



Classifying Topologies through \mathfrak{G} -Bases

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Abstract: We classify several topological properties of a Tychonoff space X by means of certain locally convex topologies \mathcal{T} with a \mathfrak{G} -base located between the pointwise topology τ_p and the bounded-open topology τ_b of the real-valued continuous function space $C(X)$.

Keywords: compact resolution; σ -compact space; hemicompact space; \mathfrak{G} -base

MSC: 54C; 54D; 54E; 46A; 46B

1. Introduction

Unless otherwise stated, X stands for an infinite Tychonoff space. We denote by $C_p(X)$ the linear space $C(X)$ of real-valued continuous functions on X equipped with the pointwise topology τ_p . The topological dual of $C_p(X)$ is denoted by $L(X)$, or by $L_p(X)$ when provided with the weak* topology. We represent by $C_k(X)$ the space $C(X)$ equipped with the compact-open topology τ_k . A family $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ of subsets of a set X is called a *resolution* for X if it covers X and verifies that $A_\alpha \subseteq A_\beta$ for $\alpha \leq \beta$. Let us recall that a Hausdorff topological space X is called a μ -space if the closure of each functionally bounded set in X is compact. Every realcompact space is a μ -space, but the converse is not true. If E is a locally convex space, the *bidual* E'' of E is the topological dual of the strong dual E'_β of E . By the strong dual E'_β of E , we mean the dual E' of E equipped with the strong topology $\beta(E', E)$ of the uniform convergence on the bounded sets in E . We denote by τ_w and τ_b the weak locally convex topology of $C_k(X)$ and the bounded-open topology of $C(X)$, respectively. If A is a nonempty set in a real linear space L , we represent by $\text{abx}(A)$ the (real) absolutely convex hull of A . The linear subspace of $C(X)$ consisting of those bounded functions is denoted by $C^b(X)$.

A base $\{U_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ of (absolutely convex) neighborhoods of the origin in a locally convex space E such that $U_\beta \subseteq U_\alpha$ if $\alpha \leq \beta$ is called a \mathfrak{G} -base. Let us mention that the notion of a \mathfrak{G} -base, originally introduced in the realm of locally convex spaces [1], has been extended to topological groups and general topological spaces, sometimes under the name of ω^ω -base, by some authors (see [2,3]). However, in this paper we keep the original name. Trivially, if E is a metrizable locally convex space with a decreasing base $\{V_n : n \in \mathbb{N}\}$ of locally convex neighborhoods of the origin, the family $\{U_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$, where $U_\alpha = V_{\alpha(1)}$ for all $\alpha \in \mathbb{N}^{\mathbb{N}}$ is a \mathfrak{G} -base for E . The converse is not true, i.e., a locally convex space with a \mathfrak{G} -base need not be metrizable (see p. 107, [4]).

A locally convex space E with a \mathfrak{G} -base has metrizable compact sets, since, in this case, the weak dual $(E', \sigma(E', E))$ is quasi-Suslin [5], hence trans-separable [6] (properties not defined in this paper can be found in [7–10]). Therefore, if there is a locally convex topology \mathcal{T} on $C(X)$ with a \mathfrak{G} -base such that $\tau_p \leq \mathcal{T} \leq \tau_b$, every completely regular topology τ on $C(X)$ such that $\mathcal{T} \leq \tau$ is angelic and has metrizable compact sets. Research on \mathfrak{G} -bases and their generalizations remain active since [1]. For recent results on this topic see [3,11–14]. Research on the existence of locally convex topologies on $C(X)$ stronger



Citation: Ferrando, J.C.;

López-Pellicer, M. Classifying Topologies through \mathfrak{G} -Bases. *Axioms* **2022**, *11*, 744. <https://doi.org/10.3390/axioms11120744>

Academic Editor: Hari Mohan Srivastava

Received: 5 December 2022

Accepted: 14 December 2022

Published: 19 December 2022

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than the pointwise topology has been studied in [15,16]. In this paper, we enlarge the classification of topological properties on X provided in (Theorem 3.1, [15]).

