Received: 24 February 2018 | Revised: 2 December 2018 | Accepted: 4 March 2019
DOI: 10.1002/mana.201800085

MATHEMATISCHE NACHRICHTEN

ORIGINAL PAPER

Bounded sets structure of $C_p(X)$ and quasi-(DF)-spaces

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Funding information
GAČR, Grant/Award Number: 16-34860L; RVO, Grant/Award Number: 67985840

Abstract
For wide classes of locally convex spaces, in particular, for the space $C_p(X)$ of continuous real-valued functions on a Tychonoff space X equipped with the pointwise topology, we characterize the existence of a fundamental bounded resolution (i.e., an increasing family of bounded sets indexed by the irrationals which swallows the bounded sets). These facts together with some results from Grothendieck's theory of (DF)-spaces have led us to introduce quasi-(DF)-spaces, a class of locally convex spaces containing (DF)-spaces that preserves subspaces, countable direct sums and countable products. Regular (LM)-spaces as well as their strong duals are quasi-(DF)-spaces. Hence the space of distributions $D'(\Omega)$ provides a concrete example of a quasi-(DF)-space not being a (DF)-space. We show that $C_p(X)$ has a fundamental bounded resolution if and only if $C_p(X)$ is a quasi-(DF)-space if and only if the strong dual of $C_p(X)$ is a quasi-(DF)-space if and only if X is countable. If X is metrizable, then $C_k(X)$ is a quasi-(DF)-space if and only if X is a σ-compact Polish space.

KEYWORDS
bounded resolution, class $\mathfrak{G}$, (DF)-space, free locally convex space, pointwise topology, quasi-(DF)-space

MSC (2010)
46A03, 54A25, 54D50

1 | INTRODUCTION

Several recent and earlier results involving the concepts of bounded and compact resolutions (see definitions below), especially developed for spaces $C_p(X)$ of continuous real-valued functions on a Tychonoff space X with the pointwise topology, combined with Grothendieck's ideas about (DF)-spaces, and Cascales–Orihuela concept of the class $\mathfrak{G}$, have motivated us to introduce the class of *quasi-(DF)-spaces* strictly containing (DF)-spaces. We refer the reader to the papers of Talagrand [47], Tkachuk [48], Cascales–Tkachuk–Orihuela [11], Dow–Guerrero Sanchez [12], Mercourakis–Stamati [34] (see also [30, Chapters 3–11] and references therein), where a number of locally convex spaces with certain types of resolutions have been studied both from the point of view of topology and functional analysis.

The important class of (DF)-spaces, introduced by Grothendieck in [28] (we refer also to monographs [29,36,40] for details), is defined as follows.

Definition 1.1. A locally convex space E is called a (DF)-*space* if

(1) E has a fundamental sequence of bounded sets, and

(2) E is $\aleph_0$-quasibarrelled, i.e. every bornivorous (= absorbing each bounded set) closed absolutely convex subset of E which can be represented as the intersection of a sequence of closed absolutely convex neighbourhoods of zero is itself a neighbourhoods of zero, see [40].

Countable inductive limits of normed spaces (hence normed spaces) are (DF)-spaces, as well as strong duals of metrizable locally convex space (lcs for short). Also, the strong dual of a (DF)-space is a metrizable and complete lcs (see [40, Theorem 8.3.9, Proposition 8.3.7]). The concept of (DF)-spaces has been generalized by Ruess [42,43] (under the name (gDF)-spaces) and has also intensively studied by Noureddine [38,39], who called them D_b-spaces. Their papers provide also some information about those spaces which in [29] are called (df)-spaces. Next, Adasch and Ernst [1,2] provided another line of research around (DF)-spaces in the setting of general topological vector spaces. Note however that all known generalizations of (DF)-spaces E kept up the condition (1) on E to have a fundamental sequence of bounded sets while some variations of the weak barrelledness condition (2) on E have been assumed. If $\{B_n\}_{n\in\mathbb{N}}$ is a fundamental sequence of bounded subsets of an lcs E and for each $\alpha = (n_k) \in \mathbb{N}^{\mathbb{N}}$ we set $B_\alpha := B_{n_1}$, the family $\mathcal{B} = \{B_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ satisfies the following conditions:

(i) every set B_α is bounded in E;

(ii) $\mathcal{B}$ is a resolution in E (i.e., $\mathcal{B}$ covers E and $B_\alpha \subseteq B_\beta$ if $\alpha \leq \beta$, $\alpha, \beta \in \mathbb{N}^{\mathbb{N}}$);

(iii) every bounded subset of E is contained in some B_α.

If an lcs E is covered by a family $\{B_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ of bounded sets satisfying conditions (i) and (ii) [and (iii)] we shall say that E has a [*fundamental*] *bounded resolution* (see also [32] for more details). Let us recall that every metrizable lcs E has a fundamental bounded resolution (see, for example, Corollary 2.3 below), while E has a fundamental sequence of bounded sets if and only if E is normable.

In the present paper, starting with a preliminary study of the structure of a fundamental bounded resolution for an lcs E and, in particular, for the space $C_p(X)$, we propose another very natural generalization of the concept of (DF)-space by replacing the quite strong and demanding condition on E to have a fundamental sequence of bounded sets by the weaker one to have a fundamental bounded resolution and assuming a natural extra property on the weak*-dual of E (which holds for (DF)-spaces), see Definition 1.3. Our definition of a quasi-(DF)-space involves also the following concept due to Cascales and Orihuela (see [10]).

Definition 1.2 (Cascales–Orihuela)**.** An lcs E belongs to *the class* $\mathfrak{G}$ if there is a resolution $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ in the weak*-dual $(E', \sigma(E', E))$ of E such that each sequence in any A_α is equicontinuous.

Particularly, every set A_α is relatively $\sigma(E', E)$-countably compact. The class $\mathfrak{G}$ is indeed large and contains "almost all" important locally convex spaces (including (DF)-spaces and even dual metric spaces). Furthermore the class $\mathfrak{G}$ preserves subspaces, completions, Hausdorff quotients, countable direct sums and countable products, every precompact set in an lcs in the class $\mathfrak{G}$ is metrizable (see [10] or [30]).

Being motivated by the aforementioned concepts and results we are ready to define quasi-(DF)-spaces.

Definition 1.3. An lcs E is called a *quasi-(DF)-space* if

(i) E admits a fundamental bounded resolution;

(ii) E belongs to the class $\mathfrak{G}$.

The most evident examples of quasi-(DF)-spaces not being (DF)-spaces are metrizable non-normable lcs; less trivial examples are provided by the strong dual of strict (LF)-spaces (which need not to be (DF)-spaces as shown in [8]); consequently the space $D'(\Omega)$ of distributions over a nonempty open subset Ω of $\mathbb{R}^n$ provides a very concrete example of a quasi-(DF)-space not being a (DF)-space, see Example 4.5. Although the conditions (1) and (2) from Definition 1.1 may lack for quasi-(DF)-spaces, an essential property of (DF)-spaces: *metrizability of precompact sets* is still kept up for quasi-(DF)-spaces.

Observe that conditions (i) and (ii) are independent in the sense that there exist lcs E in the class $\mathfrak{G}$ without a fundamental bounded resolution, and there exist lcs E not being in the class $\mathfrak{G}$ but having a fundamental bounded resolution, see Examples 4.2 and 4.3 below. In addition, the above definition seems to be optimal because the class of quasi-(DF)-spaces preserves subspaces, countable direct sums and countable products (see Theorem 4.1), while infinite products of (DF)-spaces are not of that type.

The detailed organization of the paper goes as follows. Section 2 provides a dual characterization of lcs with fundamental bounded resolutions, see Theorem 2.7. This result nicely applies to show that every regular (LM)-space (for the definition see below) has a fundamental bounded resolution.

Main two results of Section 3 (which have motivated us to introduce the class of quasi(DF)-spaces) state that $C_p(X)$ has a fundamental bounded resolution if and only if X is countable if and only if the strong dual of $C_p(X)$ has a fundamental bounded resolution, and $C_k(X)$ is metrizable if and only if $C_k(X)$ is Fréchet–Urysohn and has a fundamental bounded resolution, see Theorem 3.3 and Proposition 3.10.

Last section collects together some properties of quasi-(DF)-spaces (see Theorem 4.1) and extends a result of Corson in [35]. The previous sections apply to conclude that $C_p(X)$ is a quasi-(DF)-space if and only if $C_p(X)$ has a fundamental bounded resolution if and only if X is countable if and only if the strong dual of $C_p(X)$ is a quasi-(DF)-space. The latter result combined with Theorem 3.3 shows that, for spaces $C_p(X)$, both conditions (i) and (ii) from Definition 1.3 fail for uncountable X. Recall here that $C_p(X)$ is a (DF)-space only if X is finite (see [46]). On the other hand, if X is a metrizable space, then $C_k(X)$ is a quasi-(DF)-space if and only if X is a Polish σ-compact space, see Proposition 4.9. This shows also an essential difference between quasi-(DF)-spaces and (DF)-spaces $C_k(X)$. Moreover, it is well known that a linear map ξ on a (DF)-space E to a Banach space F is continuous provided the restriction of ξ to any bounded set of E is continuous. However, in Example 4.10 we construct a quasi-(DF)-space E such that there exists a discontinuous linear functional whose restriction to any bounded set of E is continuous. We provide many concrete examples in order to highlight the differences among some properties or between the concepts of (DF)-spaces and quasi-(DF)-spaces. For instance, concrete examples of lcs having a bounded resolution but not admitting a fundamental bounded resolution will be examined. Open questions are also provided.

2 | [FUNDAMENTAL] BOUNDED RESOLUTIONS AND $\mathfrak{G}$-BASES; GENERAL CASE

In what follows we shall denote by E a locally convex space over the field of real or complex numbers and represent by $C_p(X)$ or $C_k(X)$ the space $C(X)$ of all continuous real-valued functions on a Tychonoff space X endowed with the pointwise topology or the compact-open topology, respectively.

In this section we obtain dual characterizations of the existence of a [fundamental] bounded resolutions in a lcs E.

