Existence of periodic solutions for a scalar differential equation modelling optical conveyor belts

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Abstract

We study a one-dimensional ordinary differential equation modelling optical conveyor belts, showing in particular cases of physical interest that periodic solutions exist. Moreover, under rather general assumptions it is proved that the set of periodic solutions is bounded.

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1 Introduction

Electromagnetic fields can exert forces on micro and nano-particles which are the result of radiation scattering produced by the own particle. As it was demonstrated by Ashkin [2], these electromagnetic forces can be used to trap and manipulate the matter. Since then, electromagnetic microparticle manipulation has become an important technique in a wide range of fields like biology, colloid science or microfluidics.

In the case of small particles (radius of a few nanometers), assuming that we are working far from the resonance of nano-particles region, the main optical techniques for micromanipulation like optical tweezers, tractor beams or conveyor belts are based on the gradient forces [3, 5, 6, 7] generated by the interaction of the particles with an spatially inhomogeneous optical beams like Gaussian beams or Bessel beams [13].

In this work, we are going to study the particular case of an optical conveyor belt [9, 10], where the gradient forces that act on the z-axially confined particles are obtained by means of the superposition of two temporally dephased counter propagating complex electromagnetic fields of frequency ω. Thus, by assuming linearly polarized beams along the x-axis, the electric field is given by:

$$\vec{E}(t,z) = \big(e_0 f_0(z)\exp(ikz) + e_0 f_0(z)\exp(-ikz)\exp(ibt)\big)\exp(i\omega t), \tag{1}$$

where $k = n\omega/c$ is the wavenumber in a medium with refractive index n, b is the parameter that makes the conveyor work by controlling the relative phase of counter propagating fields [6], e_0 is the amplitude of the electric field and $f_0(z)$ takes into account the spatial variation of the electric field amplitude.

If particles are small enough (Rayleigh regimen), the matter-radiation interaction can be modelled by electric dipoles [3]:

$$\vec{p} = 4\pi\epsilon_0\alpha\vec{E}, \tag{2}$$

where ϵ_0 is the permittivity of the free space and α is the polarizability of the particle.

The dipole particle dynamic inside the media is governed by the differential equation:

$$m\, z''(t) = -\gamma\, z'(t) + F(t,z) \tag{3}$$

where γ is the friction coefficient of the medium where, particles of mass m are immersed, and $F(t,z)$ correspond to the time average axial force that acts on a Rayleigh particle inside of the electromagnetic field

(1), that it is given by [3]:

$$F(t,z) = 4\pi\epsilon_0 \mathcal{R}[\alpha] \frac{\partial |\vec{E}(t,z)|^2}{\partial z}, \tag{4}$$

where $\mathcal{R}[\cdot]$ denote the real part. For obtaining the last expression, as it has been previously mentioned, we have neglected the imaginary part of dipole moment assuming that the working wavelength of the electromagnetic fields are far from the particles resonance.

For Rayleigh particles, differential equation (3) is over-damped [12] ($m\, z''(t) \approx 0$), so the particle dynamic inside the media of refractive index n is governed by the non autonomous differential equation:

$$z' = \frac{F(t,z)}{\gamma} = F_z(t,z). \tag{5}$$

Introducing equation (1) in (4) and (5), it can be observed that the force F_z is a gradient one, i.e., $F_z = \frac{\partial V(t,z)}{\partial z}$, and then the particles in this field are immersed in a potential energy given by:

$$V(t,z) = \frac{4\pi\epsilon_0 \mathcal{R}[\alpha]}{\gamma} |\vec{E}|^2 = F_0\, f(z) \cos\left(kz - \frac{bt}{2}\right)^2 \tag{6}$$

where we have introduced the constant $F_0 = \frac{4\pi\epsilon_0 \mathcal{R}[\alpha] e_0^2}{\gamma}$ and the function $f(z) = |f_0(z)|^2$. Constant F_0 is proportional to total intensity of the electromagnetic wave. The periodic function that depends on the phase difference of the electromagnetic field give rise to the optical conveyor, whose continuous variation move the trapped particles along the z-axis.

Then, the dynamic of particles in an axial optical conveyor belt can be modeled by the differential equation:

$$z' = F_z(t,z), \tag{7}$$

where

$$F_z(t,z) = \frac{\partial V(t,z)}{\partial z}, \tag{8}$$

$$V(t,z) = F_0\, f(z) \cos\left(kz - \frac{bt}{2}\right)^2, \tag{9}$$

with $F_0, k, b > 0$.

As we have mentioned, the potential $V(t,z)$ is directly proportional to the conveyor's axial intensity, so the $f(z)$ function determines the axial region where the conveyor belt has enough strength to move the particles. We are going to analyze three different cases for the function $f(z)$:

1. Constant axial region strength $f(z) = 1$.

   In this case the conveyor strength is not spatially limited and can be physically obtained by means of the interference of two dephased counter propagating plane waves.

2. Lorentzian axial region strength $f(z) = \frac{1}{1+(z/z_0)^2},\ z_0 > 0$.

   This behavior can be physically obtained through the interference of two dephased counter propagating Gaussian beams. In this case the parameter z_0 is known as the Rayleigh range of the Gaussian beams, and the axial region strength has its maximum at $z = 0$ position and drops gradually as $|z|$ increases, reaching half its peak value at $z = \pm z_0$, i.e. ($f(\pm z_0) = \frac{1}{2}$) [13].

3. Gaussian axial region strength $f(z) = \exp(-2\frac{z^2}{z_0^2}),\ z_0 > 0$.

   In this case we have also the axial region strength at $z = 0$ and f drops monotonically as $|z|$ increases. The meaning of z_0 is different to the previous case since $f(z_0) = \exp(-2)$. Also, it is more difficult to obtain physically this kind of conveyor belt than previous ones; however, similar methodologies to those described in [16, 17] could be used in order to obtain it.

The main goal of this paper is to establish the existence of periodic solutions for equation (7) in the cases of Lorentzian and Gaussian axial region strengths. This result was suggested in [8] by numerical simulations for a four-dimensional model, and now we give a rigorous mathematical proof of this statement in a simpler one-dimensional model of optical conveyor belts.

This type of problem is interesting for both physical and mathematical points of view, as the existence of periodic solutions for equations of this kind is not known in the mathematical literature as far as we know. In order to prove it we have used an abstract result about the existence of solutions of boundary value problems given in [4]. We observe that standard techniques such as obtaining an invariant region in order to use the fixed point theorem (see e.g. [11, 14, 18]), the use of a Lyapunov function [15] or degree theory [1] seem to be uselees for our problem. Thus, a very specific argument has been developed for our particular type of equations.

2 Non-plane waves: $f(z) \neq 1$

Let us consider the differential equation

$$z' = F_z(t, z) \tag{10}$$

with F_z defined by (8)–(9) and with $f(z)$ not identically equal to 1. In fact, $f(z)$ will be a function satisfying the asymptotic behaviour $f(z) \to 0$ as $|z| \to \infty$.

First, we prove that under rather general assumptions on the function f the set of periodic solutions with period $T = \frac{4\pi}{b}$ is bounded.

Second, we obtain the existence of periodic solutions with period $T = \frac{4\pi}{b}$ for the particular cases $f(z) = \frac{1}{1+(z/z_0)^2}$ and $f(z) = exp(-2\frac{z^2}{z_0^2})$.

