Doctoral Program in Statistics, Optimization and Applied Mathematics

CONTRIBUTIONS TO COST ALLOCATION PROBLEMS AND SCARCE RESOURCES

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DOCTORAL DISSERTATION

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This Doctoral Thesis, entitled **"Contributions to cost allocation problems and scarce resources"**, it is presented under the modality of **thesis by compendium** of the following **publications**:

- **On how to allocate the fixed cost of transport systems.** Estañ, T., Llorca, N., Martínez, R., & Sánchez-Soriano, J. (2021). Annals of Operations Research, 301, 81-105;https://doi.org/10.1007/s10479-020-03645-1.

- **Manipulability in the cost allocation of transport systems.** Estañ, T., Llorca, N., Martínez, R., & Sánchez-Soriano, J. (2020). (ThE Papers 20/08). Department of Economic Theory and Economic History of the University of Granada.

- **On the difficulty of budget allocation in claims problems with indivisible items and prices.** Estañ, T., Llorca, N., Martínez, R., & Sánchez-Soriano, J. (2021). Group Decision and Negotiation, 30(5), 1133-1159. https://doi.org/10.1007/s10726-021-09750-1

The Dr./*"Joaquín Sánchez Soriano"*, director, and the Dr./*"Ricardo Martínez Rico"*, codirector of the thesis entitled *"Contributions to cost allocation problems and scarce resources"*.

Inform,

That Mrs./*"María Teresa Estañ Pereña"* has carried out under our supervision the thesis entitled *"Contributions to cost allocation problems and scarce resources"* in accordance with the terms and conditions defined in its Research Plan and in accordance with the Code of Good Practices of the Universidad Miguel Hernández de Elche, fulfilling the objectives satisfactorily for its public defense as a doctoral thesis.

We sign for the appropriate purposes, in Elche, June 13, 2022.

The director, The codirector,

Dr./ Joaquín Sánchez Soriano. Dr./ Ricardo Martínez Rico.

Dr./ *"Domingo Morales González"*, coordinator of the academic committee of the PhD program in Statistics, Optimization and Applied Mathematics.

Inform,

That Mrs./*"María Teresa Estañ Pereña"* has carried out under the supervision of our Doctoral Program the thesis entitled *"Contributions to cost allocation problems and scarce resources"* in accordance with the terms and conditions defined in its Research Plan and in accordance with the Code of Good Practices of the Universidad Miguel Hernández de Elche, fulfilling the objectives satisfactorily for its public defense as a doctoral thesis.

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The coordinator of the Academic Committee of the PhD program in Statistics,
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Chapter 1

Acknowledgments

1.1 Agradecimientos

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Contents

Chapter 2

Resumen

Esta tesis está enmarcada dentro de la Teoría de Juegos, disciplina Matemática de gran relevancia en Economía por su alto grado de aplicabilidad en situaciones reales, como por ejemplo las derivadas del reparto de costes y/o beneficios o la distribución de recursos escasos, entre muchas otras. Una de las grandes referencias que da origen a esta rama de las Matemáticas es el libro *"Theory of Games and Economic Behavior"* de Oskar Morgenstern y John Von Neumann (Morgenstern and Von Neumann (1953)) al cual contribuyó de manera seminal con el desarrollo de los juegos múltiples el Premio Nobel John Nash.

El objetivo de esta tesis no es analizar cómo los individuos o agentes del juego toman sus decisiones sino proporcionar soluciones a los problemas planteados empleando procedimientos matemáticos que nos permiten diseñar diferentes mecanismos o reglas que satisfacen uno o un conjunto de propiedades, también llamadas axiomas, que caracterizan cada una de las reglas planteadas.

En este capítulo presentamos un resumen de todos los artículos que conforman esta tesis: **On how to allocate the fixed cost of transport systems** (Estañ et al. (2021a)),

Manipulability in the cost allocation of transport systems (Estañ et al. (2020)) y **On the difficulty of budget allocation in claims problems with indivisible items and prices** (Estañ et al. (2021b)), los dos primeros se centran en el estudio del reparto de costes y el tercero en la distribución de recursos escasos.

2.1 Problemas de reparto de costes

El primer bloque de esta tesis está formado por los artículos **On how to allocate the fixed cost of transport systems** (Estañ et al. (2021a)) y **Manipulability in the cost allocation of transport systems** (Estañ et al. (2020)). En ambos realizamos el estudio axiomático de un problema específico de reparto de costes, concretamente y como novedad nos centramos en el reparto del coste fijo derivado de una línea recta de tren formada por diferentes estaciones que pertenecen a un único municipio y además, si tenemos dos estaciones colindantes que pertenecen al mismo municipio entonces entre ellas no puede existir ninguna otra estación que pertenezca a un municipio distinto.

Como bien es conocido, cualquier construcción, como lo es en este caso la construcción de una línea de tren, tiene asociados unos costes como pueden ser los costes de construcción o una vez construida la red los costes de mantenimiento de dicha línea, entre otros. Estos costes pueden ser divididos en dos tipos: costes fijos y costes variables. Como su nombre indica los costes variables son aquellos que cambian en función de diversos elementos implícitos en la red, como por ejemplo podrían ser aquellos que derivan del uso de la red de transporte o del tamaño de la misma. Por otro lado, el coste fijo sería aquel que es invariable, pues no depende de los elementos de la red sino de la existencia de la misma, es decir, sería el coste que siempre existe y no cambia, como por ejemplo el coste de las cocheras donde se guardan los trenes.

En *Estañ et al. (2021a)* y *Estañ et al. (2020)* nos centramos en el reparto del coste fijo derivado de la línea de tren entre los diferentes municipios que la conforman desde el punto de vista axiomático. Los elementos de los que disponemos en nuestro modelo matemático son: el número de municipios $M = \{1, \ldots, m\}$ ($m \geq 3$), el número de estaciones $S = \{s_1, \ldots, s_n\}$, el flujo de pasajeros entre estaciones representado en una matriz OD y el montante del coste fijo a repartir $C \in \mathbb{R}_+$. Por tanto nuestro problema de reparto viene dado por la tupla $a = (M, S, OD, C)$.

Nuestro objetivo es diseñar mecanismos de reparto que sean lo más justos y sensatos posible en referencia a las diferentes propiedades planteadas dentro del marco del problema objeto de estudio. En concreto, hemos propuesto reglas de reparto basadas en el criterio de proporcionalidad, no obstante, como veremos más adelante no todas satisfacen los mismos axiomas y se requieren diferentes combinaciones de ellos para su caracterización.

La primera regla considerada es la **regla uniforme**, la cual sólo tiene en cuenta el coste fijo a repartir y el número total de municipios que participan en el juego, es decir, reparte el coste fijo de la línea de forma proporcional al número de municipios.

Regla uniforme. Para cada $a \in \mathbb{A}$ y cada $i \in M$,

$$U_i(a) = \frac{C}{m}.$$

Las dos siguientes reglas (**regla proporcional basada en el uso de las estaciones** y **regla proporcional basada en el uso de los tramos**) consideran el uso de la red para realizar la distribución del coste entre los agentes. En el caso de la **regla proporcional basada en el uso de las estaciones** el reparto se realiza proporcional al número de pasajeros que usan cada estación de un municipio y en el caso de la **regla proporcional basada en el uso de los tramos** asignamos el coste proporcionalmente al número de pasajeros que utiliza cada tramo de la red, para ello dividimos cada pasajero en tantas partes como tramos utiliza en su trayecto y cada una de esas partes es repartida entre las dos estaciones que delimitan cada tramo.

Regla proporcional basada en el uso de las estaciones. Para cada $a \in \mathbb{A}$ y cada $i \in M$,

$$SP_i(a) = \frac{C}{2\Omega(OD)} \cdot \Omega_i(OD).$$

Regla proporcional basada en el uso de los tramos. Para cada $a \in \mathbb{A}$ y cada $i \in M$,

$$TP_i(a) = \frac{C}{\Omega(OD)} \cdot \sum_{s_f \in S_i} \sum_{\substack{s_g, s_h \in S, s_g \neq s_h \\ f \in [g,h] \text{ or } f \in [h,g]}} \frac{\omega_{gh}}{\left(2 - \left\lceil \frac{|f-g| \cdot |h-f|}{(h-g)^2} \right\rceil\right) |h-g|},$$

donde $\lceil z \rceil = min\, \{k \in \mathbb{Z} : k \geq z\}$.

La última regla considerada (**regla proporcional al número de estaciones**) tiene presente el coste fijo y el número total de estaciones existentes en la red, así pues, distribuye el coste entre los jugadores proporcionalmente al número de estaciones que cada municipio posee.

Regla proporcional al número de estaciones. Para cada $a \in \mathbb{A}$ y cada $i \in M$,

$$R_i^{SP}(a) = \frac{C}{n} \cdot |S_i|.$$

En la literatura podemos encontrar un número extenso de requisitos que hacen que una regla sea más o menos deseable por los agentes. En los dos estudios que analizamos en esta sección: *Estañ et al. (2021a)* y *Estañ et al. (2020)*, proporcionamos diversas propiedades diferenciadas en cuatro bloques.

En el primer bloque incluimos todos aquellos requisitos basados en el principio de **justicia**. En primer lugar, el axioma **municipio nulo** establece que un municipio es nulo en el pago si ninguna de sus estaciones es utililzada, si además de lo anterior requerimos que ningún tren circule por dichas estaciones obtenemos la segunda propiedad llamada **municipio nulo débil**. Obviamente **municipio nulo** implica **municipio nulo débil** pero no al contrario.

Municipio nulo. Para cada $a \in \mathbb{A}$ y cada $i \in M$, si $\omega_{gh} = \omega_{hg} = 0$ para todo $s_g \in S_i$ y todo $s_h \in S$, entonces $R_i(a) = 0$.

Municipio nulo débil. Para cada $a \in \mathbb{A}$ y cada $i \in M$, si se cumple cualesquiera de las dos condiciones siguientes:

- $\omega_{gh} = \omega_{hg} = 0$, para todo $j \leq i$, para todo $s_g \in S_j$, y todo $s_h \in S$;
- $\omega_{gh} = \omega_{hg} = 0$, para todo $j \geq i$, para todo $s_g \in S_j$, y todo $s_h \in S$;

entonces $R_i(a) = 0$.

El segundo bloque está formado por aquellas propiedades que satisfacen el criterio de **equidad**. En el caso de **simetría** establecemos que dos municipios que tienen el mismo tráfico (flujo total de pasajeros) deberían contribuir igual, además si todo el tráfico se concentra solo en dos estaciones adyacentes que pertenecen a municipios diferentes la aportación al coste de dichos municipios será la misma tal y como establece la propiedad de **simetría adyacente**. Del mismo modo, **simetría en estaciones** establece que dos municipios que tienen el mismo número de estaciones se consideran simétricos y por ende su participación en el pago será la misma.

Simetría. Para cada $a \in \mathbb{A}$ y cada $\{i,j\} \subseteq M$, si $\Omega_{ii}(OD) = \Omega_{jj}(OD)$, y $\Omega_{ik}(OD) = \Omega_{jk}(OD)$, y $\Omega_{ki}(OD) = \Omega_{kj}(OD))$, para todo $k \in M \setminus \{i,j\}$. Entonces $R_i(a) = R_j(a)$.

Simetría adyacente. Para cada $a \in \mathbb{A}$ y cada $\{i,j\} \subseteq M$, si $\omega_{gh} + \omega_{hg} = \Omega(OD)$, tal que $|g-h| = 1$, y $g \in S_i, h \in S_j$, entonces $R_i(a) = R_j(a)$.

Simetría en estaciones. Para cada $a \in \mathbb{A}$ y cada $\{i,j\} \subseteq M$,si $|S_i| = |S_j|$. Entonces $R_i(a) = R_j(a)$.

El tercer bloque de axiomas está formado por características que requieren cierto tipo de **consistencia**. La propiedad **consistencia bilateral en ratio** establece que la ratio entre los pagos de dos municipios siempre es la misma, es decir, en el supuesto en el que todos los municipios salvo dos dejen de pertenecer a dicha línea entonces si reformulamos el problema teniendo en cuenta que ahora el conjunto de municipios solo está formado por dos, entonces la ratio entre los pagos de cada uno de estos dos municipios en el problema original es equivalente a la ratio entre los pagos de ambos en el problema reformulado. Ahora bien, ¿que ocurriría si se tuviese en cuenta si un pasajero decide realizar su trayecto en tramos, en vez de realizarlo directamente? Si una regla satisface el requisito de **descomposición del trayecto**, entonces el coste no se vería afectado si un trayecto largo se divide en pequeños tramos. Y si decidimos realizar el reparto de costes mensualmente, ¿el computo anual de las mensualidades sería el mismo que si realizamos el reparto por anualidades? la respuesta a esta pregunta es sí para todas aquellas reglas que satisfagan **aditividad** y/o **aditividad ponderada**, pues no dependerán del flujo.

Consistencia bilateral en ratio. Para cada $a = (M, S, OD, C) \in \mathbb{A}$ y cada par de municipios $\{i, j\} \subseteq M$ tenemos que

$$\frac{R_i(a)}{R_j(a)} = \frac{R_i(a_{\{i,j\}})}{R_j(a_{\{i,j\}})},$$

donde $a_{\{i,j\}} = \big(\{i, j\}, S_i \cup S_j, OD_{\{i,j\}}, C\big)$.

Descomposición del trayecto. Para cada $(M, S, OD, C), (M, S, OD', C) \in \mathbb{A}$. Si $s_g, s_h \in S$, son estaciones tales que $h - g > 1$, y dado

1. $\omega'_{g(g+1)} = \omega_{g(g+1)} + \frac{\omega_{gh}}{|h-g|}; \omega'_{(g+1)(g+2)} = \omega_{(g+1)(g+2)} + \frac{\omega_{gh}}{|h-g|}, \ldots, \omega'_{(h-1)h} = \omega_{(h-1)h} + \frac{\omega_{gh}}{|h-g|}$; y $\omega'_{gh} = 0$;

2. $\omega'_{ef} = \omega_{ef}$, si $(ef) \neq (gh)$,

o

1. $\omega''_{h(h-1)} = \omega_{h(h-1)} + \frac{\omega_{hg}}{|h-g|}; \omega''_{(h-1)(h-2)} = \omega_{(h-1)(h-2)} + \frac{\omega_{hg}}{|h-g|}, \ldots, \omega'_{(g+1)g} = \omega_{(g+1)g} + \frac{\omega_{hg}}{|h-g|}$; and $\omega''_{hg} = 0$;

2. $\omega''_{ef} = \omega_{ef}$, si $(ef) \neq (hg)$,

entonces, $R(M, S, OD, C) = R(M, S, OD', C)$ y $R(M, S, OD, C) = R(M, S, OD'', C)$.

Aditividad. Para cada $(M, S, OD, C) \in \mathbb{A}$ y cada $i \in M$,

$$R_i\left(M, S, OD, C\right) = \sum_{t=1}^{T} R_i\left(M, S, OD_t, C_t\right),$$

donde $OD = \sum_{t=1}^{T} OD_t$ y $C = \sum_{t=1}^{T} C_t$.

Aditividad ponderada. Para cada $(M, S, OD, C) \in \mathbb{A}$ y cada $i \in M$,

$$\frac{\Omega(OD)}{C} R_i\left(M, S, OD, C\right) = \sum_{t=1}^{T} \frac{\Omega(OD_t)}{C_t} R_i\left(M, S, OD_t, C_t\right),$$

donde $OD = \sum_{t=1}^{T} OD_t$ y $C = \sum_{t=1}^{T} C_t$.