Let I be a partially ordered set. A family $\mathcal{A} = \{A_i\}_{i\in I}$ of subsets of a set Ω is called *I-increasing* (*I-decreasing*) if $A_i \subseteq A_j$ ($A_i \supseteq A_j$, respectively) for every $i \le j$ in I. We say that the family $\mathcal{A}$ *swallows* a family $\mathcal{B}$ of subsets of Ω if for every $B \in \mathcal{B}$ there is an $i \in I$ such that $B \subseteq A_i$. An $\mathbb{N}^{\mathbb{N}}$-increasing family of subsets of Ω is called a *resolution* in Ω if it covers Ω. A resolution in a topological space X is called *compact* (respectively, *fundamental compact*) if all its elements are compact subsets of X (and it swallows compact subsets of X, respectively).

We start with the following general observation.

Proposition 2.1. *If a lcs E admits an I-decreasing base at zero, then E has an $\mathbb{N}^I$-increasing bounded resolution swallowing bounded subsets of E. Consequently E'_β has an $\mathbb{N}^I$-decreasing base at zero.*

Proof. Let $\mathcal{U} := \{U_i : i \in I\}$ be an I-decreasing base at zero in E. For every $\alpha \in \mathbb{N}^I$, set

$$B_\alpha := \bigcap_{i\in I} \alpha(i) U_i,$$

and set $\mathcal{B} := \left\{B_\alpha : \alpha \in \mathbb{N}^I\right\}$. Clearly, $\mathcal{B}$ is $\mathbb{N}^I$-increasing bounded resolution in E. To show that $\mathcal{B}$ swallows the bounded sets of E, fix a bounded subset B of E. For every $i \in I$, choose a natural number $\alpha(i)$ such that $B \subseteq \alpha(i)U_i$ and set $\alpha := \big(\alpha(i)\big) \in \mathbb{N}^I$. Clearly, $B \subseteq B_\alpha$. □

In [9] the following important subclass of the class $\mathfrak{G}$ is introduced and studied.

Definition 2.2. An lcs E is said to have a $\mathfrak{G}$-*base* if E admits a base $\left\{U_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\right\}$ of neighborhoods of zero such that $U_\alpha \subseteq U_\beta$ for all $\beta \le \alpha$.

In [9, Lemma 2] it is proved that a quasibarrelled lcs E has a $\mathfrak{G}$-base if and only if E is in class $\mathfrak{G}$. However, under $\aleph_1 < \mathfrak{b}$ there is a (DF)-space (hence belonging to the class $\mathfrak{G}$) which does not admit a $\mathfrak{G}$-base (see [30] or Example 4.6 below). Note also that every (LM)-space E has a $\mathfrak{G}$-base, so E is in the class $\mathfrak{G}$.

Since a metrizable lcs has an $\mathbb{N}$-decreasing base, Proposition 2.1 implies:

Corollary 2.3. *If E is a metrizable lcs, then E has a fundamental bounded resolution and hence the space E'_β has a $\mathfrak{G}$-base.*

Example 2.4. For every uncountable cardinal κ, the space $\mathbb{R}^\kappa$ does not have bounded resolution. Indeed, assuming the converse we obtain that the complete space $\mathbb{R}^\kappa$ has a fundamental bounded resolution by Valdivia's theorem [30, Theorem 3.5]. But then the closures of the sets of this latter family compose a fundamental compact resolution. Now Tkachuk's theorem [30, Theorem 9.14] implies that κ is countable, a contradiction.

Next proposition gathers the most important stability properties of spaces with a fundamental bounded resolution.

Proposition 2.5. *The class of locally convex spaces with a fundamental bounded resolution is closed under taking (i) subspaces, (ii) countable direct sums, and (iii) countable products.*

Proof. (i) is clear. To prove (ii) and (iii) we shall use the following encoding operation of elements of $\mathbb{N}^\mathbb{N}$. We encode each $\alpha \in \mathbb{N}^\mathbb{N}$ into a sequence $\{\alpha_i\}_{i\in\mathbb{N}}$ of elements of $\mathbb{N}^\mathbb{N}$ as follows. Consider an arbitrary decomposition of $\mathbb{N}$ onto a disjoint family $\{N_i\}_{i\in\mathbb{N}}$ of infinite sets, where $N_i = \left\{n_{k,i}\right\}_{k\in\mathbb{N}}$ for $i \in \mathbb{N}$. Now for $\alpha = (\alpha(n))_{n\in\mathbb{N}}$ and $i \in \mathbb{N}$, we set $\alpha_i = \left(\alpha_i(k)\right)_{k\in\mathbb{N}}$, where $\alpha_i(k) := \alpha(n_{k,i})$ for every $k \in \mathbb{N}$. Conversely, for every sequence $\{\alpha_i\}_{i\in\mathbb{N}}$ of elements of $\mathbb{N}^\mathbb{N}$, we define $\alpha = (\alpha(n))_{n\in\mathbb{N}}$ setting $\alpha(n) := \alpha_i(k)$ if $n = n_{k,i}$.

(ii) Let $E = \prod_{i\in\mathbb{N}} E_i$, where every E_i has a fundamental bounded resolution $\left\{B^i_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\right\}$. For every $\alpha \in \mathbb{N}$, we define $B_\alpha := \prod_{i\in\mathbb{N}} B^i_{\alpha_i}$ and set $\mathcal{B} := \left\{B_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\right\}$. Since a subset of E is bounded if and only if its projection onto E_i is bounded in E_i for every $i \in \mathbb{N}$, it is easy to see that $\mathcal{B}$ is a fundamental bounded resolution in E.

(iii) Let $E = \bigoplus_{i\in\mathbb{N}} E_i$, where every E_i has a fundamental bounded resolution $\left\{B^i_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\right\}$. For every $\alpha \in \mathbb{N}^\mathbb{N}$, set $\alpha^* := \left(\alpha(k+1)\right)_{k\in\mathbb{N}}$ and $B_\alpha := \prod_{i=1}^{\alpha(1)} B^i_{\alpha^*_i}$. Since every bounded subset of E is contained and bounded in $\bigoplus_{i=1}^m E_i$ for some $m \in \mathbb{N}$ (see Proposition 24 of Chapter 5 of [41]), we obtain that the family $\left\{B_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\right\}$ is a fundamental bounded resolution in E. □

Example 2.6. Let E be an infinite-dimensional Banach space and let B be the closed unit ball of E. Then its dimension is uncountable. Choose a Hamel basis $H = \{x^*_i : i \in I\}$ from its topological dual E'. Then the map $x \to \left(x^*_i(x)\right)$ from E_w into $\mathbb{R}^H$ is an embedding with dense image. Since E is a Banach space, the sequence $\{nB\}_{n\in\mathbb{N}}$ is a fundamental bounded sequence in E_w. However, since H is uncountable, the completion $\mathbb{R}^H$ of E_w does not have even a bounded resolution by Example 2.4. Hence, the completion of a lcs with a fundamental bounded resolution in general does not have a bounded resolution.

For a subset A of an lcs E, we denote by A° the polar of A in E'. Below we give a dual characterization of lcs with a fundamental bounded resolution. Recall that a lcs E is a *quasi-(LB)-space* [49] if E admits a resolution consisting of Banach discs of E. Recall also that an lcs E is *locally complete* if every closed disc A in E is a Banach disc, i.e., the space E_A is a Banach space, see [7, Definition 2.10.16] or [37].

Theorem 2.7. *For an lcs E the following assertions are equivalent:*

(i) E has a fundamental bounded resolution;

(ii) the strong dual E'_β of E has a $\mathfrak{G}$-base;

(iii) the weak bidual $\left(E'', \sigma\left(E'', E'\right)\right)$ is a quasi-(LB)-space.*

If in addition the space E is locally complete, then (i)-(iii) are equivalent to

(iv) E has a bounded resolution;

(v) E is a quasi-(LB)-space.

Proof. (i)⇒(ii) Let $\left\{A_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\right\}$ be a fundamental bounded resolution. Then, as easily seen, the family $\left\{A^\circ_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\right\}$ is a $\mathfrak{G}$-base of neighborhoods of the strong topology on E'.

(ii)⇒(i) If $\left\{V_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\right\}$ is a $\mathfrak{G}$-base of neighborhoods of $\beta\left(E', E\right)$, the polars $\left\{V^\circ_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\right\}$ of the sets V_α in the bidual E'' of E compose a compact resolution on E'' for the weak* topology. Setting $B_\alpha := V^\circ_\alpha \cap E$ for every $\alpha \in \mathbb{N}^\mathbb{N}$, we can see that the family $\left\{B_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\right\}$ is a bounded resolution for $\left(E, \sigma\left(E, E'\right)\right)$, hence for E. If Q is a closed absolutely convex bounded subset of E, the polar Q° of Q in E' is a neighborhood of the origin in $\beta\left(E', E\right)$ and consequently there exists $\beta \in \mathbb{N}^\mathbb{N}$ such that $V_\beta \subseteq Q^\circ$. This implies that $Q^{\circ\circ} \subseteq V^\circ_\beta$, the polars being taken in E''. Hence

$$Q = \overline{Q}^{\sigma\left(E,E'\right)} = Q^{\circ\circ} \cap E \subseteq B_\beta,$$

which means that the family $\left\{B_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\right\}$ swallows the bounded sets of E.

(i)⇒(iii) If $\{A_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ is a fundamental bounded resolution of E, it is clear that

$$E'' = \bigcup \{A_\alpha^{\circ\circ} : \alpha \in \mathbb{N}^\mathbb{N}\}.$$

Since each $A_\alpha^{\circ\circ}$ is absolutely convex and weak* compact, it is a Banach disc. So $(E'', \sigma(E'', E'))$ has a resolution consisting of weak* compact Banach discs, which means that the weak* bidual $(E'', \sigma(E'', E'))$ is a quasi-(LB)-space.

(iii)⇒(i) If $(E'', \sigma(E'', E'))$ is a quasi-(LB)-space, then, by [30, Theorem 3.5], there is a resolution $\{D_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ for $(E'', \sigma(E'', E'))$ consisting of Banach discs that swallows the Banach discs of $(E'', \sigma(E'', E'))$. If Q is a bounded subset of E, then $Q^{\circ\circ}$ is a weak* compact Banach disc in $(E'', \sigma(E'', E'))$. Hence, there is $\gamma \in \mathbb{N}^\mathbb{N}$ such that $Q^{\circ\circ} \subseteq D_\gamma$, so that $Q \subseteq D_\gamma \cap E$. Setting $B_\alpha := D_\alpha \cap E$ for each $\alpha \in \mathbb{N}^\mathbb{N}$, the family $\{B_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ is a fundamental bounded resolution for E.