2.1 Boundedness of the set of periodic solutions

We assume that f satisfies the following conditions:

1. $f \in C^1(\mathbb{R})$.
2. $f(z) \to 0$ if $|z| \to \infty$.
3. $f'(z) \to 0$ if $|z| \to \infty$.
4. If $z_n, v_n \to \infty$, $\frac{v_n}{z_n} \to 1$, as $n \to \infty$, and $f'(z_n) \neq 0$, then
$$\frac{f^2(v_n)}{f'(z_n)} \to 0, \ \frac{(f'(v_n))^2}{f'(z_n)} \to 0. \tag{11}$$
5. The flow generated by equation (10) has no fixed points.

We note that

$$F_z(t, z) = -kF_0 f(z) \sin(2kz - bt) + F_0 \left(\cos\left(kz - \frac{bt}{2}\right)\right)^2 f'(z).$$

The function $t \mapsto F_z(t, z)$ is periodic with period $T = \frac{4\pi}{b}$. Hence, we look for periodic solutions of (10) with the same period T, which is equivalent to solving (10) in the interval $[0, T]$ with the following boundary condition:

$$z(0) - z(T) = 0. \tag{12}$$

Denote by $\|\cdot\|_C = \max_{t\in[0,T]} |x(t)|$ the norm in the space $C([0, T], \mathbb{R})$.

We will establish that the set of solutions of problem (10)–(12), assuming that it is non-empty, is bounded in the space $C([0, T], \mathbb{R})$.

Theorem 1 *There exists a constant $D > 0$ such that every solution $z(\cdot)$ of problem (10)–(12) satisfies*

$$\|z\|_C \leq D. \tag{13}$$

Proof. On the one hand, making use of the equality

$$F_z\left(t,z\left(t\right)\right)z'=\frac{\mathrm{d}V}{\mathrm{d}t}(t,z\left(t\right))-\frac{\partial V}{\partial t}(t,z\left(t\right)),$$

multiplying (10) by z' and integrating over $(0,T)$ we have

$$\begin{aligned}\int_0^T\left|z'\right|^2\mathrm{d}t&=\int_0^T\left|F_z(t,z\left(t\right))\right|^2\mathrm{d}t\\&=V\left(T,z\left(T\right)\right)-V\left(0,z\left(0\right)\right)-\int_0^T\frac{\partial V}{\partial t}(t,z\left(t\right))\mathrm{d}t\\&=-\int_0^T\frac{\partial V}{\partial t}(t,z\left(t\right))\mathrm{d}t,\end{aligned}\tag{14}$$

due to the boundary conditions (12).

On the other hand, since

$$\frac{\partial V}{\partial t}(t,z)=\frac{b}{2}F_0f\left(z\right)\sin\left(2kz-bt\right),$$

we have

$$F_z\left(t,z\right)=-\frac{2k}{b}\frac{\partial V}{\partial t}(t,z)+F_0\left(\cos\left(kz-\frac{bt}{2}\right)\right)^2f'\left(z\right),$$

so that integrating (10) over $(0,T)$ we deduce that

$$\begin{aligned}&\frac{2k}{b}\int_0^T\frac{\partial V}{\partial t}(t,z\left(t\right))\mathrm{d}t\\&=z\left(0\right)-z\left(T\right)+F_0\int_0^T\left(\cos\left(kz\left(t\right)-\frac{bt}{2}\right)\right)^2f'\left(z\left(t\right)\right)\mathrm{d}t\\&=F_0\int_0^T\left(\cos\left(kz\left(t\right)-\frac{bt}{2}\right)\right)^2f'\left(z\left(t\right)\right)\mathrm{d}t.\end{aligned}$$

Substituting this expression in (14) we have

$$\begin{aligned}&\int_0^T\left|F_z(t,z\left(t\right))\right|^2\mathrm{d}t\\&=\int_0^T\left(-kF_0f\left(z(t)\right)\sin\left(2kz(t)-bt\right)+F_0\left(\cos\left(kz(t)-\frac{bt}{2}\right)\right)^2f'\left(z(t)\right)\right)^2\mathrm{d}t\\&=-\frac{bF_0}{2k}\int_0^T\left(\cos\left(kz\left(t\right)-\frac{bt}{2}\right)\right)^2f'\left(z\left(t\right)\right)\mathrm{d}t.\end{aligned}\tag{15}$$

We state first that that there exists $D_1>0$ such that for any solution $z(\cdot)$ of problem (10)–(12) there exists t_z for which

$$|z(t_z)|\leq D_1.\tag{16}$$

Otherwise, there would exist a sequence of solutions $z_n(\cdot)$ such that

$$\inf_{t\in[0,T]}|z_n(t)|\to\infty.\tag{17}$$

Then integrating equation (10) over (s,t) we obtain

$$\begin{aligned}|z_n(t)-z_n(s)|&\leq\int_s^t|F_z(r,z_n(r))|\,\mathrm{d}r\\&\leq TF_0\left(k\sup_{r\in[0,T]}|f(z_n(r))|+\sup_{r\in[0,T]}|f'(z_n(r))|\right).\end{aligned}$$

In view of (17) and the properties of the function f we have

$$\sup_{t,s\in[0,T]} |z_n(t) - z_n(s)| \to 0 \text{ as } n \to \infty. \tag{18}$$

Further, we will analyze each term in equality (15) in order to get a contradiction.

Using the second mean value theorem for integrals it follows the existence of $\overline{z}_n \in \{z_n(t),\ t \in [0,T]\}$ such that

$$-\frac{bF_0}{2k}\int_0^T \left(\cos\left(kz_n(t) - \frac{bt}{2}\right)\right)^2 f'(z_n(t))\,\mathrm{d}t = -\frac{bF_0}{2k} f'(\overline{z}_n)\int_0^T \left(\cos\left(kz_n(t) - \frac{bt}{2}\right)\right)^2 \mathrm{d}t.$$

Hence, (15) yields

$$\begin{aligned}
&\int_0^T \left(\cos\left(kz_n(t) - \frac{bt}{2}\right)\right)^2 \mathrm{d}t \qquad (19)\\
&\leq \frac{2k}{bF_0}\frac{1}{|f'(\overline{z}_n)|}\int_0^T \left(-kF_0 f(z_n(t))\sin(2kz_n(t) - bt) + F_0\left(\cos\left(kz_n(t) - \frac{bt}{2}\right)\right)^2 f'(z_n(t))\right)^2 \mathrm{d}t\\
&\leq \frac{4kF_0}{b}\frac{1}{|f'(\overline{z}_n)|}\int_0^T \left(k^2(f(z_n(t)))^2\sin^2(2kz_n(t) - bt) + \left(\cos\left(kz_n(t) - \frac{bt}{2}\right)\right)^4 (f'(z_n(t)))^2\right)\mathrm{d}t\\
&\leq \frac{4kF_0}{b}\frac{1}{|f'(\overline{z}_n)|}\int_0^T \left(k^2(f(z_n(t)))^2 + (f'(z_n(t)))^2\right)\mathrm{d}t\\
&= \frac{4kF_0}{b}\frac{1}{|f'(\overline{z}_n)|}\left(k^2(f(\widetilde{z}_n))^2 + (f'(\widetilde{z}_n))^2\right),
\end{aligned}$$

for some $\widetilde{z}_n \in \{z_n(t),\ t \in [0,T]\}$. We observe that $f'(\overline{z}_n) \neq 0$, because otherwise from (15) we would obtain that

$$\int_0^T |F_z(t, z_n(t))|^2\,\mathrm{d}t = \int_0^T |z_n'(t)|^2\,\mathrm{d}t = 0,$$

so $z_n'(t) \equiv 0$ and $z_n(\cdot)$ would be a fixed point, which is impossible by assumption.