Para finalizar proponemos un cuarto bloque compuesto de dos propiedades que garantizan que una regla es inmune frente a la manipulación de los agentes, es decir, si los municipios deciden unirse y actuar como uno solo, **no manipulabilidad por unión**, o por el contrario, su dual, si deciden separase y actuar como varios municipios, **no manipulabilidad por separación**. Estos requisitos han sido utilizados en varios trabajos como por ejemplo *de Frutos (1999)*, *Ju et al. (2007)* y *Moulin (2008).*

No manipulabilidad por unión: Para cada par M, M' tal que $M' \subset M$, cada $(M, S, OD, C) \in \mathbb{A}$, y cada $(M', S', OD, C) \in \mathbb{A}$. Si exite $i \in M'$ tal que $S'_i = S_i \cup \bigcup_{j \in M \setminus M'} S_j$, y para cada $j \in M' \setminus \{i\}$, $S'_j = S_j$, entonces $R_i(M', S', OD, C) \geq R_i(M, S, OD, C) + \sum_{j \in M \setminus M'} R_j(M, S, OD, C)$.

No manipulabilidad por separación: Para cada par M, M' tal que $M' \subset M$, cada $(M, S, OD, C) \in \mathbb{A}$, y cada $(M', S', OD, C) \in \mathbb{A}$. Si existe $i \in M'$ tal que $S'_i = S_i \cup \bigcup_{j \in M \setminus M'} S_j$, y para cada $j \in M' \setminus \{i\}$, $S'_j = S_j$, entonces $R_i(M', S', OD, C) \leq R_i(M, S, OD, C) + \sum_{j \in M \setminus M'} R_j(M, S, OD, C)$.

A continuación listamos los resultados obtenidos tras el estudio de la caracterización de las reglas planteadas en los trabajos de reparto de costes *Estañ et al. (2021a)* y *Estañ et al. (2020)* cuyas demostraciones podemos ver de manera detallada en los propios artículos, los cuales se encuentran anexados en el Apéndice.

El primer teorema establece que si requerimos las propiedades de **simetría** y **aditividad**, entonces el coste ha de repartirse uniformemente entre los municipios.

Teorema 1. *Una regla satisface simetría y adivitividad si y solo si es la regla uniforme.*

Sin embargo, si requerimos que una regla satisfaga las propiedades de **simetría**, **consistencia bilateral en ratio** y **aditividad ponderada**, entonces la única regla que satisface este conjunto de axiomas es aquella que **reparte el coste equitativamente entre los municipios** sin considerar otros factores del problema como, por ejemplo, el uso o el número de estaciones como así se expone en el siguiente teorema.

Teorema 2. *Una regla satisface simetría, consistencia bilateral en ratio y aditividad ponderada si y solo si es la regla uniforme.*

El tercer resultado obtenido establece que la única regla que satisface el conjunto de axiomas de **municipalidad nula**, **simetría** y **simetría ponderada** es la **regla que divide el coste proporcionalmente al flujo de pasajeros por municipios**.

Teorema 3. *Una regla satisface municipio nulo, simetría y simetría ponderada si y solo si es la regla proporcional basada en el uso de las estaciones.*

El siguiente teorema establece que la **regla proporcional basada en el uso de los tramos** se caracteriza por los axiomas siguientes: **simetría adyacente**, **municipio nulo débil**, **descomosición del trayecto** y **aditividad ponderada**.

Teorema 4. *La única regla que satisface simetría adyacente, municipio nulo débil, descomposición del trayecto y aditividad ponderada es la regla proporcional basada en el uso de los tramos.*

El último resultado de este bloque, establece que el conjunto de propiedades formado por **simetría en estaciones**, **no manipulabilidad por unión** y **no manipulabilidad por separación** conducen a una distribución del coste que es proporcional al número de estaciones que tiene cada ciudad.

Teorema 5. *Una regla satisface simetría en estaciones, no manipulabilidad por unión y no manipulabilidad por separación si y solo si es la regla proporcional al número de estaciones.*

2.2 Problemas de distribución de recursos escasos

El segundo bloque de esta tesis se enmarca dentro de los problemas de asignación de recursos escasos. En nuestro trabajo **On the difficulty of budget allocation in claims problems with indivisible items and prices** (Estañ et al. (2021b)) presentamos una nueva situación: el estudio de la clase de problemas de reparto de recursos escasos donde consideramos que el monto a dividir es perfectamente divisible y se realizan demandas de unidades indivisibles de varios artículos. Cada artículo tiene un precio y la cantidad disponible es insuficiente para poder cubrir todas las demandas a los precios indicados.

En el modelo matemático que presentamos en este trabajo, el problema a resolver representa una situación en la que una cantidad perfectamente divisible, $E \in \mathbb{R}_{++}$ (llamada **presupuesto**) debe distribuirse entre los agentes en N de acuerdo con sus demandas. Esas demandas vienen dadas en una matriz de demandas $c \in \mathbb{Z}_+$ que tiene tantas filas como agentes y tantas columnas como elementos:

$$c = \begin{pmatrix} c_{11} & c_{12} & \dots & c_{1h} \\ c_{21} & c_{22} & \dots & c_{2h} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \dots & c_{nh} \end{pmatrix},$$

donde $c_{ig} \in \mathbb{Z}_+$ indica la cantidad del artículo g reclamado por el agente i.

En cualquier problema de bancarrota, el presupuesto o bien a repartir no es suficiente para cubrir completamente todas las demandas, es decir, $\sum_{i=1}^{n} \sum_{g=1}^{h} c_{ig} p_g \geq E$.

Por tanto, en nuestro trabajo *Estañ et al. (2021b)*, el problema que presentamos viene dado por la tupla $a = (N, H, p, c, E)$, donde $N = \{1, \dots, n\}$ es el conjunto de agentes, $H = \{1, \dots, h\}$ es el conjunto de elementos posibles cuyos precios vienen dados por $p = (p_1, \dots, p_h) \in \mathbb{R}_+^h$, c es la matriz de demandas y E es el presupuesto. Dado que los elementos N, H y p son fijos en todo el documento, cuando no surja confusión alguna, simplemente escribiremos el problema de distribución de recursos escasos como el problema reducido $a = (c, E)$. La novedad que presentamos en este trabajo es la respuesta a la pregunta ¿por qué deberíamos gastar todo el presupuesto que tenemos? Obsérvese que en los modelos clásicos sobre problemas de bancarrota se impone este hecho, es decir, hay que agotar todo el presupuesto de que se dispone.

Sea $\mathbb{A}$ el conjunto de todos los problemas:

$$\mathbb{A} = \left\{ a = (c, E) \in \mathbb{Z}_+^{n \times h} \times \mathbb{R}_{++} : \|c \cdot p\| = \sum_{i=1}^{n} \sum_{g=1}^{h} c_{ig} p_g \geq E \right\}.$$

Una **asignación o reparto** para $a \in \mathbb{A}$ es una distribución del presupuesto entre los agentes que especifica cuántos artículos de cada precio se otorgan a cada agente. Por lo tanto, es una matriz $x \in \mathbb{Z}_+^{n \times h}$ que satisface las siguientes dos condiciones:

(a) Cada agente recibe una cantidad no negativa de cada tipo de artículo, que no es mayor que su demanda:

$$0 \leq x_{ig} \leq c_{ig} \quad \text{para todo } i \in N \text{ y todo } g \in H.$$

(b) El coste total no excede el presupuesto disponible:

$$\|x \cdot p\| = \sum_{i=1}^{n} \sum_{g=1}^{h} x_{ig} p_g \leq E.$$

Para solucionar este problema proponemos varias reglas. En particular, una **regla** es una correspondencia, $R : \mathbb{A} \rightrightarrows \mathbb{Z}_+^{n \times h}$, que selecciona, para cada problema $a \in \mathbb{A}$, un subconjunto no vacío de asignaciones $R(a) \subseteq X(a)$.

La primera regla que introducimos es útil desde un punto de vista teórico y establece que la cantidad que recibe cada agente es cero.

Regla nula, R^N. Para cada $a \in \mathbb{A}$ y cada $x \in X(a)$,

$$x \in R^N(a) \;\Leftrightarrow\; x_{ig} = 0 \; \forall i \in N \text{ y } \forall g \in H.$$

La siguiente regla que proponemos es el caso contrario a la anterior, ya que selecciona todo el conjunto de asignaciones $X(a)$.

Regla laxa, R^G. Para cada $a \in \mathbb{A}$,

$$R^G(a) = X(a).$$

A continuación enumeramos dos reglas que se basan en la **prioridad**. En la primera de estas asignaciones, **regla de prioridad en la llegada agente-artículo**, los agentes con mayor prioridad se satisfacen antes que los de menor prioridad. Además, para cada agente, los artículos más relevantes se dan primero en su totalidad. Es decir, sea $\succ_N$ un orden en el conjunto de agentes N, donde $i \succ_N j$ significa que i tiene prioridad sobre j y sea $\succ_H$ un orden en el conjunto de elementos H, donde $f \succ_H g$ significa que f tiene prioridad sobre g. Esta regla es tal que los agentes demandan según el orden $\succ_N$ comenzando cada uno con los elementos con la prioridad más alta en $\succ_H$. Este proceso continúa hasta que, eventualmente, el presupuesto se agota. La segunda de estas reglas se llama **regla de prioridad en la llegada del agente** y consiste en: dado un orden $\succ_N$ en el conjunto de agentes, los agentes son satisfechos de acuerdo a dicho orden. El primer agente selecciona el conjunto de artículos que maximiza el valor de su elección sujeto al presupuesto limitado dado por E. Sea E^1 el presupuesto restante. Ahora, el segundo agente selecciona el conjunto de artículos para maximizar el valor de su elección sujeto al presupuesto restringido dado por E^1. Continuamos el proceso hasta que el presupuesto, eventualmente, se agota.

Regla de prioridad en la llegada agente-artículo, R^{AIPA}. Para cada $a \in \mathbb{A}$ y cada $x \in X(a)$,

$$x \in R^{AIPA}(a) \;\Leftrightarrow\; [x_{ig} > 0 \Rightarrow x_{if} = c_{if} \, \forall f \succ_H g \text{ y } x_{jf} = c_{jf} \, \forall j \succ_N i \, \forall f \in H] \, .$$

Regla de prioridad en la llegada del agente, R^{APA}. Para cada $a \in \mathbb{A}$ y cada $x \in X(a)$,

$$x \in R^{APA}(a) \;\Leftrightarrow\; [x_{jg} > 0 \Rightarrow x_{ig} = c_{ig} \, \forall i \succ_N j] \, .$$

En este punto presentamos dos reglas que son un **proceso de dos pasos**: la **regla de igualdad por artículo** y la **regla de igualdad por agente**.

En la **regla de igualdad por artículo**, primero, el presupuesto se divide en partes iguales entre los artículos ($\frac{E}{h}$ para cada uno). Y en segundo lugar, para cada artículo, se asignan cantidades lo más iguales posible a todos los demandantes, sujeto a que nadie reciba más que su reclamación. Obsérvese que el segundo paso de este procedimiento está estrechamente relacionado con algunas otras extensiones en entornos con indivisibilidades de las llamadas *reglas de premios iguales restringidas (CEA)* (Herrero and Martínez (2008a) y Chen (2015)).

La segunda regla, **regla de igualdad por agente** se obtiene aplicando el proceso de dos pasos de la regla de premios iguales restringida por artículo pero a los agentes. Para cada agente, seleccionamos un conjunto de artículos cuyo precio es menor o igual a $\frac{E}{n}$ y tal que al agregar un nuevo artículo el precio es mayor que $\frac{E}{n}$. Posteriormente, el presupuesto restante se asigna a cualquier conjunto de agentes que gaste tanto como sea posible.

Regla de igualdad por artículo, R^{EI}. Para cada $a \in \mathbb{A}$ y cada $x \in X(a)$,

$$x \in R^{EI}(a) \;\Leftrightarrow\; \begin{cases} |x_{ig} - x_{jg}| \leq 1 \text{ para todo } i, j \in N \\ p_g \left(\sum_{i=1}^{n} x_{ig}\right) \leq \frac{E}{h} \\ p_g \left(1 + \sum_{i=1}^{n} x_{ig}\right) > \frac{E}{h}. \end{cases}$$

Regla de igualdad por agente, R^{EA}. Para cada $a \in \mathbb{A}$ y cada $x \in X(a)$,

$$x = z + y \in R^{EA}(a) \;\Leftrightarrow\; \begin{cases} z, y \in \mathbb{Z}_+^{n\times h} \\ \sum_{g=1}^{h} p_g z_{ig} \leq \frac{E}{n}, \forall i \in N \\ \sum_{g=1}^{h} p_g z'_{ig} > \min\{\sum_{g=1}^{h} p_g c_{ig}, \frac{E}{n}\}, \forall z' > z, \forall i \in N \\ \sum_{i=1}^{n} \sum_{g=1}^{h} p_g y_{ig} \leq E - \|z \cdot p\| \\ \sum_{i=1}^{n} \sum_{g=1}^{h} p_g y_{ig} \geq \sum_{i=1}^{n} \sum_{g=1}^{h} p_g y'_{ig}, \forall y' \in X(a'), \end{cases}$$

donde $z' > z$ significa que hay al menos una celda ig tal que $z'_{ig} > z_{ig}$, y los demás son mayores o iguales y $a' = (N, H, p, c - z, E - \|x \cdot p\|)$.

A continuación presentamos tres requisitos mínimos que debe satisfacer una regla, los cuales son bastante estándar en la literatura sobre problemas de reparto de recursos escasos donde las restricciones que se imponen en este trabajo (Estañ et al. (2021b)) suelen ser compatibles. La primera propiedad, **no despilfarro**, estipula que en una situación de racionamiento debemos desperdiciar lo menos posible. La segunda propiedad, **débil trato igualitario de iguales**, es un criterio mínimo de equidad y establece que los agentes con iguales derechos deben recibir el mismo trato y finalmente, la última propiedad, **no manipulabilidad por fusión o división**, hace que la regla sea inmune a ciertas manipulaciones por parte de los agentes. En resumen, **la eficiencia, la equidad y la no manipulabilidad** serán los requisitos básicos que imponemos como punto de partida.

No despilfarro. Para cada $a \in \mathbb{A}$, si $x \in R(a)$, entonces no hay otro reparto $x' \in X(a)$ tal que $E - \|x' \cdot p\| < E - \|x \cdot p\|$.

Débil trato igualitario de iguales. Para cada $a \in \mathbb{A}$ y cada $\{i, j\} \subseteq N$, si $c_{ig} = c_{jg} \forall g \in H$, entonces para todo $x \in R(a)$ tenemos

- para todo $g \in H$, $|x_{ig} - x_{jg}| \leq 1$, y
- para cada $g \in H$, existe $x' \in R(a)$, tal que $x'_{ig} = x_{jg}$, $x'_{jg} = x_{ig}$ y el resto de celdas x' son iguales que x.

No manipulabilidad por fusión o división. Para cada $(N, c, E), (N', c', E) \in \mathbb{A}$ con $N' \subset N$, si existe $i \in N'$ tal que satisface las dos condiciones siguientes

1. $c'_{ig} = c_{ig} + \sum_{j \in N \setminus N'} c_{jg}$ para todo $g \in H$
2. $c'_{jg} = c_{jg}$ para todo $j \in N' \setminus \{i\}$ y para todo $g \in H$,

entonces

(a) $\forall x' \in R(N', c', E)$ existe $x \in R(N, c, E)$ tal que $x'_{ig} = x_{ig} + \sum_{j \in N \setminus N'} x_{jg}$ $\forall g \in H$.