Assume that E is locally complete. Clearly, (i) implies (iv). Let us show that (iv) implies (v). Since E is locally complete, the absolutely convex closed envelope of any bounded set of E is a Banach disc by Proposition 5.1.6 of [40]. Thus the space E is a quasi-(LB)-space.

(v)⇒(i) By Valdivia's theorem [30, Theorem 3.5], there exists another quasi-(LB)-representation $\mathcal{B} = \{B_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ of E swallowing all Banach discs of E. Since each bounded set of E is contained in a Banach disc and each Banach disc is bounded, we obtain that $\mathcal{B}$ is a bounded resolution which swallows all bounded sets of E. □

Below we apply Theorem 2.7 to function spaces $C_k(X)$ and $C_p(X)$.

Corollary 2.8. *Let X be such that $C_k(X)$ is locally complete (for instance, X is a $k_\mathbb{R}$-space). Then the following assertions are equivalent:*

(i) $C_k(X)$ has a bounded resolution;

(ii) $C_k(X)$ has a fundamental bounded resolution.

If in addition X is metrizable, then (i)–(ii) are equivalent to

(iii) $C_p(X)$ has a bounded resolution;

(iv) X is σ-compact.

Proof. The equivalences (i)–(ii) follow from Theorem 2.7. The equivalence (iii)⇔(iv) is Corollary 9.2 of [30], and the implication (i)⇒(iii) is trivial. The implication (iv)⇒(i) follows from Corollary 2.10 of [22] which states that $C_k(X)$ has even a fundamental compact resolution. □

Corollary 2.9. *Let X be such that $C_k(X)$ is locally complete. If X has an increasing sequence of functionally bounded subsets which swallows the compact sets of X, then the strong dual of $C_k(X)$ has a $\mathfrak{G}$-base.*

Proof. By Theorem 3.1(ii) of [14], there is a metrizable locally convex topology $\mathcal{T}$ on $C(X)$ stronger than the compact-open topology, and hence $(C(X), \mathcal{T})$ has a fundamental bounded resolution by Corollary 2.3. So $C_k(X)$ has a bounded resolution and Corollary 2.8 applies. □

Recall that a lcs E is called *quasibarrelled* if every closed absolutely convex bornivorous subset of E is a neigbourhood of zero, see [40] or [7,29]. Trivially, every metrizable lcs, as well as, any (LM)-space is quasibarrelled.

We supplement Theorem 2.7 with the following fact. Recall that E is *dual locally complete* if $(E', \sigma(E', E))$ is locally complete, see [45]. Note that a lcs E is barrelled if and only if E is a quasibarelled dual locally complete space, [40].

Proposition 2.10. *The following statements hold true.*

(i) Let E be dual locally complete. If E has a $\mathfrak{G}$-base, then E'_β has a fundamental bounded resolution.

(ii) Let E be a quasibarrelled space. Then the strong dual E'_β of E has a fundamental bounded resolution if and only if E has a $\mathfrak{G}$-base.

Proof. (i) Let $\{U_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ be a $\mathfrak{G}$-base in E. Then the polar sets $W_\alpha := U_\alpha^\circ$ are weakly*-compact and absolutely convex, hence Banach discs, so $(E', \sigma(E', E))$ is a quasi-(LB)-space. Again Valdivia's theorem [30, Theorem 3.5] applies to get that $(E', \sigma(E', E))$ has a fundamental resolution $\mathcal{B} = \{B_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ consisting of Banach discs. As $(E', \sigma(E', E))$ is locally complete, every $\sigma(E', E)$-bounded set B is included in a Banach disc by Proposition 5.1.6 of [40]. Since Banach discs also are $\beta(E', E)$-bounded, the family $\mathcal{B}$ is a fundamental bounded resolution.

(ii) Assume that E'_β has a fundamental bounded resolution. Then the strong bidual space E''_β of E has a $\mathfrak{G}$-base by Theorem 2.7. Since E is quasibarrelled, E is a subspace of E''_β by Theorem 15.2.3 of [37]. Therefore E has a $\mathfrak{G}$-base. Conversely, assume that E has a $\mathfrak{G}$-base $\{U_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$. Then the polar sets $W_\alpha := U_\alpha^\circ$ are weakly*-compact and absolutely convex, and hence W_α are $\beta(E', E)$-bounded by Theorem 11.11.5 of [37]. To show that $\{W_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ is a fundamental bounded resolution in E', fix a strongly bounded subset B of E'. Since E is quasibarrelled, W_α is equicontinuous by Theorem 11.11.4 of [37]. So there is $\alpha \in \mathbb{N}^\mathbb{N}$ such that $B \subseteq W_\alpha$. □

A subset A of a Tychonoff space X is *b-bounding* if for every bounded subset B of $C_k(X)$ the number $\sup\{|f(x)| : x \in A, f \in B\}$ is finite. The space X is called a *W-space* if every b-bounding subset of X is relatively compact.

Corollary 2.11. *Let X be a W-space (for example, X is realcompact). Then the strong dual E of $C_k(X)$ has a fundamental bounded resolution if and only if X has a fundamental compact resolution.*

Proof. Note that $C_k(X)$ is quasibarrelled by Theorem 10.1.21 of [40]. Therefore, by Proposition 2.10, E has a fundamental bounded resolution if and only if the space $C_k(X)$ has a $\mathfrak{G}$-base. But $C_k(X)$ has a $\mathfrak{G}$-base if and only if X has a fundamental compact resolution by [16]. □

By an (LM)-space $E := (E, \tau)$ we mean a lcs which is the countably inductive limit of an increasing sequence (E_n, τ_n) of metrizable lcs such that $E = \bigcup_n E_n$ and $\tau|_{E_n} \le \tau_n$ for each $n \in \mathbb{N}$. The inductive limit topology τ of E is the finest locally convex topology on E such that $\tau|_{E_n} \le \tau_n$ for each $n \in \mathbb{N}$. If each step (E_n, τ_n) is a Fréchet lcs, i.e. a metrizable and complete lcs, we call E an (LF)-space. Moreover, if additionally $\tau_{n+1}|_{E_n} = \tau_n$ for each $n \in \mathbb{N}$ the inductive limit space E is called *strict*. The latter case implies that every bounded set in E is contained and bounded in some E_n. Recall that (LM)-spaces enjoying this property are called *regular.* We refer the reader to [40, Definition 8.5.11] or to [29] for details.

Next Proposition 2.12 provides fundamental bounded resolutions for regular (LM)-spaces.

Proposition 2.12. *Let E be an (LM)-space. Then E has a bounded resolution. If in addition E is regular, then E has a fundamental bounded resolution, and consequently, E'_β has a $\mathfrak{G}$-base.*

Proof. Let (E_i, τ_i) be an increasing sequence of metrizable lcs generating the inductive limit space $E = (E, \tau)$, i. e., $\tau_{i+1}|_{E_i} \le \tau_i$ and $\tau|_{E_i} \le \tau_i$ for any $i \in \mathbb{N}$. For each $i \in \mathbb{N}$, let $\{U_k^i\}_{k\in\mathbb{N}}$ be a decreasing base of neighbourhoods of zero for E_i. For every $i \in \mathbb{N}$ and $\alpha \in \mathbb{N}^\mathbb{N}$, set $W_\alpha^i := \bigcap_{k\in\mathbb{N}} \alpha(k) U_k^i$. Any W_α^i is bounded in τ_i, consequently in τ too. It is easy to see that the family $\{W_\alpha^i : \alpha \in \mathbb{N}^\mathbb{N}\}$ is a fundamental bounded resolution in E_i. Now, for every $\alpha = (\alpha(k)) \in \mathbb{N}^\mathbb{N}$, set $\alpha^* := (\alpha(k+1))$ and $B_\alpha := W_{\alpha^*}^{\alpha(1)}$. Clearly, the family $\{B_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ is a desired bounded resolution in E.

Assume that E is regular. Then every τ-bounded set is contained in some E_i and is τ_i-bounded. Therefore the bounded resolution $\{B_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ is fundamental. Finally, the space E'_β has a $\mathfrak{G}$-base by Theorem 2.7. □

Corollary 2.13. *Let E be a locally complete lcs which is an image of an infinite-dimensional metrizable topological vector space under a continuous linear map. Then every precompact set in E'_β is metrizable.*

Proof. Let T be a continuous linear map from a metrizable tvs H onto E. It is well-known that H has a bounded resolution $\{B_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ (cf. the proof of Theorem 2.1). Then $\{T(B_\alpha) : \alpha \in \mathbb{N}^\mathbb{N}\}$ is a bounded resolution on E. Now Theorem 2.7 implies that E'_β has a $\mathfrak{G}$-base. Therefore every precompact set in E'_β is metrtizable by Cascales–Orihuela's theorem, see [30]. □

Remark 2.14. Analysing the proof of Proposition 1 in [15] one can prove the following result: A lcs E has a bounded resolution if and only if E_w is K-analytic-framed in $\mathbb{R}^\kappa$ for some cardinal κ, that is there exists a K-analytic space H such that $E \subseteq H \subseteq \mathbb{R}^\kappa$.

Remark 2.15. Christensen's theorem [30, Theorem 6.1] states that a metrizable space X has a fundamental compact resolution if and only if X is Polish. Therefore every *separable* infinite-dimensional Banach space E has a fundamental compact resolution, and this resolution is not a fundamental bounded resolution (otherwise, the closed unit ball B of E would be compact). On the other hand, every *nonseparable* infinite-dimensional metrizable space E has a fundamental *bounded* resolution by Corollary 2.3, but E does not have a fundamental *compact* resolution since the space E is not Polish.

3 | MORE ABOUT [FUNDAMENTAL] BOUNDED RESOLUTIONS FOR SPACES $C_p(X)$ AND $C_k(X)$ AND THEIR DUALS

Tkachuk proved (see [30, Theorem 9.14]) that the space $C_p(X)$ has a fundamental compact resolution if and only if X is countable and discrete. Hence, if for an infinite compact space K, the space $C_p(K)$ is K-analytic, then $C_p(K)$ has a compact resolution but it does not have a fundamental compact resolution. Also, if we consider the case $C([0,1])$, then $C_w([0,1])$ does not have a fundamental compact resolution, see [34, Corollary 1.10]. Below we prove an analogous result for $C_p(X)$ having a fundamental bounded resolution.

