earth (see [11], [18]) have discontinuities of this type. This inclusion is also important because it is the limit of a sequence of Chaffee-Infante problems which has undergone all the bifurcation cascade of this type of problems (see [2] for more details), so it is natural to expect that it should inherit the structure of the attractor. In fact, in the autonomous case (that is, when the functions $b(\cdot), \omega(\cdot)$ are constants), it is proved in [2] that (1) has an infinite, but countable, number of equilibria (each of which is related to a corresponding one in the Chaffee-Infante equation) and that the attractor is described by them and their heteroclinic connections. Moreover, when $\omega$ is equal to 0 it is shown that some of the connections of the Chaffee-Infante equation are also true for problem (1). The hypothesis is that the connections are the same and then the structure of the attractors coincide. However, this still remains an open problem. We observe also that for a related inclusion modeling the climate on Earth several results on bifurcations of steady states were proved in [5], [6].

In this paper, we continue the study of the structure of the pullback attractor for problem (1) initiated in [14], where the existence of two special non-degenerate bounded complete trajectories containing the pullback attractor was established (the positive and negative nonautonomous equilibria). More precisely, we give a complete characterization of the pullback attractor in the positive cone, that is, when we consider only non-negative solutions.

First, using the maximum principle as the main tool we prove the following important facts:

- For any non-negative initial condition not equal to zero there exists a unique non-negative solution, which becomes positive instantaneously.
- If the initial condition is equal to zero, then the unique non-negative solution backwards in time is the zero solution.
- If the initial condition is non-negative and not equal to zero, belongs to the space $H_0^1(0,1)$ and is not positive, then there cannot exist a non-negative solution backwards in time.
- If the initial condition is non-negative and belongs to $L^2(0,1) \setminus H_0^1(0,1)$, then there cannot exist a non-negative solution backwards in time.

Second, using these results we obtain that the pullback attractor in the positive cone consists of two nonautonomous equilibria (the zero solution and the positive nonautonomous equilibrium denoted by $\xi_M$) and all the bounded complete trajectories which connect them heteroclinically. Moreover, these connections always go from 0 to $\xi_M$ and have the form

$$\varphi(t) = \begin{cases} 0 \text{ if } t \leq \tau, \\ u(t) \text{ if } t \geq \tau, \end{cases}$$

where $u(\cdot)$ is the unique solution to the associated linear problem of (1) in which the right-hand side is equal to $b(t) + \omega(t)u(t)$ and we choose the initial condition $u_\tau = 0$. It is important to point out that these results are also new in the autonomous case. We observe also that the structure of the pullback attractor in the positive cone is the same as in the associated ordinary differential inclusion in which the diffusion term $-\dfrac{\partial^2 u}{\partial x^2}$ is replaced by the linear function $\lambda u$, $\lambda > 0$ [13].

Additionally, we obtain the following properties of the positive nonautonomous equilibrium $\xi_M$, which were left as an open problem in [14, Remark 2]:

- The solution starting at any point $\xi_M(s)$, $s \in \mathbb{R}$, is unique in the class of all non-negative solutions. As a consequence, $\xi_M$ is the unique non-negative non-degenerate bounded complete trajectory at $-\infty$.
- If $b, \omega$ are more regular ($b, \omega \in W^{1,2}_{loc}(\mathbb{R})$), then the solution starting at any point $\xi_M(s)$, $s \in \mathbb{R}$, is unique in the class of all solutions. Thus, $\xi_M$ is the unique non-degenerate bounded complete trajectory at $-\infty$.

Moreover, increasing the regularity of the functions $b, \omega$ we increase the regularity (in both time and space) of non-negative solutions as well. In this way, we are able to prove that the attractor in the positive cone is as regular as we desire.

Third, we obtain the existence of other types of nonautonomous attractors such as the cocycle attractor, the uniform attractor and the skew product semiflow attractor and describe their relationship. In particular, we obtain that the associated skew product semiflow possesses in the positive cone a global attractor with a gradient structure.

# 2 Setting of the problem

We consider the problem

$$\begin{cases} \dfrac{\partial u}{\partial t} - \dfrac{\partial^2 u}{\partial x^2} \in b(t)H_0(u) + \omega(t)u, \text{ on } (\tau, \infty) \times (0,1), \\ u(t,0) = u(t,1) = 0, \\ u(\tau, x) = u_\tau(x), \ x \in (0,1), \end{cases} \tag{3}$$

where $b : \mathbb{R} \to \mathbb{R}^+$, $\omega : \mathbb{R} \to \mathbb{R}^+$ are continuous functions such that

$$0 < b_0 \leq b(t) \leq b_1, \ 0 \leq \omega_0 \leq \omega(t) \leq \omega_1, \tag{4}$$

and $H_0$ is the Heaviside function given in (2).

Let $H = L^2(0,1)$, $V = H^1_0(0,1)$. The norm in $H$ will be denoted by $\|\cdot\|$, whereas the norm in any other Banach space $X$ will be denoted by $\|\cdot\|_X$. $L^p_{loc}(\tau, +\infty; X)$, $p \geq 1$ (respectively, $W^{k,p}_{loc}(\tau, +\infty; X)$, $p \geq 1$, $k \in \mathbb{N}$) will stand for the space of functions $g$ such that $g \in L^p(\tau, T; X)$ (respectively, $W^{k,p}(\tau, T; X)$) for any $T > \tau$.

For a metric space $Y$ denote by $dist_Y(A, B) = \sup_{a \in A} \inf_{b \in B} \rho_Y(a, b)$ the Hausdorff semidistance from the set $A$ to the set $B$, where $\rho_Y$ is the metric in $Y$.

Let $A : D(A) \to H$, $D(A) = H^2(0,1) \cap V$, be the operator $A = -\dfrac{d^2}{dx^2}$ with Dirichlet boundary conditions. This operator is the generator of a $C_0$-semigroup $T(t) = e^{-At}$. Also, we define the multivalued operator $R : \mathbb{R} \times H \to 2^H$ (with $2^H$ being the set of all subsets of $H$) by

$$R(t, u) = \{y \in H : \ y(x) \in b(t)H_0(u(x)) + \omega(t)u(x), \text{ a.e. on } (0,1)\}.$$

It is known [14, Lemma 2] that this operator has nonempty, closed, bounded and convex values.

We rewrite (3) in the abstract form

$$\begin{cases} \dfrac{du}{dt} + Au \in R(t, u(t)), \ t \in (\tau, +\infty), \\ u(\tau) = u_\tau. \end{cases}$$

**Definition 1** *(Strong solution) Let $u_\tau \in H$. The function $u \in C([\tau, +\infty), H)$ is said to be a strong solution to problem (3) if:*

1. *$u(\tau) = u_\tau$;*

*2. $u(\cdot)$ is absolutely continuous on $[T_1, T_2]$ for any $\tau < T_1 < T_2$ and $u(t) \in D(A)$ for a.a. $t \in (T_1, T_2)$;*

*3. There exists a function $r \in L^2_{loc}(\tau, +\infty; H)$ such that $r(t) \in R(t, u(t))$ for a.a. $t \in (\tau, +\infty)$ and*

$$\frac{du}{dt} + Au(t) = r(t) \text{ for a.a. } t \in (\tau, +\infty), \tag{5}$$

*where the equality is understood in the sense of the space $H$.*

**Definition 2** *(Mild solution) Let $u_\tau \in H$. The function $u \in C([\tau, +\infty), H)$ is said to be a mild solution to problem (3) if there exists a function $r \in L^2_{loc}(\tau, +\infty; H)$ such that $r(t) \in R(t, u(t))$ for a.a. $t \in (\tau, +\infty)$ and*

$$u(t) = e^{-A(t-\tau)}u_\tau + \int_\tau^t e^{-A(t-s)}r(s)ds \text{ for } \tau \leq t < +\infty.$$

It is known [14, Theorem 1] that for any $u_\tau \in H$ there exists at least one strong solution to problem (3). For any $u_\tau \in H$ the function $u \in C([\tau, +\infty), H)$ is a strong solution to problem (3) if and only if it is a mild solution [14, Corollary 4]. Therefore, from now on we will speak just about solutions of (3).

The main aim of this paper is to establish a precise characterization of the pullback attractor for non-negative solutions to problem (3) in terms of the bounded complete trajectories connecting the stationary solutions.

# 3 About positive solutions

An element $v \in H$ is said to be non-negative (denoted by $v \geq 0$) if $v(x) \geq 0$ for a.a. $x \in (0, 1)$. Let us consider non-negative solutions to problem (3). Under assumption (4) it is known [14, Corollary 5] that for any $u_\tau \in H$ satisfying $u_\tau \geq 0$ there exists at least one strong solution $u(\cdot)$ such that $u(t) \geq 0$ for any $t \geq \tau$.

An element $v \in V$ is said to be positive if $v(x) > 0$ for all $x \in (0, 1)$. We shall show in this section that for initial conditions $u_\tau$ that are non-negative and not identically equal to zero any solution $u$ to problem (3) satisfies that $u(t)$ is positive for any $t > 0$. Also, for such initial conditions the solution is unique among non-negative solutions. Moreover, we establish that if either the initial condition is not positive and not equal to zero or it belongs to $H \backslash V$, then the solution does not exist backwards in time.

We consider the following auxiliary problem:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \dfrac{\partial^2 u}{\partial x^2} = f(t), \text{ on } (\tau, \infty) \times (0, 1), \\ u(t, 0) = u(t, 1) = 0, \\ u(\tau, x) = u_\tau(x), \ x \in (0, 1), \end{cases} \tag{6}$$

where $f \in L^1_{loc}(\tau, T; H)$.

For $u_\tau \in H$ the function $u \in C([\tau, +\infty), H)$ is called a strong solution to problem (6) if $u(\tau) = u_\tau$, $u$ is absolutely continuous on any compact interval of $(\tau, +\infty)$, $u(t) \in D(A)$ for a.a. $t \in (\tau, +\infty)$ and

$$\frac{du}{dt} + Au(t) = f(t) \text{ in } H \text{ for a.a. } t \in (\tau, +\infty).$$

The following lemma follows from a general result for equations governed by subdifferential maps given for example in [7, Theorem 3.6] or [3, p.189].

**Lemma 3** *For any $f(\cdot) \in L^2_{loc}(\tau, +\infty; H)$, $u_\tau \in H$, there exists a unique strong solution to problem (6) satisfying*

$$\sqrt{t}\frac{du}{dt} \in L^2_{loc}(\tau, +\infty; H), \ u \in L^2_{loc}(\tau, +\infty; V). \tag{7}$$

*Also, the map $t \mapsto \|u(t)\|_V$ is absolutely continuous on any compact interval of $(\tau, +\infty)$.*

*If, moreover, $u_\tau \in V$, then $\dfrac{du}{dt} \in L^2_{loc}(\tau, +\infty; H)$ and $t \mapsto \|u(t)\|_V$ is absolutely continuous on any compact interval of $[\tau, +\infty)$.*

**Remark 4** *We observe that $u \in C([T_1, T_2], H) \cap L^{\infty}(T_1, T_2; V)$ implies that $u \in C([T_1, T_2], V_w)$, where $V_w$ stands for the weak continuity in $V$ [39, Lemma 1.4, p.263]. Then, as weak continuity in $V$ together with continuity in norm gives strong continuity in the Hilbert space $V$, $u \in C([T_1, T_2], H)$ and $\|u(t)\|_V \in C([T_1, T_2])$ imply that $u \in C([T_1, T_2], V)$. Hence, if $u_\tau \in H$ (respectively, $u_\tau \in V$), then $u \in C((\tau, +\infty), V)$ (respectively, $u \in C([\tau, +\infty), V)$).*

For $u_\tau \in H$ the function $u \in C([\tau, +\infty), H)$ is called a mild solution to problem (6) if

$$u(t) = e^{-A(t-\tau)}u_\tau + \int_\tau^t e^{-A(t-s)} f(s)ds \text{ for } \tau \leq t < +\infty.$$

By [14, Lemma 4] the unique strong solution given in Lemma 3 is the unique mild solution to problem (6).

We observe that by Remark 4 any solution $u$ to problem (3) satisfies $u \in C((\tau, +\infty), V)$, so $V \subset C([0,1])$ gives $u \in C((\tau, +\infty), C([0,1]))$. Therefore, the solutions are continuous in $(\tau, +\infty) \times [0,1]$. If we take an initial condition in $V$, then $u \in C([\tau, +\infty), V)$, so the solutions are continuous in $[\tau, +\infty) \times [0,1]$. Also, the derivative $\dfrac{du}{dt}$ exists in the classical sense for a.a. $t > \tau$.

**Lemma 5** *Let $u(\cdot)$ be a non-negative solution to problem (3) with initial condition $u_\tau \in H$. Then*

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} \geq 0 \text{ for a.a. } t > 0,\ x \in (0,1). \tag{8}$$

*Hence, $\dfrac{\partial u}{\partial t} - \dfrac{\partial^2 u}{\partial x^2} \geq 0$ in the sense of distributions as well.*

**Proof.** The function $u$ is the unique solution to problem (6) for some $f \in L^2_{loc}(\tau, +\infty; H)$ such that $f(t,x) \in b(t)H_0(u(t,x)) + \omega(t)u(t,x)$ for a.a. $(t,x)$. If we prove for a.a. $t > 0$ that $f(t,x) \geq 0$ for a.a. $x \in (0,1)$, then (8) follows.

We fix $t > 0$ such that the derivative $\dfrac{du}{dt}(t)$ exists in the classical sense and $f(t, \cdot) \in H$. Denote $A_t = \{x \in [0,1] : u(t,x) > 0\}$, $B_t = \{x \in [0,1] : u(t,x) = 0\}$, $A_{qt} = A_t \cap \mathbb{Q}$. Since $u(t) \in V \subset C([0,1])$, for any $x \in A_{qt}$ there exists a maximal interval $I_x = [x_1, x_2]$, $x_1 < x_2$, $x \in I_x$, such that $u(y) > 0$, for any $y \in (x_1, x_2)$, and $u(x_1) = u(x_2) = 0$. Again by continuity, for any $x \in A_t$ there exists $\overline{x} \in A_{qt}$ such that $x \in int\ I_{\overline{x}}$. Hence, $A_t = \cup_{x \in A_{qt}} int\ I_x$, so $A_t$ is the countable union of open intervals $I_i = (x_i, x_{i+1})$, where $u(t,x) > 0$ for $x \in (x_i, x_{i+1})$ and $u(t, x_i) = u(t, x_{i+1}) = 0$. Therefore, $B_t$ is the countable union of closed intervals $\widetilde{I}_j = [\widetilde{x}_j, \widetilde{x}_{j+1}]$, where $u(t,x) = 0$ for $x \in \widetilde{I}_j$. It is possible in this last case that $x_i = x_{i+1}$. We split $B_t$ into two sets:

$$B_t = B_t^1 \cup B_t^2,$$

$$B_t^1 = \{x \in B_t : x \in (\widetilde{x}_i, \widetilde{x}_{i+1}),\ \widetilde{x}_i < \widetilde{x}_{i+1},\ u(t,y) = 0\ \forall y \in [\widetilde{x}_i, \widetilde{x}_{i+1}]\},$$
$$B_t^2 = \{x \in B_t : u(t,x) = 0, \text{ there is no } \varepsilon > 0 \text{ such that } u(t,y) = 0 \text{ for } y \in [x - \varepsilon, x + \varepsilon]\}.$$

The set $B_t^2$ has zero measure. Thus, it is enough to check that $f(t,x) \geq 0$ for a.a. $x \in A_t \cup B_t^1$.

If $x \in A_t$, then $H_0(u(t,x)) = 1$, so that $f(t,x) = b(t) + \omega(t)u(t,x) > 0$.