(b) $\forall x \in R(N, c, E)$ existe $x' \in R(N', c', E)$ tal que $x'_{ig} = x_{ig} + \sum_{j \in N \setminus N'} x_{jg}$ $\forall g \in H$.

En este trabajo exploramos la compatibilidad de las tres propiedades presentadas.

Dado un problema $a \in \mathbb{A}$, consideremos el siguiente problema de programación lineal entera (PLE):

$$\left.\begin{aligned} &\min_{x \in \mathbb{Z}_+^{n \times h}} E - \sum_{i=1}^{n} \sum_{g=1}^{h} p_g x_{ig} \\ &\text{s.a.:} \ \sum_{i=1}^{n} \sum_{g=1}^{h} p_g x_{ig} \leq E \\ &0 \leq x_{ig} \leq c_{ig}, \ \forall i \in N, \forall g \in H \end{aligned}\right\}$$

o equivalentemente,

$$\left.\begin{aligned} &\max_{x \in \mathbb{Z}_+^{n \times h}} \sum_{i=1}^{n} \sum_{g=1}^{h} p_g x_{ig} \\ &\text{s.a.:} \ \sum_{i=1}^{n} \sum_{g=1}^{h} p_g x_{ig} \leq E \\ &0 \leq x_{ig} \leq c_{ig}, \ \forall i \in N, \forall g \in H \end{aligned}\right\} \tag{2.1}$$

Denotemos por $PLE(a)$ el conjunto de todas las soluciones óptimas para el problema de programación lineal entera definido por (2.1) que pertenece a la clase de problemas de mochila acotados. Observamos que una regla R satisface **no despilfarro** si es una selección de soluciones óptimas del problema que presentamos anteriormente, es decir, $R(a) \subseteq PLE(a)$ para todo $a \in \mathbb{A}$. Observe que las restricciones $0 \leq x_{ig} \leq c_{ig}$, $\forall i \in N, \forall g \in H$ restringen los posibles valores de las variables de decisión y, por lo tanto, el problema de la mochila está acotado.

Los resultados obtenidos tras el estudio son los siguientes:

Teorema 1. *Hay reglas que satisfacen no despilfarro, débil trato igualitario de iguales y no manipulabilidad por fusión o división, si y solo si, hay reglas que satisfacen esas propiedades para la subclase de problemas con* $|H| = 1$.

Teorema 2. *No existe una regla que satisfaga no despilfarro, débil trato igualitario de iguales y no manipulabilidad por fusión o división.*

Proposición 1. *Hay reglas que satisfacen no despilfarro y débil trato igualitario de iguales.*

Proposición 2. *Hay reglas que satisfacen no despilfarro y no manipulabilidad por fusión o división.*

Proposición 3. *Hay reglas que satisfacen débil trato igualitario de iguales y no manipulabilidad por fusión o división a la vez.*

Por otro lado, pensamos que podría ser interesante estudiar la compatibilidad entre la condición de **no despilfarro** y otras propiedades estándar requeridas para resolver problemas de reparto (distribución) de recursos escasos. En particular, nos centramos en aquellos requisitos que protegen a los pequeños agentes, para ello hemos seleccionado cinco propiedades que consideramos relevantes para este estudio: **exención**, **compensación total condicional**, **aseguramiento**, **aseguramiento débil** y la última propiedad **autodualidad**.

Exención. Para cada $a \in \mathbb{A}$ y cada $i \in N$, si

$$n \cdot \left(\sum_{g=1}^{h} p_g c_{ig} \right) \leq E,$$

entonces, para cualquier $x \in R(a)$, $x_{ig} = c_{ig}$ $\forall g \in H$.

Compensación total condicional. Para cada $a \in \mathbb{A}$ y cada $i \in N$, si

$$\sum_{j \in N_i^-} \sum_{g=1}^{h} p_g c_{jg} + (n - |N_i^-|) \sum_{g=1}^{h} p_g c_{ig} \leq E,$$

entonces, para cualquier $x \in R(a)$, $x_{ig} = c_{ig}$ $\forall g \in H$, donde $N_i^- = \left\{ j \in N : \sum_{g=1}^{h} p_g c_{jg} < \sum_{g=1}^{h} p_g c_{ig} \right\}$.

Observamos que la **exención** implica la **compensación total condicional** y ambas propiedades coinciden cuando $|N| = 2$.

Aseguramiento. Para cada $a \in \mathbb{A}$, cada $x \in R(a)$, y cada $i \in N$

$$\sum_{g=1}^{h} p_g x_{ig} \geq \frac{1}{n} \min \left\{ \sum_{g=1}^{h} p_g c_{ig}, E \right\}, \quad \forall i \in N.$$

Aseguramiento débil. Para cada $a \in \mathbb{A}$, cada $x \in R(a)$, y cada $i \in N$

$$\sum_{g=1}^{h} p_g x_{ig} \geq \max_{y \in X(a)} \left\{ \sum_{g=1}^{h} p_g y_{ig} \,\middle|\, \sum_{g=1}^{h} p_g y_{ig} \leq \frac{1}{n} \min \left\{ \sum_{g=1}^{h} p_g c_{ig}, E \right\} \right\}, \quad \forall i \in N.$$

Auto-dualidad. Para cada $a \in \mathbb{A}$ tenemos que $R(a) = c - R(a^d)$.

Los resultados obtenidos cuando exploramos la combinación de **no despilfarro** y los requisitos mencionados anteriormente son los siguientes:

Teorema 3. *No existe una regla que satisfaga no despilfarro y compensación total condicional.*

El siguiente resultado muestra que cuando se requiere **no despilfarro** junto con **seguridad débil** también surge una *imposibilidad.*

Teorema 4. *No existe una regla que satisfaga simultánemente no despilfarro y seguridad débil.*

Proposición 4. *Si una regla satisface la propiedad de auto-dualidad, agota el presupuesto.*

El recíproco de la Proposición 4 no es cierto en general. Por ejemplo, si consideramos la clase de problemas con $H = \{1\}$ y $p_1 = 1$ y E un número entero positivo, entonces la *regla discreta de premios iguales restringidos* (véase, por ejemplo, Herrero and Martínez (2008a)) siempre agota el presupuesto pero no satisface **auto-dualidad**. Una consecuencia inmediata de la Proposición 4 es que no pueden existir reglas que satisfagan la propiedad de **auto-dualidad**, ya que ninguna regla, en general, puede agotar siempre el presupuesto. Nótese que, a diferencia de los otros resultados de esta sección, la falta de reglas auto-duales es absoluta, y el principio de **no despilfarro** no juega ningún papel en eso, sin embargo la propiedad de **no despilfarro** no es compatible con la *eficiencia*, propiedad clásica utilizada en los problemas de racionalización. Por esta razón presentamos una alternativa a **no despilfarro**: **Pareto eficiencia**. En contraste con no despilfarro, esta propiedad se centra en las asignaciones de los agentes más que en el gasto del presupuesto. Una asignación satisface **Pareto eficiencia** si no hay otra asignación en la que algún otro individuo esté mejor y ningún individuo esté en peor situación.

Pareto eficiencia. Dado $a \in \mathbb{A}$, si $x \in R(a)$ entonces no hay otra asignación $x' \in X(a)$ tal que $\sum_{g \in H} p_g x'_{ig} \geq \sum_{g \in H} p_g x_{ig}, \forall i \in N$, con al menos una desigualdad estricta.

Dado $a \in \mathbb{A}$, denotamos por $P(a) \subset X(a)$ el conjunto de soluciones que cumplen **Pareto eficiencia**.

Es obvio que **no despilfarro** implica **Pareto eficiencia**, pero lo contrario no es cierto. Aunque estas dos propiedades no son equivalentes en general, no es difícil probar que coinciden cuando $|H| = 1$. Como consecuencia, podemos reemplazar **no despilfarro** por **Pareto eficiencia** en el Teorema 1, lo que implica que el **trato débil igualitario de iguales** y la **no manipulabilidad por fusión o división** son incompatibles con la propiedad **Pareto eficiencia**. El siguiente resultado es análogo al Teorema 2.

Teorema 5. *No existe una regla que satisfaga la Pareto eficiencia, el trato débil igualitario de iguales y no manipulabilidad por fusión o división.*

Dado que la **Pareto eficiencia** es más débil que el **no despilfarro**, obtenemos las contrapartes de las Proposiciones 1 y 2.

Proposición 5. *Hay reglas que satisfacen la Pareto eficiencia y la igualdad de trato débil de los iguales.*

Proposición 6. *Hay reglas que satisfacen la Pareto eficiencia y la no manipulabilidad por fusión o división.*

Con respecto a la **auto-dualidad**, es evidente que no será compatible con la **Pareto eficiencia**, ya que esta última no garantiza que el presupuesto se agote por completo. Los Teoremas 3 y 4 establecen que la **compensación total condicional** y la **seguridad débil** son incompatibles con la de **no despilfarro**. Sin embargo, los dos resultados siguientes muestran que, si el último requisito se debilita a la **Pareto eficiencia**, entonces surge la *compatibilidad*.

Teorema 6. *Hay reglas que satisfacen la Pareto eficiencia y la compensación total condicional.*

Teorema 7. *Hay reglas que satisfacen la Pareto eficiencia y el aseguramiento débil simultáneamente.*

Por lo tanto, la **Pareto eficiencia** es una propiedad menos exigente y puede ser compatible con otras propiedades razonables. Además, podemos definir reglas que satisfagan varias de las propiedades presentadas en esta sección. Por ejemplo, la regla R^{CS} definida a continuación

$$R^{CS}(a) = x^0 + R^S(a'), \forall a \in \mathbb{A},$$

donde $a' = (N, H, p, c - x^0, E - \|x^0 \cdot p\|)$, satisface **Pareto eficiencia**, **compensación total condicional** y **aseguramiento débil**.

O como por ejemplo la regla R^{CES} definida a continuación.

Para cada, $a \in \mathbb{A}$,

$$R^{CES}(a) = R^{CS} \cap E(a)$$

Esta regla satisface la **Pareto eficiencia**, el **trato igualitario débil de iguales**, la **compensación total condicional** y el **aseguramiento débil**.

El siguiente resultado establece que cualquier regla que satisfaga la **Pareto eficiencia**, el **trato igual débil de iguales**, la **compensación total condicional** y **aseguramiento débil** debe ser una subselección de R^{CES}.

Teorema 8. *Si una regla R satisface la Pareto eficiencia, la igualdad de trato débil de iguales, la compensación total condicional y el aseguramiento débil, entonces $R(a) \subset R^{CES}(a)$, $\forall a \in \mathbb{A}$.*

Chapter 3

Summary

This thesis is framed within Game Theory, a Mathematical discipline of great relevance in Economics due to its high degree of applicability in real situations, such as for example, those derived from the allocation of costs and/or benefits or the distribution of scarce resources, among others. One of the great references that gives rise to this branch of Mathematics is the book *"Theory of Games and Economic Behavior"* by Oskar Morgenstern and John Von Neumann (Morgenstern and Von Neumann (1953)) to which the Nobel Laureate John Nash contributed with the development of multiple games.

The objective of this thesis is not to analyze how the individuals or agents of the game make their decisions, otherwise to provide solutions to the problems proposed using mathematical procedures that allow us to design different mechanisms or rules that satisfy one or a set of properties, also called axioms, that characterize each one of the proposed rules.

In this chapter we present a summary of all the papers that make up this thesis: **On how to allocate the fixed cost of transport systems** (Estañ et al. (2021a)), **Manipulability in the cost allocation of transport systems** (Estañ et al. (2020)) and **On the difficulty of budget allocation in claims problems with indivisible items and prices** (Estañ et al. (2021b)), the first two are focused on the study of cost sharing and the third on the distribution of scarce resources.

3.1 Cost sharing problems

The first block of this thesis is made up of the papers **On how to allocate the fixed cost of transport systems** (Estañ et al. (2021a)) and **Manipulability in the cost allocation of transport systems** (Estañ et al. (2020)). In both, we carry out the axiomatic study of a specific problem of cost distribution, specifically and as a novelty, we focus on the distribution of the fixed cost derived from a straight tram line formed by different stations belonging to a single municipality and also, if we have two adjacent stations that belong to the same municipality, then between them there cannot be any other station that belongs to a different municipality.

As it is well known, any construction, such as the construction of a tram line in this case, has associated costs such as construction costs or, once built, the maintenance costs of the line, among others. These costs can be divided into two types: fixed costs and variable costs. As its name indicates, variable costs are those that change as a function of various elements implicit in the network, such as, for example, they could be those that derive from the use of the transport network or its size. On the other hand, the fixed cost is invariable, that is, the cost that does not depend on the elements of the network but on its existence, such as the cost of the garages where the trams are stored.

In *Estañ et al. (2021a)* and *Estañ et al. (2020)* we are focused on the distribution of the fixed cost derived from the tram line between the different municipalities that comprise it from the axiomatic point of view.

The elements of our mathematical model are: the number of municipalities $M = \{1, \dots, m\}$ $(m \geq 3)$, the number of stations $S = \{s_1, \dots, s_n\}$, the passengers flow between stations represented in a matrix OD and the amount of the fixed cost to be shared $C \in \mathbb{R}_+$. Therefore our distribution problem is given by $a = (M, S, OD, C)$.

Our objective is to design mechanisms to distribute the cost as fair and feasible as possible in reference to the different properties raised within the framework of the problem under study. Specifically, we have proposed distribution rules based on the proportionality criterion, however, as we will see later, not all of them satisfy the same axioms and different combinations of them are required for their characterization.

The first rule considered is the **uniform rule**, which only takes into account the fixed cost to be distributed and the total number of municipalities that participate in the game, that is, it distributes the fixed cost of the line proportional to the number of municipalities.

Uniform rule. For each $a \in \mathbb{A}$ and each $i \in M$,

$$U_i(a) = \frac{C}{m}.$$

The following two rules (**station-based proportional rule** and **track-based proportional rule**) consider the use of the network to allocate the cost among the agents, in the case of **proportional rule based on the use of stations** the distribution is proportional to the number of passengers that use each station in a municipality and in the case of the **proportional rule based on the use of the sections** we assign the cost proportionally to the number of passengers that use each section of the network, to do this, we divide each passenger into as many parts as sections used in their journey and each of these parts is distributed between the two stations that define each section.

Station-based proportional rule. For each $a \in \mathbb{A}$ and each $i \in M$,

$$SP_i(a) = \frac{C}{2\Omega(OD)} \cdot \Omega_i(OD).$$

Track-based proportional rule. For each $a \in \mathbb{A}$ and each $i \in M$,

$$TP_i(a) = \frac{C}{\Omega(OD)} \cdot \sum_{s_f \in S_i} \sum_{\substack{s_g, s_h \in S, s_g \neq s_h \\ f \in [g,h] \text{ or} f \in [h,g]}} \frac{\omega_{gh}}{\left(2 - \left\lceil \frac{|fg| \cdot |hf|}{(h-g)^2} \right\rceil\right) |hg|},$$

where $\lceil z \rceil = min \{k \in \mathbb{Z} : k \geq z\}$

The last rule considered (**station proportional rule**) takes into account the fixed cost and the total number of the stations in the network, thus it distributes the cost between the players proportionally to the number of the stations that each municipality has.