Let X be a Tychonoff space. Denote by $\mathfrak{F}(X)$ and $\mathcal{K}(X)$ the families of all finite subsets and all compact subsets of X, respectively. For $\mathcal{T} = \tau_p$ or $\mathcal{T} = \tau_k$, we denote by $C_{\mathcal{T}}(X)$ the space $C_p(X)$ or the space $C_k(X)$ and set $\mathcal{T}(X) = \mathfrak{F}(X)$ or $\mathcal{T}(X) = \mathcal{K}(X)$, respectively. We need the following notion. For every $\alpha \in \mathbb{N}^{\mathbb{N}}$ and each $k \in \mathbb{N}$, set

$$I_k(\alpha) := \{\beta \in \mathbb{N}^{\mathbb{N}} : \ \beta(1) = \alpha(1), \dots, \beta(k) = \alpha(k)\}.$$

Let $\mathcal{B}$ and $\mathcal{C}$ be two families of subsets of a set Ω. We shall say that $\mathcal{B}$ *swallows* $\mathcal{C}$ if for every $C \in \mathcal{C}$ there is a $B \in \mathcal{B}$ such that $C \subseteq B$.

Definition 3.1. Let X be a Tychonoff space and $\mathcal{T} \in \{\tau_p, \tau_k\}$. A family $\{U_{\alpha,n} : (\alpha, n) \in \mathbb{N}^{\mathbb{N}} \times \mathbb{N}\}$ of closed subsets of X is called $\mathcal{T}$*-framing* if

(1) for each $\alpha \in \mathbb{N}^{\mathbb{N}}$, the sequence $\{U_{\alpha,n} : n \in \mathbb{N}\}$ is increasing and swallows $\mathcal{T}(X)$,
(2) for every $n \in \mathbb{N}$, $U_{\beta,n} \subseteq U_{\alpha,n}$ whenever $\alpha \leq \beta$.

Recall that a family $\mathcal{N}$ of subsets of a topological space X is called a cs^**-network at a point* $x \in X$ if for each sequence $\{x_n\}_{n\in\mathbb{N}}$ in X converging to x and for each neighborhood O_x of x there is a set $N \in \mathcal{N}$ such that $x \in N \subseteq O_x$ and the set $\{n \in \mathbb{N} : x_n \in N\}$ is infinite; $\mathcal{N}$ is a cs^**-network* in X if $\mathcal{N}$ is a cs^*-network at each point $x \in X$, see [6].

Proposition 3.2. *Let X be a Tychonoff space and let $\mathcal{T} \in \{\tau_p, \tau_k\}$. If the space $C_{\mathcal{T}}(X)$ has a fundamental bounded resolution, then $C_{\mathcal{T}}(X)$ has a countable cs^*-network at zero.*

Proof. Let $\{B_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ be a fundamental bounded resolution in $C_{\mathcal{T}}(X)$. We prove the proposition in four steps. First we note the following simple observation.

Step 1. A subset Q of $C_{\mathcal{T}}(X)$ is bounded if and only if there exists an increasing sequence $\{V_n : n \in \mathbb{N}\}$ consisting of closed subsets of X and swallowing $\mathcal{T}(X)$ such that

$$\sup_{f\in Q} |f(x)| \leq n \ \textit{ for all } \ x \in V_n.$$

Indeed, assume that Q is a bounded subset of $C_{\mathcal{T}}(X)$. For every $n \in \mathbb{N}$, set

$$V_n = \left\{ x \in X : \sup_{f\in Q} |f(x)| \leq n \right\}.$$

Clearly, all V_n are closed, $V_n \subseteq V_{n+1}$ for each $n \in \mathbb{N}$, and $\sup_{f\in Q} |f(x)| \leq n$ for all $x \in V_n$. If $K \in \mathcal{T}(X)$, the $\mathcal{T}$-boundedness of Q implies that there is an $m \in \mathbb{N}$ such that $Q \subseteq [K; m] := \{f : |f(x)| < m \text{ for all } x \in K\}$, so that $K \subseteq V_m$. This shows that $\{V_n : n \in \mathbb{N}\}$ swallows $\mathcal{T}(X)$.

The converse assertion is clear.

Step 2. There exists a $\mathcal{T}$-framing family $\{U_{\alpha,n} : (\alpha, n) \in \mathbb{N}^{\mathbb{N}} \times \mathbb{N}\}$ in X *enjoying the following property: if $\{V_n : n \in \mathbb{N}\}$ is an increasing sequence consisting of closed subsets of X and swallowing $\mathcal{T}(X)$, then there exists a $\gamma \in \mathbb{N}^{\mathbb{N}}$ such that $U_{\gamma,n} \subseteq V_n$ for all $n \in \mathbb{N}$.*

Indeed, for each $\alpha \in \mathbb{N}^{\mathbb{N}}$ and every $n \in \mathbb{N}$, set

$$U_{\alpha,n} = \left\{ x \in X : \sup_{f\in B_\alpha} |f(x)| \leq n \right\}.$$

Then, for each $\alpha \in \mathbb{N}^{\mathbb{N}}$, the family $\{U_{\alpha,n} : n \in \mathbb{N}\}$ is an increasing sequence consisting of closed subsets of X and such that $U_{\beta,n} \subseteq U_{\alpha,n}$ whenever $\alpha \leq \beta$, and in addition $\sup_{f\in B_\alpha} |f(x)| \leq n$ for every $x \in U_{\alpha,n}$ and $n \in \mathbb{N}$. By Step 1, the sequence $\{U_{\alpha,n} : n \in \mathbb{N}\}$ swallows $\mathcal{T}(X)$. Therefore the family $\mathcal{U} := \{V_{\alpha,n} : (\alpha, n) \in \mathbb{N}^{\mathbb{N}} \times \mathbb{N}\}$ is framing.

We claim that $\mathcal{U}$ satisfies the stated property. Indeed, fix an increasing sequence $\{V_n : n \in \mathbb{N}\}$ consisting of closed subsets of X and swallowing $\mathcal{T}(X)$. Set

$$P := \left\{ f \in C(X) : \sup_{x \in V_n} |f(x)| \le n \text{ for all } n \in \mathbb{N} \right\}.$$

Then P is a bounded subset of $C_{\mathcal{T}}(X)$ by Step 1. Since $\{B_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is a fundamental bounded resolution for $C_{\mathcal{T}}(X)$, there exists $\delta \in \mathbb{N}^{\mathbb{N}}$ such that $P \subseteq B_\delta$. To prove the claim we show that $U_{\delta,n} \subseteq V_n$ for every $n \in \mathbb{N}$. Take arbitrarily $x \in U_{\delta,n}$. Then

$$\sup_{f \in P} |f(x)| \le \sup_{f \in B_\delta} |f(x)| \le n.$$

Now if $x \notin \overline{V_n} = V_n$, there is $h \in C(X)$ with $0 \le h \le n+1$ such that $h(x) = n+1$ and $h(y) = 0$ for every $y \in V_n$. By construction of h and since $\{V_n\}_n$ is increasing, we have $|h(y)| \le n$ for every $n \in \mathbb{N}$ and each $y \in V_n$. Therefore $h \in P \subseteq B_\delta$. So we have at the same time that $x \in U_{\delta,n}$ and $|h(x)| = n+1$ with $h \in B_\delta$, a contradiction. Thus $U_{\delta,n} \subseteq V_n$ for every $n \in \mathbb{N}$.

Step 3. For every $n \in \mathbb{N}$ and each $\alpha \in \mathbb{N}^{\mathbb{N}}$, set $M_n(\alpha) := \bigcup_{\beta \in I_n(\alpha)} B_\beta$ and

$$K_n(\alpha) := \bigcap_{\beta \in I_n(\alpha)} U_{\beta,n} = \left\{ x \in X : \sup_{f \in M_n(\alpha)} |f(x)| \le n \right\}.$$

Then all $K_n(\alpha)$ are closed (but can be empty) and satisfy the following conditions:

- *(i) $K_n(\alpha) \subseteq K_{n+1}(\alpha)$ for every $n \in \mathbb{N}$ and each $\alpha \in \mathbb{N}^{\mathbb{N}}$;*
- *(ii) $K_n(\alpha) \supseteq K_n(\beta)$ for every $n \in \mathbb{N}$ whenever $\alpha \le \beta$;*
- *(iii) for each $\alpha \in \mathbb{N}^{\mathbb{N}}$, the sequence $\{K_n(\alpha)\}_{n \in \mathbb{N}}$ swallows $\mathcal{T}(X)$;*
- *(iv) for every increasing sequence $\{V_n : n \in \mathbb{N}\}$ consisting of closed subsets of X and swallowing $\mathcal{T}(X)$ there exists $\gamma \in \mathbb{N}^{\mathbb{N}}$ such that $K_n(\gamma) \subseteq V_n$ for all $n \in \mathbb{N}$.*

Moreover, the family $\mathcal{K} := \{K_n(\alpha) : n \in \mathbb{N}, \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is countable.

Indeed, (i) and (ii) are clear. To prove (iii), suppose for a contradiction that there are $\alpha \in \mathbb{N}^{\mathbb{N}}$ and $K \in \mathcal{T}(X)$ such that $K \nsubseteq K_n(\alpha)$ for every $n \in \mathbb{N}$. For every $n \in \mathbb{N}$, choose $\beta_n \in I_n(\alpha)$ and $x_n \in K$ such that $x_n \notin U_{\beta_n,n}$. Set $\gamma := \sup\{\beta_n : n \in \mathbb{N}\}$. Then, for every $n \in \mathbb{N}$, $\beta_n \le \gamma$ and hence $x_n \notin U_{\gamma,n}$ since $U_{\gamma,n} \subseteq U_{\beta_n,n}$ by the definition of a $\mathcal{T}$-framing family. Therefore $K \nsubseteq U_{\gamma,n}$ for every $n \in \mathbb{N}$, a contradiction. Thus (iii) holds. Now we prove (iv). By Step 2, there is a $\gamma \in \mathbb{N}^{\mathbb{N}}$ such that $U_{\gamma,n} \subseteq V_n$ for all $n \in \mathbb{N}$. Then $K_n(\gamma) \subseteq V_n$ since $K_n(\gamma) \subseteq U_{\gamma,n}$ for all $n \in \mathbb{N}$. Finally, the family $\mathcal{K}$ is countable since, by construction, the set $K_n(\alpha)$ depends only on $\alpha(1), \dots, \alpha(n)$.