If $x \in (\widetilde{x}_i, \widetilde{x}_{i+1})$, where $(\widetilde{x}_i, \widetilde{x}_{i+1})$ is an interval given in the definition of $B_t^1$, then the derivative $\dfrac{\partial^2 u}{\partial x^2}(t,x)$ exists in the classical sense and is equal to zero. This means that

$$\frac{\partial u}{\partial t}(t,x) = f(t,x) \text{ for a.a. } x \in (\widetilde{x}_i, \widetilde{x}_{i+1}).$$

Since $\dfrac{du}{dt}(t)$ exists in the classical sense, $\dfrac{u(t+h) - u(t)}{h} - \dfrac{du}{dt}(t) \to 0$ in $H$, so $\dfrac{u(t+h_n, x) - u(t,x)}{h_n} - \dfrac{\partial u}{\partial t}(t,x) \to 0$ for a.a. $x \in (0,1)$ and some subsequence. For a.a. $x \in (\widetilde{x}_i, \widetilde{x}_{i+1})$ we have that $\dfrac{u(t+h,x) - u(t,x)}{h} \geq 0$ (as $u(t+h,x) \geq 0$), and then $\dfrac{\partial u}{\partial t}(t,x) \geq 0$ for a.a. $x \in (\widetilde{x}_i, \widetilde{x}_{i+1})$. Thus, $f(t,x) \geq 0$ for a.a. $x \in (\widetilde{x}_i, \widetilde{x}_{i+1})$. ■

**Lemma 6** *Assume that (4) holds. Let $u_\tau \in H$ be such that $u_\tau \geq 0$ but $u_\tau \not\equiv 0$ and let $u(\cdot)$ be a non-negative solution to problem (3). Then the solution $u(\cdot)$ is unique in the class of non-negative solutions and $u(t)$ is positive for any $t > \tau$.*

**Proof.** We will use the minimum principle for non-smooth functions proved in [26] (see the appendix) in order to prove that $u(t)$ is positive for any $t > \tau$. By contradiction we assume the existence of $t_0 > \tau$ and $x_0 \in (0,1)$ such that $u(t_0, x_0) = 0$. Let

$$\mathcal{O} = [\tau, t_0] \times [0,1],$$
$$Q_{\rho,\sigma} = \{(t,x) : t \in (t_0 - \sigma, t_0), \left| x - \frac{1}{2} \right| < \rho\},$$

where $\sigma = t_0 - \tau$. Choosing $\rho = \frac{1}{2}$ we have that

$$\inf_{(t,x) \in Q_{\nu\rho,\sigma_1}} u(t,x) = 0, \tag{9}$$

for some $0 < \nu < 1$ and any $0 < \sigma_1 < \sigma$. By (8) and Theorem 40 we obtain that $u(t,x) = 0$ for a.a. $(t,x) \in Q_{\frac{1}{2},\sigma}$, which is not possible.

It remains to prove the uniqueness. Since $u(t)$ is positive for $t > \tau$, $u$ has to be the solution of problem (6) with $f(t,x) = b(t) + \omega(t) u(t,x)$. Let there exist two solutions $u_1, u_2$. Then the difference $v = u_1 - u_2$ satisfies the equality

$$\frac{\partial v}{\partial t} - \frac{\partial^2 v}{\partial x^2} = \omega(t) v.$$

Multiplying by $v$ and making use of Gronwall's lemma we get

$$\|v(t)\|^2 \leq e^{2\int_\tau^t \omega(s) ds} \|v(\tau)\|^2 = 0.$$

■

**Lemma 7** *Assume that (4) holds. If $u_\tau \equiv 0$, then, apart from the zero solution, all the possible non-negative solutions are of the following type:*

$$u(t) = \begin{cases} 0 \text{ if } \tau \leq t \leq t_0, \\ u_{t_0}(t) \text{ if } t \geq t_0, \end{cases} \tag{10}$$

*where $u_{t_0}(\cdot)$ is the unique solution to the problem*

$$\begin{cases} \dfrac{\partial u}{\partial t} - \dfrac{\partial^2 u}{\partial x^2} = b(t) + \omega(t) u(t), \text{ on } (t_0, \infty) \times (0,1), \\ u(t,0) = u(t,1) = 0, \\ u(t_0, x) = 0, \end{cases} \tag{11}$$

*and $u(t)$ is positive for all $t > t_0$.*

**Proof.** By comparison, it is clear that the function given in (10) is a non-negative solution to problem (3) for any $t_0 \geq \tau$. It remains to check that these are the only ones and that $u(t)$ is positive for $t > t_0$.

Since any solution $u(\cdot)$ to problem (3) with $u_\tau \equiv 0$ satisfies that $u \in C([\tau, +\infty), C([0,1]))$, if $u(\bar{t})$ is not identically equal to 0, then there exists an interval $(t_0, t_1)$ containing $\bar{t}$ such that $u(t)$ is not identically equal to 0 for all $t \in (t_0, t_1)$. By Lemma 6 the element $u(t)$ is positive for any $t > t_0$. Thus, if $u(\bar{t}) \not\equiv 0$ for some $\bar{t} > \tau$, then $u(t)$ is positive for any $t \geq \bar{t}$. This implies that any non-negative solution not being identically equal to 0 has to be of the type (10) and $u(t)$ is positive for any $t > t_0$. Obviously, $u_{t_0}(\cdot)$ has to be the solution to problem (11). ■

**Corollary 8** *Assume that (4) holds. If $u_\tau \in V$ be such that $u_\tau \geq 0$, $u_\tau(x_0) = 0$ at some $x_0 \in (0,1)$ but $u_\tau \not\equiv 0$, then there cannot exist a non-negative solution backwards in time.*

**Proof.** By contradiction let there be a non-negative solution $u(\cdot)$ to problem (3) in some interval $[t_0, +\infty)$ with $t_0 < \tau$ such that $u(\tau) = u_\tau$. By Lemmas 6 and 7 we have that $u(\tau)$ needs to be positive, so $u_\tau(x_0) > 0$. ■

**Remark 9** *Assume that (4) holds. If $u_\tau \in H$ but $u_\tau \notin V$, then there cannot exist a non-negative solution $u(\cdot)$ backwards in time as well. This follows from the fact that $u \in C((\tau, +\infty), V)$.*

**Corollary 10** *Assume that (4) holds. If $u_\tau \equiv 0$, then the unique non-negative solution backwards in time is the zero solution, that is, $u(t) \equiv 0$ for $t \leq \tau$.*

Adding some extra conditions on the functions $b, \omega$, we will establish that the solutions are more regular. First, we state a technical lemma about the derivatives of the product of two functions.

**Lemma 11** *Let $S = (T_1, T_2) \subset \mathbb{R}$, $h \in W^{1,2}(S)$, $v \in W^{1,2}(S; H)$. Then $hv \in W^{1,2}(S; H)$ and*

$$\frac{d}{dt}(hv) = h\frac{dv}{dt} + \frac{dh}{dt}v. \tag{12}$$

*If $h \in W^{k,2}(S)$, $v \in W^{k,2}(S; H)$, $k \in \mathbb{N}$, then $hv \in W^{k,2}(S; H)$ and*

$$\frac{d^j}{dt^j}(hv) = \sum_{i=0}^{j} a_i \frac{d^i h}{dt^i}\frac{d^{j-i}v}{dt^{j-i}}, \text{ for } j = 1, ..., k, \tag{13}$$

*where $a_i$ are the coefficients of the Tartaglia/Pascal triangle.*

**Proof.** Since $W^{1,2}(S) \subset L^\infty(S)$, we get that $hv \in L^2(S; H)$. Let $h_n \in C^1(\overline{S})$, $v_n \in C^1(\overline{S}, H)$ be such that $h_n \to h$ in $W^{1,2}(S)$, $v_n \to v$ in $W^{1,2}(S; H)$ (see [22, Chapter IV]). Then

$$\frac{d}{dt}(h_n v_n) = h_n \frac{dv_n}{dt} + \frac{dh_n}{dt}v_n \to h\frac{dv}{dt} + \frac{dh}{dt}v \text{ in } L^1(S; H).$$

Thus, (12) follows and $h \in L^\infty(S)$, $v \in C(\overline{S}, H) \subset L^\infty(S; H)$ imply that $hv \in W^{1,2}(S; H)$.

Let $h \in W^{k,2}(S)$, $v \in W^{k,2}(S; H)$, $k \in \mathbb{N}$. By induction assume that $h \in W^{m,2}(S)$, $v \in W^{m,2}(S; H)$ and (13) holds for $m \in \{1, ..., j\}$ with $j \in \{1, ..., k-1\}$. Since $\frac{d^i h}{dt^i} \in W^{1,2}(S)$, $\frac{d^{j-i}v}{dt^{j-i}} \in W^{1,2}(S; H)$ for any $i \in \{0, ..., j\}$, the previous result gives that $\frac{d^i h}{dt^i}\frac{d^{j-i}v}{dt^{j-i}} \in W^{1,2}(S; H)$ and

$$\frac{d}{dt}\left(\frac{d^i h}{dt^i}\frac{d^{j-i}v}{dt^{j-i}}\right) = \frac{d^{i+1}h}{dt^{i+1}}\frac{d^{j-i}v}{dt^{j-i}} + \frac{d^i h}{dt^i}\frac{d^{j-i+1}v}{dt^{j-i+1}}. \tag{14}$$

Hence, $hv \in W^{j+1,2}(S; H)$. For any $i \in \{1, ..., j\}$ we have

$$\frac{d}{dt}\left(\frac{d^{i-1}h}{dt^{i-1}}\frac{d^{j-i+1}v}{dt^{j-i+1}}\right) = \frac{d^i h}{dt^i}\frac{d^{j-i+1}v}{dt^{j-i+1}} + \frac{d^{i-1}h}{dt^{i-1}}\frac{d^{j-i+2}v}{dt^{j-i+2}}, \tag{15}$$

so the second term in (14) and the first term in (15) are equal. Thus,

$$\frac{d^{j+1}}{dt^{j+1}}(hv) = a_0 h\frac{d^{j+1}v}{dt^{j+1}} + \sum_{i=1}^{j}(a_{i-1} + a_i)\frac{d^i h}{dt^i}\frac{d^{j-i+1}v}{dt^{j-i+1}} + a_j\frac{d^{j+1}h}{dt^{j+1}}v,$$

proving formula (13) for $j + 1$. ■

**Lemma 12** *Assume that (4) holds and, additionally, that $b, \omega \in W^{1,2}_{loc}(\mathbb{R})$. Let $u_\tau \in H$ be such that $u_\tau \geq 0$ and $\|u_\tau\| > 0$. Then the unique non-negative solution $u(\cdot)$ to problem (3) satisfies that*

$$u \in C((\tau, +\infty), H^3(0,1)) \cap C^1((\tau, +\infty), V), \tag{16}$$
$$\frac{d^2u}{dt^2} \in L^2_{loc}(\tau + \varepsilon, +\infty; H) \text{ for all } \varepsilon > 0.$$

*The partial derivatives $u_t, u_{xx}$ exist in the classical sense and are continuous on $(\tau, +\infty) \times [0, 1]$.*

*If $b, \omega \in W^{k+1,2}_{loc}(\mathbb{R})$, where $k \in \mathbb{N}$, then*

$$u \in C^{k+1}((\tau, +\infty), V),\ u \in \cap_{j=0}^{k} C^j \left( (\tau, +\infty), H^{2(k-j)+3}(0, 1) \right), \tag{17}$$
$$\frac{d^{k+2}u}{dt^{k+2}} \in L^2_{loc}(\tau + \varepsilon, +\infty; H) \text{ for all } \varepsilon > 0,\ u \in C^{k+1}((\tau, +\infty) \times [0, 1]).$$

**Proof.** By definition $u$ is the unique solution of problem (6) with $f(t, x) \in b(t)H_0(u(t, x)) + \omega(t)u(t, x)$ for a.a. $(t, x)$. In view of Lemma 6 $u(t)$ is positive for any $t > \tau$, so $H_0(u(t, x)) = 1$ for a.a. $(t, x) \in (\tau, +\infty) \times (0, 1)$. Since $u \in C((\tau, +\infty), V)$, $\frac{du}{dt} \in L^2_{loc}(\tau + \varepsilon, +\infty; H)$, for all $\varepsilon > 0$, and $b, \omega \in W^{1,2}_{loc}(\mathbb{R})$, we obtain by Lemma 11 that

$$f \in L^2_{loc}(\tau, +\infty; H) \cap C((\tau, +\infty), H^1(0, 1)) \cap W^{1,2}_{loc}(\tau + \varepsilon, +\infty; H) \text{ for all } \varepsilon > 0. \tag{18}$$

The first statement follows from Proposition 41 and Corollary 42 (see the appendix).

Let now $b, \omega \in W^{k+1,2}_{loc}(\mathbb{R})$ with $k \in \mathbb{N}$. By induction assume that for $j \in \{1, ..., k\}$ we have that $u \in C^j((\tau, +\infty), V)$ and

$$u \in \cap_{i=0}^{j-1} C^i \left( (\tau, +\infty), H^{2(j-i)+1}(0, 1) \right),$$
$$\frac{d^{j+1}u}{dt^{j+1}} \in L^2_{loc}(\tau + \varepsilon, +\infty; H) \text{ for all } \varepsilon > 0.$$

By Lemma 11 we get $f \in W^{j+1,2}_{loc}(\tau + \varepsilon, +\infty; H)$, for all $\varepsilon > 0$, and

$$f \in \cap_{i=0}^{j} C^i \left( (\tau, +\infty), H^{2(j-i)+1}(0, 1) \right).$$

Now, Lemma 43 and Corollary 44 yield

$$\frac{d^{j+2}u}{dt^{j+2}} \in L^2_{loc}(\tau + \varepsilon, +\infty; H) \text{ for all } \varepsilon > 0,\ u \in C^{j+1}((\tau, +\infty), V),$$
$$u \in \cap_{i=0}^{j} C^i \left( (\tau, +\infty), H^{2(j-i)+3}(0, 1) \right),\ u \in C^{j+1}((\tau, +\infty) \times [0, 1]).$$

■

Adding some extra conditions on the initial condition and the functions $b, \omega$, the solution will now be proved to be unique in the class of all solutions.

Let $V^{2r} = D(A^r)$. Take $u_\tau \in V^{2r}$ with $\frac{3}{4} < r < 1$. Hence, $u \in C([\tau, +\infty), V)$ implies that $u$ is a solution of problem (6) with $f \in L^\infty_{loc}(\tau, +\infty; L^\infty(0, 1))$, so Lemma 42.7 in [38] implies that

$$u \in C([\tau, +\infty), V^{2r}).$$

We observe that $H^s(0, 1)$ is continuously embedded in $C^1([0, 1])$ if $s > \frac{3}{2}$ [23, Lemma 4.4]. This fact, together with $D(A^r) = H^{2r}(0, 1) \cap H^1_0(0, 1)$ for $\frac{3}{4} < r < 1$ [21, Theorem 1], implies that

$$u \in C([\tau, +\infty), C^1([0, 1])). \tag{19}$$

**Lemma 13** *Let $u_\tau \in V^{2r}$, $\frac{3}{4} < r < 1$, be positive and such that $\frac{d}{dx}u_\tau(0) > 0$, $\frac{d}{dx}u_\tau(1) < 0$. Assume that (4) holds and, additionally, that $b, \omega \in W^{1,2}_{loc}(\mathbb{R})$. Then there is a unique solution $u(\cdot)$ to problem (3), which satisfies that $u(t)$ is positive for any $t \geq \tau$ and (16) as well. Moreover, $u_x(t, 0) > 0$, $u_x(t, 1) < 0$ for any $t \geq \tau$ and the partial derivatives $u_t, u_{xx}$ exists in the classical sense and are continuous on $(\tau, +\infty) \times [0, 1]$.*

*If, moreover, $b, \omega \in W^{k+1,2}_{loc}(\mathbb{R})$, , where $k \in \mathbb{N}$, then additionally $u$ satisfies (17).*

**Proof.** Let $u(\cdot)$ be an arbitrary solution to problem (3). First we will show that there exists an interval $[\tau, t_1]$, $t_1 > \tau$, such that $u(t)$ is positive and $u_x(t,0) > 0$, $u_x(t,1) < 0$, for any $t \in [\tau, t_1]$. Since $u_x$ is jointly continuous by (19), there exist $t_1 > \tau$, $0 < x_0 < x_1 < 1$ and $\alpha_0 > 0$ such that

$$u_x(t,x) \geq \alpha_0,\ \forall\, t \in [\tau, t_1],\ x \in [0, x_0]$$
$$u_x(t,1) \leq -\alpha_0,\ \forall\, t \in [\tau, t_1],\ x \in [x_1, 1].$$

Hence,

$$u(t,x) \geq \alpha_0 x \text{ for } x \in [0, x_0],\ t \in [\tau, t_1],$$
$$u(t,x) \geq \alpha_0(1-x) \text{ for } x \in [x_1, 1],\ t \in [\tau, t_1].$$

Finally, by the joint continuity of $u$ there exist $\tau < t_2 \leq t_1$ and $\alpha_1 > 0$ such that

$$u(t,x) \geq \alpha_1 \text{ for } x \in [x_0, x_1],\ t \in [\tau, t_2].$$

We state that in fact these properties hold in any interval $[\tau, t]$.