Station proportional rule. For each $a \in \mathbb{A}$ and each $i \in M$,

$$R_i^{SP}(a) = \frac{C}{n} \cdot |S_i|.$$

In the literature we can find an extensive number of requirements that decide if a rule is more or less desirable by the agents. In the two papers that we study in this section: *Estañ et al. (2021a)* and *Estañ et al. (2020)*, we provide several properties that have been distributed in four blocks.

In the first block we include all those requirements based on the principle of **justice**. In the first place, the axiom **null municipality** establishes that a municipality is null in the payment if none of its stations is used, if in addition to the above we require that no tram circulates through these stations, we obtain the second property named **weak null municipality**. Obviously **null municipality** implies **weak null municipality** but not in the other way.

Null municipality. For each $a \in \mathbb{A}$ and each $i \in M$, if $\omega_{gh} = \omega_{hg} = 0$ for all $s_g \in S_i$ and all $s_h \in S$, then $R_i(a) = 0$.

Weak null municipality. For each $a \in \mathbb{A}$ and each $i \in M$, if one of the following two conditions holds

- $\omega_{gh} = \omega_{hg} = 0$, for all $j \leq i$, for all $s_g \in S_j$, and all $s_h \in S$;
- $\omega_{gh} = \omega_{hg} = 0$, for all $j \geq i$, for all $s_g \in S_j$, and all $s_h \in S$;

then $R_i(a) = 0$.

The second block of axioms include those properties that satisfy an **equity** criterion. In the case of **symmetry** we say that two municipalities that have the same traffic (total passenger flow) should contribute the same, also if all the traffic is concentrated only in two adjacent stations that belong to different municipalities, the contribution to the cost of those municipalities will be the same as it is established by **adjacent symmetry**. In the same way, **symmetry in stations** says that two municipalities that have the same number of stations are considered symmetrical and therefore their participation in the payment will be the same.

Symmetry. For each $a \in \mathbb{A}$ and each $\{i, j\} \subseteq M$, if $\Omega_{ii}(OD) = \Omega_{jj}(OD)$, and $\Omega_{ik}(OD) = \Omega_{jk}(OD)$, and $\Omega_{ki}(OD) = \Omega_{kj}(OD))$, for all $k \in M \setminus \{i, j\}$. Then $R_i(a) = R_j(a)$.

Adjacent symmetry. For each $a \in \mathbb{A}$ and each $\{i, j\} \subseteq M$, if $\omega_{gh} + \omega_{hg} = \Omega(OD)$, such that $|gh| = 1$, and $g \in S_i, h \in S_j$, then $R_i(a) = R_j(a)$.

Symmetry in stations. For each $a \in \mathbb{A}$ and each $\{i, j\} \subseteq M$, if $|S_i| = |S_j|$. Then $R_i(a) = R_j(a)$.

The third block of axioms contains the properties that require a certain type of **consistency**. The **bilateral ratio consistency** property establishes that the ratio between the payments of two municipalities is always the same. That is, in the assumption in which all the municipalities except two cease to belong to the line, if we reformulate the problem taking into account that now the set of municipalities is formed only by two, then the ratio between the payments of each of these two municipalities in the original problem is equivalent to the ratio between the payments of both in the reformulated problem. Now, what would happen if we have into account whether a passenger decides to do his journey by sections, instead of doing it directly? If a rule satisfies the axiom **trip decomposition**, then the cost would not be affected if a long journey is divided into small ones. And if we decide to do the cost sharing monthly, would the annual calculation be the same as if we carry out the allocation by annuities? The answer to this question is yes for all those rules that satisfy **additivity** or **weighted additivity**, since they will not depend on the flow.

Bilateral ratio consistency. For each $a = (M, S, OD, C) \in \mathbb{A}$ and each pair of municipalities $\{i, j\} \subseteq M$ we have that

$$\frac{R_i(a)}{R_j(a)} = \frac{R_i(a_{\{i,j\}})}{R_j(a_{\{i,j\}})},$$

where $a_{\{i,j\}} = \big(\{i, j\}, S_i \cup S_j, OD_{\{i,j\}}, C\big)$.

Trip decomposition. For each $(M, S, OD, C), (M, S, OD', C) \in \mathbb{A}$. If $s_g, s_h \in S$, are stations such that $h - g > 1$, and either

1. $\omega'_{g(g+1)} = \omega_{g(g+1)} + \frac{\omega_{gh}}{|h-g|}; \omega'_{(g+1)(g+2)} = \omega_{(g+1)(g+2)} + \frac{\omega_{gh}}{|h-g|}, \ldots, \omega'_{(h-1)h} = \omega_{(h-1)h} + \frac{\omega_{gh}}{|h-g|}$; and $\omega'_{gh} = 0$;
2. $\omega'_{ef} = \omega_{ef}$, if $(ef) \neq (gh)$,

or

1. $\omega''_{h(h-1)} = \omega_{h(h-1)} + \frac{\omega_{hg}}{|h-g|}; \omega''_{(h-1)(h-2)} = \omega_{(h-1)(h-2)} + \frac{\omega_{hg}}{|h-g|}, \ldots, \omega'_{(g+1)g} = \omega_{(g+1)g} + \frac{\omega_{hg}}{|h-g|}$; and $\omega''_{hg} = 0$;
2. $\omega''_{ef} = \omega_{ef}$, if $(ef) \neq (hg)$,

then, $R(M, S, OD, C) = R(M, S, OD', C)$ and $R(M, S, OD, C) = R(M, S, OD'', C)$

Additivity. For each $(M, S, OD, C) \in \mathbb{A}$ and each $i \in M$,

$$R_i(M, S, OD, C) = \sum_{t=1}^{T} R_i(M, S, OD_t, C_t),$$

where $OD = \sum_{t=1}^{T} OD_t$ and $C = \sum_{t=1}^{T} C_t$.

Weighted additivity. For each $(M, S, OD, C) \in \mathbb{A}$ and each $i \in M$,

$$\frac{\Omega(OD)}{C} R_i(M, S, OD, C) = \sum_{t=1}^{T} \frac{\Omega(OD_t)}{C_t} R_i(M, S, OD_t, C_t),$$

where $OD = \sum_{t=1}^{T} OD_t$ and $C = \sum_{t=1}^{T} C_t$.

Finally, we propose a fourth block composed of two properties that guarantee that a rule is immune from manipulation by agents, that is, if the municipalities decide to join together and act as one, **non-manipulability via merging**, or conversely, its dual, if they decide to separate and act as several municipalities, **non-manipulability via splitting**. These requirements have been used in several works such as *de Frutos (1999)*, Ju et al. (2007) and *Moulin (2008)*.

Non-manipulability via merging: For each pair M, M' such that $M' \subset M$, each $(M, S, OD, C) \in \mathbb{A}$, and each $(M', S', OD, C) \in \mathbb{A}$. If there is $i \in M'$ such that $S'_i = S_i \cup \bigcup_{j \in M \setminus M'} S_j$, and for each $j \in M' \setminus \{i\}$, $S'_j = S_j$, then $R_i(M', S', OD, C) \geq R_i(M, S, OD, C) + \sum_{j \in M \setminus M'} R_j(M, S, OD, C)$.

Non-manipulability via splitting: For each pair M, M' such that $M' \subset M$, each $(M, S, OD, C) \in \mathbb{A}$, and each $(M', S', OD, C) \in \mathbb{A}$. If there is $i \in M'$ such that $S'_i = S_i \cup \bigcup_{j \in M \setminus M'} S_j$, and for each $j \in M' \setminus \{i\}$, $S'_j = S_j$, then $R_i(M', S', OD, C) \leq R_i(M, S, OD, C) + \sum_{j \in M \setminus M'} R_j(M, S, OD, C)$.

Next we list the results obtained after the study of the characterization of the rules proposed in the cost sharing papers *Estañ et al. (2021a)* and *Estañ et al. (2020)* whose demonstrations we can see more detailed in the articles, which are attached in the Appendix section.

The first theorem states that if we require the properties of **symmetry** and **additivity**, then the cost must be shared evenly among the municipalities.

Theorem 1. *A rule satisfies symmetry and additivity if and only if it is the uniform rule.*

However, if we require a rule that satisfies the properties of **symmetry**, **bilateral ratio consistency** and **weighted additivity**, then the unique rule that satisfies this set of axioms is that distributes the cost equally among the municipalities without considering other factors of the problem, such as the use or the number of stations, as shown in the following result.

Theorem 2. *A rule satisfies symmetry, bilateral ratio consistency and weighted additivity if and only if it is the uniform rule.*

The second result, states that the unique rule that satisfies the set of axioms of **null municipality**, **symmetry** and **weighted symmetry** is the rule that divides the cost in proportion to the flow of passengers by municipalities.

Theorem 3. *A rule satisfies null municipality, symmetry and weighted additivity if and only if it is the station-based proportional rule.*

The next theorem states that the **track-based proportional rule** is characterized by the axioms: **adjacent symmetry**, **weak null municipality**, **trip decomposition** and **weighted additivity**.

Theorem 4. *The unique rule that satisfies adjacent symmetry, weak null municipality, trip decomposition and weighted additivity is the track-based proportional rule.*

The last result of this block, states that the set of properties formed by **symmetry in stations**, **non-manipulability via merging** and **non-manipulability via splitting** lead to a cost distribution that is proportional to the number of stations each city has.

Theorem 5. *A rule satisfies symmetry in station, non-manipulability via merging and non-manipulability via splitting if and only if it is the proportional station rule.*

3.2 Scarce resource distribution problems

The second part of this thesis is framed within the problems of allocation of scarce resources. In our paper **On the difficulty of budget allocation in claims problems with indivisible items and prices** (Estañ et al. (2021b)) we present a new situation: the study of the class of claims problems where we consider that the amount to be divided is perfectly divisible and claims are made for indivisible units of various items. Each item has a price and the available quantity is not enough to cover all the demands at the indicated prices.

In the mathematical model that we present in this study the problem represents a situation in which a perfectly divisible quantity, $E \in \mathbb{R}_{++}$ (called **estate**) must be distributed among agents in N according to their demands. Those demands are described by a matrix of claims $c \in \mathbb{Z}_+$ that has as many rows as agents, and as many columns as items :

$$c = \begin{pmatrix} c_{11} & c_{12} & \dots & c_{1h} \\ c_{21} & c_{22} & \dots & c_{2h} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \dots & c_{nh} \end{pmatrix},$$

where $c_{ig} \in \mathbb{Z}_+$ indicates the amount of item g claimed by agent i. In any claims problem, the estate falls short to fully cover all the demands, that is, $\sum_{i=1}^{n} \sum_{g=1}^{h} c_{ig} p_g \geq E$.

Therefore, a problem is given by a tuple $a = (N, H, p, c, E)$, where $N = \{1, \ldots, n\}$ is the set of agents, $H = \{1, \ldots, h\}$ is the set of possible items, whose prices are given by $p = (p_1, \ldots, p_h) \in \mathbb{R}^h_+$, c is the matrix of claims, and E is the estate. Since the elements N, H, and p are fixed throughout the paper, when no confusion arises we simply write the claims problem as $a = (c, E)$. The novelty that we present in this study is the answer to the question: Why we should spend all the budget (estate) we have? In the classical models on claims problems this fact is imposed, that is, we must spend all the estate.

Let $\mathbb{A}$ be the set of all problems:

$$\mathbb{A} = \left\{ a = (c, E) \in \mathbb{Z}^{n \times h}_+ \times \mathbb{R}_{++} : \|c \cdot p\| = \sum_{i=1}^{n} \sum_{g=1}^{h} c_{ig} p_g \geq E \right\}.$$

An **allocation** for $a \in \mathbb{A}$ is a distribution of the estate among the agents that specifies how many items of each price are awarded to each agent. Thus, it is a matrix $x \in \mathbb{Z}^{n \times h}_+$ that satisfies the following two conditions:

(a) Each agent receives a non-negative amount of each type of item, which is not larger than her claim:

$$0 \leq x_{ig} \leq c_{ig} \quad \text{for all } i \in N \text{ and all } g \in H.$$

(b) The overall cost does not exceed the available estate:

$$\|x \cdot p\| = \sum_{i=1}^{n} \sum_{g=1}^{h} x_{ig} p_g \leq E.$$

To solve these problems we propose several rules. In our setting a **rule**, it is a correspondence, $R : \mathbb{A} \rightrightarrows \mathbb{Z}^{n \times h}_+$, that selects, for each problem $a \in \mathbb{A}$, a non-empty subset of allocations $R(a) \subseteq X(a)$.

The first rule that we introduce is useful from a theoretical point of view, it states that no agent receives anything.

Null rule, R^N. For each $a \in \mathbb{A}$ and each $x \in X(a)$,

$$x \in R^N(a) \;\Leftrightarrow\; x_{ig} = 0 \;\forall i \in N \text{ and } \forall g \in H.$$

The next rule that we propose is the opposite case to the previous one, since it selects the whole set of allocations $X(a)$.

Greedy rule, R^G. For each $a \in \mathbb{A}$,

$$R^G(a) = X(a).$$

Next, we list two rules that are based on the **priority**. In the first of these allocations, **agent-item priority arrival rule**, agents with higher priority are satisfied before those with lower priority. Besides, for each agent the more relevant items are fully served first. That is, let $\succ_N$ be an ordering on the set of claimants N, where $i \succ_N j$ means i has priority over j and let $\succ_H$ be an ordering on the set of items H, where $f \succ_H g$ means f has priority over g, consider now a rule as the following procedure: the agents arrive one at a time in the ordering $\succ_N$, and try to fully satisfy them, starting with the items with the highest priority in $\succ_H$. This process continues until, eventually, the estate runs out. The second rule is called **agent priority arrival rule** and consist in: given an ordering $\succ_N$ on the set of claimants, agents arrive one at a time in the ordering. The first agent in the ordering selects the set of items so that she maximizes the value of her choice subject to the budget constrained given by E. Let E^1 be the remaining estate. Now, the second agent in the ordering selects the set of items so that she maximizes the value of her choice subject to the budget constrained given by E^1. We continue the process until the estate, eventually, runs out.

Agent-item priority arrival rule, R^{AIPA}. For each $a \in \mathbb{A}$ and each $x \in X(a)$,

$$x \in R^{AIPA}(a) \;\Leftrightarrow\; \left[x_{ig} > 0 \Rightarrow x_{if} = c_{if}\, \forall f \succ_H g \text{ and } x_{jf} = c_{jf}\, \forall j \succ_N i\, \forall f \in H\right].$$

Agent priority arrival rule, R^{APA}. For each $a \in \mathbb{A}$ and each $x \in X(a)$,

$$x \in R^{APA}(a) \;\Leftrightarrow\; \left[x_{jg} > 0 \Rightarrow x_{ig} = c_{ig}\, \forall i \succ_N j\right].$$

Now we present two rules that are a **two-step process**. In **equal-by-item rule**, first, the estate is equally divided among the items ($\frac{E}{h}$ for each one). And second, for each item, amounts as equal as possible are assigned to all claimants subject to no-one receiving more than her claim, this step is closely related to some other extensions in settings with indivisibilities of the so called *constrained equal awards rule* (Herrero and Martínez (2008a) and Chen (2015)).The second rule, **equal-by-agent rule** is obtained by applying the two step process of the equal by item rule but to agents. For each agent we select a set of items whose price is smaller or equal than $\frac{E}{n}$ and such that adding a new item the price is larger than $\frac{E}{n}$. Later the remaining budget is assigned to any set of agents spending as much as possible.