Step 4. For every $m, n \in \mathbb{N}$ and each $\alpha \in \mathbb{N}^{\mathbb{N}}$, set

$$N_{mn}(\alpha) := \left\{ f \in C(X) : |f(x)| \le \frac{1}{m} \text{ for all } x \in K_n(\alpha) \right\}$$

(if $K_n(\alpha)$ is empty we set $N_{mn}(\alpha) := \{0\}$). We claim that the family

$$\mathcal{N} := \left\{ N_{mn}(\alpha) : m, n \in \mathbb{N} \text{ and } \alpha \in \mathbb{N}^{\mathbb{N}} \right\}$$

is a countable cs^*-network at $0 \in C_{\mathcal{T}}(X)$.

Indeed, the family $\mathcal{N}$ is countable because the family $\mathcal{K}$ is countable. To show that $\mathcal{N}$ is a cs^*-network at $0 \in C_{\mathcal{T}}(X)$, let $S = \{g_n : n \in \mathbb{N}\}$ be a null-sequence in $C_{\mathcal{T}}(X)$ and let U be a standard neighborhood of zero in $C_{\mathcal{T}}(X)$ of the form

$$U = [F, \varepsilon] := \{ f \in C(X) : |f(x)| < \varepsilon \text{ for all } x \in F \},$$

where $F \in \mathcal{T}(X)$ and $\varepsilon > 0$. Fix arbitrarily an $m \in \mathbb{N}$ such that $m > 1/\varepsilon$. For every $n \in \mathbb{N}$, set

$$T_n := \bigcap_{i \ge n} R_i, \quad \text{where } R_i := \left\{ x \in X : |g_i(x)| \le \frac{1}{m} \right\}.$$

It is clear that $\{T_n\}_{n\in\mathbb{N}}$ is an increasing sequence of closed subsets of X. Moreover, since $g_n \to 0$ in $C_{\mathcal{T}}(X)$ we obtain that the sequence $\{T_n\}_{n\in\mathbb{N}}$ swallows $\mathcal{T}(X)$. Therefore, by (iv) of Step 3, there is a $\gamma \in \mathbb{N}^{\mathbb{N}}$ such that

$$K_n(\gamma) \subseteq T_n \quad \text{for every } n \in \mathbb{N}. \tag{3.1}$$

Now, by (iii) of Step 3, choose an $l \in \mathbb{N}$ such that $F \subseteq K_l(\gamma)$. Then (3.1) implies

$$\{g_i\}_{i\geq l} \subseteq N_{ml}(\gamma) \subseteq U = [F, \varepsilon].$$

Thus $\mathcal{N}$ is a countable cs^*-network at zero of $C_{\mathcal{T}}(X)$. □

Now we are ready to prove the main result of this section.

Theorem 3.3. *Let X be a Tychonoff space.*

(i) The space $C_p(X)$ is metrizable if and only if $C_p(X)$ admits a fundamental bounded resolution (so exactly when X is countable).

(ii) The space $C_k(X)$ is metrizable if and only if $C_k(X)$ is Fréchet–Urysohn and admits a fundamental bounded resolution.

Proof. (i) If X is countable, the space $C_p(X)$ has a fundamental bounded resolution by Corollary 2.3. If $C_p(X)$ has a fundamental bounded resolution, then X is countable by Proposition 3.2 and [44] (or [21], recall that $C_p(X)$ is b-Baire-like for every Tychonoff space X).

(ii) The necessity follows from Corollary 2.3. To prove sufficiency we note that every Fréchet–Urysohn group with countable cs^*-character is metrizable by [6] and Proposition 3.2 applies. □

Remark 3.4. (1) Note that the condition on $C_k(X)$ of being Fréchet–Urysohn in (ii) of Theorem 3.3 is essential. Indeed, $C_k(\mathbb{Q})$ has a fundamental bounded resolution by (iv) of Corollary 2.8, but $C_k(\mathbb{Q})$ is not metrizable.

(2) The metrizability of $C_k(X)$ for the case when $C_k(X)$ is Fréchet–Urysohn and admits a fundamental sequence of bounded sets (S_n) can be proved shorter. Indeed, since every Fréchet–Urysohn lcs is bornological, see [30, Lemma 14.4.3], the original topology of $C_k(X)$ (as easily seen) is the inductive limit topology of the increasing sequence of normed spaces (generated by the liner span of each S_n endowed with the Minkowski norm topology). But then $C_k(X)$ is in class $\mathfrak{G}$, see [30, p. 244] and [30, Theorem 15.1.3] applies to get the metrizability of $C_k(X)$.

Example 3.5. Let X be an uncountable pseudocompact space. Then the space $C_p(X)$ has a bounded resolution but it does not have a fundamental bounded resolution. Indeed, as X is pseudocompact, the sets $B_n = \{f \in C(X) : |f(x)| \leq n \text{ for all } x \in X\}$ form a bounded resolution (even a bounded sequence) in $C_p(X)$. The second assertion follows from Theorem 3.3.

For P-spaces we obtain even a stronger result.

Proposition 3.6. *Let X be a P-space. Then $C_p(X)$ has a bounded resolution if and only if X is countable and discrete.*

Proof. Assume that $C_p(X)$ has a bounded resolution. Since X is a P-space, $C_p(X)$ is sequentially complete (equivalently locally complete) by [18]. So, by Valdivia's theorem [30, Theorem 3.5] there is a fundamental bounded resolution (consisting of Banach disks) in $C_p(X)$. Consequently, by Theorem 3.3 the space X is countable. But every countable P-space is discrete. The converse is obvious. □

Remark 3.7. Proposition 3.2 suggests the following question posed by the referee: *Assume that an lcs L has a fundamental bounded resolution (or is a quasi-(DF)-space). Has L a countable cs^*-network at zero*? We remark the following:

(1) Let E be the James tree space and let $L := E_w$. Clearly, L has even a fundamental bounded sequence. But since E does not contain a subspace linearly isomorphic to ℓ_1 and E' is not separable, the space L does not have countable cs^*-character by Theorem 1.8 of [25].

(2) If E is a quasibarrelled quasi-(DF)-space, then E has a countable cs^*-network at zero. Indeed, since a quasibarrelled space in the class $\mathfrak{G}$ must have a $\mathfrak{G}$-base (see [30, Lemma 15.2]), the assertion follows from Theorem 3.12 of [26] which states that every topological group with a $\mathfrak{G}$-base has a countable cs^*-network at the identity.

Recall (see [35]) that a Tychonoff space X is called a *cosmic space* (an *$\aleph_0$-space*) if X is an image of a separable metric space under a continuous (respectively, compact-covering) map.

Theorem 3.8. *Let $E = C_k(C_k(X))$. Then:*

(i) If X is metrizable, then E has a $\mathfrak{G}$-base if and only if X is σ-compact. In this case E is barrelled.

(ii) If X is an $\aleph_0$-space, then the strong dual E'_β of E has a $\mathfrak{G}$-base if and only if X is finite. In particular, if X is infinite, then E is not a regular (LM)-space.

(iii) If X is a μ-space and E'_β has a $\mathfrak{G}$-base, then X has a fundamental compact resolution, so that $C_k(X)$ has a $\mathfrak{G}$-base. But the converse is not true in general.

Consequently, if X is an infinite metrizable σ-compact space, then E is a barrelled space with a $\mathfrak{G}$-base whose strong dual E'_β does not have a $\mathfrak{G}$-base.

Proof. (i) If E has a $\mathfrak{G}$-base, then $C_k(X)$ has a fundamental compact resolution by Theorem 2 of [16]. Therefore X is σ-compact by Corollary 9.2 of [30]. Conversely, if X is σ-compact, then $C_k(X)$ has a fundamental compact resolution by Corollary 2.10 of [22]. Once again applying Theorem 2 of [16], we obtain that E has a $\mathfrak{G}$-base.

To prove the last assertion we note that any metrizable σ-compact space is an $\aleph_0$-space, and hence $C_k(X)$ is Lindelöf by [35]. So $C_k(X)$ is a μ-space and the space E is barrelled by the Nachbin–Shirota theorem.

(ii) Assume that the strong dual of $C_k(C_k(X))$ has a $\mathfrak{G}$-base. Then, by Theorem 2.7, the space $C_k(C_k(X))$ and hence also $C_p(C_k(X))$ have a bounded resolution. Since $C_k(X)$ is also cosmic by Proposition 10.3 of [35], Corollary 9.1 of [30] implies that the space $C_k(X)$ is σ-compact. Therefore $C_p(X)$ is also σ-compact. Now Velichko's theorem [30, Theorem 9.12] implies that X is finite. Conversely, if X is finite and $|X| = n$, then $E = C_k(\mathbb{R}^n)$ is a Fréchet space and Theorem 2.7 applies.

The last assertion follows from this result and Proposition 2.12.

(iii) Let $M_c(X)$ be the topological dual of $C_k(X)$. Denote by $\mathcal{T}_k$ and $\mathcal{T}_\beta$ the compact-open topology induced from E and the strong topology $\beta\left(M_c(X), C(X)\right)$ on $M_c(X)$, respectively. Clearly, $\mathcal{T}_k \le \mathcal{T}_\beta$. Set $G := \left(M_c(X), \mathcal{T}_k\right)'$ and $F := \left(M_c(X), \mathcal{T}_\beta\right)'$. Then $C(X) \subseteq G \subseteq F$, algebraically. Hence every $\sigma(M_c(X), F)$-bounded subset of $M_c(X)$ is also $\sigma\left(M_c(X), G\right)$-bounded, and therefore

$$\beta\big(F, M_c(X)\big)|_G \le \beta\big(G, M_c(X)\big). \tag{3.2}$$

Observe that $C_k(X)$ is barrelled by the Nachbin–Shirota theorem, and hence, by Theorem 15.2.3 of [37], the space $C_k(X)$ is a subspace of its strong bidual space $\left(F, \beta\big(F, M_c(X)\big)\right)$. Therefore

$$\tau_k = \beta\big(F, M_c(X)\big)|_{C(X)}. \tag{3.3}$$

As $\left(M_c(X), \mathcal{T}_k\right)$ is a closed subspace of E we obtain $G = \left(M_c(X), \mathcal{T}_k\right)' = E'/M_c(X)^\perp$, algebraically. Denote by $\mathcal{T}_q$ the quotient topology of the strong dual E'_β of E on G. We claim that the strong topology $\beta\left(G, M_c(X)\right)$ on G is coarser than $\mathcal{T}_q$. Indeed, if j is the canonical inclusion of $\left(M_c(X), \mathcal{T}_k\right)$ into E, the adjoint map j^* is strongly continuous, see [37, Theorem 8.11.3]. Then the claim follows from the fact that j^* is raised to the continuous map from the quotient $E'_\beta/M_c(X)^\perp$ to the strong dual of $\left(M_c(X), \mathcal{T}_k\right)$. Now the claim and (3.2) and (3.3) imply

$$\tau_k = \beta\big(F, M_c(X)\big)|_{C(X)} \le \beta\big(G, M_c(X)\big)|_{C(X)} \le \mathcal{T}_q|_{C(X)}. \tag{3.4}$$

Suppose for a contradiction that E'_β has a $\mathfrak{G}$-base. Then the quotient topology $\mathcal{T}_q$ and hence $\mathcal{T}_q|_{C(X)}$ also have a $\mathfrak{G}$-base. Therefore, by (3.4), there exists a locally convex topology $\mathcal{T} := \mathcal{T}_q|_{C(X)}$ with a $\mathfrak{G}$-base on $C(X)$ such that $\tau_k \le \mathcal{T}$. According to [14, Corollary 2.3] applied to the family $\mathcal{S} = \mathcal{K}(X)$ of all compact subsets of X and $C(X)$, we obtain that X has a functionally bounded resolution swallowing the compact sets of X. Finally, since X is a μ-space, it follows that X admits a fundamental compact resolution.