We observe first that there cannot be a time $t_0 > \tau$ and $x_0 \in (0,1)$ such that

$$\begin{aligned} u(t,x) &> 0\ \forall\, \tau \leq t < t_0,\ x \in [0,1], \\ u(t_0,x) &\geq 0,\ \forall x \in [0,1],\ u(t_0,x_0) = 0. \end{aligned} \tag{20}$$

In such a case, $u(t_0,x_0)$ is the minimum of the function $u$ in the region $\Omega = \{(t,x) : \tau < t \leq t_0,\ 0 \leq x \leq 1\}$.

Since by Lemma 6 $u(\cdot)$ is the unique non-negative solution in the interval $[\tau, t_0]$, Lemma 12 implies that the derivatives $u_t, u_{xx}$ exist in the classical sense and are continuous on $\Omega$. Then, as $u(t_0,x_0)$ is the minimum in $\Omega$, we have

$$u_t(t_0,x_0) \leq 0,\ u_{xx}(t_0,x_0) \geq 0.$$

Since

$$u_t(t,x) - u_{xx}(t,x) = b(t) + \omega(t)u(t,x) \text{ for } t \in (\tau, t_0),\ x \in (0,1),$$

we obtain by continuity that

$$0 \geq u_t(t_0,x_0) - u_{xx}(t_0,x_0) = b(t_0) + \omega(t_0)u(t_0,x_0) > 0,$$

which is a contradiction. Therefore, (20) cannot happen.

Further, we will prove that while $u(t)$ is positive the spatial derivatives at the boundary cannot vanish. By contradiction assume for example that for some $t_0 > \tau$ one has

$$u_x(t_0,0) = 0,$$
$$u_x(t,0) > 0, \text{ for } t \in [\tau, t_0),$$
$$u(t,x) > 0 \text{ for } t \in [\tau, t_0],\ x \in (0,1).$$

As we have seen before, the derivatives $u_t, u_{xx}$ exist in the classical sense and are continuous on $(\tau, t_0] \times [0,1]$. It is not possible that $u_{xx}(t_0,x) < 0$ for $x$ in some interval $[0, \varepsilon]$ as in such a case we would have that

$$u_x(t_0,x) - u_x(t_0,0) = u_x(t_0,x) = \int_0^x u_{yy}(t_0,y)dy < 0, \text{ for } x \in (0,\varepsilon],$$

and then

$$u(t_0,\varepsilon) = \int_0^\varepsilon u_y(t_0,y)dy < 0,$$

which is false. Therefore, there has to exist a sequence $u_{xx}(t_0,x_n) \geq 0$ with $x_n \to 0^+$. Thus,

$$u_t(t_0,x_n) = b(t_0) + \omega(t_0)\, u(t_0,x_n) + u_{xx}(t_0,x_n) \geq b_0 > 0.$$

By the uniform continuity of $u_t$ on the compact set $[\widetilde{\tau}, t_0] \times [0,1]$, where $\widetilde{\tau} > \tau$, there exists $\delta > 0$ such that

$$u_t(t,x_n) \geq \frac{b_0}{2} \text{ for all } n,\ t \in [t_0 - \delta, t_0].$$

Hence,

$$u(t_0,x_n)=u(t_0-\delta,x_n)+\int_{t_0-\delta}^{t_0}u_t(t,x_n)dt\geq\frac{\delta b_0}{2}\text{ for all }n,$$

so

$$u(t_0,x_n)\rightarrow u(t_0,0)=0$$

gives a contradiction. By the same argument $u_x$ cannot vanish at $x=1$.

The only possibility left is that $u$ becomes 0 near the boundary instantaneously at some moment of time, that is, there exist for example a time $t_0>\tau$ and sequences $t_n>t_0,\ x_n>0$ such that $t_n\rightarrow t_0,\ x_n\rightarrow 0$ and

$$\begin{aligned} u(t,x) &> 0 \text{ for } t\in[\tau,t_0],\ x\in(0,1)\,, \\ u(t_n,x_n) &= 0. \end{aligned} \tag{21}$$

We have seen that $u_x(t,0)>0$ for any $t\in[\tau,t_0]$. By the joint continuity of $u_x$ there exist $t_1>t_0,\ x_0>0$ and $\alpha_0>0$ such that $u_x(t,x)\geq\alpha_0$ for any $t\in[t_0,t_1],\ x\in[0,x_0]$. Thus, $u(t,x)\geq\alpha_0 x$ for $t\in[t_0,t_1]$, $x\in[0,x_0]$, which contradicts (21). In the same way one can show that $u$ cannot become 0 instantaneously near $x=1$.

We have proved that $u(t)$ is positive and $u_x(t,0)>0,\ u_x(t,1)<0$ for any $t\geq\tau$. Thus, Lemma 12 implies that the solution $u(\cdot)$ is unique in the class of all solutions and the regularity of $u$ as well. ∎

# 4 Structure of the pullback attractor in the positive cone

In this section, we will apply the previous results in order to study the structure and regularity of the pullback attractor for the multivalued process generated by the solutions of problem (3) in the positive cone.

## 4.1 The autonomous case

We start with the autonomous case, that is, we assume that $b(t)\equiv b>0,\ \omega(t)\equiv\omega\geq 0$. Hence, problem (3) becomes

$$\begin{cases} \dfrac{\partial u}{\partial t}-\dfrac{\partial^2 u}{\partial x^2}\in bH_0(u)+\omega u,\ \text{ on } (0,\infty)\times(0,1), \\ u(t,0)=u(t,1)=0, \\ u(0,x)=u_0(x),\ x\in(0,1)\,. \end{cases} \tag{22}$$

Additionally, in order to guarantee the existence of the global attractor, we will suppose throughout this section that

$$\omega<\pi^2, \tag{23}$$

where $\pi^2$ is the first eigenvalue of the operator $-\frac{\partial^2 u}{\partial x^2}$ in $V$.

Let us denote by $\mathcal{D}(u_0)$ the set of all solutions of the autonomous problem (22) with initial condition $u_0$ at $t=0$. Let $\mathcal{R}_0=\cup_{u_0\in H}\mathcal{D}(u_0)$ be the set of all solutions. Denoting by $P(H)$ the set of all non-empty subsets of $H$, the multivalued map $G:\mathbb{R}^+\times H\rightarrow P(H)$ is defined by

$$G(t,u_0)=\{u(t):u\in\mathcal{D}(u_0)\}.$$

We recall [43, 44] that this map satisfies that $G(0,\cdot)$ is the identity map and $G(t+s,u_0)=G(t,G(s,u_0))$ for all $t,s\in\mathbb{R}^+$, $u_0\in H$, that is, it is a strict multivalued semiflow, and that it possesses a global compact connected invariant attractor $\mathcal{A}$ in the phase space $H$, which means that:

- $\mathcal{A}$ is compact in $H$;
- $\mathcal{A}$ is connected in $H$;
- $\mathcal{A}=G(t,\mathcal{A})$ for all $t\in\mathbb{R}^+$ (that is, it is strictly invariant);

- $dist_H(G(t,B),\mathcal{A}) \to 0$ as $t \to +\infty$ for any set $B$ bounded in $H$ (that is, it is attracting).

The function $\varphi : \mathbb{R} \to H$ is said to be a complete trajectory of $\mathcal{R}_0$ if $\varphi(\cdot + h)\mid_{t\geq 0} \in \mathcal{R}_0$ for any $h \in \mathbb{R}$. It is bounded if the set $\cup_{t\in\mathbb{R}}\varphi(t)$ is bounded. The global attractor $\mathcal{A}$ consists of the union of the elements of all the bounded complete trajectories [2], that is,

$$\mathcal{A} = \{\varphi(0) : \varphi \text{ is a bounded complete trajectory of } \mathcal{R}_0\}.$$

Moreover, each bounded complete trajectory satisfies that

$$\varphi(t) \to z_1 \text{ as } t \to +\infty,$$
$$\varphi(t) \to z_2 \text{ as } t \to -\infty,$$

where $z_i$ are fixed points such that $E(z_1) < E(z_2)$ for the Lyapunov energy functional $E : V \to \mathbb{R}$ given by $E(u) = \frac{1}{2}\int_0^1 \left|\frac{du}{dx}\right|^2 dx - \int_0^1 |u|\, dx$. Also, the stationary points were fully described in [2], where it was shown that there is an infinite but countable number of them, denoted by $v_0 = 0,\ v_1^{\pm},\ v_2^{\pm}, ..., v_n^{\pm}, ...$, and that they are ordered by the functional $E$, that is, $E(0) < E\left(v_1^{\pm}\right) < E\left(v_2^{\pm}\right) < ...$ The functions $v_n^{\pm}$ have exactly $n-1$ zeros in $(0,1)$ and $v_n^{+} = -v_n^{-}$. In particular, $v_1^{\pm}$ have no zeros in $(0,1)$ and they are given by

$$v_1^{+}(x) = \frac{b}{\omega}\cos\left(\sqrt{\omega}x\right) + \frac{b\left(1-\cos\sqrt{\omega}\right)}{\omega\sin\left(\sqrt{\omega}\right)}\sin\left(\sqrt{\omega}x\right) - \frac{b}{\omega} \text{ if } 0 < \omega < \pi^2,$$
$$v_1^{+}(x) = -\frac{x^2}{2} + \frac{x}{2} \text{ if } \omega = 0.$$

It is clear that $v_1^{\pm} \in C^{\infty}([0,1])$. By Lemma 13 the solutions starting at $v_1^{\pm}$ are unique. In addition, we know from [14, Section 3] that

$$v_1^{-} \leq z \leq v_1^{+} \text{ for any } z \in \mathcal{A}.$$

In relation with the regularity of the attractor, it was proved in [2, Theorem 3.1 and Corollary 3.2] that the global attractor is compact in $W^{2-\delta,p}(0,1)$ for any $\delta > 0$, $1 \leq p < \infty$ and that it is bounded in $L^{\infty}(0,1)$. In our particular one dimensional situation, the last result is also consequence of the continuous embedding $H_0^1(0,1) \subset L^{\infty}(0,1)$. Let us extend these results to the $V^{2r}$ spaces.

**Lemma 14** *Let (23) hold. The global attractor $\mathcal{A}$ is compact in $V^{2r}$ for any $0 \leq r < 1$.*

**Proof.** For any $z \in \mathcal{A}$ the invariance of $\mathcal{A}$ implies the existence of $u \in \mathcal{D}_0(u_0)$, $u_0 \in \mathcal{A}$, such that $z = u(1)$ and $u(t) \in \mathcal{A}$ for all $t \geq 0$. As $u$ is a mild solution, by the variation of constants formula we get

$$z = e^{-A}u_0 + \int_0^1 e^{-A(1-s)}f(s)ds,$$

where $f(s,x) \in bH_0(u(t,x)) + \omega u(t,x)$ for a.a. $(t,x)$. By the boundedness of $\mathcal{A}$ in $H$, there exists a universal constant such that

$$\|u_0\| \leq C,\ \|f\|_{L^{\infty}(0,1;H)} \leq C,$$

where $C$ does not depend on the chosen $z$. Then by standard estimates of the norm of $e^{-At}$ in $V^{2r}$ [38, Theorem 37.5] for some constants $M_r > 0$, $a \in \mathbb{R}$ we have

$$\|A^r z\| \leq \left\|A^r e^{-A}u_0\right\| + \int_0^1 \left\|A^r e^{-A(1-s)}f(s)\right\| ds$$
$$\leq M_r e^{-a}C + M_r C\int_0^1 (1-s)^{-r}\, ds \leq C_r.$$

Therefore, $\mathcal{A}$ is bounded in $V^{2r}$ if $r < 1$.

The compact embedding $V^{\alpha} \subset V^{\beta}$ for $\alpha > \beta$ implies that $\mathcal{A}$ is relatively compact in $V^{2r}$ for any $r < 1$. Since $\mathcal{A}$ is closed in $H$, it is closed in $V^{2r}$ as well, so $\mathcal{A}$ is compact in $V^{2r}$ for any $0 \leq r < 1$. ■

Let us consider non-negative solutions to problem (3). We have seen that for any $u_0 \in H$ satisfying $u_0 \geq 0$ there exists at least one solution $u(\cdot)$ such that $u(t) \geq 0$ for any $t \geq 0$. We denote by $\mathcal{D}^+(u_0)$ the set of all non-negative solutions of problem (3) with initial condition $u_0 \geq 0$ at time $t = 0$ and let $\mathcal{R}_0^+ = \cup_{u_0 \in H} \mathcal{D}^+(u_0)$. Also, let $H^+$ be the positive cone of $H$, that is,

$$H^+ = \{v \in H : v(x) \geq 0 \text{ for a.a. } x \in (0,1)\}.$$

We define the strict multivalued semiflow $G^+ : \mathbb{R}^+ \times H^+ \to P(H^+)$ given by

$$G^+(t, u_0) = \{u(t) : u \in \mathcal{D}_0^+(u_0)\}.$$

The bounded complete trajectories of $\mathcal{R}_0^+$ are all the bounded complete trajectories of $\mathcal{R}_0$ such that $\varphi(t) \geq 0$ for any $t \in \mathbb{R}$. The semiflow $G^+$ possesses a global compact connected invariant attractor, denoted by $\mathcal{A}^+$ and

$$\mathcal{A}^+ = \{\varphi(0) : \varphi \text{ is a bounded complete trajectory of } \mathcal{R}_0^+\}.$$

In the positive cone the only fixed points are 0 and $v_1^+$. Therefore, any bounded complete trajectory different from 0 or $v_1^+$ has to satisfy that

$$\begin{aligned} &\varphi(t) \to v_1^+ \text{ as } t \to +\infty, \\ &\varphi(t) \to 0 \text{ as } t \to -\infty. \end{aligned} \tag{24}$$

Also,

$$0 \leq z \leq v_1^+ \text{ for any } z \in \mathcal{A}^+.$$

We observe that by Lemma 14 the convergence in (24) is true in the $V^{2r}$ spaces with $r < 1$ and by the compact embedding $V^{2r} \subset C^1([0,1])$, $r > \frac{3}{4}$, in the space $C^1([0,1])$ as well.

Let us consider the set $D = \{v \in V : v(x) > 0 \text{ for } x \in (0,1)\}$.