Equal-by-item rule, R^{EI}. For each $a \in \mathbb{A}$ and each $x \in X(a)$,

$$x \in R^{EI}(a) \;\Leftrightarrow\; \begin{cases} |x_{ig} - x_{jg}| \leq 1 \text{ for all } i, j \in N \\ p_g \left(\sum_{i=1}^{n} x_{ig}\right) \leq \frac{E}{h} \\ p_g \left(1 + \sum_{i=1}^{n} x_{ig}\right) > \frac{E}{h}. \end{cases}$$

Equal-by-agent rule, R^{EA}. For each $a \in \mathbb{A}$ and each $x \in X(a)$,

$$x = z + y \in R^{EA}(a) \;\Leftrightarrow\; \begin{cases} z, y \in \mathbb{Z}_+^{n\times h} \\ \sum_{g=1}^{h} p_g z_{ig} \leq \frac{E}{n}, \forall i \in N \\ \sum_{g=1}^{h} p_g z'_{ig} > \min\{\sum_{g=1}^{h} p_g c_{ig}, \frac{E}{n}\}, \forall z' > z, \forall i \in N \\ \sum_{i=1}^{n} \sum_{g=1}^{h} p_g y_{ig} \leq E - \|z \cdot p\| \\ \sum_{i=1}^{n} \sum_{g=1}^{h} p_g y_{ig} \geq \sum_{i=1}^{n} \sum_{g=1}^{h} p_g y'_{ig}, \forall y' \in X(a'), \end{cases}$$

where $z' > z$ means that there is at least one cell ig such that $z'_{ig} > z_{ig}$, and the others are greater or equal; and $a' = (N, H, p, c - z, E - \|x \cdot p\|)$.

Now we present three minimal requirements that a rule should satisfy, which are quite standard in the literature on claims problems. The restrictions they impose are so slight that they are usually compatible. The first property, **non-wastefulness**, stipulates that in a rationing situation we should waste as little as possible. The second property, **weak equal treatment of equals**, is a minimal criterion on fairness, and states that agents with equal claims should be equally treated. Finally, the last property, **non-manipulability by merging or splitting**, makes the rule immune to certain manipulations by the agents. To summarize, **efficiency, fairness, and non-manipulability** will be the core requirements we impose as starting point.

Non-wastefulness. For each $a \in \mathbb{A}$, if $x \in R(a)$, then there is no other allocation $x' \in X(a)$ such that $E - \|x' \cdot p\| < E - \|x \cdot p\|$.

Weak equal treatment of equals. For each $a \in \mathbb{A}$ and each $\{i, j\} \subseteq N$, if $c_{ig} = c_{jg} \forall g \in H$, then for all $x \in R(a)$ it holds that

- for all $g \in H$, $|x_{ig} - x_{jg}| \leq 1$, and
- for each $g \in H$, there is $x' \in R(a)$, such that $x'_{ig} = x_{jg}$, $x'_{jg} = x_{ig}$ and the rest of cells of x' are the same as in x.

Non-manipulability by merging or splitting. For each $(N, c, E), (N', c', E) \in \mathbb{A}$ with $N' \subset N$, if there is $i \in N'$ such that the following two conditions hold

1. $c'_{ig} = c_{ig} + \sum_{j \in N \setminus N'} c_{jg}$ for all $g \in H$
2. $c'_{jg} = c_{jg}$ for all $j \in N' \setminus \{i\}$ and for all $g \in H$,

then

(a) $\forall x' \in R(N', c', E)$ there exists $x \in R(N, c, E)$ such that $x'_{ig} = x_{ig} + \sum_{j \in N \setminus N'} x_{jg}$ $\forall g \in H$.

(b) $\forall x \in R(N, c, E)$ there exists $x' \in R(N', c', E)$ such that $x'_{ig} = x_{ig} + \sum_{j \in N \setminus N'} x_{jg}$ $\forall g \in H$.

In this paper we explore the compatibility of the aforementioned basic properties.

Given a problem $a \in \mathbb{A}$. Consider the following integer linear programming problem (ILP, for short):

$$\left.\begin{aligned} &\min_{x \in \mathbb{Z}_+^{n \times h}} E - \sum_{i=1}^{n} \sum_{g=1}^{h} p_g x_{ig} \\ &\text{s.t.: } \sum_{i=1}^{n} \sum_{g=1}^{h} p_g x_{ig} \leq E \\ &\qquad 0 \leq x_{ig} \leq c_{ig},\ \forall i \in N, \forall g \in H \end{aligned}\right\}$$

or equivalently,

$$\left.\begin{aligned} &\max_{x \in \mathbb{Z}_+^{n \times h}} \sum_{i=1}^{n} \sum_{g=1}^{h} p_g x_{ig} \\ &\text{s.t.: } \sum_{i=1}^{n} \sum_{g=1}^{h} p_g x_{ig} \leq E \\ &\qquad 0 \leq x_{ig} \leq c_{ig},\ \forall i \in N, \forall g \in H \end{aligned}\right\} \tag{3.1}$$

Let us denote by $ILP(a)$ the set of all optimal solutions for the program in (3.1) that belongs to the class of bounded knapsack problems. We observe that a rule R satisfies non-wastefulness if it is a selection of solutions of the optimization problem that we present above, i.e., $R(a) \subseteq ILP(a)$ for all $a \in \mathbb{A}$. Notice that the constrains $0 \leq x_{ig} \leq c_{ig}$, $\forall i \in N, \forall g \in H$ restrict the possible values of the optimization variables, and therefore the knapsack problem is bounded.

The results that we have obtained are the following:

Theorem 1. *There are rules that satisfy non-wastefulness, weak equal treatment of equals and non-manipulability by merging or splitting if and only if there are rules that satisfy those properties for the subclass of problems with* $|H| = 1$.

Theorem 2. *There is no rule that satisfies non-wastefulness, weak equal treatment of equals and non-manipulability by merging or splitting.*

Proposition 1. *There are rules that satisfy non-wastefulness and weak equal treatment of equals together.*

Proposition 2. *There are rules that satisfy non-wastefulness and non-manipulability by merging or splitting together.*

Proposition 3. *There are rules that satisfy weak equal treatment of equals and non-manipulability by merging or splitting together.*

We thought that should be interesting study the compatibility between the **non-wastefulness** condition and other standard properties required when solving claims problems. In particular, we focus on requirements that protect small claimants, to do that we choose five conditions that we consider that are relevant for this study: **exemption**, **conditional full compensation**, **securement** , **weak securement**, **self-duality**.

Exemption. For each $a \in \mathbb{A}$ and each $i \in N$, if

$$n \cdot \left(\sum_{g=1}^{h} p_g c_{ig} \right) \leq E,$$

then, for any $x \in R(a)$, $x_{ig} = c_{ig}$ $\forall g \in H$.

Conditional full compensation. For each $a \in \mathbb{A}$ and each $i \in N$, if

$$\sum_{j \in N_i^-} \sum_{g=1}^{h} p_g c_{jg} + (n - |N_i^-|) \sum_{g=1}^{h} p_g c_{ig} \leq E,$$

then, for any $x \in R(a)$, $x_{ig} = c_{ig}$ $\forall g \in H$, where $N_i^- = \left\{ j \in N : \sum_{g=1}^{h} p_g c_{jg} < \sum_{g=1}^{h} p_g c_{ig} \right\}$.

Notice that exemption implies conditional full compensation, and both properties coincide when $|N| = 2$.

Securement. For each $a \in \mathbb{A}$, each $x \in R(a)$, and each $i \in N$

$$\sum_{g=1}^{h} p_g x_{ig} \geq \frac{1}{n} \min \left\{ \sum_{g=1}^{h} p_g c_{ig}, E \right\}, \quad \forall i \in N.$$

Weak securement. For each $a \in \mathbb{A}$, each $x \in R(a)$, and each $i \in N$

$$\sum_{g=1}^{h} p_g x_{ig} \geq \max_{y \in X(a)} \left\{ \sum_{g=1}^{h} p_g y_{ig} \,\middle|\, \sum_{g=1}^{h} p_g y_{ig} \leq \frac{1}{n} \min \left\{ \sum_{g=1}^{h} p_g c_{ig}, E \right\} \right\}, \quad \forall i \in N.$$

Self-duality. For each $a \in \mathbb{A}$ it holds that $R(a) = c - R(a^d)$.

The results obtained when we explore the combination of non-wastefulness and the requirements mentioned above are the following:

Theorem 3. *There is no rule that satisfies non-wastefulness and conditional full compensation together.*

The next result shows that, however, when non-wastefulness is required in conjunction with weak securement, an impossibility emerges.

Theorem 4. *There is no rule that satisfies non-wastefulness and weak securement together.*

Proposition 4. *If a rule satisfies the self-duality property then it exhausts the estate.*

The converse of *Proposition 4* is not true in general. For example, if we consider the class of problems with $H = \{1\}$ and $p_1 = 1$ and E a positive integer number, then the discrete constrained equal awards rule (see, for example, Herrero and Martínez (2008a)) always exhaust the estate but does not satisfies **self-duality**.
An immediate consequence of *Proposition 4* is that there can be no rules that satisfy the property of **self-duality**, since no rule can always exhaust the estate, in general. Notice that, unlike the other results in this section, the lack of self-dual rules is absolute, and the principle of **non-wastefulness** plays no role in that.

As we can observe the condition of **non-wastefulness** is not compatible with the classical properties used in claims problems, because is the counterpart of the *efficiency*. For this reason we present an alternative to **non-wastefulness**: **Pareto efficiency**.

In contrast with **non-wastefulness**, this property focuses on the agents' allocations rather than on the expenditure of the budget. An allocation is **Pareto efficient** if there is no other allocation in which some other individual is better off and no individual is worse off.

Pareto efficiency. For $a \in \mathbb{A}$, if $x \in R(a)$ then there is no other allocation $x' \in X(a)$ such that $\sum_{g \in H} p_g x'_{ig} \geq \sum_{g \in H} p_g x_{ig}, \forall i \in N$, with at least one strict inequality.

Given $a \in \mathbb{A}$, we denote by $P(a) \subset X(a)$ the set of all allocations which are Pareto efficient.

Notice that it is glaringly obvious that **non-wastefulness** implies **Pareto efficiency**, but the converse is not true. Even though these two properties are not equivalent in general, it is not difficult to prove that they coincide when $|H| = 1$. As a consequence, we can replace **non-wastefulness** by **Pareto efficiency** in Theorem 1, which implies that **weak equal treatment of equals** and **non-manipulability by merging or splitting** together are incompatible with **Pareto efficiency**. This result is the analogous to Theorem 2.

Theorem 5. *There is no rule that satisfies Pareto efficiency, weak equal treatment of equals and non-manipulability by merging or splitting.*

Since **Pareto efficiency** is milder than **non-wastefulness**, we obtain the counterparts of Propositions 1 and 2.

Proposition 5. *There are rules that satisfy Pareto efficiency and weak equal treatment of equals together.*

Proposition 6. *There are rules that satisfy Pareto efficiency and non-manipulability by merging or splitting together.*

With regard to **self-duality**, it is evident that it will not be compatible with **Pareto efficiency** since the latter does not guarantee that the estate is fully exhausted. Theorems 3 and 4 state that **conditional full compensation** and **weak securement** are incompatible with **non-wastefulness**. However, the next two results show that, if the latter requirement is weakened to **Pareto efficiency**, then the possibility emerges.

Theorem 6. *There are rules that satisfies Pareto efficiency and conditional full compensation together.*

Theorem 7. *There are rules that satisfies Pareto efficiency and weak securement together.*

Therefore, **Pareto efficiency** is a sufficiently less demanding property to be compatible with other reasonable properties. Furthermore, we can define rules that satisfy several of the properties introduced in this paper. For example, the following rule

$$R^{CS}(a) = x^0 + R^S(a'), \forall a \in \mathbb{A},$$

where $a' = (N, H, p, c - x^0, E - \|x^0 \cdot p\|)$, satisfies **Pareto efficiency**, **conditional full compensation** and **weak securement**.

Consider the rule R^{CES} defined as follows. For each, $a \in \mathbb{A}$,

$$R^{CES}(a) = R^{CS} \cap E(a)$$

This rule satisfies **Pareto efficiency**, **weak equal treatment of equals**, **conditional full compensation**, and **weak securement**. The converse is not true, there are rules different from R^{CES} that also fulfill these four properties. However, any rule that satisfies Pareto efficiency, weak equal treatment of equals, conditional full compensation, and weak securement must be a subselection of R^{CES}.

Theorem 8. *If a rule R satisfies Pareto efficiency, weak equal treatment of equals, conditional full compensation, and weak securement, then $R(a) \subset R^{CES}(a)$, $\forall a \in \mathbb{A}$.*

Chapter 4

Introduction

In real-life different situations occur in which there is a resource or a cost that has to be shared among different agents. How to carry out this distribution depends on different factors derived from the problem we are addressing. This type of problem can be solved from the axiomatic or methodological point of view in Game Theory, discipline of Mathematics that studies, models and solves conflict situations (games) among different agents (players) whose decisions in the game may have repercussions on other agents. Depending on whether the agents involved in the game are able to reach binding agreements or not we have a cooperative or non-cooperative game. A reference to start in the study of cooperative games can be the second version of the book *Introduction to the theory of cooperative games* by Peleg and Shuldhöter (Peleg and Sudhölter (2007)) and, of course, an important manuscript to get into a non-cooperative games is the book *Non-Cooperative games* by John Nash (Nash (1951)).

Although it was born earlier, Game Theory became more relevant in 1944 with the book *Theory of Games and Economic Behavior* by John von Neumann and Oskar Morgenstern (see Morgenstern and Von Neumann (1953)). This branch of Mathematics has been and continues to be applied in a multitude of real situations such as those raised in the papers studied in this thesis (see Estañ et al. (2021a), Estañ et al. (2020) and Estañ et al. (2021b)), which are framed within cost sharing and claims problems.

One of the most relevant transformation in the world is the globalization. This allows us, among other things, to be more connected, that is, we are able to, for example create traffic networks between countries, cities in the same country, municipalities in the same community or province. That allow us to travel long distances in increasingly shorter periods of time, which leads to an increase in the profitability or productivity of certain jobs that require the use of this type of network. A clear example of these networks is the high-speed train created in Spain and known as AVE, which connects Spanish capitals in a short time and whose future projection is the connection with other countries such as France. This type of infrastructure entails different associated costs, such as the costs related to the construction of the network or, on the other hand, those derived from the maintenance of the infrastructure once it is in operation. Maintenance costs can be divided into two types: variable costs (those depend, for example, on the use of the network) and fixed costs (those costs that are constant, that is, they do not depend on the use, such as the costs derived from the garages where the trams are stored). How to distribute these costs among the different countries, cities or municipalities or the agents involved in the network, is a clear example of what is known as cost sharing in Game Theory.

In Estañ et al. (2021a) and Estañ et al. (2020), we reduce this problem and focus on studying the distribution of the fixed cost of maintaining a linear network between municipalities. In this way we provide solutions that perhaps could be extrapolated to larger networks. We would like to remark that we study the fixed cost, and that is a novelty.

In the literature we can find numerous articles that study the distribution of the variable costs derived from a network, among them see Sánchez-Soriano et al. (2002), Ni and Wang (2007), or Kuipers et al. (2013), for instance.

Another relevant issue nowadays is the distribution of a resource. On certain occasions a good (perfectly divisible or not) has to be distributed among different agents, the problem arises when the available quantity of the good is not enough to satisfy the demands of the agents. These types of problems are framed within as bankruptcy problems.Bankruptcy problems are studied from the point of view of Game Theory in O'Neil (1982), among others.