Observe that for $X = \mathbb{R}$ the space X is even hemicompact, but E'_β does not have a $\mathfrak{G}$-base by (ii). □

Following Markov [33], the *free lcs* $L(X)$ over a Tychonoff space X is a pair consisting of a lcs $L(X)$ and a continuous mapping $i : X \to L(X)$ such that every continuous mapping f from X to a lcs E gives rise to a unique continuous linear operator $\bar{f} : L(X) \to E$ with $f = \bar{f} \circ i$. The free lcs $L(X)$ always exists and is unique. The set X forms a Hamel basis for $L(X)$, and the mapping i is a topological embedding. Denote by $L_p(X)$ the free lcs $L(X)$ endowed with the weak topology.

Following [4], a Tychonoff space X is called an *Ascoli space* if every compact subset $\mathcal{K}$ of $C_k(X)$ is equicontinuous (see [20]). By Ascoli's theorem [37], each k-space is Ascoli. The following proposition complements [22, Theorem 3.2] and [5, Theorem 6.5]. In fact the equivalence between (i) and (iii) below has been proved in [5, Theorem 6.5] (see also [20, Theorem 1.2]). The essential part is the direct proof of the implication (i) ⇒ (ii).

Proposition 3.9. *For an Ascoli space X the following assertions are equivalent:*

(i) $L(X)$ *has a* $\mathfrak{G}$*-base;*

(ii) $C_k(X)$ *has a fundamental compact resolution;*

(iii) $C_k\big(C_k(X)\big)$ *has a* $\mathfrak{G}$*-base.*

In particular, any item above implies that every compact subset of X is metrizable.

Proof. (i)⇒(ii) Let $\mathcal{U} = \{U_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ be a $\mathfrak{G}$-base in $L(X)$. Then the family $\mathcal{U}^\circ = \{U_\alpha^\circ : \alpha \in \mathbb{N}^{\mathbb{N}}\}$, where the polars are taken in the dual space $L(X)'$ of $L(X)$, is a compact resolution of $\big(L(X)', \sigma(L(X)', L(X))\big)$. It is well-known and easy to see that the dual space $L(X)'$ of $L(X)$ can be identified with the space $C(X)$ under the restriction map (recall that X is a Hamel base for $L(X)$)

$$L(X)' \ni \chi \mapsto \chi|_X \in C(X).$$

It is clear that the weak* topology $\sigma(L(X)', L(X))$ on $L(X)'$ induces the pointwise topology τ_p on $C(X)$. Therefore, for every $\alpha \in \mathbb{N}^{\mathbb{N}}$, the set $K_\alpha := \{\chi|_X : \chi \in U_\alpha^\circ\}$ is closed also in the compact-open topology τ_k on $C(X)$. We claim that K_α is a compact subset of $C_k(X)$. For this, by the Ascoli theorem [37, Theorem 5.10.4], it is sufficient to check that K_α is pointwise bounded and equicontinuous. Clearly, K_α is pointwise bounded. To show that K_α is equicontinuous fix an $x \in X$. Then for every $y = x + t \in (x + U_\alpha) \cap X$ (recall that X is a subspace of $L(X)$), we obtain

$$|\chi|_X(y) - \chi|_X(x)| = |\chi(t)| \le 1, \quad \text{for all } \chi|_X \in K_\alpha.$$

Thus K_α is a compact subset of $C_k(X)$. Consequently, the family $\mathcal{K} := \{K_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is a compact resolution in $C_k(X)$.

Take arbitrarily a compact subset K of $C_k(X)$. Then the polar K° of K in $C_k\big(C_k(X)\big)$ is a neighborhood of zero in $C_k\big(C_k(X)\big)$. Since X is Ascoli, the space $L(X)$ is a subspace of $C_k\big(C_k(X)\big)$ by Theorem 1.2 of [20]. So there is $\alpha \in \mathbb{N}^{\mathbb{N}}$ such that $U_\alpha \subset K^\circ \cap L(X)$. Hence $K \subseteq K^{\circ\circ} \subseteq U_\alpha^\circ$. Thus $\mathcal{K}$ swallows the compact sets of $C_k(X)$.

(ii)⇒(iii) follows from Theorem 2 of [16], and (iii) implies (i) since $L(X)$ is a subspace of $C_k\big(C_k(X)\big)$ by Theorem 1.2 of [20].

The last assertion follows from the fact that X is a subspace of $L(X)$ and Cascales–Orihuela's theorem [10, Theorem 11] (which states that every compact subset of an lcs with a $\mathfrak{G}$-base is metrizable). □

The previous theorem may suggest the following problem: Characterize in terms of X those spaces $C_k(X)$ with a fundamental bounded resolution.

We propose also the following

Proposition 3.10. *The following assertions are equivalent:*

(i) The strong dual space $L(X)_\beta$ *of* $C_p(X)$ *has a bounded resolution.*

(ii) $L(X)_\beta$ *has a fundamental bounded resolution.*

(iii) X *is countable.*

Proof. (i)⇒(iii) Assume that $L(X)_\beta$ has a bounded resolution $\{B_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$. Since $C_p(X)$ is quasibarrelled, its strong dual $L(X)_\beta$ is feral, see p. 392 of [17] (recall that following [31], a lcs E is called *feral* if every bounded set of E is finite-dimensional). If X has an uncountable number of points, some set B_α of the resolution would contain infinitely many points of X by [30, Proposition 3.7]. Since X is a Hamel basis for $L(X)$, the set B_α would be infinite-dimensional, a contradiction. Thus X must be countable.

(iii)⇒(ii) If X is countable, $C_p(X)$ is metrizable. Thus E has a fundamental bounded resolution by Proposition 2.10. Finally, the implication (ii)⇒(i) is trivial. □

The following item characterizes those μ-spaces X for which the weak* dual $L_p(X)$ of $C_p(X)$ has a fundamental bounded resolution. Recall also that the topology of $L_p(X)$ coincides with the weak topology of the free lcs $L(X)$.

Proposition 3.11. *Let X be a μ-space. Then X has a fundamental compact resolution if and only if $L_p(X)$ has a fundamental bounded resolution.*

Proof. Denote by $\delta : X \to L_p(X)$ the canonical embedding map and let E be the topological dual of $C_k(X)$. First we observe that if X is a μ-space, then $C_k(X)$ is the strong dual of $L_p(X)$. Indeed, since X is a μ-space, the barrelledness of $C_k(X)$ yields $\tau_k = \beta(C(X), E)$. Since clearly $\tau_k \leq \beta(C(X), L(X)) \leq \beta(C(X), E)$, it follows that $\tau_k = \beta(C(X), L(X))$.

We claim that if A is a bounded subset of $L_p(X)$, there is a compact set K in X and $n \in \mathbb{N}$ such that $A \subseteq n \cdot \overline{\operatorname{acx}(\delta(K))}^{L_p(X)}$. Indeed, by the previous observation, if A is a bounded set in $L_p(X)$ there are $n \in \mathbb{N}$ and a compact subset K of X such that

$$\left\{ f \in C(X) : \sup_{x \in K} |f(x)| \leq n^{-1} \right\} \subseteq A^\circ.$$

So, if u_f denotes the (unique) continuous linear extension of f to $L_p(X)$, we have

$$\delta(K)^\circ = \left\{ f \in C(X) : \sup_{x \in K} \left| \left\langle \delta_x, u_f \right\rangle \right| \leq 1 \right\} \subseteq nA^\circ.$$

Hence $A \subseteq A^{\circ\circ} \subseteq n\,\delta(K)^{\circ\circ}$, where the bipolar is taken in E. Thus $A \subseteq n \cdot \overline{\operatorname{acx}(\delta(K))}^{L_p(X)}$.

Assume that X has a fundamental compact resolution $\mathcal{K} = \left\{ K_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}} \right\}$. For every $\alpha = (\alpha(i)) \in \mathbb{N}^{\mathbb{N}}$, set $\alpha^* := (\alpha(i+1))$ and

$$B_\alpha := \alpha(1) \cdot \overline{\operatorname{acx}\left(\delta\left(K_{\alpha^*}\right)\right)}^{L_p(X)}.$$

Clearly the family $\mathcal{B} := \left\{ B_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}} \right\}$ consists of bounded sets in $L_p(X)$ and satisfies that $B_\alpha \subseteq B_\beta$ whenever $\alpha \leq \beta$. Moreover, according to the claim and the fact that the family $\mathcal{K}$ is fundamental, we obtain that $\mathcal{B}$ is a fundamental bounded resolution for $L_p(X)$.

Conversely, let $\left\{ A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}} \right\}$ be a fundamental bounded resolution for $L_p(X)$. For each $\alpha \in \mathbb{N}^{\mathbb{N}}$, set $M_\alpha := A_\alpha \cap \delta(X)$. Then $\left\{ M_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}} \right\}$ is a resolution for $\delta(X)$ consisting of functionally bounded sets that swallows the functionally bounded subsets of $\delta(X)$. Since X is a μ-space, then the family $\left\{ \overline{\delta^{-1}\left(M_\alpha\right)}^X : \alpha \in \mathbb{N}^{\mathbb{N}} \right\}$ is a fundamental compact resolution for X. □

The condition that X is a μ-space cannot be removed from Proposition 3.11, as the following example shows.