**Lemma 15** *Let (23) hold. The fixed point $v_1^+$ is the unique bounded complete trajectory $\varphi(\cdot)$ for which there exists a time $t_0$ satisfying that $\varphi(t) \in D$ for any $t \leq t_0$.*

**Proof.** Let there exist another bounded complete trajectory $\varphi(\cdot)$ for which there exists a time $t_0$ satisfying that $\varphi(t) \in D$ for any $t \leq t_0$. Then they are solutions in any interval $[\tau, t_0]$ of the problem

$$\begin{cases} \dfrac{\partial u}{\partial t} - \dfrac{\partial^2 u}{\partial x^2} \in b + \omega u, \text{ on } (\tau, t_0) \times (0,1), \\ u|_{\partial\Omega} = 0, \\ u(\tau, x) = u_\tau(x), \end{cases} \tag{25}$$

with $u_\tau = v_1^+$ and $u_\tau = \varphi(\tau)$, respectively. Taking the difference of the two equations and multiplying by $v = \varphi - v_1^+$ we obtain that

$$\frac{1}{2}\frac{d}{dt}\|v\|^2 + \left(\pi^2 - \omega\right)\|v\|^2 \leq 0,$$

so

$$\|v(t)\|^2 \leq e^{-2\left(\pi^2-\omega\right)(t-\tau)}\|v(\tau)\|^2 \to 0 \text{ as } \tau \to -\infty \text{ for all } t \leq t_0. \tag{26}$$

Hence, $\varphi(t) = v_1^+$ for all $t \leq t_0$. Since the solution of the autonomous problem (3) with $u_\tau = v_1^+$ is unique by Lemma 13, we have that $\varphi \equiv v_1^+$. ■

**Corollary 16** *Let (23) hold. If $\varphi$ is a bounded complete trajectory such that $\varphi(t) \to 0$ as $t \to -\infty$, then for any $t_1$ there exists a time $t_0 \leq t_1$ such that $\varphi(t_0) \notin D$.*

It is clear that the functions

$$\varphi(t) = \begin{cases} 0 \text{ if } t \leq \tau, \\ u(t) \text{ if } t \geq \tau, \end{cases} \tag{27}$$

where $u(\cdot)$ is the unique solution to problem (25) in $[\tau, +\infty)$ with $u_\tau = 0$, are bounded complete trajectories of $\mathcal{R}_0^+$ connecting 0 and $v_1^+$, as in this case the convergence in (26) holds true for $t \to +\infty$. In fact, these are the unique possible connections.

**Theorem 17** *Let (23) hold. Any bounded complete trajectory of $\mathcal{R}_0^+$ distinct from* 0 *and $v_1^+$ is of the type (27).*

**Proof.** In view of Corollary 8, if $v \in \mathcal{A}^+$, then either $v \equiv 0$ or $v$ is positive. Let $\varphi(\cdot)$ be a bounded complete trajectory of $\mathcal{R}_0^+$ distinct from 0 and $v_1^+$, so there is $t_1 \in \mathbb{R}$ such that $\varphi(t_1)$ is positive. By Lemmas 15 and 6 there must be a first time $\tau < t_1$ for which $\varphi(\tau) \equiv 0$ and $\gamma(t) > 0$ for all $t > \tau$, and Corollary 10 implies then that $\varphi(t) \equiv 0$ for all $t \leq \tau$. Hence, $\varphi$ is of the type (27). ■

We obtain also that the trajectories inside the global attractor are regular.

**Proposition 18** *Let (23) hold. For any bounded complete trajectory $\varphi$ of $\mathcal{R}_0^+$ which is not a fixed point, that is, of the type (27), we have:*

1. *For any $t_0 > \tau$ the function $u = \varphi\mid_{t \geq t_0}$ is the unique non-negative solution to problem (3) in $[t_0, +\infty)$ with $u_{t_0} = \varphi(t_0)$;*
2. *$\varphi \in C^\infty((\tau, +\infty) \times [0,1])$ and $\varphi \in C((-\infty, +\infty), C^1([0,1]))$;*
3. *There exists a time $t_0 > \tau$ such that $u = \varphi\mid_{t \geq t_0}$ is the unique solution to problem (3) in $[t_0, +\infty)$ with $u_{t_0} = \varphi(t_0)$.*

**Proof.** The first two statements are a direct consequence of Lemmas 6, 12 and (19).

Taking into account that $V^{2r} \subset C^1([0,1])$ for $r > \frac{3}{4}$, we obtain that

$$\varphi(t) \to v_1^+ \text{ in } C^1([0,1]) \text{ as } t \to +\infty.$$

Thus, there exist $t_0$ such that $\varphi_x(t,0) > 0$, $\varphi_x(t,1) < 0$, $\varphi(t,x) > 0$ if $x \in (0,1)$, for all $t \geq t_0$. The result follows then from Lemma 13. ■

**Corollary 19** *Let (23) hold. Then, $\mathcal{A}^+ \subset C^\infty([0,1])$.*

## 4.2 The nonautonomous case

Throughout this section we assume that the functions $b(\cdot), \omega(\cdot)$ satisfy (4) and, additionally, that

$$\omega_1 < \pi^2. \tag{28}$$

Let $\mathcal{D}_\tau(u_\tau)$ be the set of all solutions of problem (3) with initial condition $u_\tau$ at time $\tau$ and let $\mathcal{R}_\tau = \cup_{u_\tau \in H} \mathcal{D}_\tau(u_\tau)$, $\mathcal{R} = \cup_{\tau \in \mathbb{R}} \mathcal{R}_\tau$. Denoting $\mathbb{R}^2_\geq = \{(t,s) \in \mathbb{R}^2 : t \geq s\}$ we define the family of operators $U : \mathbb{R}^2_\geq \times H \to P(H)$ given by

$$U(t,s,x) = \{u(t) : u(\cdot) \in \mathcal{D}_s(x)\}.$$

$U$ is a strict multivalued process, that is, $U(t,t,\cdot)$ is the identity map and $U(t,s,U(s,\tau,x)) = U(t,\tau,x)$ for all $t \geq s \geq \tau$ and $x \in H$. Under assumptions (4), (28) this process has a compact strictly invariant pullback attractor $\{\mathcal{A}(t)\}_{t \in \mathbb{R}}$ [14, Theorem 5], which means that:

- The sets $\mathcal{A}(t)$ are compact for any $t$;
- $\mathcal{A}(t) = U(t,s,\mathcal{A}(s))$ for all $t \geq s$ (strict invariance);
- $dist(U(t,s,B), \mathcal{A}(t)) \to 0$ as $s \to -\infty$ for any bounded set $B \subset H$ (pullback attraction);

- $\{\mathcal{A}(t)\}_{t\in\mathbb{R}}$ is the minimal pullback attracting family, that is, if $\{\mathcal{K}(t)\}_{t\in\mathbb{R}}$ is a family of closed pullback attracting sets, then $\mathcal{A}(t)\subset\mathcal{K}(t)$ for any $t$.

The function $\gamma:\mathbb{R}\to H$ is called a complete trajectory of $\mathcal{R}$ if

$$\varphi=\gamma\mid_{[\tau,+\infty)}\in\mathcal{R}_\tau \text{ for any } \tau\in\mathbb{R}.$$

A complete trajectory $\gamma$ is bounded if the set $\cup_{t\in\mathbb{R}}\gamma(t)$ is bounded in $H$. The pullback attractor $\mathcal{A}(t)$ consists of the union of the elements of all the bounded complete trajectories [14, Lemma 6], that is,

$$\mathcal{A}(t)=\{\gamma(t):\gamma \text{ is a bounded complete trajectory of } \mathcal{R}\}.$$

Moreover, it is known [14, Theorem 5 and Corollary 6] that $\cup_{t\in\mathbb{R}}\mathcal{A}(t)$ is bounded in $V$ (so $\overline{\cup_{t\in\mathbb{R}}\mathcal{A}(t)}$ is compact in $H$) and that the sets $\mathcal{A}(t)$ are compact in $V$. Let us prove that in fact the union $\cup_{t\in\mathbb{R}}\mathcal{A}(t)$ is relatively compact in $V^{2r}$ for $0\leq r<1$.

**Lemma 20** *Assume that (4), (28) hold. Then the set $\cup_{t\in\mathbb{R}}\mathcal{A}(t)$ is relatively compact in $V^{2r}$ for any $0\leq r<1$.*

**Proof.** For any $z\in\mathcal{A}(t)$ the equality $\mathcal{A}(t)=U(t,t-1,\mathcal{A}(t-1))$ implies the existence of $u(\cdot)\in\mathcal{D}_{t-1}(u_{t-1})$, $u_{t-1}\in\mathcal{A}(t-1)$, such that $z=u(t)$ and that $u(s)\in\mathcal{A}(s)$ for any $s\geq t-1$. As $u$ is a mild solution, by the variation of constants formula we get

$$z=e^{-A}u_{t-1}+\int_{t-1}^{t}e^{-A(t-s)}f(s)ds,$$

where $f(s,x)\in b(t)H_0(u(t,x))+\omega(t)u(t,x)$ for a.a. $(t,x)$. By the boundedness of $\cup_{s\in\mathbb{R}}\mathcal{A}(s)$ in $H$, there exists an universal constant $C>0$ such that

$$\|u_{t-1}\|\leq C,\ \|f\|_{L^\infty(t-1,t;H)}\leq C,$$

where $C$ does not depend either on the chosen $z$ or $t$. Then using the estimates of the norm of $e^{-At}$ in $V^{2r}$ [38, Theorem 37.5] there exist $M_r>0$, $a\in\mathbb{R}$ such that

$$\begin{aligned}\|A^r z\| &\leq \left\|A^r e^{-A}u_{t-1}\right\|+\int_{t-1}^{t}\left\|A^r e^{-A(t-s)}f(s)\right\|ds\\ &\leq M_r e^{-a}C+M_r C\int_{t-1}^{t}(t-s)^{-r}\,ds\leq C_r.\end{aligned}$$

Therefore, $\mathcal{A}(t)$ is bounded in $V^{2r}$ if $r<1$ uniformly in $t\in\mathbb{R}$. The compact embedding $V^\alpha\subset V^\beta$ for $\alpha>\beta$ implies that $\cup_{t\in\mathbb{R}}\mathcal{A}(t)$ is relatively compact in $V^{2r}$ for any $0\leq r<1$. ■

By [14, Theorem 6] we know that under assumptions (4), (28) there exists a bounded complete trajectory $\xi_M$ such that:

1. $\xi_M(t)$ is positive for any $t\in\mathbb{R}$.
2. $-\xi_M(t)\leq\gamma(t)\leq\xi_M(t)$ for all $t\in\mathbb{R}$ and any bounded complete trajectory $\gamma$.
3. $w^+_{b_0,\omega_0}\leq\xi_M(t)\leq w^+_{b_1,\omega_1}$, where $w^+_{b_i,\omega_i}$ denote the positive fixed point $v_1^+$ of the autonomous problem (3) with $b=b_i,\ \omega=\omega_i$.
4. $\xi_M$ is the unique bounded complete trajectory such that $\xi_M(t)\in D$ for all $t\in\mathbb{R}$, where we recall that $D=\{v\in V:v(x)>0 \text{ for } x\in(0,1)\}$.

From these properties the following lemma follows immediately.

**Lemma 21** *Assume that (4), (28) hold. If $\gamma$ is a bounded complete trajectory such that $\gamma(t)\to 0$ as $t\to-\infty$, then there exists a time $t_0$ such that $\gamma(t_0)\notin D$.*

**Proof.** By contradiction let $\gamma$ be a bounded complete trajectory such that $\gamma(t) \to 0$ as $t \to -\infty$ and $\gamma(t) \in D$ for any $t \in \mathbb{R}$. Hence, $\gamma \equiv \xi_M$. However, $\xi_M(t) \geq w^+_{b_0,\omega_0}$, so $\gamma(t)$ cannot converge to 0. ■

Now we are in position to improve the fourth point above and obtain that in fact $\xi_M$ is the unique non-degenerate bounded complete solution at $-\infty$, which means that there exists a time $t_0$ such that $\xi_M(t) \in D$ for any $t \leq t_0$. This question was left as an open problem in [14, Remark 2].

**Lemma 22** *Assume that (4), (28) hold. Then:*

- *If $\gamma(\cdot)$ is a non-negative bounded complete trajectory such that for some $t_0$ we have that $\gamma(t) \in D$ for all $t \leq t_0$, then $\gamma(t) = \xi_M(t)$ for all $t \in \mathbb{R}$, that is, $\xi_M$ is the unique non-negative bounded complete trajectory which is non-degenerate at $-\infty$. Also, for any $\tau \in \mathbb{R}$ the function $u = \xi_M |_{t \geq \tau}$ is the unique non-negative solution to problem (3) with $u_\tau = \xi_M(\tau)$.*
- *Assume, additionally, that $b, \omega \in W^{1,2}_{loc}(\mathbb{R})$. Then $\xi_M$ is the unique bounded complete trajectory which is non-degenerate at $-\infty$. Also, for any $\tau \in \mathbb{R}$ the function $u = \xi_M |_{t \geq \tau}$ is the unique solution to problem (3) with $u_\tau = \xi_M(\tau)$.*

**Proof.** Let us consider the first statement. Since $\gamma$ is non-negative, Lemma 6 implies that for all $\tau \in \mathbb{R}$ the function $u = \gamma |_{t \geq \tau}$ is the unique non-negative solution to problem (3) and $\gamma(\tau) \in D$. Thus, by the previous results it follows that $\gamma(t) = \xi_M(t)$ for all $t \in \mathbb{R}$.

We prove further the second statement. As $\mathcal{A}(t) \subset C^1([0,1])$ by Lemma 20 and the embedding $V^{2r} \subset C^1([0,1])$ for $r > \frac{3}{4}$, it follows from $w^+_{b_0,\omega_0} \leq \xi_M(t)$, $\frac{d}{dx} w^+_{b_0,\omega_0}(0) > 0$, $\frac{d}{dx} w^+_{b_0,\omega_0}(1) < 0$ that $\frac{\partial}{\partial x}\xi_M(t,0) > 0$, $\frac{\partial}{\partial x}\xi_M(t,1) < 0$ for any $t \in \mathbb{R}$. Hence, from Lemma 13 and $w^+_{b_0,\omega_0} \in D$ we have that for any $\tau \in \mathbb{R}$ the function $u = \xi_M |_{t \geq \tau}$ is the unique solution to problem (3) with $u_\tau = \xi_M(\tau)$. Let now $\gamma$ be a bounded complete trajectory which is non-degenerate at $-\infty$. Thus, by Corollary 8 in [14] there exists a time $t_0$ such that $\gamma(t) = \xi_M(t)$ for all $t \leq t_0$. By the uniqueness of the solution $u = \xi_M |_{t \geq t_0}$ with $u_{t_0} = \xi_M(t_0)$ we get that $\gamma(t) = \xi_M(t)$ for any $t \in \mathbb{R}$. ■

**Corollary 23** *Assume that (4), (28) hold. Then if $\gamma$ is a non-negative bounded complete trajectory such that $\gamma(t) \to 0$ as $t \to -\infty$, for any $t_1$ there exists $t_0 \leq t_1$ such that $\gamma(t_0) \notin D$.*

*If, additionally, $b, \omega \in W^{1,2}_{loc}(\mathbb{R})$, then the result is valid for any bounded complete trajectory $\gamma$ such that $\gamma(t) \to 0$ as $t \to -\infty$.*

**Proof.** If there is $t_1$ such that $\gamma(t) \in D$ for all $t \leq t_1$, then Lemma 22 implies that $\gamma \equiv \xi_M$, so that the convergence $\gamma(t) \to 0$, as $t \to -\infty$, is not possible. The same proof is valid for the second statement. ■

We denote by $\mathcal{D}^+_\tau(u_\tau)$ the set of all non-negative solutions of problem (3) with initial condition $u_\tau \in H^+$ at time $\tau$ and let $\mathcal{R}^+_\tau = \cup_{u_\tau \in H} \mathcal{D}^+_\tau(u_\tau)$, $\mathcal{R}^+ = \cup_{\tau \in \mathbb{R}} \mathcal{R}^+_\tau$. We define the map $U^+ : \mathbb{R}_\geq \times H^+ \to P(H^+)$ given by

$$U^+(t, \tau, u_\tau) = \{u(t) : u \in \mathcal{D}^+_\tau(u_\tau)\}.$$

$U^+$ is a strict multivalued process, which follows easily by the translation and concatenation properties of solutions (see [14, Section 4]). The bounded complete trajectories of $\mathcal{R}^+$ are all the bounded complete trajectories $\gamma(\cdot)$ of $\mathcal{R}$ such that $\gamma(t) \geq 0$ for any $t \in \mathbb{R}$. The process $U^+$ possesses a compact strictly invariant pullback attractor, denoted by $\{\mathcal{A}^+(t)\}_{t \in \mathbb{R}}$, and

$$\mathcal{A}^+(t) = \{\gamma(t) : \gamma \text{ is a bounded complete trajectory of } \mathcal{R}^+\}.$$

As $\mathcal{A}^+(t) \subset \mathcal{A}(t)$ for all $t$, $\{\mathcal{A}^+(t)\}_{t \in \mathbb{R}}$ inherits the regularity properties of $\{\mathcal{A}(t)\}_{t \in \mathbb{R}}$ described above.