The rigth-hand side of (19) converges to 0 in light of assumption (11) as (18) implies that $\frac{\widetilde{z}_n}{\overline{z}_n} \to 1$. But we can prove that

$$\int_0^T \left(\cos\left(kz_n(t) - \frac{bt}{2}\right)\right)^2 \mathrm{d}t \to \frac{T}{2},$$

obtaining in this way the desired contradiction. Indeed, it is clear that

$$\int_0^T \left(\cos\left(kz_n(0) - \frac{bt}{2}\right)\right)^2 \mathrm{d}t = \frac{T}{2},$$

and (18) implies that

$$\left|\left(\cos\left(kz_n(t) - \frac{bt}{2}\right)\right)^2 - \left(\cos\left(kz_n(0) - \frac{bt}{2}\right)\right)^2\right| \to 0 \text{ uniformly in } [0,T],$$

so

$$\begin{aligned}
&\left|\int_0^T \left(\cos\left(kz_n(t) - \frac{bt}{2}\right)\right)^2 dt - \frac{T}{2}\right|\\
&= \left|\int_0^T \left(\left(\cos\left(kz_n(t) - \frac{bt}{2}\right)\right)^2 - \left(\cos\left(kz_n(0) - \frac{bt}{2}\right)\right)^2\right) dt\right| \to 0.
\end{aligned}$$

Once we have proved that (16) holds true, the statement of the theorem is deduced easily. Integrating equation (10) over the interval (t_z, t) we get

$$\begin{aligned} |z(t)| &\leq |z(t_z)| + \int_{t_z}^{t} |F_z(s, z(s))| \, \mathrm{d}s \\ &\leq D_1 + F_0 T \left(k \sup_{r \in \mathbb{R}} |f(r)| + \sup_{r \in \mathbb{R}} |f'(r)| \right) = D_2, \end{aligned}$$

for any $t \in [t_z, T]$. Finally, since $z(0) = z(T)$, for $t \in [0, t_z]$ we get

$$\begin{aligned} |z(t)| &\leq |z(0)| + \int_{0}^{t} |F_z(s, z(s))| \, \mathrm{d}s \\ &\leq D_2 + F_0 T \left(k \sup_{r \in \mathbb{R}} |f(r)| + \sup_{r \in \mathbb{R}} |f'(r)| \right) = D, \end{aligned}$$

which concludes the proof. ■

Corollary 2 *Every periodic solution with period T belongs to the ball of radius D centered at 0 of the space of continuous bounded functions $C_{\mathrm{b}}(\mathbb{R})$, whose norm is $\|u\|_{C_{\mathrm{b}}} = \sup_{t \in \mathbb{R}} |u(t)|$.*

The following criterion is useful in order to check that the equation has no fixed points.

Lemma 3 *$\overline{z}$ is a fixed point of (10) if and only if*

$$f(\overline{z}) = f'(\overline{z}) = 0. \tag{20}$$

Proof. Let $\overline{z}$ be a fixed point, so $F_z(s, \overline{z}) = 0$ for any s, that is,

$$k f(\overline{z}) \sin(2k\overline{z} - bt) = \left(\cos\left(k\overline{z} - \frac{bt}{2} \right) \right)^2 f'(\overline{z}) \ \ \forall t.$$

Assume that (20) is not true. If $f(\overline{z}) = 0$, $f'(\overline{z}) \neq 0$ (respectively, $f(\overline{z}) \neq 0$, $f'(\overline{z}) = 0$), then $\cos\left(k\overline{z} - \frac{bt}{2}\right) = 0$ for any t (respectively, $\sin(2k\overline{z} - bt) = 0$), which is not possible. If $f(\overline{z}) \neq 0$, $f'(\overline{z}) \neq 0$, then $\tan\left(k\overline{z} - \frac{b}{2}t\right) = \frac{f'(\overline{z})}{2kf(\overline{z})}$ for any t, which, again, cannot occur. Therefore, if $\overline{z}$ is a fixed point, then (20) holds.

The converse statement is straightforward. ■

2.2 Existence of periodic solutions

We will prove the existence of periodic solutions in the aforementioned particular cases.

2.2.1 Case 1: $f(z) = \frac{z_0^2}{z_0^2 + z^2}$

We shall prove in this section that problem (10), (12) is solvable in the particular case where $f(z) = \frac{z_0^2}{z_0^2 + z^2}$, where $z_0 > 0$.

First of all, let us define the functions $p \colon C([0,T], \mathbb{R}) \times C([0,T], \mathbb{R}) \to C([0,T], \mathbb{R})$, $l \colon C([0,T], \mathbb{R}) \times C([0,T], \mathbb{R}) \to \mathbb{R}$ by

$$\begin{aligned} p(z, y)(t) &= -y(t), \ \forall t \in [0, T], \\ l(z, y) &= y(0) - y(T). \end{aligned} \tag{21}$$

It is obvious that the maps $p(z,\cdot)\colon C([0,T],\mathbb{R}) \to C([0,T],\mathbb{R})$, $l(z,\cdot)\colon C([0,T],\mathbb{R}) \to \mathbb{R}$ are linear for any fixed $z \in C([0,T],\mathbb{R})$. Also, for any $y, z \in C([0,T],\mathbb{R})$ and $t \in [0,T]$ we have

$$|p(z,y)(t)| \le |y(t)|,$$
$$|l(z,y)| \le 2\|y\|_C .$$

Consider the following problem

$$\frac{\mathrm{d}y}{\mathrm{d}t} = p(z,y)(t) + q(t),\ \ l(z,y) = c_0, \tag{22}$$

where $q \in C([0,T],\mathbb{R})$ and $c_0 \in \mathbb{R}$. Every solution of this problem satisfies

$$y(t) = \exp(-t)\, y(0) + \int_0^t \exp(-(t-s))\, q(s)\mathrm{d}s. \tag{23}$$

Using the boundary conditions we have

$$y(T) = \exp(-T)\,(y(T) + c_0) + \int_0^T \exp(-(T-s))\, q(s)\mathrm{d}s,$$

so

$$|y(T)| \le \frac{1}{1-\exp(-T)}\left(|c_0| + \int_0^T |q(s)|\,\mathrm{d}s\right)$$

and

$$|y(0)| \le |y(T)| + |c_0| \le \frac{1}{1-\exp(-T)}\left(|c_0| + \int_0^T |q(s)|\,\mathrm{d}s\right) + |c_0|.$$

Therefore, by (23) there exists $\beta > 0$ such that

$$\|y\|_C \le \beta(|c_0| + \|q\|_{L^1(0,T;\mathbb{R})}). \tag{24}$$

Further, we study the boundary-value problem

$$\begin{cases} \dfrac{\mathrm{d}z}{\mathrm{d}t} = p(z,z)(t) + \lambda\left(F_z(t,z(t)) - p(z,z)(t)\right), \\ z(0) - z(T) = 0, \end{cases} \tag{25}$$

for $\lambda \in (0,1)$.