2. A Preliminary Result

A resolution $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ for X is *functionally bounded* if it consists of functionally bounded sets A_α in X . A space X is said to be *strictly angelic* [17] if it is angelic and all separable compact subsets of X are first countable. A classification of topologies on X by locally convex topologies on $C_p(X)$ is provided in [15] (see also (Theorem 98, [4])).

Theorem 1 (Ferrando–Gabrielyan–Kąkol (Theorem 3.1, [15])). *If X is a Tychonoff space, the following properties hold*

1. *There exists a metrizable locally convex topology \mathcal{T} on $C(X)$ such that $\tau_p \leq \mathcal{T} \leq \tau_k$ if and only if X is a σ -compact space.*
2. *There exists a metrizable locally convex topology \mathcal{T} on $C(X)$ such that $\tau_k \leq \mathcal{T} \leq \tau_b$ if and only if there is an increasing sequence $\{A_n : n \in \mathbb{N}\}$ of functionally bounded subsets of X swallowing the compact sets of X .*
3. *There exists a metrizable locally convex topology \mathcal{T} on $C(X)$ such that $\tau_p \leq \mathcal{T} \leq \tau_b$ if and only if there is an increasing sequence $\{A_n : n \in \mathbb{N}\}$ of functionally bounded subsets of X covering X , or equivalently, if and only if vX is σ -compact.*
4. *There is a metrizable locally convex topology \mathcal{T} on $C(X)$ such that $\tau_p \leq \mathcal{T} \leq \tau_w$ if and only if X is countable.*
5. *There is a locally convex topology \mathcal{T} on $C(X)$ with a \mathfrak{G} -base such that $\tau_p \leq \mathcal{T} \leq \tau_k$ if and only if X has a compact resolution.*
6. *There is a locally convex topology \mathcal{T} on $C(X)$ with a \mathfrak{G} -base such that $\tau_k \leq \mathcal{T} \leq \tau_b$ if and only if X has a functionally bounded resolution swallowing the compact sets.*
7. *There exists a locally convex topology \mathcal{T} on $C(X)$ with a \mathfrak{G} -base such that $\tau_p \leq \mathcal{T} \leq \tau_b$ if and only if X has a functionally bounded resolution. Equivalently, if and only if vX is K -analytic. In this case $(C(X), \tau_b)$ is strictly angelic.*
8. *There is a locally convex topology \mathcal{T} on $C(X)$ with a \mathfrak{G} -base such that $\tau_p \leq \mathcal{T} \leq \tau_w$ if and only if X is countable.*

3. A More Complete Classification

To enlarge the previous classification, we need the following result.

Lemma 1. *Let X be completely regular. If Q is a metrizable and compact subspace of X , there exists a continuous linear extender map $\varphi : C_k(Q) \rightarrow C_k(X)$, i.e., such that $\varphi(f)|_Q = f$ for every $f \in C(Q)$.*

Proof. Since Q is (homeomorphic to) a metrizable compact subspace of the Stone–Čech compactification βX of X , according to [18] (see also (Proposition 4.1, [19])) there exists a continuous linear map $\psi : C_p(Q) \rightarrow C_p(\beta X)$ such that $\psi(f)|_Q = f$ which embeds $C_p(Q)$ in $C_p(\beta X)$ (as a closed subspace). If $\phi : C_p(\beta X) \rightarrow C_p^b(X)$ is the restriction map $\phi(g) = g|_X$, it turns out that $\varphi := \phi \circ \psi$ is a continuous linear map from $C_p(Q)$ into $C_p^b(X)$ such that $\varphi(f)|_Q = \phi(\psi(f))|_Q = \psi(f)|_Q = f$ for every $f \in C(Q)$.