We have seen in Lemma 22 that the function $\xi_M$ is the unique non-negative bounded complete trajectory which is non-degenerate at $-\infty$. This function plays the role of a positive nonautonomous equilibrium. As in the autonomous case, we will obtain a precise characterization of the pullback attractor in the positive cone in terms of the bounded complete trajectories connecting 0 and $\xi_M$.

As in (27), we define the map

$$\varphi(t) = \begin{cases} 0 \text{ if } t \leq \tau, \\ u(t) \text{ if } t \geq \tau, \end{cases} \tag{29}$$

where $u(\cdot)$ is the unique solution to problem

$$\begin{cases} \dfrac{\partial u}{\partial t} - \dfrac{\partial^2 u}{\partial x^2} \in b(t) + \omega(t)u, \text{ on } (\tau, +\infty) \times (0,1), \\ u|_{\partial\Omega} = 0, \\ u(\tau, x) = 0, \end{cases} \tag{30}$$

It is clear that a function of the type (29) is a bounded complete trajectory of $\mathcal{R}^+$. Alos, it is easy to see that $\|u(t) - \xi_M(t)\| \to 0$ as $t \to +\infty$. Indeed, for $v(t) = u(t) - \xi_M(t)$ we have that

$$\frac{d}{dt} \|v\|^2 + 2\pi^2 \|v\|^2 \leq 2\omega(t) \|v\|^2 \leq 2\omega_1 \|v\|^2.$$

We denote $\delta = \pi^2 - \omega_1 > 0$. Then

$$\|v(t)\|^2 \leq e^{-2\delta(t-\tau)} \|v(\tau)\|^2 \to 0 \text{ as } t \to +\infty.$$

Obviously, $u(t) \to 0$ as $t \to -\infty$. Hence, $\varphi(\cdot)$ is a connection from 0 to $\xi_M$, that is,

$$u(t) \to 0 \text{ as } t \to -\infty,$$
$$u(t) - \xi_M(t) \to 0 \text{ as } t \to +\infty.$$

We will establish that these are the only possible bounded complete trajectories of $\mathcal{R}^+$.

**Theorem 24** *Assume that (4), (28) hold. Then any bounded complete trajectory of $\mathcal{R}^+$ distinct from 0 and $\xi_M$ is of the type (29).*

**Proof.** By Corollary 8, if $v \in \mathcal{A}^+(t)$, then either $v \equiv 0$ or $v$ is positive. We take a bounded complete trajectory $\gamma(\cdot)$ of $\mathcal{R}^+$ different from 0 and $\xi_M$, so there exists $t_1 \in \mathbb{R}$ such that $\gamma(t_1) > 0$. By Lemmas 6 and 22 there exists a first time $\tau < t_1$ for which $\gamma(\tau) \equiv 0$ and $\gamma(t) > 0$ for all $t > \tau$. Thus, Corollary 10 implies that $\gamma(t) \equiv 0$ for all $t \leq \tau$, and then $\gamma$ is of the type (29). ■

**Remark 25** *The structure of the pullback attractor in the positive cone is the same as for the ordinary differential inclusion*

$$\begin{cases} \dfrac{du}{dt} + \lambda u \in b(t)H_0(u), \text{ on } (\tau, +\infty), \\ u(\tau) = u_\tau, \end{cases}$$

*where $\lambda > 0$, which was studied in [13].*

We conclude this section by obtaining some regularity results for the solutions inside the pullback attractor. The following proposition is proved in the same way as Proposition 18 but using that any bounded complete trajectory $\gamma$ of the type (29) satisfies that

$$\|\gamma(t) - \xi_M(t)\|_{C^1([0,1])} \to 0 \text{ as } t \to +\infty.$$

**Proposition 26** *Assume that (4), (28) hold. Then any bounded complete trajectory $\gamma$ of $\mathcal{R}^+$ different from 0 and $\xi_M$, that is, of the type (29), satisfies that for any $t_0 > \tau$ the function $u = \gamma|_{t \geq t_0}$ is the unique non-negative solution to problem (3) in $[t_0, +\infty)$ with $u_{t_0} = \gamma(t_0)$. If, additionally, $b, \omega \in W^{k,2}_{loc}(\mathbb{R})$, $k \in \mathbb{N}$, then:*

1. *$\gamma \in C^k((\tau, +\infty), V)$, $\gamma \in \cap_{j=0}^{k-1} C^j\left((\tau, +\infty), H^{2(k-j)+1}(0,1)\right)$ and $\gamma \in C^k((\tau, +\infty) \times [0,1])$.*
2. *There exists a time $t_0 > \tau$ such that $u = \gamma|_{t \geq t_0}$ is the unique solution to problem (3) in $[t_0, +\infty)$ with $u_{t_0} = \gamma(t_0)$.*

*Moreover, if $b, \omega \in W^{k,2}_{loc}(\mathbb{R})$ for any $k \in \mathbb{N}$, then $\gamma \in C^\infty((\tau, +\infty) \times [0,1])$.*

In the case of the nonautonomous equilibrium $\xi_M$, the regularity is extended to the whole line $(-\infty,+\infty)$.

**Proposition 27** *Assume that (4), (28) hold. If* $b,\omega\in W^{k,2}_{loc}(\mathbb{R})$, $k\in\mathbb{N}$, *then*

$$\xi_M\in C^k((-\infty,+\infty),V),\ \xi_M\in\cap_{j=0}^{k-1}C^j\left((-\infty,+\infty),H^{2(k-j)+1}(0,1)\right),\ \xi_M\in C^k((-\infty,+\infty)\times[0,1]).$$

*Moreover, if* $b,\omega\in W^{k,2}_{loc}(\mathbb{R})$ *for any* $k\in\mathbb{N}$, *then* $\xi_M\in C^\infty((-\infty,+\infty)\times[0,1])$.

**Corollary 28** *Assume that (4), (28) hold. We have:*

1. *If* $b,\omega\in W^{k,2}_{loc}(\mathbb{R})$, *where* $k\in\mathbb{N}$, *then* $\cup_{t\in\mathbb{R}}\mathcal{A}^+(t)\subset H^{2k+1}(0,1)\subset C^{2k}([0,1])$.
2. *If* $b,\omega\in W^{k,2}_{loc}(\mathbb{R})$ *for any* $k\in\mathbb{N}$, *then* $\cup_{t\in\mathbb{R}}\mathcal{A}^+(t)\subset C^\infty([0,1])$.

# 5 Comparison of different non-autonomous attractors

In this section we are going to consider different type of attractors like the uniform attractor, the cocycle attractor, the skew-product semiflow attractor and, of course, the pullback attractor in order to establish the relationship between them. Moreover, we will prove that the attractor of the skew-product semiflow in the positive cone has a gradient structure.

For this aim we need to consider not only problem (3) itself but all the problems generated by the translations of the functions $b(\cdot)$, $\omega(\cdot)$, that is, by the hull of these functions. In order to define this hull properly, along with conditions (4), (28), we need the following extra assumption:

$$\text{the functions } b \text{ and } \omega \text{ are uniformly continuous in } (-\infty,+\infty). \tag{31}$$

For a sequence $t_n\in\mathbb{R}$ we define the functions $\sigma_n(\cdot)=(b(\cdot+t_n),\omega(\cdot+t_n))\in C(\mathbb{R},\mathbb{R}^2)$. As usual, the space $C(\mathbb{R},\mathbb{R}^2)$ is equipped with the topology of uniform convergence on compact sets of $\mathbb{R}$. The sequence $\sigma_n$ is uniformly bounded and uniformly (in $(-\infty,+\infty)$) equicontinuous. Therefore, by applying the Ascoli-Arzelà theorem and a diagonal argument there exists a subsequence $\sigma_{n'}$ and a uniformly continuous function $\sigma(\cdot)=\left(\overline{b}(\cdot),\overline{\omega}(\cdot)\right)$ satisfying (4), (28) as well (with the same constants $b_0,\ b_1,\ \omega_1$) such that

$$\sigma_n\to\sigma \text{ in } C(\mathbb{R},\mathbb{R}^2).$$

Thus, we define the hull of $(b(\cdot),\omega(\cdot))$ by

$$\Sigma=cl_{C(\mathbb{R},\mathbb{R}^2)}\{(b(\cdot+t),\omega(\cdot+t)):t\in\mathbb{R}\},$$

which is a compact set of the metrizable space $C(\mathbb{R},\mathbb{R}^2)$, and consider the family of problems (3) given by each element $\sigma(\cdot)=\left(\overline{b}(\cdot),\overline{\omega}(\cdot)\right)\in\Sigma$. We define the translation operator $\theta_s:\Sigma\to\Sigma,\ s\in\mathbb{R}$, given by $\theta_s\sigma(\cdot)=\sigma(\cdot+s)$. Clearly, $\theta_0$ is the identity operator and $\theta_{s+r}=\theta_s\circ\theta_r$, so $\theta$ is a group, which is called the driving group in $\Sigma$. Also, it is not difficult to see that the map $(t,\sigma)\mapsto\theta_t\sigma$ is continuous and that $\theta_t\Sigma=\Sigma$ for any $t\in\mathbb{R}$.

Let now $\mathcal{D}_{\tau,\sigma}(u_\tau)$ be the set of all solutions of problem (3) with initial condition $u_\tau$ at time $\tau$ with symbol $\sigma$ and let $\mathcal{R}_{\tau,\sigma}=\cup_{u_\tau\in H}\mathcal{D}_{\tau,\sigma}(u_\tau)$, $\mathcal{R}_\sigma=\cup_{\tau\in\mathbb{R}}\mathcal{R}_{\tau,\sigma}$. For each $\sigma\in\Sigma$ the operator $U_\sigma:\mathbb{R}^2_\geq\times H\to P(H)$ given by

$$U_\sigma(t,\tau,x)=\{u(t):u(\cdot)\in\mathcal{D}_{\tau,\sigma}(x)\}$$

is a strict multivalued process. We obtain then a family of multivalued processes. Additionally, we will need the following translation property:

$$U_\sigma(t+h,\tau+h,x)=U_{\theta_h\sigma}(t,\tau,x) \text{ for all } x\in H,\ \sigma\in\Sigma,\ h\in\mathbb{R},\ (t,\tau)\in\mathbb{R}^2_\geq. \tag{32}$$

**Lemma 29** *Let (4), (28), (31) hold. Then the family* $\{U_\sigma\}$ *satisfies (32).*

**Proof.** If $y \in U_\sigma(t+h, \tau+h, x)$, there is $u \in \mathcal{D}_{\tau+h,\sigma}(x)$ such that $y = u(t+h)$, where $\sigma = (\overline{b}, \overline{\omega})$. Using the concept of mild solution we have a selection $r \in L^2_{loc}(\tau+h, +\infty; H)$ of the map $R(t, u(t))$ such that

$$u(t+h) = e^{-A(t-\tau)}x + \int_{\tau+h}^{t+h} e^{-A(t+h-s)}r(s)ds \text{ for } \tau \leq t < +\infty.$$

Let $v(\cdot) = u(\cdot + h)$, $r_h(\cdot) = r(\cdot + h) = \theta_h r \in L^2_{loc}(\tau, +\infty; H)$. Then

$$\begin{aligned} v(t) &= e^{-A(t-\tau)}x + \int_{\tau}^{t} e^{-A(t-s)}r(s+h)ds \\ &= e^{-A(t-\tau)}x + \int_{\tau}^{t} e^{-A(t-s)}r_h(s)ds \text{ for all } \tau \leq t < +\infty. \end{aligned}$$

As $r_h(t,x) \in \overline{b}(t+h)H_0(v(t,x)) + \overline{\omega}(t+h)v(t,x)$ for a.a. $(t,x)$, we infer that $v(\cdot)$ is a mild solution on $(\tau, +\infty)$ for the symbol $\theta_h\sigma$, so that $v \in \mathcal{D}_{\tau,\theta_h\sigma}(x)$. Hence, $y \in U_{\theta_h\sigma}(t, \tau, x)$.

The converse inclusion is proved in a similar way. ■

By the results in the previous section we known that each $U_\sigma$ possesses a compact strictly invariant pullback attractor $\{\mathcal{A}_\sigma(t)\}_{t\in\mathbb{R}}$, which can be characterized in terms of the bounded complete trajectories of $\mathcal{R}_\sigma$ :

$$\mathcal{A}_\sigma(t) = \{\gamma(t) : \gamma \text{ is a bounded complete trajectory of } \mathcal{R}_\sigma\}.$$

We can extend Lemma 20 by taking the union of all the attractors over $\Sigma$.

**Lemma 30** *Assume that (4), (28), (31) hold. Then the set* $\cup_{\sigma\in\Sigma, t\in\mathbb{R}}\mathcal{A}_\sigma(t)$ *is relatively compact in* $V^{2r}$ *for any* $0 \leq r < 1$.

**Proof.** The proof is the same as in Lemma 20 by taking into account that the constant $C$ in the estimate $\|f\|_{L^\infty(t-1,t;H)} \leq C$ is independent of $\sigma \in \Sigma$. ■

As before, we extend the above results when we restrict ourselves to non-negative solutions. Let $\mathcal{D}^+_{\tau,\sigma}(u_\tau)$ be the set of all non-negative solutions of problem (3) with initial condition $u_\tau$ at time $\tau$ with symbol $\sigma$ and let $\mathcal{R}^+_{\tau,\sigma} = \cup_{u_\tau\in H}\mathcal{D}^+_{\tau,\sigma}(u_\tau)$, $\mathcal{R}^+_\sigma = \cup_{\tau\in\mathbb{R}}\mathcal{R}_{\tau,\sigma}$. For each $\sigma \in \Sigma$ the operator $U^+_\sigma : \mathbb{R}^2_\geq \times H^+ \to P(H^+)$ is given by

$$U^+_\sigma(t, \tau, x) = \{u(t) : u(\cdot) \in \mathcal{D}^+_{\tau,\sigma}(x)\}.$$

$U^+_\sigma$ is a strict multivalued process and satisfies the translation property (32). The process $U^+_\sigma$ possesses a compact strictly invariant pullback attractor $\{\mathcal{A}^+_\sigma(t)\}_{t\in\mathbb{R}}$ and

$$\mathcal{A}^+_\sigma(t) = \{\gamma(t) : \gamma \text{ is a bounded complete trajectory of } \mathcal{R}^+_\sigma\}.$$

As proved in the previous section, for each $\sigma$ the attractor $\{\mathcal{A}^+_\sigma(t)\}$ consists of the equilibria 0 and $\xi_{M,\sigma}(t)$ and the bounded complete trajectories, given by (29), which connect them.

## 5.1 The cocycle attractor

We define for the family $\{U_\sigma\}$ the map $\phi : \mathbb{R}^+ \times \Sigma \times H \to P(H)$ given by

$$\phi(t, \sigma, x) = U_\sigma(t, 0, x) \text{ for all } t \geq 0,\ \sigma \in \Sigma,\ x \in H,$$

which is a strict multivalued cocycle, that is, $\phi(0, \sigma, \cdot)$ is the identity map and we have that $\phi(t+s, \sigma, x) = \phi(t, \theta_s\sigma, \phi(s, \sigma, x))$ for all $t \geq s \geq 0$, $\sigma \in \Sigma$, $x \in H$ [17, Proposition 1].