Theorem 4 *There exists $\rho > 0$ such that for any $\lambda \in (0,1)$ and any solution $z_\lambda(\cdot)$ to problem (25) it holds that*

$$\|z_\lambda\|_C \le \rho. \tag{26}$$

Proof. First, we shall prove the existence of $\rho' > 0$ such that for any $\lambda \in (0,1)$ and any solution $z_\lambda(\cdot)$ to problem (25) the estimate

$$\min_{t\in[0,T]} |z_\lambda(t)| < \rho' \tag{27}$$

is satisfied. By contradiction, if this is not true, then there exists a sequence of solutions $z_{\lambda_n}(\cdot)$ such that

$$\min_{t\in[0,T]} |z_{\lambda_n}(t)| \to \infty.$$

Without loss of generality we can assume that $\min_{t\in[0,T]} z_{\lambda_n}(t) \to +\infty$.

Multiplying the equation in (25) by $z_0^2 + z_{\lambda_n}^2(t)$ and integrating over $(0,T)$ we have

$$\begin{aligned} &-(1-\lambda_n)\int_0^T p(z_{\lambda_n}, z_{\lambda_n})(t)(z_0^2 + z_{\lambda_n}^2(t))\mathrm{d}t \\ &= -kF_0 z_0^2 \lambda_n \int_0^T \sin(2kz_{\lambda_n}(t) - bt)\,\mathrm{d}t + \lambda_n F_0 \int_0^T \left(\cos\left(kz_{\lambda_n}(t) - \frac{bt}{2}\right)\right)^2 f'(z_{\lambda_n}(t))\,(z_0^2 + z_{\lambda_n}^2(t))\mathrm{d}t. \end{aligned}$$

There exist $z^*_{\lambda_n} = z_{\lambda_n}(t^*_n)$, $\overline{z}_{\lambda_n} = z_{\lambda_n}(\overline{t}_n)$ such that

$$\begin{aligned}
&T(1-\lambda_n)z^*_{\lambda_n}(z_0^2+(z^*_{\lambda_n})^2)\\
&= -kF_0\lambda_n z_0^2\int_0^T \sin\left(2kz_{\lambda_n}(t)-bt\right)\mathrm{d}t + \lambda_n F_0 f'\left(\overline{z}_{\lambda_n}\right)\left(z_0^2+(\overline{z}_{\lambda_n})^2\right)\int_0^T\left(\cos\left(kz_{\lambda_n}(t)-\frac{bt}{2}\right)\right)^2\mathrm{d}t\\
&= -kF_0\lambda_n z_0^2\int_0^T \sin\left(2kz_{\lambda_n}(t)-bt\right)\mathrm{d}t - 2\lambda_n F_0 z_0^2\frac{\overline{z}_{\lambda_n}}{z_0^2+(\overline{z}_{\lambda_n})^2}\int_0^T\left(\cos\left(kz_{\lambda_n}(t)-\frac{bt}{2}\right)\right)^2\mathrm{d}t,
\end{aligned}$$

so

$$\begin{aligned}
&T(1-\lambda_n)\frac{z^*_{\lambda_n}}{\overline{z}_{\lambda_n}}(z_0^2+(z^*_{\lambda_n})^2)(z_0^2+(\overline{z}_{\lambda_n})^2) \\
&= -kF_0\lambda_n z_0^2\frac{z_0^2+(\overline{z}_{\lambda_n})^2}{\overline{z}_{\lambda_n}}\int_0^T \sin\left(2kz_{\lambda_n}(t)-bt\right)\mathrm{d}t - 2\lambda_n F_0 z_0^2\int_0^T\left(\cos\left(kz_{\lambda_n}(t)-\frac{bt}{2}\right)\right)^2\mathrm{d}t.
\end{aligned}\tag{28}$$

Integrating the equation in (25) over $(0,T)$ and using $z_\lambda(T) = z_\lambda(0)$ we obtain for any λ that

$$-(1-\lambda)\int_0^T p(z_\lambda, z_\lambda)(s)\mathrm{d}s = \lambda\int_0^T F_z(s, z_\lambda(s))\mathrm{d}s. \tag{29}$$

Since $z_{\lambda_n}(t) > 0$ and $p(z_{\lambda_n}, z_{\lambda_n})(s) = -z_{\lambda_n}(s)$, we have

$$\begin{aligned}
-(1-\lambda_n)\int_0^T p(z_{\lambda_n}, z_{\lambda_n})(s)\mathrm{d}s &= (1-\lambda_n)\int_0^T z_{\lambda_n}(s)\mathrm{d}s\\
&= (1-\lambda_n)\int_0^T \left|p(z_{\lambda_n}, z_{\lambda_n})(s)\right|\mathrm{d}s = \lambda_n\int_0^T F_z(s, z_{\lambda_n}(s))\mathrm{d}s.
\end{aligned}\tag{30}$$

Therefore, we deduce from (25) and (30) that

$$\begin{aligned}
\left|z_{\lambda_n}(t)-z_{\lambda_n}(0)\right| &\le \int_0^T \left|(1-\lambda_n)p(z_{\lambda_n}, z_{\lambda_n})(s)+\lambda_n F_z(s, z_{\lambda_n}(s))\right|\mathrm{d}s\\
&\le 2\int_0^T \left|F_z(s, z_{\lambda_n}(s))\right|\mathrm{d}s\\
&\le 2\int_0^T \left(kF_0\left|f\left(z_{\lambda_n}(s)\right)\right| + F_0\left|f'\left(z_{\lambda_n}(s)\right)\right|\right)\mathrm{d}s\\
&= 2F_0T(k\left|f\left(\widetilde{z}_{\lambda_n}\right)\right| + \left|f'\left(\widetilde{z}_{\lambda_n}\right)\right|)\\
&= 2F_0Tz_0^2\left(\frac{k}{z_0^2+(\widetilde{z}_{\lambda_n})^2}+\frac{2\widetilde{z}_{\lambda_n}}{\left(z_0^2+(\widetilde{z}_{\lambda_n})^2\right)^2}\right),\ \forall t\in[0,T],
\end{aligned}\tag{31}$$

for some $\widetilde{z}_{\lambda_n} = z_{\lambda_n}(\widetilde{t}_n)$. This implies, in particular, that

$$\sup_{t,s\in[0,T]}\left|z_{\lambda_n}(t)-z_{\lambda_n}(s)\right| \to 0, \tag{32}$$

as $n\to\infty$.

If we proved that

$$\frac{z_0^2+(\overline{z}_{\lambda_n})^2}{\overline{z}_{\lambda_n}}\int_0^T \sin\left(2kz_{\lambda_n}(t)-bt\right)\mathrm{d}t \to 0, \tag{33}$$

$$\int_0^T\left(\cos\left(kz_{\lambda_n}(t)-\frac{bt}{2}\right)\right)^2\mathrm{d}t \to \frac{T}{2}, \tag{34}$$

then (28) would imply that

$$\frac{1-\lambda_n}{\lambda_n}\frac{z^*_{\lambda_n}}{\overline{z}_{\lambda_n}}(z_0^2+(z^*_{\lambda_n})^2)(z_0^2+(\overline{z}_{\lambda_n})^2) \to -F_0z_0^2,$$

which is not possible as $\frac{1-\lambda_n}{\lambda_n}\frac{z^*_{\lambda_n}}{\overline{z}_{\lambda_n}}(z_0^2+(z^*_{\lambda_n})^2)(z_0^2+(\overline{z}_{\lambda_n})^2)>0$ for all n. Thus, (27) would be true.