Hence, $\varphi : C(Q) \rightarrow C^b(X)$ has a closed graph when both spaces are regarded as Banach spaces. This implies in particular that $\varphi : C_k(Q) \rightarrow C_k^b(X)$ is continuous, since the supremum norm topology on $C^b(X)$ is stronger than the relative compact-open topology of $C_k(X)$. Thus, $\varphi : C_k(Q) \rightarrow C_k(X)$ is a continuous linear extender map, as stated. \square

Theorem 2. *If X is a Tychonoff space, the following properties hold*

1. *The compact-open topology $\mathcal{T} = \tau_k$ on $C(X)$ is metrizable if and only if X is a hemicompact space (Arens' theorem (Theorem 7, [20]), see also (Theorem 2.5, [21])).*

2. The bounded-open topology $\mathcal{T} = \tau_b$ on $C(X)$ is metrizable if and only if there is an increasing sequence $\{A_n : n \in \mathbb{N}\}$ of functionally bounded subsets of X swallowing the functionally bounded sets of X .
3. The weak topology $\mathcal{T} = \tau_w$ on $C(X)$ is metrizable if and only if X is countable and compact sets in X are finite.
4. The pointwise topology $\mathcal{T} = \tau_p$ on $C(X)$ is metrizable if and only if X is countable.
5. The compact-open topology $\mathcal{T} = \tau_k$ on $C(X)$ has a \mathfrak{G} -base if and only if X has a compact resolution that swallows the compact sets (Theorem 2, [22]).
6. The bounded-open topology $\mathcal{T} = \tau_b$ on $C(X)$ has a \mathfrak{G} -base if and only if X has a functionally bounded resolution that swallows the functionally bounded sets.
7. The weak topology $\mathcal{T} = \tau_w$ on $C(X)$ has a \mathfrak{G} -base if and only if X is countable and compact sets in X are finite.
8. The pointwise topology $\mathcal{T} = \tau_p$ on $C(X)$ has a \mathfrak{G} -base if and only if X is countable (Corollary 15.2, [7]).

Proof. The proof of statements (2) and (6) is similar (for (6) see (Theorem 12, [23])).

Let us prove statement (3). If $C_k(X)$ is weakly metrizable, the weakly bounded sets are metrizable, so X is countable by (Theorem 2.3, [21]). This fact also follows from statement (4) of Theorem 1. We claim that, in addition, the compact sets of X are finite. Otherwise, there exists an infinite compact set Q in X . However, since X is countable, Q is metrizable. By Lemma 1 there is a continuous linear extender φ from $C_k(Q)$ into $C_k(X)$, i.e., such that $\varphi(f)|_Q = f$ for every $f \in C(Q)$. If $\varphi(f_d) \rightarrow g$ in $C_k(X)$, given $\epsilon > 0$ there is $h \in D$ with $\sup_{x \in Q} |\varphi(f_d)(x) - g(x)| < \epsilon$ for every $d \geq h$, so $\sup_{x \in Q} |f_d(x) - g(x)| < \epsilon$ for $d \geq h$. Thus, $f_d \rightarrow f := g|_Q$ in $C_k(Q)$, and hence $\varphi(f_d) \rightarrow \varphi(f)$ in $C_k(X)$, i.e., $g = \varphi(f)$, which means that φ embeds the Banach space $C_k(Q)$ in $C_k(X)$ as a closed subspace. Since the weak topology is inherited by linear subspaces, the space $C_w(Q)$ is linearly homeomorphic to a subspace of $C_w(X)$. In other words, the Banach space $C_k(Q)$ is weakly metrizable. As the weak topology of a Banach space is metrizable if and only if it is finite-dimensional, it turns out that Q must be finite, a contradiction. Conversely, if X is both countable and has finite compact sets, the former statement guarantees that $C_p(X)$ is metrizable whereas the latter implies that $C_p(X) = C_k(X)$ coincides with $C_w(X)$. Hence, the weak topology τ_w is metrizable.

The proof of statement (7) is similar to that of statement (3). The only difference is that the weak topology of the Banach space $C_k(Q)$ now carries a \mathfrak{G} -base. However, if a locally convex space in its weak topology has a \mathfrak{G} -base, it is countable-dimensional (Proposition 11.2, [7]). Thus, $C(Q)$ must be countable-dimensional, so finite-dimensional by the Baire category theorem. This ensures that the compact set Q must be finite. \square

Remark 1. In (Theorem 2.3, [21]), it is shown that if X is countable, the bounded sets of $C_k(X)$ are weakly metrizable. According to statement (3) of the previous theorem, if X contains an infinite compact set, then $C_k(X)$ cannot be weakly metrizable. Thus, if X is countable and contains an infinite compact set, then $C_w(X)$ is not metrizable but has metrizable bounded sets.