By [17, Corollary 4] the family $\{\mathcal{A}(\sigma)\}_{\sigma\in\Sigma}$ defined by

$$\mathcal{A}(\sigma) = \mathcal{A}_\sigma(0),$$

where $\{\mathcal{A}_\sigma(t)\}_{t\in\mathbb{R}}$ is the pullback attractor of $U_\sigma$, is a compact strictly invariant cocycle attractor for $\phi$, which means that:

- The sets $\mathcal{A}(\sigma)$ are compact for any $\sigma \in \Sigma$.
- $\mathcal{A}(\theta_t\sigma) = \phi(t, \sigma, \mathcal{A}(\sigma))$ for all $\sigma \in \Sigma$, $t \geq 0$ (strict invariance).
- $\lim_{t\to+\infty} dist(\phi(t, \theta_{-t}\sigma, B), \mathcal{A}(\sigma)) = 0$ for any bounded set $B \subset H$ and all $\sigma \in \Sigma$ (pullback attraction).
- $\{\mathcal{A}(\sigma)\}_{\sigma\in\Sigma}$ is minimal, that is, if $\{\mathcal{A}'(\sigma)\}_{\sigma\in\Sigma}$ is a family of closed sets satisfying the pullback attraction property, then $\mathcal{A}(\sigma) \subset \mathcal{A}'(\sigma)$ for any $\sigma \in \Sigma$.

Moreover,

$$\mathcal{A}(\theta_t\sigma) = \mathcal{A}_\sigma(t) \text{ for all } \sigma \in \Sigma,\ t \in \mathbb{R},$$

so

$$\mathcal{A}(\theta_t\sigma) = \{\gamma(t) : \gamma \text{ is bounded complete trajectory of } \mathcal{R}_\sigma\}.$$

Also, Lemma 30 implies that $\cup_{\sigma\in\Sigma}\mathcal{A}(\sigma)$ is relatively compact in $V^{2r}$ for any $0 \leq r < 1$.

In the same way, we define the cocycle for non-negative solutions $\phi^+ : \mathbb{R}^+ \times \Sigma \times H^+ \to P(H^+)$ by

$$\phi^+(t, \sigma, x) = U_\sigma^+(t, 0, x) \text{ for all } t \geq 0,\ \sigma \in \Sigma,\ x \in H,$$

which possesses a compact strictly invariant cocycle attractor $\{\mathcal{A}_\sigma^+(t)\}_{t\in\mathbb{R}}$ which satisfies

$$\mathcal{A}^+(\theta_t\sigma) = \mathcal{A}_\sigma^+(t) \text{ for all } \sigma \in \Sigma,\ t \in \mathbb{R},$$

and

$$\mathcal{A}^+(\theta_t\sigma) = \{\gamma(t) : \gamma \text{ is a bounded non-negative complete trajectory of } \mathcal{R}_\sigma\}.$$

We finish this subsection by noticing (see the proof of Lemma 29) that $\gamma$ is a complete trajectory of $\mathcal{R}_\sigma$ if and only if

$$\gamma(\cdot + s)\mid_{[0,+\infty)} \in \mathcal{R}_{0,\theta_s\sigma} \text{ for any } s \in \mathbb{R}.$$

## 5.2 The skew-product semiflow attractor

We will denote by $\mathcal{X}$ the product space $H \times \Sigma$ with the metric $\rho_{\mathcal{X}}$ given by

$$\rho_{\mathcal{X}}((x_1, \sigma_1), (x_2, \sigma_2)) = \|x_1 - x_2\| + \rho(\sigma_1, \sigma_2),$$

where $\rho$ is a metric in the space $C(\mathbb{R}, \mathbb{R}^2)$. Also, let $\mathcal{P}_H : \mathcal{X} \to H$ be the projector onto $H$, that is, for a subset $C \subset \mathcal{X}$ we put

$$\mathcal{P}_H(C) = \{u \in H : (u, \sigma) \in C \text{ for some } \sigma \in \Sigma\}.$$

From the cocycle $\phi$ we define the skew product semiflow $\Pi : \mathbb{R}^+ \times \mathcal{X} \to P(\mathcal{X})$ given by

$$\Pi(t, (x, \sigma)) = (\phi(t, \sigma, x), \theta_t\sigma).$$

Since $\phi$ is a strict cocycle it is easy to check that $\Pi$ is a strict multivalued semiflow, which means that $\Pi(0, \cdot)$ is the identity map and $\Pi(t, y) = \Pi(t, \Pi(s, y))$ for any $y \in \mathcal{X}$, $t, s \geq 0$.

We need to prove some properties of $\Pi$ leading to the existence of a global attractor. After that we will establish the relationship with the cocycle attractor and study its structure.

**Lemma 31** *Let (4), (28), (31) hold. Then $\Pi$ possesses a compact absorbing set $\mathbb{K}$, which means that for any bounded set $\mathbb{B}$ there exists $T(\mathbb{B})$ such that $\Pi(t, \mathbb{B}) \subset \mathbb{K}$ if $t \geq T$.*

**Proof.** In a standard way (see [14, Lemma 5]) for any solution to problem (3) in $[0, +\infty)$ we obtain the estimates

$$\|u(t)\|^2 \leq e^{-\delta t}\|u(0)\|^2 + \frac{C_1}{\delta} \text{ for all } t \geq 0, \tag{33}$$

$$\int_{t-\alpha}^{t} \left\|\frac{\partial u}{\partial x}\right\|^2 dr \leq \frac{\pi^2 C_1}{\delta} + \frac{\pi^2}{\delta}\|u(t-\alpha)\|^2, \tag{34}$$

where $\alpha \in (0,1]$ is arbitrary and $C, \delta > 0$ are universal constants which are independent of $\sigma \in \Sigma$ (they depend only on the constants $b_1, \omega_1$ from (4)).

Further, we multiply (5) by $\dfrac{du}{dt}$ and use Corollary 1 in [14] to obtain that

$$\left\| \frac{du}{dt}(t) \right\|^2 + \frac{1}{2}\frac{d}{dt}\left\| \frac{\partial u}{\partial x} \right\|^2 \leq b_1 \left\| \frac{du}{dt}(t) \right\| + \omega_1 \|u(t)\| \left\| \frac{du}{dt}(t) \right\| \\ \leq b_1^2 + \omega_1^2 \|u(t)\|^2 + \frac{1}{2}\left\| \frac{du}{dt}(t) \right\|^2. \tag{35}$$

For $0 \leq t - \alpha \leq r \leq t$ we integrate over the interval $(r,t)$. Hence, by (33) we have

$$\left\| \frac{\partial u}{\partial x}(t) \right\|^2 \leq \left\| \frac{\partial u}{\partial x}(r) \right\|^2 + 2b_1^2 + 2\omega_1^2 e^{-\delta(t-\alpha)} \|u(0)\|^2 + \frac{2C_1\omega_1^2}{\delta}.$$

Integrating now with respect to the variable $r$ over the interval $(t-\alpha, t)$ and using (33) and (34) we get

$$\alpha \left\| \frac{\partial u}{\partial x}(t) \right\|^2 \leq \frac{\pi^2 C_1}{\delta} + \frac{\pi^2}{\delta} \|u(t-\alpha)\|^2 + 2b_1^2 + 2\omega_1^2 e^{-\delta(t-\alpha)} \|u(0)\|^2 + \frac{2C_1\omega_1^2}{\delta} \\ \leq C\left(1 + e^{-\delta(t-\alpha)} \|u(0)\|^2\right), \tag{36}$$

where $C > 0$ is a constant.

We take $\alpha = 1$ and define the set $K = \{v \in V : \|v\|_V^2 \leq 2C\}$. The compact embedding $V \subset H$ implies that $K$ is relatively compact in $H$. Also, as $K$ is weakly closed in $V$, it is closed in $H$. Thus, $K$ is compact in $H$. From (36) we obtain that for any bounded set $B \subset H$ there exists $T(B)$ (independent of $\sigma \in \Sigma$) such that $\phi(t, \sigma, B) \subset K$ for all $t \geq T(B)$ and any $\sigma \in \Sigma$. Let $\mathbb{K} = K \times \Sigma$, which is compact in $\mathcal{X}$. Any bounded set $\mathbb{B} \subset \mathcal{X}$ satisfies that $\mathbb{B} \subset \mathcal{P}_H\mathbb{B} \times \Sigma$, where $\mathcal{P}_H\mathbb{B}$ is bounded in $H$. Then we have

$$\Pi(t, \mathbb{B}) \subset \Pi(t, \mathcal{P}_H\mathbb{B} \times \Sigma) \subset \cup_{\sigma \in \Sigma} \left(\phi\left(t, \sigma, \mathcal{P}_H\mathbb{B}\right), \theta_t\sigma\right) \subset \mathbb{K} \text{ for } t \geq T(\mathbb{B}).$$

■

**Lemma 32** *Let $h_n \to h$ weakly in $L^1(t_1, t_2; H)$ and let for a.a. $(t,x)$ there is $N(t,x)$ such that $h_n(t,x)$ belong to the closed convex set $C(t,x)$ for all $n \geq N$. Then $h(t,x) \in C(t,x)$ for a.a. $(t,x)$.*

**Proof.** By [42, Proposition 1.1] for a.a. $t \in (t_1, t_2)$ there is a sequence of convex combinations of $\{h_n(t)\}$ given by

$$y_n(t) = \sum_{i=1}^{M_n} \lambda_i h_{k_i}(t), \ \sum_{i=1}^{M_n} \lambda_i = 1, \ k_i \geq n,$$

such that $y_n(t) \to h(t)$ in $H$. Since for a.a. $(t,x)$ $h_{k_i}(t,x) \in C(t,x)$ if $n \geq N(t,x)$, we get by the convexity of $C(t,x)$ that $h(t,x) \in C(t,x)$ for a.a. $(t,x)$. ■

**Lemma 33** *Let (4), (28), (31) hold. Let $u_0^n \to u_0$ in $H$ and $\sigma_n = (b_n, \omega_n) \to \sigma = (\overline{b}, \overline{\omega})$ in $\Sigma$. Then for any $u_n \in \mathcal{D}_{0,\sigma_n}(u_0^n)$ there exists a subsequence $u_{n'}$ and $u \in \mathcal{D}_{0,\sigma}(u_0)$ such that*

$$u_{n'} \to u \text{ in } C([0,+\infty), H).$$

**Proof.** By (36) we have that for any $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$\|u_n(t)\|_V \leq C_\varepsilon \text{ for all } t \geq \varepsilon \text{ and any } n. \tag{37}$$

We fix an arbitrary $T > \varepsilon$. Integrating over $(\varepsilon, T)$ in (35) and using (37) we obtain the existence of $D_1 = D_1(\varepsilon, T)$ such that

$$\int_\varepsilon^T \left\| \frac{du}{dt} \right\|^2 dt \leq D_1. \tag{38}$$

In view of (37) and the compact embedding $V \subset H$, the sequence $u_n(t)$ is relatively compact in $H$ for all $t \in [\varepsilon, T]$. Also, it follows easily from (38) that the functions $u_n : [\varepsilon, T] \to H$ are equicontinuous. Hence, (37), (38) and the Ascoli-Arzelà theorem imply that for any $0 < \varepsilon < T$ up to a subsequence the following convergences hold:

$$u_n \to u \text{ weakly star in } L^\infty(\varepsilon, T; V),$$
$$\frac{du_n}{dt} \to \frac{du}{dt} \text{ weakly in } L^2(\varepsilon, T; H),$$
$$u_n \to u \text{ in } C([\varepsilon, T], H).$$

Since $u_n(\cdot)$ satisfies (5), where $r_n \in L^2_{loc}(0, +\infty; H)$ is such that $r_n(t,x) \in b_n(t) H_0(u_n(t,x)) + \omega_n(t) u_n(t,x)$ for a.a. $(t,x)$, there exists a sequence $h_n \in L^2_{loc}(0, +\infty; H)$ such that $h_n(t,x) \in H_0(u_n(t,x))$, for a.a. $(t,x)$, and

$$-Au_n(t) = \frac{du_n}{dt}(t) - b_n(t) h_n(t) - \omega_n(t) u_n(t) \text{ for a.a. } t.$$

As up to a subsequence $h_n$ converges weakly in $L^2_{loc}(0, +\infty; H)$ to some $h$, we have

$$-Au_n \to \frac{du}{dt} - \bar{b}h - \overline{\omega}u \text{ weakly in } L^2(\varepsilon, T; H) \text{ for all } 0 < \varepsilon < T.$$

Hence, as $Au_n = -\dfrac{\partial^2 u_n}{\partial x^2}$ converges to $Au = -\dfrac{\partial^2 u}{\partial x^2}$ in the sense of distributions, we have

$$\frac{du}{dt} + Au(t) = \bar{b}(t) h(t) + \overline{\omega}(t) u(t) \text{ for a.a. } t \in (0, +\infty).$$

We need to show that $h(t,x) \in H_0(u(t,x))$ for a.a. $(t,x)$. We observe that for a.a. $(t,x)$ there is $N(t,x)$ such that $h_n(t,x) \in H_0(u(t,x)))$ if $n \geq N$. Indeed, let $(t_0, x_0) \in A^c$, where $A$ is a set of measure 0 such that $u_n(t,x) \to u(t,x)$ for any $(t,x) \in A^c$. If $u(t_0, x_0) > 0$ $(< 0)$, then $u_n(t_0, x_0) > 0$ $(< 0)$ for $n$ large enough, so $h_n(t_0, x_0) = 1$ $(-1)$. Hence, $h_n(t_0, x_0) \in H_0(u(t,x))$. If $u(t_0, x_0) = 0$, then $h_n(t_0, x_0) \in [-1, 1] = H_0(u(t,x))$. Then the assertion follows from Lemma 32.

It remains to show that $u(t) \to u(0)$ as $t \to 0^+$ and that $u_n(t_n) \to u(0)$ as $t_n \to 0$.

Let $v_n(t) = u_n(t) - \widehat{u}(t)$, where $\widehat{u}$ is the unique solution of the linear problem

$$\begin{cases} \dfrac{\partial u}{\partial t} - \dfrac{\partial^2 u}{\partial x^2} = \overline{\omega}(t) u, \\ u(t,0) = u(t,1) = 0, \\ u(0,x) = u_0(x). \end{cases}$$

Fix some $T > 0$. Then in a standard way we obtain that

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|v_n\|^2 + \|v_n(t)\|_V^2 &\leq b_n(t)\|v_n(t)\| + \omega_n(t)\|v_n(t)\|^2 + |\omega_n(t) - \overline{\omega}(t)| \|\widehat{u}(t)\| \|v_n(t)\| \\ &\leq \frac{b_1^2}{2\varepsilon_0} + (\omega_1 + \varepsilon_0)\|v_n(t)\|^2 + \alpha_n, \text{ for a.a. } t \in (0,T), \end{aligned}$$

where $\varepsilon_0 > 0$ is such that $\omega_1 + \varepsilon_0 < \pi^2$ and $\alpha_n = \frac{1}{2\varepsilon_0} \sup_{t \in [0,T]} \left( |\omega_n(t) - \overline{\omega}(t)|^2 \|\widehat{u}(t)\|^2 \right) \to 0$ as $n \to \infty$. Then

$$\|v_n(t)\|^2 \leq \|v_n(0)\|^2 + Ct,$$

where $\frac{b_1^2}{\varepsilon_0} + 2\alpha_n \leq C$ for any $n$. It follows that $\|u(t) - \widehat{u}(t)\|^2 = \lim_{n\to\infty} \|v_n(t)\|^2 \leq Ct$ for $t > 0$. Thus,

$$\|u(t) - u_0\|^2 \leq 2\|u(t) - \widehat{u}(t)\|^2 + 2\|\widehat{u}(t) - u_0\|^2 \to 0 \text{ as } t \to 0^+.$$

Therefore, $u \in \mathcal{D}_{0,\sigma}(u_0)$.

Finally, if $t_n \to 0^+$, then

$$\begin{aligned} \|u_n(t_n) - u_0\|^2 &\leq 2\|v_n(t_n)\|^2 + 2\|\widehat{u}(t_n) - u_0\|^2 \\ &\leq 2\|v_n(0)\|^2 + 2Ct_n + 2\|\widehat{u}(t_n) - u_0\|^2 \to 0, \end{aligned}$$

so $u_n \to u$ in $C([0,+\infty),H)$. ■

For any $\sigma \in \Sigma$ and $u \in \mathcal{R}_{0,\sigma}$ let $y : \mathbb{R}^+ \to \mathcal{X}$ be given by $y(\cdot) = (u(\cdot), \theta.\sigma)$ and denote by $\mathcal{K} \subset C([0,+\infty),\mathcal{X})$ the set of all functions $y(\cdot)$ of this type. We consider the following standard axiomatic properties:

$(K1)$ For any $y_0 = (u_0,\sigma) \in \mathcal{X}$ there exists $y \in \mathcal{K}$ such that $y(0) = y_0$.