In order to check (33), making use of the fact that

$$\int_0^T \sin\left(2kz_{\lambda_n}(0)-bt\right)\mathrm{d}t=0,$$

we obtain by (31) that

$$\begin{aligned}
&\frac{z_0^2+(\overline{z}_{\lambda_n})^2}{\overline{z}_{\lambda_n}}\left|\int_0^T \sin\left(2kz_{\lambda_n}(t)-bt\right)\mathrm{d}t\right| \\
&\le \frac{z_0^2+(\overline{z}_{\lambda_n})^2}{\overline{z}_{\lambda_n}}\int_0^T \left|\sin\left(2kz_{\lambda_n}(t)-bt\right)-\sin\left(2kz_{\lambda_n}(0)-bt\right)\right|\mathrm{d}t \\
&\le \frac{z_0^2+(\overline{z}_{\lambda_n})^2}{\overline{z}_{\lambda_n}}\int_0^T \left|\cos(\beta_n(t))\right| 2k\left|z_{\lambda_n}(t)-z_{\lambda_n}(0)\right|\mathrm{d}t \\
&\le \frac{z_0^2+(\overline{z}_{\lambda_n})^2}{\overline{z}_{\lambda_n}}4kT^2F_0z_0^2\left(\frac{k}{z_0^2+(\widetilde{z}_{\lambda_n})^2}+\frac{2\widetilde{z}_{\lambda_n}}{\left(z_0^2+(\widetilde{z}_{\lambda_n})^2\right)^2}\right)\to 0,
\end{aligned}$$

as from (32) we can easily see that

$$\begin{aligned}
\frac{z_0^2+(\overline{z}_{\lambda_n})^2}{z_0^2+(\widetilde{z}_{\lambda_n})^2}&\to 1,\\
\frac{\widetilde{z}_{\lambda_n}}{\overline{z}_{\lambda_n}}&\to 1.
\end{aligned}$$

In order to obtain (34) it suffices to see that

$$\int_0^T\left(\cos\left(kz_{\lambda_n}(0)-\frac{bt}{2}\right)\right)^2\mathrm{d}t=\frac{T}{2}$$

and that (32) implies

$$\left|\left(\cos\left(kz_{\lambda_n}(t)-\frac{bt}{2}\right)\right)^2-\left(\cos\left(kz_{\lambda_n}(0)-\frac{bt}{2}\right)\right)^2\right|\to 0 \text{ uniformly in } [0,T],$$

so

$$\begin{aligned}
&\left|\int_0^T\left(\cos\left(kz_{\lambda_n}(t)-\frac{bt}{2}\right)\right)^2\mathrm{d}t-\frac{T}{2}\right| \\
&=\left|\int_0^T\left(\left(\cos\left(kz_{\lambda_n}(t)-\frac{bt}{2}\right)\right)^2-\left(\cos\left(kz_{\lambda_n}(0)-\frac{bt}{2}\right)\right)^2\right)\mathrm{d}t\right|\to 0.
\end{aligned}$$

Finally, let us prove that (27) implies (26).

It follows from (27) that for every solution $z_\lambda(\cdot)$ of problem (25) there exists a moment of time t_{z_λ} such that $|z_\lambda(t_{z_\lambda})|<\rho'$. Integrating the equation in (25) over (t_{z_λ},t) we have

$$z_\lambda(t)=z_\lambda(t_{z_\lambda})+\int_{t_{z_\lambda}}^t\left(-\left(1-\lambda\right)z_\lambda(s)+\lambda F_z(t,z_\lambda\left(s\right))\right)\mathrm{d}s.$$

Hence,

$$\begin{aligned}
|z_\lambda(t)| &\le \rho' + \int_{t_{z_\lambda}}^{t} \left((1-\lambda)\,|z_\lambda(s)| + \lambda\,|F_z(t, z_\lambda(s))|\right) \mathrm{d}s \\
&\le \rho' + \int_{t_{z_\lambda}}^{t} \left(|z_\lambda(s)| + kF_0 \max_{u\in\mathbb{R}} |f(u)| + F_0 \max_{u\in\mathbb{R}} |f'(u)|\right) \mathrm{d}s \\
&\le \rho' + T\left(kF_0 \max_{u\in\mathbb{R}} |f(u)| + F_0 \max_{u\in\mathbb{R}} |f'(u)|\right) + \int_{t_{z_\lambda}}^{t} |z_\lambda(s)|\,\mathrm{d}s \\
&= R + \int_{t_{z_\lambda}}^{t} |z_\lambda(s)|\,\mathrm{d}s.
\end{aligned}$$

Applying Gronwall's lemma we get

$$|z_\lambda(t)| \le Re^{t-t_{z_\lambda}} \le Re^T \text{ for all } t \in [t_{z_\lambda}, T].$$

Noting that $z_\lambda(T) = z_\lambda(0)$ implies

$$|z_\lambda(0)| = |z_\lambda(T)| \le Re^T,$$

we can repeat the same argument in order to show the existence of $\rho \ge Re^T$ such that

$$|z_\lambda(t)| \le \rho \text{ for all } t \in [0, t_{z_\lambda}].$$

The proof of (26) is now complete. ■

Let us recall a general result on existence of solutions for boundary-value problems, which was proved in [4].

Let $I = [a,b] \subset \mathbb{R}$, $g\colon C(I,\mathbb{R}^n) \to L^1(I,\mathbb{R}^n)$, $h\colon C(I,\mathbb{R}^n) \to \mathbb{R}^n$ be continuous operators such that for any $\rho > 0$ one has

$$\sup\{\|g(x)(\cdot)\|_{\mathbb{R}^n} : x \in C(I,\mathbb{R}^n),\ \|x\|_C \le \rho\} \in L^1(I,\mathbb{R}^n), \tag{35}$$

$$\sup\{\|h(x)\|_{\mathbb{R}^n} : x \in C(I,\mathbb{R}^n),\ \|x\|_C \le \rho\} < \infty, \tag{36}$$

where, as before, $\|x\|_C = \max_{t\in I} \|x(t)\|_{\mathbb{R}^n}$, and consider the boundary-value problem

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = g(x)(t), \\ h(x) = 0. \end{cases} \tag{37}$$

We also consider a pair (p, l) of continuous operators $p\colon C(I,\mathbb{R}^n) \times C(I,\mathbb{R}^n) \to L^1(I,\mathbb{R}^n)$, $l\colon C(I,\mathbb{R}^n) \times C(I,\mathbb{R}^n) \to \mathbb{R}^n$ satisfying:

1. for any fixed $x \in C(I,\mathbb{R}^n)$ the operators $p(x,\cdot)\colon C(I,\mathbb{R}^n) \to L^1(I,\mathbb{R}^n)$, $l(x,\cdot)\colon C(I,\mathbb{R}^n) \to \mathbb{R}^n$ are linear;

2. for any $x, y \in C(I,\mathbb{R}^n)$ we have

$$\|p(x,y)(t)\|_{\mathbb{R}^n} \le \alpha(t, \|x\|_C)\,\|y\|_C\,,$$

$$\|l(x,y)(t)\|_{\mathbb{R}^n} \le \alpha_0(\|x\|_C)\,\|y\|_C\,,$$

   with $\alpha_0 : \mathbb{R}^+ \to \mathbb{R}^+$ being nondecreasing and $\alpha : \mathbb{R}^+ \to \mathbb{R}^+$ being nondecreasing in the second argument and integrable with respect to the first one;