Example 1. As \mathbb{Q} is a countable space with infinite compact sets which is not hemicompact, neither τ_k nor τ_w are metrizable, the former statement by Arens' theorem and the latter by the previous remark, but τ_p is metrizable. Of course, τ_w has no \mathfrak{G} -base. In fact, since \mathbb{Q} is not a Polish space, Christensen's theorem (Theorem 94, [4]) prevents \mathbb{Q} from having a compact resolution that swallows the compact sets. This also implies that $C_k(\mathbb{Q})$ has no \mathfrak{G} -base, by statement (5) of Theorem 2.

Example 2. Let \mathbb{N} be equipped with the discrete topology and choose $p \in \beta\mathbb{N} \setminus \mathbb{N}$. Then, $X := \mathbb{N} \cup \{p\}$ provided with the relative topology of $\beta\mathbb{N}$ is not discrete but has finite compact sets. Thus, the weak topology τ_w on $C(X)$ is metrizable. In fact, clearly, $\tau_p = \tau_k$, although $C_p(X) \neq \mathbb{R}^X$.

Example 3. \mathbb{R} is an uncountable hemicompact space, hence τ_k is metrizable, but τ_w and τ_p are not.

Example 4. If K is a countable infinite compact set, obviously both τ_p and τ_k are metrizable, but τ_w is not.

Example 5. The Sorgenfrey line \mathbb{S} is a Lindelöf space which is not σ -compact, so $\tau_k = \tau_b$ and by statement (1) of Theorem 1 there is no metrizable, locally convex topology \mathcal{T} on $C(\mathbb{S})$ such that $\tau_p \leq \mathcal{T} \leq \tau_k$.

Example 6. The space $C_k(\mathbb{N}^{\mathbb{N}})$ has a \mathfrak{G} -base by statement (5) of Theorem 2, but there is no metrizable topology \mathcal{T} such that $\tau_p \leq \mathcal{T} \leq \tau_k$, since $\mathbb{N}^{\mathbb{N}}$ is not σ -compact.

4. The Interval $\tau_w \leq \mathcal{T} \leq \tau_k$

In this section, we deal with the interval $\tau_w \leq \mathcal{T} \leq \tau_k$. Before stating our main result for this case, we need to establish two auxiliary results concerning the weak* dual $L_p(X)$ of the space $C_p(X)$. We regard (the canonical homeomorphic copy $\delta(X)$ of) X in $L_p(X)$ as a Hamel basis of $L(X)$, and denote by δ_x the image of $x \in X$ in $\delta(X)$. If $f \in C(X)$ and $u \in L(X)$, we write $\langle f, u \rangle$ to represent the action of the linear functional u on f , in particular $\langle f, \delta_x \rangle = f(x)$.

Lemma 2. Let E denote the dual of $C_k(X)$. If X is a μ -space, then E coincides with the bidual of $L_p(X)$.

Proof. If X is a μ -space, then $C_k(X)$ is the strong dual of $L_p(X)$ (see (Lemma 2.2, [24])). Therefore, $C_k(X) = L(X)'_{\beta}$ and hence, E coincides with the bidual $L_p(X)''$ of $L_p(X)$. \square

Lemma 3. Let X be a μ -space. If A is a bounded set in $L_p(X)$, there are a compact set K_A in X and a real number ϵ_A with $0 < \epsilon_A < 1$ such that $A \subseteq 2\epsilon_A^{-1}\text{abx}(K_A)$.

Proof. Since X is a μ -space, if A is a bounded set in $L_p(X)$, then A^0 is a neighborhood of the origin in $C_k(X)$, so there is $0 < \epsilon_A < 1$ and a compact set K_A in X such that

$$U_A := \{f \in C(X) : \sup_{x \in K_A} |f(x)| \leq \epsilon_A\} \subseteq A^0$$

which implies that $A \subseteq A^{00} \subseteq U_A^0$, where the bipolar of A is taken in $L(X)$ as well as the polar of U_A . The fact that $U_A = \epsilon_A K_A^0$ yields $\epsilon_A A \subseteq K_A^{00}$ so that $\epsilon_A A \subseteq \overline{\text{abx}(K_A)}^{L_p(X)}$. We claim that $\overline{\text{abx}(K_A)}^{L_p(X)} \subseteq 2\text{abx}(K_A)$.