Example 3.12. Let X be the ordinal space $[0, \omega_1)$, where ω_1 is the first uncountable ordinal, and set $Y := [0, \omega_1]$. Since X is pseudocompact, the restriction map $Tf = f|_X$ maps continuously $C_p(Y)$ onto $C_p(X)$. Consequently, $L_p(X)$ is topologically isomorphic to a linear subspace of $L_p(Y)$. Since Y is a compact set, according to Proposition 3.11 the space $L_p(Y)$ has a (countable) fundamental bounded resolution, which implies that $L_p(X)$ also has a fundamental bounded resolution. However, under $MA + \neg CH$ the space $X = [0, \omega_1)$ even does not have a compact resolution [48, Theorem 3.6]. Observe that X is not a μ-space.

Next example shows that there is no natural relationship between the existence of fundamental bounded resolutions for different natural topologies.

Example 3.13. If X is an uncountable Polish space, then the strong dual of $C_p(X)$ does not have a fundamental bounded resolution by Proposition 3.10, but the weak* dual of $C_p(X)$ has a fundamental bounded resolution by Proposition 3.11 and Christensen's theorem (see, [30, Theorem 6.1]). On the other hand, if X is a countable but non-Polish metrizable space (for example, $X = \mathbb{Q}$), then the strong dual of $C_p(X)$ has a fundamental bounded resolution by Proposition 3.10, however the weak* dual of $C_p(X)$ does not have a fundamental bounded resolution by Proposition 3.11 and Christensen's theorem.

4 | QUASI-(DF)-SPACES

In the Introduction we formally defined the class of quasi-(DF)-spaces. The previous sections apply to gather a few fundamental properties of quasi-(DF)-spaces. We refer the readers to corresponding facts dealing with (DF)-spaces, see [29] and [40]. Note however that quasi-(DF)-spaces are stable by taking countable products although this property fails for (DF)-spaces.

Theorem 4.1. *The following statements hold true.*

(i) Every regular (LM)-space E is a quasi-(DF)-space. In particular, every infinite-dimensional metrizable non-normable lcs E is a quasi-(DF)-space not being a (DF)-space.

(ii) The strong dual E'_β of a regular (LM)-space is a quasi-(DF)-space.

(iii) Countable direct sums of quasi-(DF)-spaces are quasi-(DF)-spaces.

(iv) A subspace of a quasi-(DF)-space is a quasi-(DF)-space.

(v) A countable product E of quasi-(DF)-spaces is a quasi-(DF)-space.

(vi) Every precompact set of a quasi-(DF)-space is metrizable.

Proof. (i) Every (LM)-space belongs to the class $\mathfrak{G}$, see [30, Section 11.1]. Being regular, E has a fundamental bounded resolution by Proposition 2.12. Thus E is a quasi-(DF)-space. In particular, if E is an infinite-dimensional metrizable non-normable lcs, then E is a quasi-(DF)-space which is not a (DF)-space.

(ii) Since E is regular, the space E'_β has a $\mathfrak{G}$-base by Proposition 2.12. Therefore E'_β is in the class $\mathfrak{G}$ (recall that every lcs E with a $\mathfrak{G}$ base $\{U_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ belongs to the class $\mathfrak{G}$ since the family of polars $\{U_\alpha^\circ : \alpha \in \mathbb{N}^\mathbb{N}\}$ is a $\mathfrak{G}$-representation of E). As E has a $\mathfrak{G}$-base and is quasibarrelled (E is even bornological, see [29, Corollary 13.1.5]), the space E'_β has a fundamental bounded resolution by Proposition 2.10. Therefore E'_β is a quasi-(DF)-space.

(iii)–(v) follow from [30, Proposition 11.1] and Proposition 2.5, and (vi) follows from [30, Theorem 11.1]. □

Next two examples show the independence of conditions (i) and (ii) appearing in Definition 1.3 of quasi-(DF)-spaces.

Example 4.2. There exist lcs E not being in class $\mathfrak{G}$ but having a fundamental bounded resolution. Indeed, if E is an infinite-dimensional Banach space, then E_w is not in class $\mathfrak{G}$ (see [30]) although E_w has a fundamental sequence of bounded sets: Assume E_w is in class $\mathfrak{G}$. Then, as E_w is dense in $\mathbb{R}^X$ for some X, the Baire space $\mathbb{R}^X$ belongs also to the class $\mathfrak{G}$. Now the main theorem of [32] applies to deduce that X is countable, a contradiction (since then E_w would be metrizable implying the finite-dimensionality of E).

Example 4.3. Let X be a non σ-compact Čech-complete Lindelöf space (for example, $X = \mathbb{N}^\mathbb{N}$). Then $C_k(X)$ has a $\mathfrak{G}$-base (hence is in the class $\mathfrak{G}$) and is barrelled but it does not have even a bounded resolution. Consequently, $C_k(X)$ is not a quasi-(DF)-space and the strong dual of $C_k(X)$ does not have a $\mathfrak{G}$-base.

Proof. Since X has a fundamental compact resolution by (see Fact 1 in the proof of Proposition 4.7 in [24]), the space $C_k(X)$ has a $\mathfrak{G}$-base by [16]. As X is a μ-space, $C_k(X)$ is barrelled. On the other hand, assume that $C_k(X)$ has a bounded resolution. Then $C_p(X)$ has a bounded resolution too. Since X is Čech-complete and Lindelöf, there exists (well known fact) a perfect map T from X onto a Polish space Y. As X is not σ-compact, then Y is also not σ-compact. Since T is onto, the adjoint map $T^* : C_p(Y) \to C_p(X), T^*(f) = f \circ T$, of T is an embedding. Therefore $C_p(Y)$ has a bounded resolution. But this is impossible, since then Y would be σ-compact by Corollary 9.2 of [30]. The last assertion follows from Theorem 2.7. □

In [35] it is proved that if E is a Banach space whose strong dual is separable, then E_w is an $\aleph_0$-space. In [23, Corollary 5.6] it was shown that a Banach space which does not contain an isomorphic copy of ℓ_1 has separable dual if and only if E_w is an $\aleph_0$-space. Next theorem extends this result to quasi-(DF)-spaces.

Theorem 4.4. *Let E be a quasi-(DF)-space.*

(i) If the strong dual E'_β is separable, then E_w is cosmic.

(ii) If the strong dual E'_β is separable and barrelled, then E_w is an $\aleph_0$-space.

(iii) If E is a strict (LF)-space such that E'_β is separable, then E_w is an $\aleph_0$-space.

Proof. (i) By Theorem 2.7, the space E'_β has a $\mathfrak{G}$-base $\{U_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$. Hence its polar $\{U_\alpha^\circ : \alpha \in \mathbb{N}^\mathbb{N}\}$ forms a compact resolution in $E''_w := (E'', \sigma(E'', E'))$. Since E'_β is separable, $\sigma(E'', E')$ admits a weaker metrizable topology, and hence the space E''_w is analytic by [10, Theorem 15] (i.e. E''_w is a continuous image on $\mathbb{N}^\mathbb{N}$). Therefore E''_w is a cosmic space. As E_w is a subspace of E''_w, the space E_w is cosmic as well.

(ii) As in (i), the space E''_w has a compact resolution $\{U_\alpha^\circ : \alpha \in \mathbb{N}^\mathbb{N}\}$. We show that E''_w has a fundamental compact resolution. Indeed, Theorem 2.7 and the fact that E''_w is locally complete (since E'_β is barrelled) imply that E''_w has a fundamental bounded resolution. Now, by the barrelledness of E'_β, the space E'_β has a $\mathfrak{G}$-base $\mathcal{U} = \{U_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$. Therefore the family

$\mathcal{U}^\circ := \{U^\circ_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ is a resolution for E''_w consisting of compact subsets. To check that $\mathcal{U}^\circ$ swallows the compact sets, let K be a compact subset of E''_w. As E'_β is barrelled, K is equicontinuous. So there is an $\alpha \in \mathbb{N}^\mathbb{N}$ such that $K \subseteq U^\circ_\alpha$, and hence $\mathcal{U}^\circ$ swallows the compact sets. On the other hand, E''_w is submetrizable since E'_β is separable. Now Theorem 3.6 of [11] yields that E''_w is an $\aleph_0$-space, so E_w is an $\aleph_0$-space, too.

(iii) Any strict (LF)-space E, being regular, is a quasi-(DF)-space by (i) of Theorem 4.1. Note also that any (LM)-space is quasibarrelled (even bornological). Thus to apply (ii) it is sufficient to show that E'_β is barrelled. Let $\{E_n\}_{n\in\mathbb{N}}$ be a defining sequence of Fréchet lcs for E. For each $n \in \mathbb{N}$, the strong dual $(E_n)'_\beta$ of E_n is a (DF)-space. Since E is a strict limit, the dual E'_β is linearly homeomorphic with the projective limit of the sequence $\{(E_n)'_\beta\}_{n\in\mathbb{N}}$ of complete (DF)-spaces, see [8, Preliminaries]. Therefore E'_β is also complete. On the other hand, E'_β is continuously mapped onto each $(E'_n)_\beta$, so any $(E'_n)_\beta$ is separable. By [40, Proposition 8.3.45], any E_n is distinguished. Hence applying again [8, Preliminaries (c)], the space E'_β is quasibarrelled. Since any complete quasibarrelled space is barrelled (see [29, Proposition 11.2.4]), the space E'_β is barrelled. □

Below we provide more concrete examples which clarify the fundamental differences between (DF)-spaces and quasi-(DF)-spaces.

Example 4.5. The space of distributions $D'(\Omega)$ over an open non-empty subset Ω of $\mathbb{R}^n$ has the following properties:

(i) $D'(\Omega)$ is a quasi-(DF)-space.
(ii) $D'(\Omega)$ is a weakly $\aleph_0$-space.
(iii) $D'(\Omega)$ is not a (DF)-space.
(iv) $D'(\Omega)$ is not a weakly Ascoli space.