3. there exists $\beta > 0$ such that for any $x \in C(I,\mathbb{R}^n)$, $q \in L^1(I,\mathbb{R}^n)$ and $c_0 \in \mathbb{R}^n$ every solution of the boundary-value problem

$$\frac{\mathrm{d}y}{\mathrm{d}t} = p(x,y)(t) + q(t),\ \ l(x,y) = c_0,$$

   satisfies the estimate

$$\|y\|_C \le \beta(\|c_0\|_{\mathbb{R}^n} + \|q\|_{L^1(I,\mathbb{R}^n)}).$$

Theorem 5 *[4, Theorem 1] Suppose that there exists* $\rho > 0$ *such that for any* $\lambda \in (0,1)$ *the boundary-value problem*

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = p(x,x)(t) + \lambda(g(x)(t) - p(x,x)(t)), \\ l(x,x) = \lambda(l(x,x) - h(x)), \end{cases}$$

satisfies the estimate

$$\|x\|_C \leq \rho. \tag{38}$$

Then problem (37) *possesses at least one solution.*

Applying this theorem we can prove the existence of periodic solutions for equation (10).

Theorem 6 *The boundary-value problem* (10)*,* (12) *possesses at least one solution, which can be extended to a periodic solution of equation* (10)*.*

Proof. We put $I = [0,T]$, $n = 1$, $x = z$, $h(z) = z(0) - z(T)$ and $g(z)(t) = F_z(t, z(t))$. It is not difficult to check that (35)-(36) hold true. The pair (p,l) was defined in (21) and the properties 1–3 were proved to be true above. Also, by Theorem 4 we know that (38) is satisfied. Therefore, Theorem 5 proves the solvability of problem (10), (12).

Finally, we can define the solution $z(\cdot)$ on $[0,\infty)$ by putting $z(t+nT) = z(t)$, for any $t \in [0,T]$ and $n \in \mathbb{N}$. This function is a solution of (10) as well due to the fact that the function $t \mapsto F_z(t,z)$ is periodic with period T for any fixed z. ■

Using Theorem 1 it is easy to see that the set of all periodic solutions of equation (10) is bounded.

Theorem 7 *There exists a constant* $D > 0$ *such that every solution* $z(\cdot)$ *of problem* (10)*,* (12) *satisfies*

$$\|z\|_C \leq D.$$

Proof. It suffices to verify that the assumptions of Theorem 1 are satisfied for $f(z) = \frac{z_0^2}{z_0^2+z^2}$.

Conditions 1–3 are obvious, and condition 5 follows from Lemma 3 as $f(z) \neq 0$ for all z.

It remains to check the fourth one. Let $z_n, v_n \to \infty$, $\frac{v_n}{z_n} \to 1$, as $n \to \infty$, and $f'(z_n) \neq 0$. Then

$$\frac{f^2(v_n)}{f'(z_n)} = \frac{z_0^4}{(z_0^2+v_n^2)^2} \frac{(z_0^2+z_n^2)^2}{z_0^2(-2z_n)} \to 0,$$

$$\frac{(f'(v_n))^2}{f'(z_n)} = \frac{4z_0^4 v_n^2}{(z_0^2+v_n^2)^4} \frac{(z_0^2+z_n^2)^2}{z_0^2(-2z_n)} \to 0.$$

■

Corollary 8 *Every periodic solution with period* T *belongs to the ball of radius* D *centered at* 0 *of the space of continuous bounded functions* $C_{\mathrm{b}}(\mathbb{R})$*.*

2.2.2 Case 2: $f(z) = \exp\left(-2\frac{z^2}{z_0^2}\right)$

We will consider in this section the function $f(z) = \exp\left(-2\frac{z^2}{z_0^2}\right)$, where $z_0 > 0$.

Let us prove that Theorem 4 is true in this case as well.

Theorem 9 *There exists* $\rho > 0$ *such that for any* $\lambda \in (0,1)$ *and any solution* $z_\lambda(\cdot)$ *to problem* (25) *it holds that*

$$\|z_\lambda\|_C \leq \rho. \tag{39}$$

Proof. First, we need to check the existence of $\rho' > 0$ such that for any $\lambda \in (0,1)$ and any solution $z_\lambda(\cdot)$ to problem (25) the estimate

$$\min_{t\in[0,T]} |z_\lambda(t)| < \rho' \tag{40}$$

is satisfied. If (40) is not true, then there exists a sequence of solutions $z_{\lambda_n}(\cdot)$ such that

$$\min_{t\in[0,T]} |z_{\lambda_n}(t)| \to \infty.$$

Without loss of generality we can assume that $\min_{t\in[0,T]} z_{\lambda_n}(t) \to +\infty$.

We use the same functions p, l as in the previous case. Multiplying the equation in (25) by $z_{\lambda_n}(t)\exp(2z^2_{\lambda_n}(t)/z_0^2)$ and integrating over $(0,T)$ we have

$$\begin{aligned}
&\int_0^T z'_{\lambda_n}(t) z_{\lambda_n}(t) \exp\left(2\frac{z^2_{\lambda_n}(t)}{z_0^2}\right) \mathrm{d}t \\
&= \frac{z_0^2}{4}\left(\exp\left(2\frac{z^2_{\lambda_n}(T)}{z_0^2}\right) - \exp\left(2\frac{z^2_{\lambda_n}(0)}{z_0^2}\right)\right) \\
&= (1-\lambda_n)\int_0^T p(z_{\lambda_n}, z_{\lambda_n})(t) z_{\lambda_n}(t)\exp\left(2\frac{z^2_{\lambda_n}(t)}{z_0^2}\right)\mathrm{d}t \\
&- kF_0\lambda_n\int_0^T \sin\left(2kz_{\lambda_n}(t) - bt\right) z_{\lambda_n}(t) dt - \frac{4\lambda_n F_0}{z_0^2}\int_0^T \left(\cos\left(kz_{\lambda_n}(t) - \frac{bt}{2}\right)\right)^2 z^2_{\lambda_n}(t)\mathrm{d}t.
\end{aligned}$$

There exist $z^*_{\lambda_n} = z_{\lambda_n}(t^*_n)$, $\widetilde{z}_{\lambda_n} = z_{\lambda_n}(\widetilde{t}_n)$, $\overline{z}_{\lambda_n} = z_{\lambda_n}(\overline{t}_n)$ such that

$$\begin{aligned}
&(1-\lambda_n)(z^*_{\lambda_n})^2 \exp\left(2\frac{\left(z^*_{\lambda_n}\right)^2}{z_0^2}\right) \\
&= -kF_0\lambda_n \sin\left(2k\widetilde{z}_{\lambda_n} - b\widetilde{t}_n\right)\widetilde{z}_{\lambda_n} - \frac{4\lambda_n F_0}{Tz_0^2}(\overline{z}_{\lambda_n})^2 \int_0^T \left(\cos\left(kz_{\lambda_n}(t) - \frac{bt}{2}\right)\right)^2 \mathrm{d}t,
\end{aligned}$$

so

$$\begin{aligned}
&\frac{(1-\lambda_n)}{\lambda_n}\left(\frac{z^*_{\lambda_n}}{\overline{z}_{\lambda_n}}\right)^2 \exp\left(2\frac{\left(z^*_{\lambda_n}\right)^2}{z_0^2}\right) \\
&= -kF_0 \sin\left(2k\widetilde{z}_{\lambda_n} - b\widetilde{t}_n\right)\frac{\widetilde{z}_{\lambda_n}}{(\overline{z}_{\lambda_n})^2} - \frac{4F_0}{Tz_0^2}\int_0^T \left(\cos\left(kz_{\lambda_n}(t) - \frac{bt}{2}\right)\right)^2 \mathrm{d}t.
\end{aligned} \tag{41}$$