Assume that $u \in \overline{\text{abx}(K_A)}^{L_p(X)} = \text{abx}(K_A)^{00}$ with $u \neq \mathbf{0}$. Since X is a Hamel basis of $L(X)$, we have that $u = \sum_{i=1}^m a_i \delta_{x_i}$, with $a_i \neq 0$ for $1 \leq i \leq m$. First, we show that the support of u is contained in K_A , so that $x_i \in K_A$ for $1 \leq i \leq m$. Indeed if $x_k \notin K_A$ for some $k \in \{1, \dots, m\}$, there is $f \in C(X)$ such that $f(x_k) = 2a_k^{-1}$ and $f(z) = 0$ for every $z \in K_A \cup \{x_i : 1 \leq i \leq m, i \neq k\}$. Note that $f \in \text{abx}(K_A)^0$ since if $v = \sum_{i=1}^n b_i \delta_{y_i} \in \text{abx}(K_A)$ with $\sum_{i=1}^n |b_i| \leq 1$ and $y_i \in K_A$ for $1 \leq i \leq n$, then $\langle f, v \rangle = \sum_{i=1}^n b_i f(y_i) = 0$. As $\langle f, u \rangle = 2$, this contradicts the fact that $u \in \text{abx}(K_A)^{00}$. Thus, $x_i \in K_A$ for $1 \leq i \leq m$.

Now, we show that $\sum_{i=1}^m |a_i| \leq 2$, so that $u \in 2\text{abx}(K_A)$. Since the support of u is finite and is contained in K_A , there are pairwise disjoint open sets $\{U_1, \dots, U_m\}$ in K_A with $x_i \in U_i$, so there are compactly supported continuous functions $\{f_1, \dots, f_m\}$ such that $\{x_i\} \prec f_i \prec U_i$ for $1 \leq i \leq m$. This means that $f_i \in C(X)$, f_i has compact support, $0 \leq f_i \leq 1$, $\text{supp } f_i \subseteq U_i$ and $f_i(x_i) = 1$ for $1 \leq i \leq m$. For $1 \leq i \leq m$, set $h_i = \text{sgn}(a_i) f_i$, where $\text{sgn}(a_i) \in \{-1, 1\}$ denotes the sign of the real number a_i , and define $h := \sum_{i=1}^m h_i$. Clearly $h \in C(X)$, $h(x_i) = \text{sgn}(a_i)$, and $|h(x)| \leq 1$ for every $x \in X$, because of the supports of the functions f_i are pairwise disjoint.

Since $u \in \overline{\text{abx}(K_A)}^{L_p(X)}$ and $h \in C(X)$ there exists $v = \sum_{i=1}^k c_i \delta_{z_i} \in \text{abx}(K_A)$ with $\sum_{i=1}^k |c_i| \leq 1$ and $z_i \in K_A$ for $1 \leq i \leq k$ such that $|\langle h, u - v \rangle| \leq 1$. Consequently,

$$\left| \sum_{i=1}^m a_i h(x_i) - \sum_{j=1}^k c_j h(z_j) \right| \leq 1,$$

and due to the fact that $v \in \text{abx}(K_A)$ and $a_i h(x_i) = |a_i|$ it follows that

$$\sum_{i=1}^m |a_i| = \left| \sum_{i=1}^m a_i h(x_i) \right| \leq 1 + \left| \sum_{j=1}^k c_j h(z_j) \right| \leq 1 + \sum_{j=1}^k |c_j| |h(z_j)| \leq 2$$

as required. \square

If X is a μ -space and $\text{Bound}(X)$ denotes the family of all bounded sets in $L_p(X)$, as a consequence of Lemma 2, the space E coincides with (see (23.2.(1), [8]))