In the Old Testament (I Kings 3: 16-28) we can read a well-known episode "The Judgment of Solomon", which deals with the dispute between two women for a very valuable and indivisible good (a son). These women were not able to reach an agreement, so they went to King Solomon who had to take a decision to solve the dilemma presented: to whom do I give this child? Could he be distributed?. King Solomon was known for his great wisdom, good sense and justice. Although many of us already know the end of this dispute, the first solution to the problem (demand made by two agents for a single and indivisible good) offered by King Solomon was a proportional distribution of the good without taking into account, obviously, whether the good could be partitioned or not, considering only for this solution the demands that the agents (women) raised.

Depending on the nature of the problem, game theory provides us with one or more solutions (allocation rules) from the methodological point of view. There are many different distribution rules, some of them are well known, for instance, proportional rationing. Herrero and Villar, in their article "The three musketeers: four classical solutions to bankruptcy" (Herrero and Villar (2001)), problems do a comparison of the so called basic rules in bankruptcy problems in the continuous case, where the good to be distributed is perfectly divisible. Such rules are: the constrained equal-awards rule, the constrained equal-losses rule, the proportional rule and the Talmud rule.

We can find many well known mathematical stories in the literature, such as the problem of Rabbi Abraham Ibn Ezra, which consists of the distribution of the inheritance by

Jacob (father). The problem is that each of his children demands an amount of the different inheritance and the sum of all the demands made exceed the total amount of the inheritance. There are many versions of this problem studied by different authors in which different distribution rules such as those mentioned above are applied and therefore with totally different solutions.

A situation of distribution of scarce resources could be the one experienced recently: the pandemic caused by COVID19, whose arrival shocked the whole world, because, as we have been able to observe, governments worldwide have had to reach different distribution agreements, which has not been easy. One of the hardest situations perhaps experienced was the need to expand medical resources such as hospital beds, respirators, medical personnel and even space. In order to solve this problem, many communities decided to make an extraordinary budget in order to create temporary hospitals and provide them with the necessary resources (beds, medical personnel, medical supplies, etc ...). In this case, the budget is a finite and perfectly divisible monetary amount, the question would be, with this budget, how many items of each requested resource can we acquire? It must be borne in mind that each resource has a different price associated with it and specifically all these resources cannot be partitioned. Furthermore, is it necessary to exhaust the entire extraordinary budget? Or is there a way we can meet the demands while spending the smallest budget possible? In our article Estañ et al. (2021b) we explore these questions and provide solutions from the axiomatic point of view of Game Theory and a well known Integer Linear Problem, the so called bounded knapsack problem.

Chapter 5

Objectives

The main objective of this thesis is the study of problems focused on the sharing costs and the distribution of scarce resources. To do that we have structured this thesis in two parts, the first of them is focused on the distribution of costs that we study in the papers: **On how to allocate the fixed cost of transport systems** (Estañ et al. (2021a)) and **Manipulabiltiy in the cost allocation of transport systems** (Estañ et al. (2020)). In the second part we analize a specific problem of distribution of scarce resources on the paper **On the difficulty of budget allocation in claims problems with indivisible items and prices** (Estañ et al. (2021b)). In both parts we have common objectives such as: model the problem to study and provide different solutions, as fair and feasible as possible. To this end, we propose several allocation rules and a list of axioms and we explore which property or set of axioms are satisfied by each rule, that is, we characterize the proposed rules. Obviously, the rules and the axioms, and therefore the characterizations, are based on the needs of each situation derived from the problem under study. Therefore in each manuscript the rules and the suggested properties are different. An example of this is the need to use integer linear programming tools in **On the difficulty of budget allocation in claims problems with indivisible items**

and prices (Estañ et al. (2021b)) to be able to model the problem under study in this paper. Another example is the differences between the proposed rules because in each problem we have different elements and therefore the rules are not the same.

To summarize, the objectives of this thesis are the following:

- In the first part, the objectives are structured as follows:
  - Model the problem of allocation of fixed costs in a transport network.
  - Propose the allocation rules for fixed cost allocation problems in a transport system with different levels of information.
  - Characterize the allocation rules for fixed cost allocation problems in a transport system with different levels of information.
- In the second part, the objectives are structured as follows:
  - Model the scarce resource allocation problem when there are several indivisible objects that have different prices.
  - Study the existence of rules for the allocation of scarce resources when there are several indivisible objects that have different prices that satisfy certain properties of equity, justice and fairness.
  - Propose rules for the allocation of scarce resources when there are several indivisible objects that have different prices.

Chapter 6

Methodology

The methodology used in the development of this doctoral thesis is the usual one in the field of Mathematics and Economic Theory. First, the problems on which the investigation was to be initiated were established. Next, we proceeded to the study of the state of the art, studying the main manuscripts in the literature related to the theme of the problems that we want to study. In our case the problems of cost sharing and distribution of scarce resources. Once the state of the art has been well studied, we search problems that had not yet been resolved in the literature and that were of interest from a mathematical and a socio-economic point of view. The next step was to formulate mathematically the problems, define the solutions to those problems and characterize those solutions from the axiomatic approach. Once the results were obtained, the next step was to publish the results of the research and disseminate them through specialized and general congresses.

Chapter 7

Discussion

In this chapter we present the discussion presented in the papers that make up this thesis: **On how to allocate the fixed cost of transport systems** (Estañ et al. (2021a)), **Manipulability in the cost allocation of transport systems** (Estañ et al. (2020)) and **On the difficulty of budget allocation in claims problems with indivisible items and prices** (Estañ et al. (2021b)).

7.1 Cost sharing problems discussion

We should mention that the first part of this thesis is based in ours two papers **On how to allocate the fixed cost of transport systems** (Estañ et al. (2021a)) and **Manipulability in the cost allocation of transport systems** (Estañ et al. (2020)), related to the problem of allocating the cost of building or maintaining a facility among all the agents involved in providing a service, is a classic example in cost sharing literature. We refer to the well known airport problem in which the Shapley value (Shapley (1953)) is obtained with very low computational effort (see Littlechild and Owen (1973)). In this case the costs of using an airport landing track have to be shared among the planes that will operate in the airport, taking into account their different sizes. The infrastructure of this problem is a line, as in our case, that is shared from the beginning by all the planes. Because different sizes of planes need different lengths of tracks, the value function of the associated cost game varies depending on the coalition of players. In contrast, in our model the cost is always the same and the value function is constant. We draw attention to the paper by Thomson et al. (2007) for a wide survey on airport problems. The idea of Littlechild and Owen (1973) was generalized through the so called painting stories by Maschler et al. (2010) and Bergantiños and Martínez (2014).

We should note two works associated with highway profit sharing by Kuipers et al. (2013) and Sudhölter and Zarzuelo (2017), where each agent uses consecutive sections of a highway. The difference from the case of airport problems is that the sections used by an agent need not start from the beginning of the line.

There is literature on cost sharing for the problem of cleaning a polluted river. The river is considered as a segment divided into subsegments each of which belongs to a region/municipality. There is a central agency that determines the cost of cleaning each segment. The seminal paper on this problem was Ni and Wang (2007). They propose two methods (*local responsibility sharing* and *upstream equal sharing*), which give rise to allocations that turn out to be the Shapley values of two TU-games. Van den Brink and van der Laan (2008) show that this problem is essentially an airport problem. Gómez-Rúa (2013) provides axiomatic characterizations of a family of rules using properties based on water taxes. One of these rules coincides with the weighted Shapley value. Alcalde-Unzu et al. (2015) characterizes the *upstream responsibility rule* that assigns to the region located in a segment the value of its responsibility taking as the transfer rate the mid-point between its lower and upper limits. The remaining cost is divided among the upstream regions. The problem of cleaning polluted river essentially focuses on a line along which the agents are located, as in our proposal. However, the river only flows downstream, while tram lines carry passenger in both directions. Besides, the overall cost of cleaning the river may be related with the pollution each region generates.

An interesting application of cost allocation related to public transport is provided by Sánchez-Soriano et al. (2002) that consider how to share the cost of transport for university students. There are other examples of transportation distributions. Algaba et al. (2019) study the problem of sharing revenues among transport companies in a multimodal transport system, that cooperate offering tickets which allow to use all transport means. In this model, multiple arcs between two nodes arise to represent the different companies that provide the service between each pair of stations. As in our model, the intensity in use is given by a matrix of flows. They introduce the colored egalitarian solution that turns out to be the Shapley value of a convex TU-game and it is in the core. Another compelling situation is studied in Slikker and Van Den Nouweland (2000) and Norde et al. (2002) to allocate the fixed and variable costs of a railway network among the trams that use this infrastructure. The property that guarantees that the rule is immune to a special type of manipulation has been widely used in other models (see, for example, de Frutos (1999), Ju et al. (2007), and Moulin (2008)).

The study of revenue and cost sharing in networks has also provided satisfactory results in other specific situation such as the analysis terrestrial flight telephone system (see van den Nouweland et al. (1996)) or the power networks (see Bergantiños and Martínez (2014)). A distinguishing feature of our model is that we always consider the same cost. This means that, contrary to all the aforementioned works, in our model if we consider the related TU-game, the characteristic function is constant and it does not depend on coalitions.

The results that we have obtained in the study of allocation problems in our papers **On how to allocate the fixed cost of transport systems** (Estañ et al. (2021a)) and **Manipulability in the cost allocation of transport systems** (Estañ et al. (2020))

are different.

When we focus on the study of the allocation or sharing the costs, the solutions obtained are rules based on the concept of proportionality. We show that the unique rule that fulfills null municipality, symmetry in municipalities and weighted additivity is the rule that divides the cost proportional to the number of users in each municipality (station-based proportional rule). However, if we require symmetry in municipalities, bilateral ratio consistency and weighted additivity, then the cost is equally split among all the municipalities (uniform rule), regardless the number of stations and their users. If we require adjacent symmetry, weak null municipality, trip decomposition and weighted additivity, the solution is the track-based proportional.rule. Moreover, if we require symmetry in stations and non-manipulability via merging and splitting the solution obtained is the station proportional rule.

7.2 Scarce resource distribution problems discussion

Several authors have studied allocations problems with indivisibilities. In some cases both the budget and the demands are integers (Chen (2015), Herrero and Martínez (2011), Herrero and Martínez (2008b), or Herrero and Martínez (2008a)), while in other papers the estate is indivisible but the claims are continuous (Dall'Aglio et al. (2016) or Dall'Aglio et al. (2019)).

In this context several papers have proposed different type of solutions: Moulin (2008), Herrero and Martínez (2008b), Chen (2015) use priority methods, while Giménez-Gómez and Vilella (2017) adopt a P-rights recursive process, described in Giménez-Gómez and Marco-Gil (2014), to ensure weak order preservation. Discrete claim models have been widely used to deal with scarce resources in technological problems such as mobile radio networks (Lucas-Estañ et al. (2012) and Gozálvez et al. (2012)) or social problems such as apportionment problems (Sánchez-Soriano et al. (2016)). On the other hand, in claims problems with multi-dimensional and perfectly divisible claims Calleja et al. (2005) introduce the run to the bank rule, while Bergantiños et al. (2011) present several characterizations of the constrained equal awards rule, and Moreno-Ternero (2009) studies the proportional rule.

With respect to those works, the novelty we present in our paper **On the difficulty of budget allocation in claims problems with indivisible items and prices** (Estañ et al. (2021b)) is twofold. One, the claims are on multiple items. And two, and more significant, the existence of prices, which allow us to consider and combine a continuous estate with indivisible claims. Interestingly, the finding of non-wasteful rules is closely related with a well known programming problem, the so called bounded knapsack problem. Since the seminal paper by Dantzig (1957), several extensions have been widely studied due to their practical applications (Kellerer et al. (2004)), including choice theory (Feuerman and Weiss (1973)). As examples of interest which relate to our situation, Darmann and Klamler (2014) study how to share the estate in a continuous setting by means of optimal solutions, and Arribillaga and Bergantiños (2019) analyze two rules related to the Shapley value of an optimistic game.

The results that we have obtained in the case of the allocation of scarce resources are a compatibility-incompatibility results.
In the paper **On the difficulty of budget allocation in claims problems with indivisible items and prices** (Estañ et al. (2021b)) we show that there are rules that satisfy non-wastefulness, weak equal treatment of equals and non-manipulability by merging or splitting if and only if there are rules that satisfy those properties for the subclass of problems with $|H| = 1$. This result states that, if we are able to obtain rules that satisfy the three conditions in a reduced domain of problems (with just one item), then they can be extended to the general domain. And conversely, if the three properties are not

compatible when $|H| = 1$, then they are not compatible in general.

The next result that we obtain exploits this relation to conclude that, in this setting, it is not possible to find a rule that fulfills non-wastefulness, weak equal treatment of equals and non-manipulability by merging or splitting. This theorem provides a surprising result, since it states an incompatibility among some principles that are compatible in the classical claims problem (see Thomson (2001)). Notice that none of the properties in the previous result is very demanding by itself. Indeed, the next propositions show that any pairwise combination of non-wastefulness, weak equal treatment of equals and non-manipulability by merging or splitting is feasible. Besides, the set of rules that satisfy each pairwise combination of properties is so wide that it does not seem to have a clear structure.

Another result that we have obtained is that there is no rule that satisfies non-wastefulness and conditional full compensation together and, as a consequence, neither exemption and non-wastefulness are compatible. This theorem illustrates that, for the problem of adjudicating conflicting indivisible claims with different prices, efficiency (non-wastefulness) and some protective conditions (exemption or conditional full compensation) cannot be conciliated. It is worth noting that this impossibility is a particularity of the model with several items and prices. Both when claims and estate are divisible, and when they are expressed in indivisible units these two properties are compatible (Herrero and Martínez (2008b)).

When non-wastefulness is required in conjunction with weak securement, an impossibility emerges.

We explore an alternative formulation of the efficiency principle: *Pareto efficiency*. In contrast with non-wastefulness, this property focuses on the agents' allocations rather than on the expenditure of the budget. An allocation is Pareto efficient if there is no other allocation in which some other individual is better off and no individual is worse off.

Notice that it is glaringly obvious that non-wastefulness implies Pareto efficiency, but the converse is not true. Even though these two properties are not equivalent in general, it is not difficult to prove that they coincide when $|H| = 1$. As a consequence, we can replace non-wastefulness by Pareto efficiency in the theorem which implies that weak equal treatment of equals and non-manipulability by merging or splitting together are incompatible with Pareto efficiency. This result is the analogous to the result of incompatibility.

Since Pareto efficiency is milder than non-wastefulness, we obtain the counterparts of Propositions 1 and 2.

With regard to self-duality, it is evident that it will not be compatible with Pareto efficiency, since the latter does not guarantee that the estate is fully exhausted. Theorems 3 and 4 state that conditional full compensation and weak securement are incompatible with non-wastefulness. However, we show that, if the latter requirement is weakened to Pareto efficiency, then the possibility emerges. Therefore, there are rules that satisfy Pareto efficiency and conditional full compensation together and if a rule R satisfies Pareto efficiency, weak equal treatment of equals, conditional full compensation, and weak securement, then $R(a) \subset R^{CES}(a)$, $\forall a \in \mathbb{A}$.

Chapter 8

Conclusiones

En este capítulo presentamos las conclusiones obtenidas en los trabajos que conforman esta tesis : **On how to allocate the fixed cost of transport systems** (Estañ et al. (2021a)), **Manipulability in the cost allocation of transport systems** (Estañ et al. (2020)) and **On the difficulty of budget allocation in claims problems with indivisible items and prices** (Estañ et al. (2021b)).