From (25) and (30) we obtain that

$$\begin{aligned}
|z_{\lambda_n}(t) - z_{\lambda_n}(0)| &\le \int_0^T \left|(1-\lambda_n)p(z_{\lambda_n}, z_{\lambda_n})(s) + \lambda_n F_z(s, z_{\lambda_n}(s))\right| \mathrm{d}s \\
&\le 2\int_0^T |F_z(s, z_{\lambda_n}(s))|\,\mathrm{d}s \\
&\le 2\int_0^T \left(kF_0\left|f\left(z_{\lambda_n}(s)\right)\right| + F_0\left|f'\left(z_{\lambda_n}(s)\right)\right|\right)\mathrm{d}s \\
&= 2F_0T\left(k\left|f\left(\widehat{z}_{\lambda_n}\right)\right| + \left|f'(\widehat{z}_{\lambda_n})\right|\right) \\
&= 2F_0T\left(k\exp\left(-2\frac{\widehat{z}^2_{\lambda_n}}{z_0^2}\right) + 4\frac{\widehat{z}_{\lambda_n}}{z_0^2}\exp\left(-2\frac{\widehat{z}^2_{\lambda_n}}{z_0^2}\right)\right), \ \forall t\in[0,T],
\end{aligned}$$

for some $\widehat{z}_{\lambda_n} = z_{\lambda_n}(\widehat{t}_n)$. This implies, in particular, that

$$\sup_{t,s\in[0,T]} |z_{\lambda_n}(t) - z_{\lambda_n}(s)| \to 0, \text{ as } n\to\infty. \tag{42}$$

It follows from (42) that

$$\frac{\widetilde{z}_{\lambda_n}}{\overline{z}_{\lambda_n}} \to 1.$$

Hence,

$$-kF_0 \sin\left(2k\widetilde{z}_{\lambda_n} - b\widetilde{t}_n\right) \frac{\widetilde{z}_{\lambda_n}}{\left(\overline{z}_{\lambda_n}\right)^2} \to 0,$$

so (41) and (34) (which can be proved in the same way as in Theorem 4) imply that

$$\frac{(1-\lambda_n)}{\lambda_n}\left(\frac{z^*_{\lambda_n}}{\overline{z}_{\lambda_n}}\right)^2 \exp\left(2\frac{\left(z^*_{\lambda_n}\right)^2}{z_0^2}\right) \to -\frac{2F_0}{z_0^2},$$

which is not possible as $\frac{(1-\lambda_n)}{\lambda_n}\left(\frac{z^*_{\lambda_n}}{\overline{z}_{\lambda_n}}\right)^2 \exp\left(2\frac{\left(z^*_{\lambda_n}\right)^2}{z_0^2}\right) > 0$ for all n. Thus, (40) is proved.

Finally, (39) is deduced exactly in the same way as in Theorem 4. ■

Arguing as in Theorem 6 we obtain the existence of periodic solutions for equation (10).

Theorem 10 *The boundary-value problem* (10), (12) *possesses at least one solution, which can be extended to a periodic solution of equation* (10).

Finally, using Theorem 1 we prove that the set of all periodic solutions of equation (10) is bounded.

Theorem 11 *There exists a constant $D > 0$ such that every solution $z(\cdot)$ of problem* (10), (12) *satisfies*

$$\|z\|_C \leq D.$$

Proof. It is sufficient to check that the assumptions of Theorem 1 are satisfied for $f(z) = \exp\left(-2\frac{z^2}{z_0^2}\right)$.

Conditions 1–3 are straightforward to verify. Since $f(z) \neq 0$ for all z, condition 5 follows from Lemma 3.

It remains to check the fourth one. For sequences z_n, v_n such that $z_n, v_n \to \infty$, $\frac{v_n}{z_n} \to 1$, as $n \to \infty$, and $f'(z_n) \neq 0$, we have

$$\begin{aligned}\frac{f^2(v_n)}{f'(z_n)} &= \frac{z_0^2 \exp\left(-\frac{4v_n^2}{z_0^2}\right)}{(-4z_n)\exp\left(-2\frac{z_n^2}{z_0^2}\right)} = -\frac{z_0^2}{4z_n}\exp\left(-\frac{4v_n^2}{z_0^2} + 2\frac{z_n^2}{z_0^2}\right) \\ &= -\frac{z_0^2}{4z_n}\exp\left(-\frac{2v_n^2}{z_0^2}\left(2 - \frac{z_n^2}{v_n^2}\right)\right) \to 0,\end{aligned}$$

$$\frac{(f'(v_n))^2}{f'(z_n)} = \frac{16v_n^2 \exp\left(-\frac{4v_n^2}{z_0^2}\right)}{z_0^2(-4z_n)\exp\left(-\frac{2z_n^2}{z_0^2}\right)} = -\frac{4}{z_0^2}\frac{v_n}{z_n}v_n \exp\left(-\frac{2v_n^2}{z_0^2}\left(2 - \frac{z_n^2}{v_n^2}\right)\right) \to 0.$$

■

Corollary 12 *Every periodic solution with period T belongs to the ball of radius D centered at 0 of the space of continuous bounded functions $C_b(\mathbb{R})$.*

3 Plane waves $f(z) = 1$

If we take $f(z) = 1$ in problem (7), the electromagnetic fields given by equation (1) correspond to two dephased counter-propagating plane waves as we has previously mentioned. This is a limit case when $z_0 \to \infty$ of the previously studied Gaussian and Lorentzian functions.

For plane waves, the problem (7) have an analytical solution given by:

$$z(t) = \frac{b\,t}{2k} - \frac{1}{k}\arctan(H(t)), \tag{43}$$

where

$$H(t) = \frac{2F_0k^2 + \sqrt{b^2 - 4F_0^2k^4}\,\tan\left(\frac{\sqrt{b^2-4F_0^2k^4}\;t}{2} - \arctan(\frac{2F_0k^2+b\,tan(z_ik)}{\sqrt{b^2-4F_0^2k^4}})\right)}{b} \tag{44}$$

and we have assumed the initial condition $z(0) = z_i$.

From this solution it can be deduced that the particles trapped by the intensity gradient generated by the plane waves are translated indefinitely with nearly the constant velocity phase of the conveyor $v_c = \frac{b}{2k}$ [6].

Depending on the problem parameters two situations can be observed. On the one hand, if $b \leq 2F_0k^2$ the particles trajectories correspond to a uniform rectilinear motion. On the other hand, if $b > 2F_0k^2$, the particles describe a bounded oscillating path around the straight line $z = z_i + \frac{b\,t}{2k} - \frac{1}{2k}\sqrt{b^2 - 4F_0^2k^4}t$. In both cases, as it has been previously mentioned, the particles are indefinitely translated, but plane waves have not enough intensity to move particles and other beams ($f(z) \neq 1$) must be used.

We observe that the map arctan is multivalued and when $b > 2F_0k^2$ we need to choose the branch of the function in such a way that the identity $\arctan(\tan(x)) = x$ is satisfied.