$$E = \bigcup \{ \overline{A} : A \in \text{Bound}(X) \}$$

where the closure is in E under the weak* topology $\sigma(E, C(X))$ of E . Hence, if $\mu \in E$ there is a bounded set A in $L_p(X)$ with $\mu \in \overline{A}$. However according to Lemma 3, given $A \in \text{Bound}(X)$, there is a compact S_A in X together with some $n_A \in \mathbb{N}$ with $A \subseteq n_A \text{abx}(S_A)$. Thus, if $\mathcal{B}(X)$ designates the family of all functionally bounded sets in X , then

$$E = \bigcup \{ \overline{n_A \text{abx}(S_A)} : A \in \text{Bound}(X) \} = \bigcup \{ \overline{n \text{abx}(S)} : S \in \mathcal{B}(X), n \in \mathbb{N} \}$$

where the closures are in $\sigma(E, C(X))$. Therefore, the following property holds.

Remark 2. If X is a μ -space and $\mathcal{K}(X)$ stands for the family of all compact sets in X , then

$$E = \bigcup \{ nS^{00} : S \in \mathcal{K}(X), n \in \mathbb{N} \}$$

where E is the dual of $C_k(X)$.

Proof. Because of the bipolar theorem, one has $\overline{\text{abx}(S)} = S^{00}$. \square

This motivates the following definition.

Definition 1. Let E be the dual of $C_k(X)$. We say that a countable family $\mathcal{F} = \{S_n : n \in \mathbb{N}\}$ of compact sets in X is a complete sequence if there exists a sequence $\{k_n : n \in \mathbb{N}\}$ of positive integers such that

$$E = \bigcup_{n \in \mathbb{N}} k_n S_n^{00}.$$

If X is a metrizable hemicompact space and $\mathcal{F} = \{S_n : n \in \mathbb{N}\}$ is a family of compact sets that swallows the compact sets in X , then \mathcal{F} is a complete sequence. If $\{V_j : j \in \mathbb{N}\}$ is a base of neighborhoods of the origin in $C_k(X)$, there are by Lemma 3 a compact set S in X and $k_j \in \mathbb{N}$ such that $V_j^0 \subseteq k_j \text{abx}(S)$. If $S \subseteq S_{n_j}$ then $V_j^0 \subseteq k_j S_{n_j}^{00}$ for every $j \in \mathbb{N}$. If we define $k_m = k_j$ if $m = n_j$ for some $j \in \mathbb{N}$ and $k_m = 1$ otherwise, then

$$E = \bigcup_{j \in \mathbb{N}} V_j^0 = \bigcup_{m \in \mathbb{N}} k_m S_m^{00}.$$

If $\langle F, E \rangle$ is a dual pair, a locally convex topology \mathcal{T} for F is called a polar topology if \mathcal{T} is the topology of the uniform convergence on the sets of a family \mathcal{A} of $\sigma(E, F)$ -bounded sets in E . Obviously, both the weak topology $\sigma(F, E)$ and the strong topology $\beta(F, E)$ are polar topologies for F . We are ready to establish our last classification result, which reads as follows.

Theorem 3. *Let X be a Tychonoff space. There is a metrizable polar topology \mathcal{T} on $C(X)$ such that $\tau_w \leq \mathcal{T} \leq \tau_k$ if and only if X contains a complete sequence.*

Proof. Let us denote by E the dual of the locally convex space $C_k(X)$. Assume that \mathcal{T} is a metrizable polar topology on $C(X)$ with a decreasing base $\{U_n : n \in \mathbb{N}\}$ of neighborhoods of the origin enjoying that $\tau_w \leq \mathcal{T} \leq \tau_k$. By the first statement of Theorem 1, we know that X is a σ -compact space, so a μ -space.

As \mathcal{T} is a polar topology, there exists a family \mathcal{M} of bounded sets in $L_p(X)$ such that for each $n \in \mathbb{N}$, there are some $M_n \in \mathcal{M}$ with $U_n = M_n^0$. According to Lemma 3, there is a compact set S_n in X and some $k_n \in \mathbb{N}$ such that $M_n \subseteq k_n \text{abx}(S_n)$, so that $S_n^0 \subseteq k_n U_n$. Set $\mathcal{F} := \{S_n : n \in \mathbb{N}\}$. Since $U_n^0 \subseteq k_n S_n^{00}$ for each $n \in \mathbb{N}$ and $\tau_w \leq \mathcal{T}$, we get

$$\bigcup_{n \in \mathbb{N}} k_n S_n^{00} = \bigcup_{n \in \mathbb{N}} U_n^0 = E.$$

Thus, the sequence \mathcal{F} is complete.