8.1 Problemas de distribución de costes: conclusiones

En el trabajo **On how to allocate the fixed cost of transport systems** (Estañ et al. (2021a)) presentamos un modelo que estudia el problema de dividir un coste fijo de una línea de tranvía entre los municipios que componen dicha línea. La regla de asignación depende del conjunto de municipios, las estaciones que cada municipio posee, el coste fijo de distribución y la matriz de flujo. Hemos demostrado que, si requerimos que esta regla satisfaga las nociones básicas de equidad y estabilidad, terminamos con unicidad en las soluciones. Más precisamente, hemos encontrado que municipio nulo (un municipio sin usuarios está exento de pago), simetría en los municipios (los municipios simétricos contribuyen por igual) y aditividad ponderada (la asignación es inmune a la división del problema) conducen a la regla proporcional basada en el uso de las estaciones, que divide el coste proporcionalmente al número de pasajeros que utilizan las estaciones de cada municipio. De manera similar, también hemos caracterizado la regla proporcional basada en la trayectoria en términos de simetría adyacente (simetría con respecto a las estaciones adyacentes), municipio nulo débil, descomposición del viaje (la regla no se altera al dividir un viaje largo en viajes pequeños) y aditividad ponderada. Finalmente, hemos probado que, si requerimos que la regla cumpla con simetría en municipios, consistencia bilateral en ratio (estabilidad frente a cambios en el conjunto de municipios) y aditividad ponderada, entonces debemos distribuir el coste uniformemente entre las ciudades. Como en muchos otros trabajos existentes en la literatura de costes compartidos (Sánchez-Soriano et al. (2002), Ni and Wang (2007), o Kuipers et al. (2013), por ejemplo), podríamos definir el juego de costes de la siguiente manera. El conjunto de jugadores es el conjunto de municipios M y la función de valor para cada coalición $S \subset M$ viene dada por

$$c(S) = \begin{cases} 0 & \text{if } S = \varnothing \\ C & \text{if } S \neq \varnothing \end{cases}$$

La función c es constante debido a la naturaleza de la problema que estudiamos, no importa el número de identidad de los municipios involucrados en la coalición, el montante que deben pagar siempre es C, ya que, por ejemplo, el salario del director general debe pagarse independientemente del uso o la estructura de la red. Una vez que aplicamos los conceptos básicos de solución en la literatura sobre juegos TU a nuestro caso, terminamos en asignaciones no completamente satisfactorias. El valor de Shapley y el nucléolo coinciden con la distribución uniforme el núcleo es el conjunto completo de asignaciones factibles $\left(x = \left(\frac{C}{m}, \ldots, \frac{C}{m}\right)\right)$, the core is the whole set of feasible allocations $(\{x : \sum_{i=1}^{m} x_i = C\})$.

Como muestra Shapley (Shapley (1953)) la solución única del juego de costes que satisface simetría en municipios, jugador nulo y aditividad es el valor de Shapley, que, en nuestro caso, coincide con la regla uniforme. Curiosamente, el teorema demuestra que, si estos mismos principios se trasladan a nuestro modelo, en su lugar caracterizamos la regla proporcional basada en el uso de las estaciones. La regla uniforme (o de manera equivalente, el valor de Shapley del juego de costes) también se caracterizamediante propiedades alternativas. En el artículo **Manipulability in the cost allocation of transport systems** (Estañ et al. (2020)), hemos demostrado que simetría en estaciones y no manipulabilidad (por fusión y división) caracterizan la regla proporcional al número de estaciones. Este resultado sigue la línea de algunos otros en la literatura para diferentes modelos (ver, por ejemplo,de Frutos (1999), Moreno-Ternero (2006) y Ju et al. (2007)), en los que la ausencia de manipulabilidad está íntimamente relacionada con mecanismos proporcionales. Sin embargo, en nuestro modelo, la proporcionalidad no está unívocamente determinada, ya que puede referirse a las estaciones, a los flujos o incluso a una combinación de ambos. El teorema obtenido establece que las propiedades que consideramos solo son compatibles con una de ellas.

8.2 Problemas de distribución de recursos escasos: conclusiones

Para finalizar, en el artículo **On the difficulty of budget allocation in claims problems with indivisible items and prices** (Estañ et al. (2021b)) hemos estudiado una clase particular de problemas de demandas. En nuestro modelo, un grupo de agentes demanda varias unidades de diferentes artículos, cada uno de los cuales tiene un precio. El patrimonio disponible no es suficiente para satisfacer la demanda total. Una regla es una multifunción que selecciona un conjunto de asignaciones que indican la cantidad de unidades de cada artículo que se asigna a cada agente.

A diferencia de otros modelos que implican problemas de demandas, no se puede garantizar la eficiencia. El requisito más cercano es el de no despilfarro, que establece que la regla debe desperdiciar la menor cantidad de patrimonio posible, y está estrechamente relacionado con el llamado problema de la mochila acotada, cuyas soluciones, en general, son difíciles de obtener. Sin embargo, con esta condición más suave de eficiencia, encontramos que no existe una regla que satisfaga el no despilfarro junto con otros criterios que protegen a los pequeños agentes o aseguran que los demandantes reciban una asignación mínima.

En vista de todos los resultados de imposibilidad obtenidos en este estudio, podemos observar que no es fácil conciliar la eficiencia (a través del no despilfarro) con la equidad. En este sentido, quizás, la propiedad de no despilfarro es demasiado restrictiva, porque se centra más en el uso del patrimonio que en la equidad de la asignación. Por lo tanto, se podría reconsiderar la necesidad absoluta de la propiedad de no despilfarro y simplemente garantizar que se distribuya la cantidad máxima del estado, respetando ciertas propiedades de equidad en la distribución. Esta sería la segunda mejor alternativa, lo que está más allá del propósito de este trabajo, una vez que hemos demostrado que el enfoque directo conlleva muchas dificultades.

Chapter 9

Conclusions

In this chapter we present the conclusions of the papers that make up this thesis : **On how to allocate the fixed cost of transport systems** (Estañ et al. (2021a)), **Manipulability in the cost allocation of transport systems** (Estañ et al. (2020)) and **On the difficulty of budget allocation in claims problems with indivisible items and prices** (Estañ et al. (2021b)).

9.1 Cost sharing problems conclusions

In the paper **On how to allocate the fixed cost of transport systems** (Estañ et al. (2021a)), we present a model to study the problem of dividing a fixed cost of a tram line among the municipalities along that line. The allotment rule depends on the set of municipalities, the stations in each municipality, the cost to distribute and the flow matrix. We have shown that, if we require that this rule satisfies basic notions of fairness and stability we end up with uniqueness in the solutions. More precisely, we have found out that null municipality (a municipality without users is exempted of payment), symmetry in municipalities (symmetric municipalities contributes equally), and weighted additivity (the allocation is immune to splitting of the problem) lead to the station-based proportional rule, which divides the cost proportional to the number of passengers that use the stations in each city. Similarly, we have also characterized track-based proportional rule in terms of adjacent symmetry (symmetry with respect to adjacent stations), weak null municipality, trip decomposition (the rule is not altered by splitting a long trip into small ones) and weighted additivity. Finally, we have proved that, if we require the rule to fulfill with symmetry in municipalities , bilateral ratio consistency (stability with respect to changes in the set of municipalities) and weighted additivity, then we must allocate the cost uniformly among the cities.

As it is the case of many other works in the literature of cost sharing (Sánchez-Soriano et al. (2002), Ni and Wang (2007), or Kuipers et al. (2013), for instance), we could have naturally define a cost game as follows. The set of player is the set of municipalities M and the value function for each coalition $S \subset M$ is given by

$$c(S) = \begin{cases} 0 & \text{if } S = \varnothing \\ C & \text{if } S \neq \varnothing \end{cases}$$

The function c is constant due to the nature of the problem we study, not matter the number of identity of the municipalities involved in the coalition, the cost they must afford is always C, since, for example, the salary of the CEO has to be paid regardless the use or structure of the network. Once we apply the basic solution concepts in the literature on TU-games to our case we end up in non completely satisfactory allocations. The Shapley value and the nucleolus are the uniform split $\big(x = \big(\frac{C}{m}, \ldots, \frac{C}{m}\big)\big)$, the core is the whole set of feasible allocations ($\{x : \sum_{i=1}^{m} x_i = C\}$).

As Shapley (1953) shows, the unique solution of the previous cost game that satisfies symmetry in municipalities, null player and additivity is the Shapley value, which, in our case, coincides with the uniform rule. Interestingly, Theorem proves that, if these same principles are translated into our model then we characterize the station-based proportional rule instead. The uniform rule (or equivalently, the Shapley values of the cost game) is also characterized by using alternative properties.

In the paper **Manipulability in the cost allocation of transport systems** (Estañ et al. (2020)), we have shown that symmetry in stations and non-manipulability (via merging and splitting) characterize the station proportional rule. This result follows the line of some others in the literature for different models (see, for example, de Frutos (1999), Moreno-Ternero (2006) and Ju et al. (2007)), in which the absence of manipulability is closely related with proportional mechanisms. However, in our setting, proportionality is not unambiguously determined, since it can refer to the stations, flows, or even a combination of both. The theorem obtained states that the properties we consider are only compatible with one of those.

9.2 Scarce resource distribution problems conclusions

To end, in the paper **On the difficulty of budget allocation in claims problems with indivisible items and prices** (Estañ et al. (2021b)) we have studied a particular class of claims problems. In our model a group of agents demand several units of different items, each of which has a price. The available estate is not sufficient to satisfy the aggregate claim. A rule is a multi-valued function that selects a set of allocations, which indicate the amount of units of each item that is assigned to each claimant.

In contrast with other models involving claims problems, efficiency cannot be guaranteed. The closest requirement is non-wastefulness, which states that the rule should waste as little estate as possible, and is closely related to the so called bounded knapsack problem, whose solutions, in general, are difficult to obtain. Even though, with this milder condition of efficiency, we find that there is no rule that satisfies non-wastefulness together with other criteria that protect small agents or ensure claimants receive a minimum allocation.

In view of all the impossibility results obtained in this study, we can observe that it is not easy to reconcile efficiency (via non-wastefulness) with fairness. In this sense, perhaps, the non-wastefulness property is too restrictive, because it is focused more on the use of the estate than the fairness of the allocation. Therefore, one could reconsider the absolute necessity of the non-wastefulness property and simply guarantee that the maximum amount of the state is distributed, while respecting certain properties of fairness in the distribution. This would be the second-best alternative, which is beyond the purpose of this paper, once we have proved that the straightforward approach entails many difficulties.

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Appendix A

On how to allocate the fixed cost of transport systems.

On how to allocate the fixed cost of transport systems

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Abstract
In this study, we consider different cities located along a tram line. Each city may have one or several stations and information is available about the flow of passengers between any pair of stations. A fixed cost (salaries of the executive staff, repair facilities, or fixed taxes) must be divided among the cities. This cost is independent of the number of passengers and the length of the line. We propose a mathematical model to identify suitable mechanisms for sharing the fixed cost. In the proposed model, we study, and characterize axiomatically, three rules, which include the uniform split, the proportional allocation and an intermediate situation. The analyzed axioms represent the basic requirements for fairness and elemental properties of stability.

Keywords Axiom · Cost game · Cost sharing · Fairness · Transport networks

1 Introduction

Determining how to divide the costs of constructing and maintaining different types of infrastructure has become increasingly important because it requires the cooperation of several institutions, states, regions, or countries. Transport networks are special cases, for example: railroads that are planned at European level and cross more than one country, highways that involve several regions, or metro and tram lines that span across different cities. The maintaining cost, as in general any other global cost, can be split into two parts: the variable cost (which depends on the intensity in use such as flows of passengers, length of the line, ...)

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and the fixed cost (the salaries of the executive staff, the maintenance of the railway yard, the payment of some fixed local taxes, and other expenses that do not depend on the usage). The problem of how to distribute the cost among the participating agents (countries, regions, cities, ...) has been studied in many works (see Henriet and Moulin 1996; Kuipers et al. 2013; Littlechild and Owen 1973 and Ni and Wang (2007), among others). These papers mainly focused on costs depending on certain elements of the problem, the intensity in use, or the length of the route. However, in this work we analyze the allocation of the fixed cost which is invariant.

It is also worth noting that here we consider the perspective of a central planner. Imagine that the European Union builds a high-speed line from Lisbon to Amsterdam across several nations with stations in the main cities. The EU constructs the infrastructure but the further maintenance is left to the individual countries. Once the line is made, countries cannot decide to exclude themselves from the maintenance cost. In this type of situations, participation is compulsory and we cannot address the problem from a cooperative perspective, exclusively. A superior authority chooses a scheme to determine the contribution of each agent on the basis of some fairness and stability criteria. To sum up, our goal is to develop several schemes a central planner may use in order to determine the contribution of each agent to the fixed cost of maintaining an infrastructure.

Let us assume that a tram line passes through different cities and each city may have one or several stations. Information is available about the number of passengers between any pair of stations, and thus how many people use each station (which can be treated as an indicator of its importance). Finally, a cost must be divided among the cities involved on the line. In summary, the *problem* has four elements comprising the set of cities located along the line, the sets of stations that belong to each city, a flow matrix that indicates the number of users between any pair of stations, and a cost that needs to be split (which is not a function of the previous elements, unlike the problems considered in other studies).

The present study addresses the axiomatic analysis of cost allocation rules in networks, where a *rule* is simply a mechanism for distributing the cost among the cities. Economic networks and the axiomatic methodology were surveyed by Sharkey (1995) and Thomson (2001), respectively. In the axiomatic method, the rules are justified in terms of the axioms or properties that they satisfy. In general, suitable combinations of properties are imposed as the desirable or minimal requirements that the rule must satisfy. The goal is then to identify the solutions or unique solution that satisfy these axioms. Thus, in this study, we introduce a collection of properties that are suitable for the framework considered.

The first group of properties imposes the basic requirements in terms of equity in the allocation of the cost. In particular, *null municipality* and *weak null municipality* state the conditions under which a city should be exempted from contributing. *Symmetry* and *adjacent symmetry* require that cities which can be consider as equals should pay equal amounts. The second group of properties are related to the stability of the rule with respect to changes in the problem. Thus, *additivity* and *weighted additivity* require that the final distribution of the cost is not altered if we split the problem into several subproblems (e.g., distributing the cost yearly is equivalent to spreading it monthly and then aggregate them). *Bilateral ratio consistency* requires that the relative ratio of the contributions by two cities does not change if a third leaves the consortium. Finally, *trip decomposition* requires that the distribution of the cost is not altered when passengers split the same long trip into smaller ones.

We show that if symmetry and additivity (or, alternatively, symmetry, bilateral ratio consistency, and weighted additivity) are required, then we must distribute the cost uniformly among the cites, regardless of the number of stations and the flows of passengers. We also show that a unique rule exists that is compatible with null municipality, symmetry, and weighted additivity. This rule is the *station-based proportional rule*, which divides the cost proportionally according to the number of users in each city. Finally, we prove that the *track-based proportional rule* is the only method that satisfies adjacent symmetry, weak null municipality, trip decomposition, and weighted additivity.