Moreover, this limit case can be used too to show that the solutions to problem (7) depend strongly on the parameters of the problem. For example, an approximated periodic solution can be found if $F_0 \approx 0$ in the case of plane waves ($f(z) = 1$). Using a Taylor series expansion of order one of function $H(t)$ (given by equation (44)) around $F_0 << 1$, and introducing it in solution (43), we obtain that

$$z(t) \approx z_i + \frac{F_0\,k\,(\cos(2\,z_i\,k) - \cos(2\,z_i\,k - b\,t))}{b}, \tag{45}$$

which shows that for low values of the parameter F_0 the particles describes approximately a periodic movement around its initial position.

4 Numerical simulations

In the next section we show the results obtained for the parameters $F_0 = 0.8\,pm/s$, $b = 100\,Hz$, $z_0 = 0.37\,\lambda$, $k = 2.66\,\pi/\lambda$ and $\lambda = 580\,nm$.

For the case of counter-propagating Gaussian beams ($f(z) = \frac{1}{1+(z/z_0)^2}$) theoretically analyzed in previous sections the trajectories for the given parameters are shown in Fig. 1. As it can be seen, all particles initially located in the interval $[-4.5\lambda, 4.5\lambda]$ (where the potential function $V(t,z)$ shows significant values, see Fig. 2) converge toward the same z region, showing all of them, as it can be observed in Fig. 3, a periodic behavior with the same frequency and amplitude.

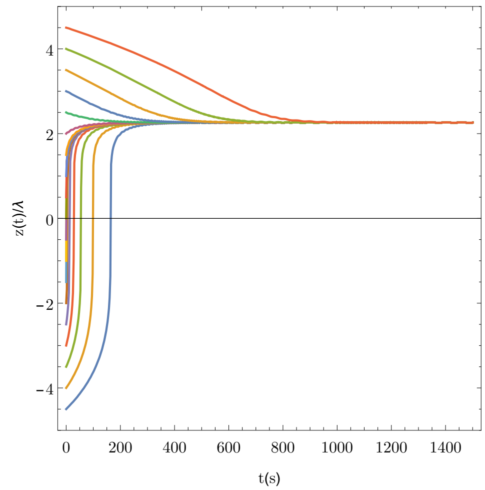

Figure 1: Particle's trajectories for different initial conditions for $f(z) = \frac{1}{1+(z/z_0)^2}$.

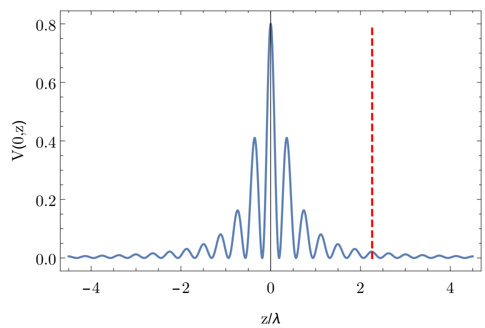

Figure 2: Potential V(0,z) for $f(z) = \frac{1}{1+(z/z_0)^2}$. The dotted line shows the z convergence position.

Fig. 4 shows the particle's trajectories of problem (7) when $f(z) = \exp(-2\frac{z^2}{z_0^2})$. As it can be observed in this case, only the particles initially located at interval $[-\lambda, \lambda]$ converge toward the same region, showing again all of them a periodic behavior with the same frequency and amplitude (see Fig. 6). The particles outside of the interval $[-\lambda, \lambda]$ remain in this case at the initial position because the potential energy is null in this region (see Fig. 5).

In the limit case $z_0 \to \infty$ of plane waves ($f(z) = 1$) and for the same value parameters, the conveyor speed is $v_c = \frac{b}{2k} = 5.98\lambda\, nm/s$, so, according to solution (43) and taking into account that $b < 2F_0k^2$, the particles will describe a rectilinear motion $\frac{z(t)}{\lambda} \approx z_i + 5.98\, t$ which implies that $\frac{z(1500)}{\lambda} \approx 9000$ and there is not an oscillating behavior.

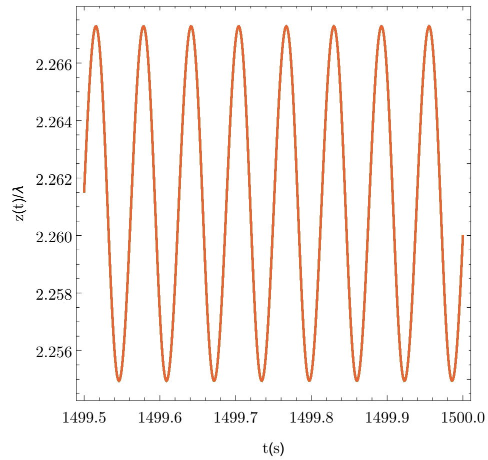

Figure 3: Particle's trajectories behavior for different initial conditions at convergence region for $f(z) = \frac{1}{1+(z/z_0)^2}$.

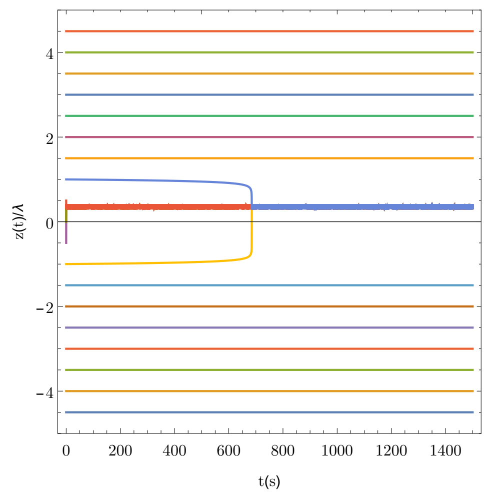

Figure 4: Particle's trajectories behavior for different initial conditions at convergence region for $f(z) = exp(-2\frac{z^2}{z_0^2})$.

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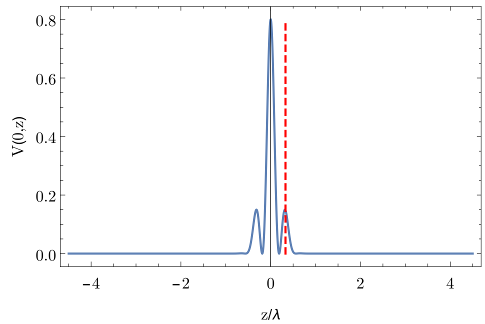

Figure 5: V(0,z) potential for $f(z) = \exp(-2\frac{z^2}{z_0^2})$. The dotted line shows the z convergence position.

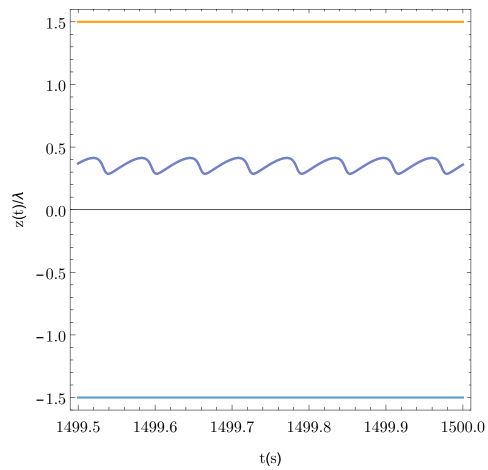

Figure 6: Particle's trajectories behavior for different initial conditions at convergence region for $f(z) = \exp(-2\frac{z^2}{z_0^2})$.

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