Assume conversely that there is a complete sequence $\mathcal{F} = \{S_n : n \in \mathbb{N}\}$ in X . We may suppose that $S_n \subseteq S_{n+1}$ as well as that $k_n \leq k_{n+1}$. If not, replace \mathcal{F} by the sequence $\mathcal{F}_1 := \{S'_n : n \in \mathbb{N}\}$ with $S'_n = \bigcup_{i=1}^n S_i$ and the sequence $\{k_n : n \in \mathbb{N}\}$ by $\{k'_n : n \in \mathbb{N}\}$ with $k'_n = \max_{1 \leq i \leq n} k_i$. Since $S_n^{00} \subseteq S_n'^{00}$ for every $n \in \mathbb{N}$, if \mathcal{F} is complete so is \mathcal{F}_1 . Then, for each $n \in \mathbb{N}$, define $V_n = k_n^{-1} S_n^0$. Clearly, $V_{n+1} \subseteq V_n$ and $\{V_n : n \in \mathbb{N}\}$ is a base of neighborhoods of the origin of a metrizable locally convex topology \mathcal{T} for $C(X)$. Observe that \mathcal{T} is stronger than τ_w , since

$$\bigcup_{n \in \mathbb{N}} V_n^0 = \bigcup_{n \in \mathbb{N}} k_n S_n^{00} = E.$$

On the other hand, as $\mathcal{F} \subseteq \mathcal{K}(X)$, the family of all compact sets in X , we see that \mathcal{T} is weaker than τ_k . \square

5. Conclusions

We enlarged the classification of some topological properties on X provided in (Theorem 3.1, [15]) by using specific locally convex topologies \mathcal{T} with a \mathfrak{G} -base lying between the pointwise topology τ_p and the bounded-open topology τ_b of the real-valued continuous function space $C(X)$. Our main results are

Theorem 4. *Let X be a Tychonoff space.*

1. *The weak topology $\mathcal{T} = \tau_w$ on $C(X)$ is metrizable if and only if it has a \mathfrak{G} -base, and if and only if X is countable and compact sets in X are finite.*
2. *There is a metrizable polar topology \mathcal{T} on $C(X)$ such that $\tau_w \leq \mathcal{T} \leq \tau_k$ if and only if X contains a complete sequence.*

In the second statement, we say that a countable family $\mathcal{F} = \{S_n : n \in \mathbb{N}\}$ of compact sets in X is a complete sequence if there exists a sequence $\{k_n : n \in \mathbb{N}\}$ of positive integers such that $\{k_n S_n^{00} : n \in \mathbb{N}\}$ covers the dual of $C_k(X)$.