$(K2)$ For any $y \in \mathcal{K}$ and $s \geq 0$, $y_s = y(\cdot{+}s) \in \mathcal{K}$.

$(K3)$ If $y_1, y_2 \in \mathcal{K}$ are such that $y_2(0) = y_1(s)$, then the composition

$$y(t) = \begin{cases} y_1(t) \text{ if } 0 \leq t \leq s, \\ y_2(t-s) \text{ if } t \geq s, \end{cases}$$

belongs to $\mathcal{K}$.

$(K4)$ If $y_n \in \mathcal{K}$ is such that $y_n(0) \to y_0$, then there exists a subsequence $\{y_{n'}\}$ and $y \in \mathcal{K}$ such that $y_{n'}(t) \to y(t)$ in $\mathcal{X}$ uniformly in compact sets of $[0,+\infty)$.

We recall that the map $\Pi(t,\cdot) : \mathcal{X} \to \mathcal{X}$ is upper semicontinuous if for any $u_0 \in \mathcal{X}$ and any neighborhood $O$ of $\Pi(t,u_0)$ there exists $\rho > 0$ such that $\Pi(t,u) \subset O$ as soon as $\rho_{\mathcal{X}}(u,u_0) < \rho$.

**Lemma 34** *Let (4), (28), (31) hold. Then properties $(K1) - (K4)$ hold. An alternative definition for the map $\Pi$ is*

$$\Pi(t,u_0) = \{\xi \in \mathcal{X} : \xi = y(t),\ y \in \mathcal{K}\} \tag{39}$$

*and for any $t \geq 0$ the map $\Pi(t,\cdot)$ is upper semicontinuous and has compact values.*

**Proof.** $(K1)$ follows from the existence of solutions to (3) for any $u_0 \in H$ and $\sigma \in \Sigma$. $(K2)$ follows from

$$y(\cdot + s) = (u(\cdot + s), \theta_{\cdot+s}\sigma) = (u_s(\cdot), \theta.\theta_s\sigma) \in \mathcal{K},$$

as $u_s(\cdot) \in \mathcal{R}_{0,\theta_s\sigma}$ (see the proof of Lemma 29). For $(K3)$ we note that $y_1(t) = (u_1(t), \theta_t\sigma_1)$, $y_2(t-s) = (u_2(t-s), \theta_{t-s}\sigma_2)$, $u_2(0) = u_1(s)$, $\sigma_2 = \theta_s\sigma_1$, so that $y(t) = (u(t), \theta_t\sigma_1)$, for any $t \geq 0$, where

$$u(t) = \begin{cases} u_1(t) \text{ if } 0 \leq t \leq s, \\ u_2(t-s) \text{ if } t \geq s. \end{cases}$$

If we prove that $u \in \mathcal{R}_{0,\sigma_1}$, then $y \in \mathcal{K}$. Indeed, if $\sigma_1(t) = (b_1(t), \omega_1(t))$, then for $t \geq s$ we have

$$\begin{aligned} u(t) &= u_2(t-s) = e^{-A(t-s)}u_1(s) + \int_0^{t-s} e^{-A(t-s-\tau)}(b_1(\tau+s)\,h_2(\tau) + \omega_1(\tau+s)\,u_2(\tau))d\tau \\ &= e^{-A(t-s)}e^{-As}u_1(0) + e^{-A(t-s)}\int_0^s e^{-A(s-\tau)}(b_1(\tau)\,h_1(\tau) + \omega_1(\tau)\,u_1(\tau))d\tau \\ &+ \int_s^t e^{-A(t-\tau)}(b_1(\tau)\,h_2(\tau-s) + \omega_1(\tau)\,u_2(\tau-s))d\tau \\ &= e^{-At}u_1(0) + \int_0^t e^{-A(t-\tau)}\left((b_1(\tau)\,h(\tau) + \omega_1(\tau)\,u(\tau))d\tau\right), \end{aligned}$$

where $h_i \in L^2_{loc}(0,+\infty;H)$ are such that $h_i(\tau) \in H_0(u_i(\tau))$ for a.a. $\tau > 0$, and

$$h(\tau) = \begin{cases} h_1(t) \text{ if } 0 \leq \tau \leq s, \\ h_2(t-s) \text{ if } \tau \geq s, \end{cases}$$

belongs to $L^2_{loc}(0,+\infty;H)$ and satisfies $h(\tau) \in H_0(u(\tau))$ for a.a. $\tau > 0$. Hence, $u$ is a mild solution for the symbol $\sigma_1$.

Property $(K4)$ follows from Lemma 33 and implies easily that $\Pi(t,\cdot)$ has compact values and is upper semicontinuous. ■

We are now in position of establishing the existence of the global attractor and its relationship with the cocycle attractor.

**Theorem 35** *The multivalued semiflow $\Pi$ possesses the global compact invariant attractor $\mathbb{A}$. Moreover,*

$$\mathbb{A} = \cup_{\sigma\in\Sigma} A(\sigma) \times \{\sigma\},$$

*where $\{\mathcal{A}(\sigma)\}_{\sigma\in\Sigma}$ is the cocycle attractor.*

**Proof.** The existence of the global compact invariant attractor follows from Lemmas 31, 34 and [32, Theorem 4 and Remark 8]. The relationship with the cocycle attractor is a consequence of [17, Corollary 1]. ■

As before, we will characterize the global attractor in terms of bounded complete trajectories. A complete trajectory of $\mathcal{K}$ is a function $\Phi : \mathbb{R} \to \mathcal{X}$ such that $\Phi(\cdot + s)\mid_{t\geq 0} \in \mathcal{K}$ for any $s \in \mathbb{R}$.

**Lemma 36** *Assume that (4), (28), (31) hold. Let $\gamma$ be a complete trajectory of $\mathcal{R}_\sigma$. Then*

$$\Phi(t) = (\gamma(t), \theta_t\sigma), \text{ for any } t \in \mathbb{R}, \tag{40}$$

*is a complete trajectory of $\mathcal{K}$. Conversely, if $\Phi$ is a complete trajectory of $\mathcal{K}$, then there exist $\sigma \in \Sigma$ and a complete trajectory $\gamma$ of $\mathcal{R}_\sigma$ such that (40) holds.*

**Proof.** Let $\gamma$ be a complete trajectory of $\mathcal{R}_\sigma$. Since $\gamma(\cdot + s) \in \mathcal{D}_{0,\theta_s\sigma}(\gamma(s))$, for any $s \in \mathbb{R}$, we have that $y(\cdot) = (\gamma(\cdot{+}s), \theta.\theta_s\sigma) \in \mathcal{K}$. Thus, $\Phi$ is a complete trajectory of $\mathcal{K}$.

Further, let $\Phi = (\gamma, \alpha)$ be a complete trajectory of $\mathcal{K}$. Then

$$(\gamma(\cdot + s), \alpha(\cdot + s))\mid_{t\geq 0} \in \mathcal{K} \text{ for any } s,$$

so $\gamma(\cdot + s) \in \mathcal{R}_{0,\alpha(s)} = \mathcal{R}_{0,\theta_s\alpha(0)}$. Therefore, as explained in the previous subsection, $\gamma$ is a complete trajectory of $\mathcal{R}_\sigma$ with $\sigma = \alpha(0)$. ■

**Theorem 37** *Let (4), (28), (31) hold. Then the global attractor $\mathbb{A}$ is given by*

$$\begin{aligned}
\mathbb{A} &= \{\Phi(0) : \Phi \textit{ is a bounded complete trajectory of } \mathcal{K}\} \\
&= \cup_{t\in\mathbb{R}}\{\Phi(t) : \Phi \textit{ is a bounded complete trajectory of } \mathcal{K}\} \\
&= \cup_{\sigma\in\Sigma}\{(\gamma(0), \sigma) : \gamma \textit{ is a bounded complete trajectory of } \mathcal{R}_\sigma\} \\
&= \cup_{\sigma\in\Sigma, t\in\mathbb{R}}\{(\gamma(t), \theta_t\sigma) : \gamma \textit{ is a bounded complete trajectory of } \mathcal{R}_\sigma\}.
\end{aligned} \tag{41}$$

**Proof.** We obtain the first two equalities from [27, Theorems 9 or 10], whereas the other ones follow from Lemma 36. ■

When we restrict the solutions to the positive cone, we obtain a Morse structure of the global attractor.

We denote by $\mathcal{X}^+$ the product space $H^+ \times \Sigma$. Let now take the skew product semiflow $\Pi^+ : \mathbb{R}^+ \times \mathcal{X}^+ \to \mathcal{P}(\mathcal{X}^+)$ given by

$$\begin{aligned}
\Pi^+(t, (x, \sigma)) &= \left(\phi^+(t, \sigma, x), \theta_t\sigma\right) \\
&= \{\xi \in \mathcal{X}^+ : \xi = y(t),\ y \in \mathcal{K}^+\},
\end{aligned}$$

where $\mathcal{K}^+$ stands for the subset of $C([0, +\infty), \mathcal{X}^+)$ of functions $y(\cdot) = (u(\cdot), \theta.\sigma)$ such that $\sigma \in \Sigma$ and $u \in \mathcal{R}^+_{0,\sigma}$. The strict multivalued semiflow $\Pi^+$ possesses the global compact invariant attractor $\mathbb{A}^+ = \mathbb{A} \cap \mathcal{X}^+$, which satisfies

$$\begin{aligned}
\mathbb{A}^+ &= \{\Phi(0) : \Phi \text{ is a bounded complete trajectory of } \mathcal{K}^+\} \\
&= \cup_{t\in\mathbb{R}}\{\Phi(t) : \Phi \text{ is a bounded complete trajectory of } \mathcal{K}^+\} \\
&= \cup_{\sigma\in\Sigma}\{(\gamma(0), \sigma) : \gamma \text{ is a bounded complete trajectory of } \mathcal{R}^+_\sigma\} \\
&= \cup_{\sigma\in\Sigma, t\in\mathbb{R}}\{(\gamma(t), \theta_t\sigma) : \gamma \text{ is a bounded complete trajectory of } \mathcal{R}^+_\sigma\}.
\end{aligned} \tag{42}$$

A set $M$ is called weakly invariant if for any $y_0 \in M$ there exist a complete trajectory $\Phi$ of $\mathcal{K}^+$ such that $\cup_{t\in\mathbb{R}}\Phi(t) \subset M$ and $\phi(0) = y_0$. It is obvious that the global attractor $\mathbb{A}^+$ is weakly invariant. A weakly invariant set $M$ is said to be isolated if there exists a neighborhood $O$ of $M$ such that $M$ is the maximal weakly invariant set in it. If $M$ is compact, this is equivalent to saying that there exists an $\varepsilon > 0$ such that $M$ is the maximal weakly invariant set in the $\varepsilon$-neighborhood $O_\varepsilon(M) = \{y \in \mathcal{X}^+ : dist_{\mathcal{X}^+}(y, M) < \varepsilon\}$. As we will consider weakly invariant sets $M$ belonging to the global attractor $\mathbb{A}^+$, they will be necessarily compact [16, Lemma 19].

A family of isolated weakly invariant sets $\mathcal{M} = \{M_1, ..., M_n\}$ in $\mathbb{A}^+$ is called disjoint if there is $\delta > 0$ such that $O_\delta(M_i) \cap O_\delta(M_j) = \varnothing$ for all $i \neq j$.

We say that the multivalued semiflow $\Pi^+$ is dynamically gradient with respect to the disjoint family of isolated weakly invariant sets $\mathcal{M} = \{M_1, ..., M_n\}$ in $\mathbb{A}^+$ if for any bounded complete trajectory $\Phi$ of $\mathcal{K}^+$ we have that either $\cup_{t\in\mathbb{R}}\Phi(t) \in M_i$ for some $i \in \{1, ..., n\}$ or

$$dist_{\mathcal{X}^+}(\Phi(t), M_i) \to 0 \text{ as } t \to +\infty, \tag{43}$$

$$dist_{\mathcal{X}^+}(\Phi(t), M_j) \to 0 \text{ as } t \to -\infty, \tag{44}$$

for some $1 \leq i < j \leq n$. We observe that convergences (43) and (44) are equivalent to saying that $\omega(\Phi) \subset M_i$ and $\alpha(\Phi) \subset M_j$, where $\omega(\Phi)$, $\alpha(\Phi)$ stand, respectively, for the omega and alpha limit sets of the bounded complete trajectory $\Phi$.

For each $\sigma$ we denote by $\xi_{M,\sigma}$ the unique positive bounded complete trajectory (the positive nonautonomous equilibrium) given in Section 4.2. We define then the following compact weakly invariant sets in $\mathbb{A}^+$:

$$\begin{aligned} M_1 &= \{(\xi_{M,\sigma}(0), \sigma) : \sigma \in \Sigma\} \\ &= \{(\xi_{M,\sigma}(t), \theta_t\sigma) : \sigma \in \Sigma,\ t \in \mathbb{R}\}, \\ M_2 &= \{0\}. \end{aligned}$$

Since $\xi_{M,\sigma}(0) \geq w^+_{b_0,\omega_0}$ for any $\sigma$, these sets are clearly disjoint. From the results in Section 4.2 and (42) we can see that, apart from $M_1$ and $M_2$, the only elements in the global attractor $\mathbb{A}^+$ are $(\gamma(t), \theta_t\sigma)$, where $\gamma$ is a bounded complete trajectory of $\mathcal{R}^+_\sigma$ of the type (29) with $u(\cdot)$ being the solution to the linear problem (30) for the symbol $\sigma = (\overline{b}, \overline{\omega})$. We know that

$$\|\gamma(t) - \xi_{M,\sigma}(t)\| \to 0 \text{ as } t \to +\infty,$$

so

$$\begin{aligned} dist((\gamma(t), \theta_t\sigma), M_1) \to 0 \text{ as } t \to +\infty, \\ dist((\gamma(t), \theta_t\sigma), M_2) \to 0 \text{ as } t \to -\infty. \end{aligned}$$

The sets $M_1, M_2$ are isolated. Indeed, we take disjoint $\varepsilon$-neighborhoods of these sets, $O_\varepsilon(M_1)$ and $O_\varepsilon(M_2)$. In $O_\varepsilon(M_1)$ the only possible complete trajectories are $\xi_{M,\sigma}$, because any other one should converge to 0 as $t \to -\infty$ and then leave the neighborhood $O_\varepsilon(M_1)$. By the same reason the only possible bounded complete trajectory in $O_\varepsilon(M_2)$ is 0, so that $M_1$ and $M_2$ are the maximal weakly invariant sets in $O_\varepsilon(M_1)$ and $O_\varepsilon(M_2)$, respectively.

All in all, we have shown the following.

**Theorem 38** *Let (4), (28), (31) hold. Then $\Pi^+$ is dynamically gradient with respect to the disjoint family of isolated weakly invariant sets $\mathcal{M} = \{M_1, M_2\}$. Hence, the global attractor $\mathbb{A}^+$ possesses a gradient structure.*

## 5.3 The uniform attractor

We finish this section by showing the relationship of the previous attractors with the uniform attractor of the cocycle $\phi$.

It follows from Theorem 5 in [17] that $\mathcal{A} = \mathcal{P}_H\mathbb{A}$ ($\mathbb{A}$ is the attractor of the skew product flow) is the uniform attractor for $\phi$, which means that:

- $\mathcal{A}$ is compact.
- $\mathcal{A}$ is uniformly attracting, that is, for any bounded set $B \subset H$ we have

$$\sup_{\sigma\in\Sigma} dist\left(\phi\left(t,\sigma,B\right),\mathcal{A}\right) \to 0, \text{ as } t \to +\infty. \tag{45}$$

- $\mathcal{A}$ is the minimal closed set satisfying (45).