Proof. Recall that the space $D'(\Omega)$ is the strong dual of the space $D(\Omega)$ of test functions which is a complete Montel (hence barrelled) strict (LF)-space of a sequence of Montel–Fréchet lcs. Therefore $D'(\Omega)$ is a quasi-(DF)-space by (ii) of Theorem 4.1. Since $D'(\Omega)$ is a Montel quasi-(DF)-space whose strong dual $D(\Omega)$ is separable and barrelled, $D'(\Omega)$ is a weakly $\aleph_0$-space by (ii) of Theorem 4.4. As $D'(\Omega)$ does not have a fundamental bounded sequence (otherwise $D(\Omega)$ would be metrizable), $D'(\Omega)$ is not a (DF)-space. Finally, Theorem 1.6 of [20] states that if E is a barrelled weakly Ascoli space, then every weak*-bounded subset of E' is finite-dimensional. Thus $D'(\Omega)$ is not weakly Ascoli since $D(\Omega)$ has an infinite-dimensional compact sets. □

Example 4.6 (Under $\aleph_1 < \mathfrak{b}$)**.** The space $C_k(\omega_1)$ is a (DF)-space by Theorem 12.6.4 of [29] which is not barrelled (recall that the ordinal space $w_1 = [0, \omega_1)$ is pseudocompact). However, $C_k(\omega_1)$ does not have a $\mathfrak{G}$-base by Proposition 16.13 of [30].

As we mentioned above, it is known that the space $C_p(X)$ is a (DF)-space if and only if X is finite. Indeed, although always $C_p(X)$ is quasibarrelled, see [29, Corollary 11.7.3], the space $C_p(X)$ admits a fundamental sequence of bounded sets only if X is finite, see [30]. For quasi-(DF)-spaces $C_p(X)$ the corresponding situation looks even more striking as the following theorem shows.

Theorem 4.7. *For $C_p(X)$ the following assertions are equivalent:*

(i) $C_p(X)$ is a quasi-(DF)-space.
(ii) $C_p(X)$ has a fundamental bounded resolution.
(iii) X is countable.
(iv) $C_p(X)$ is in the class $\mathfrak{G}$.
(v) $C_p(X)$ has a $\mathfrak{G}$-base.

Proof. (ii) follows from (i) by the definition. (iii) follows from (ii) by Theorem 3.3. The implication (iii) ⇒ (iv) is trivial. Since $C_p(X)$ is always quasibarrelled (see again [29, Corollary 11.7.3]), the implication (iv) ⇒ (v) follows from [9]. Finally, if $C_p(X)$ has a $\mathfrak{G}$-base, it is metrizable again by [9]. □

Theorem 4.7 suggests the following question:

Question 4.8. *For which Tychonoff space X, the space $C_k(X)$ is a quasi-(DF)-space?*

We know already from Theorem 3.3 that for Fréchet–Urysohn spaces $C_k(X)$ the existence of a fundamental bounded resolution implies metrizability of $C_k(X)$. Below we obtain a complete answer to Question 4.8 for metrizable spaces X.

Proposition 4.9. *Let X be a metrizable space. Then $C_k(X)$ is a quasi-(DF)-space if and only if X is a Polish σ-compact space. In particular, if X is a Polish σ-compact but non-compact space, then $C_k(X)$ is a quasi-(DF)-space which is not a (DF)-space.*

Proof. Assume that $C_k(X)$ is a quasi-(DF)-space. Since $C_k(X)$ has a fundamental bounded resolution, Corollary 2.8 implies that X is a σ-compact space. On the other hand, as $C_k(X)$ is barrelled, $C_k(X)$ belongs to the class $\mathfrak{G}$ if and only if it has a $\mathfrak{G}$-base, see Lemma 15.2 of [30]. Therefore X has a fundamental compact resolution by [16]. Thus X is Polish by the Christensen theorem [30, Theorem 6.1].

Conversely, if X is a Polish σ-compact space, then $C_k(X)$ has a $\mathfrak{G}$-base by [16] and has a fundamental bounded resolution by Corollary 2.8. Thus $C_k(X)$ is a quasi-(DF)-space.

If the Polish σ-compact X is not compact, it has a countable non relatively compact subset. Thus $C_k(X)$ is a (DF)-space by Theorem 10.1.22 of [40]. $\square$

Recall that $C_k(X)$ is a (DF)-space if and only if any countable union of compact subsets of X is a relatively compact set, see [40, Theorem 10.1.22].

Let us recall two important properties which hold for any (DF)-space. The first property is the following: If E is a (DF)-space with its fundamental sequence of bounded sets $\{S_n\}$, then a linear map ξ on E to a Banach space F is continuous provided the restriction $\xi|S_n$ is continuous for any $n \in \mathbb{N}$, see [40, Corollary 8.3.3]. Recall that a lcs E is *bornological* if for every Banach space F any linear map $\xi : E \to F$ which transforms bounded sets to bounded sets is continuous, [40]. It is clear that if E is a bornological space with a fundamental bounded resolution $\left\{B_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\right\}$, then any linear map $\xi : E \to F$ such that each $\xi|B_\alpha$ is continuous is continuous on the whole space E. This fact refers to a large class of quasi-(DF)-spaces including all regular (LM)-spaces. This suggests the following question:

(A) *Does there exist a quasi-(DF)-space E with its fundamental bounded resolution $\left\{B_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\right\}$ and a discontinuous linear functional ξ on E such that each $\xi|B_\alpha$ is continuous?*

Let's mention here the following remarkable result of Drewnowski [13] which also motivates the above question: *If E and F are metrizable and complete lcs and E has a resolution $\left\{E_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\right\}$ of subsets of E, then a linear map $\xi : E \to F$ is continuous provided each $\xi|E_\alpha$ is continuous.*

The second property characterizes quasibarrelled (DF)-spaces: A (DF)-space E is quasibarrelled if and only if E has countable tightness, see [30, Proposition 16.4 and Theorem 12.3]. This may suggest also the following question:

(B) *Is a quasi-(DF)-space with countable tightness quasibarrelled?*

Next example answers both Questions (A) and (B) in the negative (recall that every cosmic space is even hereditary countably tight, see [35]):

Example 4.10. There exists a countably-dimensional locally convex space E such that:

(i) E is a quasi-(DF)-space but not a (DF)-space;
(ii) E has a fundamental bounded sequence which is also a fundamental compact sequence;
(iii) there exists a discontinuous linear functional χ on E which is continuous on every bounded subset of E;
(iv) the strong bidual E''_β of E is a (DF)-space;
(v) E is cosmic and non-quasibarrelled.

Proof. Let $A(\mathbf{s})$ be the free abelian topological group over the convergent sequence $\mathbf{s} = \{0\} \cup \left\{\frac{1}{n} : n \in \mathbb{N}\right\} \subseteq \mathbb{R}$. Then $A(\mathbf{s})$ is algebraically the direct sum $F := \bigoplus_{n\in\mathbb{N}} \mathbb{Z}$ of the group $\mathbb{Z}$ and has a $\mathfrak{G}$-base by Theorem 4.16 of [26]. Denote by G the group F, endowed with the metrizable group topology τ whose neighborhood base at zero consists of the subgroups $2^k F$, $k \in \mathbb{N}$. Being metrizable the topological group G has a $\mathfrak{G}$-base. Therefore the diagonal group Δ of the product $A(\mathbf{s}) \times G$ also has a $\mathfrak{G}$-base, see [26, Proposition 2.10]. Moreover, as it is shown in Example 4.1 of [19], the group Δ is a non-discrete countable abelian group whose compact sets are finite. Let $\Delta = \left\{x_n : n \in \mathbb{N}\right\}$ be an enumeration of Δ. Define $E := L(\Delta)$. Let us show that E satisfies (i)–(v).

The space E has a $\mathfrak{G}$-base by [5]. Clearly, Δ is a μ-space and it has a fundamental compact sequence. Since $L(\Delta)$ and $L_p(\Delta)$ have the same bounded sets, E has a fundamental bounded resolution by Proposition 3.11. Thus E is a quasi-(DF)-space.

Every bounded subset of $E = L(\Delta)$ is finite-dimensional. Indeed, bearing in mind that the strong dual of any space $C_p(X)$ is feral (see [17]), then each $\beta(L(\Delta), C(\Delta))$-bounded set B of $L(\Delta)$ is finite-dimensional. The space $C_p(\Delta)$ is barrelled by the

Buchwalter–Schmets theorem, see [30, Proposition 2.17], so if B is a bounded set of the free locally convex space $L(\Delta)$, then it is bounded in $L_p(\Delta)$. Hence B is $\beta(L(\Delta), C(\Delta))$-bounded. This shows that B finite-dimensional. Therefore, the sequence

$$\left\{[-n,n]x_1 + \cdots + [-n,n]x_n : n \in \mathbb{N}\right\}$$

is a fundamental bounded sequence which is also a fundamental compact sequence in E. This proves (ii).

Take arbitrarily $\chi = (a_n) \in \mathbb{R}^{\mathbb{N}}$ such that $\chi \notin C(\Delta)$, such a χ exists since Δ is not discrete. Since $E' = C(\Delta)$, χ is a discontinuous linear functional on E. On the other hand, the finite-dimensionality of bounded sets in E implies that χ is continuous on every bounded subset of E. This proves (iii). Corollary 8.3.3 of [40] implies that E is not a (DF)-space.

As Δ is a μ-space, the strong dual of $E = L(\Delta)$ is $C_k(\Delta)$ (see the first paragraph of the proof of Proposition 3.11). But since Δ does not have infinite compact subsets we obtain that $C_k(\Delta) = C_p(\Delta)$ is a metrizable space. Thus E''_β is a (DF)-space and (iv) is proved. Finally, (v) follows from Corollary 5.20 and Theorem 6.4 of [27]. □

Note that the space Δ is not an Ascoli space by Proposition 5.12 of [4].

It is clear that the strong bidual of a (DF)-space is a (DF)-space. This and the previous example suggest the following

Question 4.11. *Is it true that the strong bidual of a quasi-(DF)-space is again a quasi-(DF)-space?*

Surely such an example cannot be quasibarrelled by Proposition 2.10. Note that Proposition 2.10 together with Theorem 2.7 imply that for a quasibarrelled space E the strong dual E'_β is a quasi-(DF)-space if and only if E is a quasi-(DF)-space.

ACKNOWLEDGEMENTS

The authors are grateful to the referee for valuable comments and remarks. The third named author was supported by GAČR Project 16-34860L and RVO: 67985840.

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How to cite this article: Ferrando JC, Gabriyelyan S, Kąkol J. Bounded sets structure of $C_p(X)$ and quasi-(DF)-spaces. *Mathematische Nachrichten*. 2019;292:2602–2618. https://doi.org/10.1002/mana.201800085