Author Contributions: Conceptualization, M.L.-P. and J.C.F.; formal analysis, M.L.-P. and J.C.F.; investigation, M.L.-P. and J.C.F.; writing—original draft preparation, M.L.-P. and J.C.F.; writing—review and editing, M.L.-P. and J.C.F.; visualization, M.L.-P. and J.C.F. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded in part by grant PGC2018-094431-B-I00 of Ministry of Science, Innovation and Universities of Spain.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: The authors thank the Editor of the Special Issue “Mathematical Analysis and Applications III”, Hari Mohan Srivastava, for his invitation to write this feature paper. Moreover, we thank the anonymous reviewers for their comments and suggestions to improve this paper.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Ferrando, J.C.; Kąkol, J.; López-Pellicer, M.; Saxon, S. Tightness and distinguished Fréchet spaces. *J. Math. Anal. Appl.* **2006**, *324*, 862–881. [[CrossRef](#)]
2. Banakh, T.; Kąkol, J.; Schürz, J.P. ω^ω -base and infinite-dimensional compact sets in locally convex spaces. *R. Mat. Complut.* **2022**, *35*, 599–614. [[CrossRef](#)]
3. Banakh, T.; Leiderman, A. ω^ω -dominated function spaces and ω^ω -bases in free objects of topological algebra. *Topol. Appl.* **2018**, *241*, 203–241. [[CrossRef](#)]
4. Ferrando, J.C. Descriptive topology for analysts. *RACSAM* **2020**, *114*, 107. [[CrossRef](#)]
5. Ferrando, J.C.; Kąkol, J.; López-Pellicer, M.; Saxon, S. Quasi-Suslin weak duals. *J. Math. Anal. Appl.* **2008**, *339*, 1253–1263. [[CrossRef](#)]
6. Ferrando, J.C.; Kąkol, J.; López-Pellicer, M. Necessary and sufficient conditions for precompact sets to be metrisable. *Bull. Austral. Math. Soc.* **2006**, *74*, 7–13. [[CrossRef](#)]
7. Kąkol, J.; Kubiś, J.; López-Pellicer, M. *Descriptive Topology in Selected Topics of Functional Analysis*; Springer: Berlin/Heidelberg, Germany, 2011.
8. Köthe, G. *Topological Vector Spaces I*; Springer: Berlin/Heidelberg, Germany, 1983.
9. Valdivia, M. *Topics in Locally Convex Spaces*; Math. Studies 67; North Holland: Amsterdam, The Netherlands, 1987.
10. Arkhangel'skiĭ, A.V. *Topological Function Spaces*; Math. Appl. 78; Kluwer: Dordrecht, The Netherlands, 1992.
11. Feng, Z.; Nukala, N.C.P. Sub-posets in ω^ω and the strong Pytkeev* property. *Topol. Appl.* **2021**, *300*, 107750. [[CrossRef](#)]
12. Gabrielyan, S.; Kąkol, J.; Leiderman, A. The strong Pytkeev property for topological groups and topological vector spaces. *Monatsh. Math.* **2014**, *175*, 519–542. [[CrossRef](#)]
13. Gabrielyan, S.; Kąkol, J.; Leiderman, A. On topological groups with a small base and metrizability. *Fund. Math.* **2015**, *229*, 129–157. [[CrossRef](#)]
14. Lin, F.; Ravsky, A.; Zhang, J. Countable tightness and \mathfrak{G} -bases on free topological groups. *RACSAM* **2020**, *114*, 67. [[CrossRef](#)]
15. Ferrando, J.C.; Gabrielyan, S.; Kąkol, J. Metrizable-like locally convex topologies on $C(X)$. *Topol. Appl.* **2017**, *230*, 105–113. [[CrossRef](#)]
16. Kąkol, J. C_p -spaces dominated by metrizable topologies. *Topol. Appl.* **2019**, *265*, 106837. [[CrossRef](#)]
17. Govaerts, W. A productive class of angelic spaces. *J. Lond. Math. Soc.* **1980**, *22*, 355–364. [[CrossRef](#)]
18. Arkhangel'skiĭ, A.V.; Calbrix, J. A characterization of σ -compactness of a cosmic space X by means of subspaces of \mathbb{R}^X . *Proc. Am. Math. Soc.* **1999**, *127*, 2497–2504. [[CrossRef](#)]
19. Arkhangel'skiĭ, A.V. C_p -Theory. In *Recent Progress in General Topology*; Hušek, M., Van Mill, J., Eds.; Elsevier: Amsterdam, The Netherlands, 1992; pp. 1–56.
20. Arens, R. A topology for spaces of transformations. *Ann. Math.* **1946**, *47*, 480–495. [[CrossRef](#)]
21. Ferrando, J.C.; Kąkol, J. Metrizable bounded sets in $C(X)$ spaces and distinguished $C_p(X)$ spaces. *J. Convex. Anal.* **2019**, *26*, 1337–1346.
22. Ferrando, J.C.; Kąkol, J. On precompact sets in $C_c(X)$. *Georgian Math. J.* **2013**, *20*, 247–254. [[CrossRef](#)]
23. Ferrando, J.C. Some topological properties of $C_b(X)$. *J. Funct. Spaces* **2014**, *2014*, 195262.
24. Ferrando, J.C. On Hamel basis in Banach spaces. *Stud. Math.* **2014**, *220*, 169–178. [[CrossRef](#)]