By characterization (41) we obtain that

$$\begin{aligned}\mathcal{A} &= \cup_{\sigma\in\Sigma}\left\{\gamma\left(0\right) : \gamma \text{ is a bounded complete trajectory of } \mathcal{R}_\sigma\right\} \\ &= \cup_{\sigma\in\Sigma,t\in\mathbb{R}}\{\gamma\left(t\right) : \gamma \text{ is a bounded complete trajectory of } \mathcal{R}_\sigma\}.\end{aligned}$$

This implies, in particular, that $\mathcal{A}$ is weakly invariant.

Also, by Theorem 8 in [17] we have the relationship of the uniform attractor with the cocycle and pullback attractors:

$$\mathcal{A} = \cup_{\sigma\in\Sigma}\mathcal{A}(\sigma) = \cup_{\sigma\in\Sigma}\mathcal{A}_\sigma(0).$$

In the same way, $\mathcal{A}^+ = \mathcal{P}_H\mathbb{A}^+$ is the uniform attractor for $\phi^+$ and

$$\begin{aligned}\mathcal{A}^+ &= \cup_{\sigma\in\Sigma}\left\{\gamma\left(0\right) : \gamma \text{ is a bounded complete trajectory of } \mathcal{R}^+_\sigma\right\} \\ &= \cup_{\sigma\in\Sigma,t\in\mathbb{R}}\{\gamma\left(t\right) : \gamma \text{ is a bounded complete trajectory of } \mathcal{R}^+_\sigma\},\end{aligned}$$

$$\mathcal{A}^+ = \cup_{\sigma\in\Sigma}\mathcal{A}^+(\sigma) = \cup_{\sigma\in\Sigma}\mathcal{A}^+_\sigma(0).$$

Let $\mathcal{C} = \cup_{\sigma\in\Sigma}\cup_{\gamma_\sigma\in\mathcal{L}_\sigma}\gamma_\sigma\left(0\right)$, where $\mathcal{L}_\sigma$ is the set of all bounded complete trajectory of $\mathcal{R}^+_\sigma$ of the type (29) with $u\left(\cdot\right)$ being the solution to the linear problem (30) for the symbol $\sigma = \left(\overline{b},\overline{\omega}\right)$. Hence,

$$\mathcal{A}^+ = \{0\} \cup \{\xi_{M,\sigma}\left(0\right) : \sigma \in \Sigma\} \cup \mathcal{C}.$$

# 6 Appendix

In this appendix we present some auxiliary results that are necessary for the arguments throughout this paper.

## 6.1 A maximum principle for non-smooth functions

Usually in the literature the maximum principle is stated for smooth functions (see for example [33]). However, we need in this paper a maximum principle for less regular functions. Such result is proved in [26] in a rather general setting. We describe in this appendix a particular situation which is derived from the theorems in [26].

Let $\mathcal{O}$ be a region in $\mathbb{R}^2$ and let $(t_0,x_0) \in \mathcal{O}$ and $\rho,\sigma > 0$. We denote

$$Q_{\rho,\sigma} = \{(t,x) : t \in (t_0-\sigma,t_0), |x-x_0| < \rho\},$$

where we assume that $t_0,x_0,\rho,\sigma$ are such that $\overline{Q}_{\rho,\sigma} \subset \mathcal{O}$.

We denote by $W$ the space of all functions from $L^2\left(\mathcal{O}\right)$ such that

$$\int_{\mathcal{O}}\left(\left|u\left(t,x\right)\right|^2 + \left|\frac{\partial u}{\partial x}\left(t,x\right)\right|^2\right)d\mu < +\infty.$$

As a particular case of Theorem 6.4 in [26] we obtain the following maximum and minimum principles.

**Theorem 39** *(Maximum principle) Let* $u \in W$ *be such that*

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} \leq 0$$

*in the sense of distributions. If*

$$ess \sup_{(t,x) \in Q_{\rho\nu,\sigma_1}} u(t,x) = M,$$

*for some* $\nu$*,* $0 < \nu < 1$*, and any* $\sigma_1$*, where* $0 < \sigma_1 < \sigma$*, then* $u\,(t,x) = M$ *for a.a.* $(t,x) \in Q_{\rho,\sigma}$*.*

**Theorem 40** *(Minimum principle) Let* $u \in W$ *be such that*

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} \geq 0$$

*in the sense of distributions. If*

$$ess \inf_{(t,x) \in Q_{\rho\nu,\sigma_1}} u(t,x) = M,$$

*for some* $\nu$*,* $0 < \nu < 1$*, and any* $\sigma_1$*, where* $0 < \sigma_1 < \sigma$*, then* $u\,(t,x) = M$ *for a.a.* $(t,x) \in Q_{\rho,\sigma}$*.*

## 6.2 Parabolic regularity

The regularity of solutions of the linear parabolic problem is well known in the literature (see e.g. [1], [4], [31], [38]). In this section, we consider the linear parabolic problem (6) and following the result given in Theorem 42.14 in [38] we describe some regularity results of its solutions in a suitable form for our purposes in this paper.

The following proposition is proved in the same way as Theorem 42.14 in [38]. However, as the assumptions on the function $f$ do not coincide exactly, we give a sketch of the proof.

**Proposition 41** *Let* $f \in L^2_{loc}(\tau, +\infty; H) \cap W^{1,2}_{loc}(\tau+\varepsilon, +\infty; H)$ *for all* $\varepsilon > 0$*. For any* $u_\tau \in H$*, the solution* $u\,(\cdot)$ *to problem (6) belongs to* $C^1((\tau, +\infty), V)$ *and* $C^{0,\frac{1}{2}}([\tau+\varepsilon, T], H^2(0,1))$*,* $\dfrac{d^2u}{dt^2} \in L^2(\tau+\varepsilon, T; H)$ *for all* $\tau < T$ *and* $\varepsilon > 0$*.*

*If, moreover,* $f \in C((\tau, +\infty), H^1(0,1))$*, then* $u \in C((\tau, +\infty), H^3(0,1)$*.*

**Proof.** We fix an arbitrary $T > \tau$. We know by Lemma 3 and Remark 4 that $u \in C((\tau, T], V)$ and $u \in L^2(\tau+\varepsilon, T; D(A))$ for any $\varepsilon > 0$.

We divide the proof into three steps.

**Step 1.** Let $u_\tau \in D(A)$ and $f \in W^{1,2}_{loc}(\tau, +\infty; H)$.

We put $v_\tau = f\,(\tau) - Au_\tau \in H$ and $g = \dfrac{df}{dt} \in L^2(\tau, T; H)$. Denote by $\omega\,(\cdot)$ the unique strong solution of problem

$$\begin{cases} \dfrac{\partial \omega}{\partial t} - \dfrac{\partial^2 \omega}{\partial x^2} = g(t), \text{ on } (\tau, +\infty) \times (0,1), \\ \omega(t,0) = \omega(t,1) = 0, \\ \omega(\tau, x) = v_\tau(x), \end{cases} \tag{46}$$

Again, by Lemma 3 and Remark 4 we have that $\omega \in C((\tau, T], V)$, $\omega \in L^2(\tau+\varepsilon, T; D(A))$, $\dfrac{d\omega}{dt} \in L^2(\tau+\varepsilon, T; H)$ for any $\varepsilon > 0$.

Let $z(t) = u_\tau + \int_\tau^t \omega(s)ds$. Hence, $z \in C^1((\tau, T], V)$ and

$$\|z(t) - z(s)\|_{H^2} \leq \int_s^t \|\omega(r)\|_{H^2} \, dr \leq (t-s)^{\frac{1}{2}} \int_s^t \|\omega(r)\|^2_{H^2} \, dr \text{ for any } \tau < s < t,$$

so $z \in C^{0,\frac{1}{2}}([\tau+\varepsilon, T], H^2(0,1))$. Following the proof of Theorem 42.14 in [38] we obtain that $z$ is a strong solution to problem (6) on $(\tau, T)$. By uniqueness of solutions $u = z$, so

$$u \in C^1((\tau, T], V),\ u \in C^{0,\frac{1}{2}}([\tau+\varepsilon, T], H^2(\Omega)),\ \frac{d^2u}{dt^2} \in L^2(\tau+\varepsilon, T; H) \text{ for all } \varepsilon > 0. \tag{47}$$

**Step 2.** Let $u_\tau \in H$.

Since $u \in L^2(\tau+\varepsilon, T; D(A))$ for any $\varepsilon > 0$, for any $\delta > 0$ there is $t_0 \in (\tau, \tau+\delta)$ such that $u(t_0) \in D(A)$. We know by Step 1 that $u \in C^1((t_0, T], V)$ and $u \in C^{0,\frac{1}{2}}([t_0+\varepsilon, T], H^2(0,1))$, $\dfrac{d^2u}{dt^2} \in L^2(t_0+\varepsilon, T; H)$ for all $\varepsilon > 0$. As $\delta > 0$ is arbitrary, we get (47).

**Step 3.** Let $f \in C((\tau, T], H^1(0,1))$ and $u_\tau \in H$.

In such a case we have

$$Au = f - \frac{du}{dt} \in C((0,T], H^1(0,1)),$$

so $u \in C((\tau, T], H^3(\Omega))$.

As $T > \tau$ is arbitrary, the result follows. ■

**Corollary 42** *If $f \in C((\tau, +\infty), H^1(0,1))$, $f \in L^2_{loc}(\tau, +\infty; H) \cap W^{1,2}_{loc}(\tau+\varepsilon, +\infty; H)$ for all $\varepsilon > 0$, then the partial derivatives $u_t, u_{xx}$ exists in the classical sense and are continuous on $(\tau, +\infty) \times [0,1]$.*

**Proof.** The continuous embedding $H^1(0,1) \subset C([0,1])$ implies that

$$u_{xx}, u_t \in C((\tau, +\infty), C([0,1]).$$

■

As it is known, increasing the temporal regularity of the function $f$ we can prove that the solution $u$ is as regular as we desire.

**Lemma 43** *Let $k \in \mathbb{N}$. Assume that $f \in L^2_{loc}(\tau, +\infty; H) \cap W^{k+1,2}_{loc}(\tau+\varepsilon, +\infty; H)$, for all $\varepsilon > 0$, and that*

$$f \in \cap_{j=0}^{k} C^j(\Big(\tau, +\infty), H^{2(k-j)+1}(0,1)\Big). \tag{48}$$

*Then, for any $u_\tau \in H$ the solution $u(\cdot)$ to problem (6) satisfies:*

$$u \in C^{k+1}((\tau, +\infty), V),\ u \in \cap_{j=0}^{k} C^j\left((\tau, +\infty), H^{2(k-j)+3}(0,1)\right), \tag{49}$$
$$\frac{d^{k+2}u}{dt^{k+2}} \in L^2_{loc}(\tau+\varepsilon, +\infty; H) \text{ for all } \varepsilon > 0.$$

**Proof.** First, let us consider the case $k = 1$, so $f \in L^2_{loc}(\tau, +\infty; H) \cap W^{2,2}_{loc}(\tau+\varepsilon, +\infty; H)$, for all $\varepsilon > 0$, and $f \in C^1(\big(\tau, +\infty), H^1(0,1)\big) \cap C(\big(\tau, +\infty), H^3(0,1)\big)$.

We know by Proposition 41 that $u \in C^1((\tau, +\infty), V) \cap C((\tau, +\infty), H^3(0,1))$. The function $z_1 = \dfrac{du}{dt}$ is the unique solution to the problem

$$\begin{cases} \dfrac{\partial z_1}{\partial t} - \dfrac{\partial^2 z_1}{\partial x^2} = \dfrac{df}{dt}, \text{ on } (\tau+\varepsilon, +\infty) \times (0,1), \\ z_1(t,0) = z_1(t,1) = 0, \\ z_1(\tau+\varepsilon, x) = f(\tau+\varepsilon) - Au(\tau+\varepsilon) \in H^1(0,1) \subset H, \end{cases}$$

for any $\varepsilon > 0$. Making use again of Proposition 41 and taking into account that $\varepsilon$ is arbitrarily small we infer that

$$z_1 \in C^1((\tau, +\infty), V) \cap C((\tau, +\infty), H^3(0,1)),$$
$$\frac{d^2 z_1}{d^2 t} \in L^2_{loc}(\tau+\varepsilon, +\infty; H) \text{ for all } \varepsilon > 0,$$

so

$$u \in C^2((\tau, +\infty), V) \cap C^1((\tau, +\infty), H^3(0,1)),$$
$$\frac{d^3 u}{dt^3} \in L^2_{loc}(\tau+\varepsilon, +\infty; H) \text{ for all } \varepsilon > 0.$$

Finally,

$$Au = f - \frac{du}{dt} \in C(\left(\tau, +\infty\right), H^3(0,1)) ,$$

so

$$u \in C(\left(\tau, +\infty\right), H^5(0,1)) .$$

By induction, assume that the result is true for $k-1$. Hence, $u \in \cap_{j=0}^{k-1} C^j(\left(\tau, +\infty\right), H^{2(k-j)+1}(0,1))$, $u \in C^k((\tau, +\infty), V)$ and $\frac{d^{k+1}u}{dt^{k+1}} \in L^2_{loc}(\tau + \varepsilon, +\infty; H)$ for all $\varepsilon > 0$. The function $z_k = \frac{d^k u}{dt^k}$ is the unique solution to the problem

$$\begin{cases} \dfrac{\partial z_k}{\partial t} - \dfrac{\partial^2 z_k}{\partial x^2} = \dfrac{d^k f}{dt^k}, \text{ on } (\tau + \varepsilon, +\infty) \times (0,1), \\ z_k(t,0) = z_k(t,1) = 0, \\ z_k(\tau + \varepsilon, x) = \dfrac{d^{k-1} f}{dt^{k-1}}(\tau + \varepsilon) - A\dfrac{d^{k-1}u}{dt^{k-1}}(\tau + \varepsilon) \in H^1(0,1) \subset H, \end{cases}$$

for any $\varepsilon > 0$. Thus, Proposition 41 implies that

$$z_k \in C^1((\tau, +\infty), V) \cap C((\tau, +\infty), H^3(0,1)),$$
$$\frac{d^2 z_k}{d^2 t} \in L^2_{loc}(\tau + \varepsilon, +\infty; H) \text{ for all } \varepsilon > 0,$$

and then

$$u \in C^{k+1}((\tau, +\infty), V) \cap C^k((\tau, +\infty), H^3(0,1)),$$
$$\frac{d^{k+2}u}{dt^{k+2}} \in L^2_{loc}(\tau + \varepsilon, +\infty; H) \text{ for all } \varepsilon > 0.$$

Finally, we shall prove that $u \in C^j\left((\tau, +\infty), H^{2(k-j)+3}(0,1)\right)$, for all $j \in \{0, 1, ..., k\}$ by induction. Assume that it is true for $j \leq i \leq k$, where $j \in \{1, ..., k\}$. Then

$$Az_{j-1} = \frac{d^{j-1}f}{dt^{j-1}} - \frac{dz_{j-1}}{dt} \in C\left((\tau, +\infty), H^{2(k-j)+3}(0,1)\right),$$

so

$$u \in C^{j-1}\left((\tau, +\infty), H^{2(k-(j-1))+3}(0,1)\right).$$

■

**Corollary 44** *Under the conditions of Lemma 43, the solution $u$ belongs to $C^{k+1}((\tau, +\infty) \times [0,1])$.*

**Proof.** Since $\frac{d^j u}{dt^j} \in C\left((\tau, +\infty), H^{2(k-j)+3}(0,1)\right)$, for all $j \in \{0, 1, ..., k+1\}$, we obtain that

$$\frac{\partial^{j+2(k-j)+2}u}{\partial t^j \partial x^{2(k-j)+2}} \in C((\tau, +\infty), C([0,1]).$$

As $-j + 2k + 2 \geq k + 1$ for any $j \in \{0, ..., k+1\}$, we infer that all the partial derivatives of order less or equal to $k+1$ are continuous in $(\tau, +\infty) \times [0,1]$. ■

**Corollary 45** *If $f \in C^\infty((\tau, +\infty) \times [0,1])$ and $f \in L^2_{loc}(\tau, +\infty; H)$, then the solution $u$ to problem (6) belongs to $C^\infty((\tau, +\infty) \times [0,1])$ as well.*

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