1.1 Related literature

The problem of allocating the cost of building or maintaining a facility among all agents who are involved in providing a service is a classic problem addressed in cost sharing research. In the well-known airport problem, the Shapley value (Shapley 1953) is obtained with low computational effort (see Littlechild and Owen 1973). In this problem, the costs of using an airstrip must be shared among the planes that operate in the airport while considering their different sizes. The infrastructure of this problem is a line, as found in our problem, that is shared from the start by all of the planes. The different sizes of planes need different lengths of tracks, so the value function for the associated cost game varies according to the coalition of players. In our model, by contrast, the cost is always the same and the value function is constant. A broad survey of airport problems was provided by Thomson (2007). The idea proposed by Littlechild and Owen (1973) was generalized as the so-called "painting stories" by Maschler et al. (2010) and Bergantiños et al. (2014).

The costs associated with highway profit sharing were considered by Kuipers et al. (2013) and Sudhölter and Zarzuelo (2017), where each agent used consecutive sections of a highway. This problem differs from airport problems because the sections used by an agent do not need to start from the beginning of the line. These works apply a cooperative approach. We, alternatively, consider a centralized point of view. Besides, as we explain in Sect. 3, the natural cost function in our setting is constant (the same for all the possible coalitions), while in these papers the cost each coalition has to face depends on the sections used by the agents involved.

Several studies have addressed cost sharing for the problem of cleaning a polluted river. In this problem, the river is treated as a segment divided into subsegments where each belongs to a region/municipality. A central agency determines the cost of cleaning each segment. The seminal study of this problem was conducted by Ni and Wang (2007) who proposed two methods (*local responsibility sharing* and *upstream equal sharing*) for determining the allocations, which are the Shapley values of two transferable utility (TU) games. van den Brink and van der Laan (2008) showed that this problem is essentially an airport problem. Gómez-Rúa (2011) provided axiomatic characterizations for a family of rules by using properties based on water taxes, where one of these rules matched with the weighted Shapley value. Alcalde-Unzu et al. (2015) characterized an *upstream responsibility rule* for assigning a region located in a segment with a value in terms of its responsibility by taking as the transfer rate the mid-point between its lower and upper limits. The remaining cost was divided among the upstream regions. The problem of cleaning a polluted river essentially considers the line along which the agents are located, as found in our problem. However, the river only flows downstream, whereas tram lines carry passengers in both directions. In these models, the overall cost to distribute is mainly the sum of the individual costs of each region. If a group of regions cooperate, they just face the cost to clean their corresponding segments, which, in general, is smaller that the global cost. In contrast, in our framework, such a possibility does not exist, and the fixed cost to pay is always the same, regardless the coalition structure.

An interesting application of cost allocation in public transport was provided by Sánchez-Soriano et al. (2002) who considered how to share the cost of transport for university students.

Other studies also investigated the distribution of transportation. Thus, Algaba et al. (2019) studied the problem of sharing revenues among transport companies in a multimodal transport system that cooperates by offering tickets for using all transport modes. In this model, multiple arcs between two nodes represent the different companies that provide services between each pair of stations. Similar to our model, the intensity of use is represented by a matrix of flows. They introduced the colored egalitarian solution, which is the Shapley value of a convex TU-game and it is located in the core. Another interesting situation was studied by Slikker and Nouweland (2000) and Norde et al. (2002) who investigated the allocation of the fixed and variable costs of a railway network among the trains using this infrastructure.

Research into revenue and cost sharing in networks has also provided satisfactory results in other situations such as analyses of a terrestrial flight telephone system (see van den Nouweland et al. 1996) and power networks (see Bergantiños and Martínez 2014).

Our proposal has two key features whose combination, up to our knowledge, has not been previously analyzed. One, most of the papers mentioned above consider the allocation of the variable part of the cost, the one that is directly related with the use of the infrastructure and, therefore, include rules and properties related to those variable aspects. More specifically, the amount to be allocated depends on other elements of the problem, and thus, it can vary upon the rest of the variables in the model. By contrast, we focus on the distribution of the fixed part of the cost invariant with respect all other elements of the problem, whose nature may demand for different solutions. And two, we consider the centralized approach, by which the scheme to apply is decided by a external authority not involved in the problem. This is an alternative to the cooperative point of view followed in many of the aforementioned works. Besides, unlike other aforementioned papers, if a group of municipalities decides to form a coalition, the cost the would have to face is the same as the cost of the grand coalition, since the salaries of the executive staff, the maintenance of the railway yard, etc have to be paid anyway. This fact shrinks the potential use of allocation mechanisms based on coalitional considerations.

The remainder of this paper is organized as follows. In Sect. 2, we present the model and the elements of the problem. In Sect. 3, we define the uniform rule for our setting. In Sect. 4, we introduce the basic set of properties we consider in this study. Section 5 is devoted to the characterization of the uniform rule. In Sect. 6, we define and characterize two rules that are more intense in the use of the available information. We conclude by giving some final remarks in Sect. 7. Appendix A contains the proofs of the tightness of the characterizations.

2 Mathematical model

Let $M = \{1, \ldots, m\}$ ($m \geq 3$) be the set of **municipalities**. Let $S = \{s_1, \ldots, s_n\}$ be an ordered set of **stations**, which are located on a line. For a given station s_h we assume that s_{h-1} and s_{h+1} are located to the left and right of s_h, respectively. Each station belongs to one (and only one) municipality. We denote by S_i the set of stations in municipality i. We assume that all S_i are connected with respect to the line, i.e., if two stations belong to i, then any intermediate station also belongs to i. More formally, if $s_h, s_{h+l} \in S_i$ then $s_g \in S_i$ for all $g \in [h, h+l]$.

The flows of passengers are described by a **flow matrix** (denoted by OD), which specifies the number of people that use the line between each pair of stations.

$$OD = \begin{pmatrix} 0 & \omega_{12} & \dots & \omega_{1n} \\ \omega_{21} & 0 & \dots & \omega_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \omega_{n1} & \omega_{n2} & \dots & 0 \end{pmatrix} \in \mathbb{R}_{+}^{n\times n},$$

where ω_{gh} is a measure of the number of passengers whose trip starts in station s_g and end in station s_h. We assume that at least one entry of OD is different from zero. Finally, the network has a **fixed cost**, $C \in \mathbb{R}_{+}$, which must be distributed among the municipalities in M.

The allocation problem, or simply the **problem**, is defined by the 4-tuple $a = (M, S, OD, C)$. The class of all these allocation problems is denoted by $\mathbb{A}$.

Example 1 Consider the case of a trolley line that passes across three municipalities $M = \{1, 2, 3\}$ with four stations $S = \{s_1, s_2, s_3, s_4\}$ that are distributed as follows:

$$\overbrace{s_1}^{S_1} \text{———} \overbrace{s_2 \text{———} s_3}^{S_2} \text{———} \overbrace{s_4}^{S_3}$$

The fixed cost is $C = 12$ and the flow matrix is

$$OD = \begin{pmatrix} 0 & 4 & 2 & 0 \\ 1 & 0 & 1 & 0 \\ 5 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

where $\omega_{12} = 4$ means that four people are traveling from s_1 to s_2, and $\omega_{21} = 1$ indicates that only one person is traveling from s_2 to s_1.

For a given flow matrix OD and a given pair of municipalities $\{i, j\} \subseteq M$, the number of passengers whose trip starts in one of the stations in municipality i and ends in one of the stations in municipality j is denoted by $\Omega_{ij}(OD)$, i.e.:

$$\Omega_{ij}(OD) = \sum_{s_g \in S_i} \sum_{s_h \in S_j} \omega_{gh}.$$

Note that when $i = j$, $\Omega_{ii}(OD)$ gives the number of people who travel within municipality i. Similarly, we define $\Omega_i^{+}(OD)$ and $\Omega_i^{-}(OD)$ as the number of passengers that start and end in any of the stations of municipality i, respectively:

$$\Omega_i^{+}(OD) = \sum_{s_g \in S_i} \sum_{s_h \in S} \omega_{gh} = \sum_{j \in M} \Omega_{ij}(OD)$$

and

$$\Omega_i^{-}(OD) = \sum_{s_g \in S_i} \sum_{s_h \in S} \omega_{hg} = \sum_{j \in M} \Omega_{ji}(OD).$$

Thus, the use of the stations in i is given by:

$$\Omega_i(OD) = \Omega_i^{+}(OD) + \Omega_i^{-}(OD).$$

Finally, $\Omega(OD)$ denotes the total number of passengers involved in the flow matrix:

$$\Omega(OD) = \|OD\|_1 = \frac{1}{2}\sum_{i=1}^{m}\Omega_i(OD).$$

Example 2 For Example 1, we find that $\Omega(OD) = 15$,

$$\begin{pmatrix} \Omega_{11}(OD) & \Omega_{21}(OD) & \Omega_{31}(OD) \\ \Omega_{12}(OD) & \Omega_{22}(OD) & \Omega_{32}(OD) \\ \Omega_{13}(OD) & \Omega_{23}(OD) & \Omega_{33}(OD) \end{pmatrix} = \begin{pmatrix} 0 & 6 & 0 \\ 6 & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and

i	$\Omega_i^+(OD)$	$\Omega_i^-(OD)$	$\Omega_i(OD)$
1	6	6	12
2	9	9	18
3	0	0	0

An **allocation** for $a \in \mathbb{A}$ is a distribution of the fixed cost among the municipalities, i.e., a vector $x \in \mathbb{R}_+^M$ such that $\sum_{i \in M} x_i = C$. Let $X(a)$ be the set of all allocation vectors for $a \in \mathbb{A}$. A **rule** is a procedure for selecting allocation vectors, i.e., a function, $R : \mathbb{A} \longrightarrow \bigcup_{a \in \mathbb{A}} X(a)$, that selects a unique allocation vector $R(a) \in X(a)$ for each problem $a \in \mathbb{A}$.

3 The uniform rule

As we have already mentioned in the introduction, we consider the perspective of a central authority that has to divide the cost among the municipalities, where these are obliged to participate. However, the cooperative approach may also be of interest and, perhaps, a good starting point. So, let us assume for a moment that groups or municipalities have the possibility to coordinate their actions in some way. More specifically, let us suppose that cities may voluntarily form coalitions $T \subseteq M$ whose joint cost can be represented by means of a value or cost function $c(T)$. We have to divide the cost of the grand coalition $c(M)$ taking into account the cost structure of the different collaborations given by the cost function. How should we do that? Can we distribute the cost C in a way such that all municipalities would be willing to participate in case they have the chance? The literature on cooperative game theory provides enough instruments to answer this type of question. To start this analysis we define a cost game.

As shown in many previous studies of cost sharing in very different contexts (e.g., Sánchez-Soriano et al. 2002; Ni and Wang 2007; Aparicio and Sánchez-Soriano 2008; Kuipers et al. 2013; Platz and Hamers 2015; Li et al. 2019), we can naturally define a cost game as follows. The set of players is the set of municipalities M and the value function for each coalition $T \subset M$ is given by the following:

$$c(T) = \begin{cases} 0 & \text{if } T = \varnothing \\ C & \text{if } T \neq \varnothing. \end{cases}$$

The function c is constant due to the nature of the problem considered, so regardless of the number or identity of the municipalities involved in the coalition, the cost they incur is always C, e.g., the salary of the CEO must be paid irrespective of the use or the structure of the network.

For a given cost game c, a solution is a vector $x \in \mathbb{R}^n_+$ such that $\sum_{i \in M} x_i = c(M) = C$, where x_i represents the allocation to player i. Several authors have proposed different solution concepts based on different notions of fairness. Among those, the *Shapley value* (Shapley 1953) emerges as the most prominent one. Its expression is the following

$$x_i = \sum_{T \subset M \setminus \{i\}} \frac{|T|!(m - |T| - 1)!}{m!}(c(T \cup \{i\}) - c(T)), \forall i \in M,$$

where $|T|$ is the cardinal of T.

Alternative solutions concepts are *the nucleolus* (Schmeidler 1969), *the modified nucleolus* (Sudhölter 1997), or the *Dutta-Ray solution* (Dutta and Ray 1989), among others.

If we apply the aforementioned solutions to our cost game, we obtain that they result in the equal split of the cost $\left(x = \left(\frac{C}{m}, \ldots, \frac{C}{m}\right)\right)$, obviating the rest of the available information.[1]

The core c is the set of all allocations that distributes $c(M) = C$ with the property that no coalition would be better off if it would separate and pay its cost:

$$C(c) := \left\{ x \in \mathbb{R}^m | \sum_{i \in M} x_i = c(M) \text{ and } \sum_{i \in T} x_i \leq c(T) \text{ for all } T \in 2^M \right\}.$$

In our model, the core coincides with the whole set of feasible allocations $\left(\left\{x : \sum_{i=1}^m x_i = C\right\}\right)$, and therefore it is not very informative.

Even though the *uniform rule* does not exploit all the information the central planner has, it is obviously a focal point. The reason is twofold. One, although in our setting a central authority decides the rule to apply, we have illustrated above that the equal allotment emerges as the natural solution even from a cooperative perspective. And two, it is reasonable to think that, when the cost is fixed, all municipalities must contribute equally.

Uniform rule For each $a \in \mathbb{A}$ and each $i \in M$,

$$U_i(a) = \frac{C}{m}.$$

4 Properties

Our first property imposes a minimal criterion of equity, where it requires that symmetric municipalities must contribute equally. We say that two municipalities are *symmetric* if the same number of passengers travel within them and between them and any other third municipality.

Symmetry For each $a \in \mathbb{A}$ and each $\{i, j\} \subseteq M$, if $\Omega_{ii}(OD) = \Omega_{jj}(OD)$, and $\Omega_{ik}(OD) = \Omega_{jk}(OD)$, and $\Omega_{ki}(OD) = \Omega_{kj}(OD)$), for all $k \in M \setminus \{i, j\}$. Then $R_i(a) = R_j(a)$.

Consider that, instead of allocating the fixed cost C for a whole year according to the traffic in that period, we may solve the problem month by month. Thus, for each month, we distribute $\frac{C}{12}$ by considering the passengers flows only in that month and we then aggregate for the 12 months. Additivity requires that the final allocation must be the same regardless of whether we solve the problem yearly or monthly (and then we aggregate).

[1] In fact, any symmetric solution for the cost game leads to the uniform allocation.

Additivity For each $(M, S, OD, C) \in \mathbb{A}$ and each $i \in M$,

$$R_i\,(M, S, OD, C) = \sum_{t=1}^{T} R_i\,(M, S, OD_t, C_t)\,,$$

where $OD = \sum_{t=1}^{T} OD_t$ and $C = \sum_{t=1}^{T} C_t$.

In some context additivity may be very demanding as it is the case in our context as the following proposition shows.

Proposition 1 *If a rule satisfies additivity then it does not depend on the flow matrix.*

Proof Let $OD, OD' \in \mathbb{R}^{n\times n}$ be two different flow matrices. Let us define $\overline{OD} \in \mathbb{R}^{n\times n}$ such that

$$\overline{\omega}_{gh} = \min\left\{\omega_{gh}, \omega'_{gh}\right\}.$$

Then $OD = \overline{OD} + (OD - \overline{OD})$ and $OD' = \overline{OD} + (OD' - \overline{OD})$. By applying *additivity*,

$$\begin{aligned} R(M, S, OD, C) &= R(M, S, \overline{OD}, C) + R(M, S, OD - \overline{OD}, 0) \\ &= R(M, S, \overline{OD}, C) \\ &= R(M, S, \overline{OD}, C) + R(M, S, OD - \overline{OD'}, 0) \\ &= R(M, S, OD', C) \end{aligned}$$

and thus, the allocation is the same for any flow matrix. □

The previous result illustrates the strength of additivity in our setting. If it is required, then the information on the flows must be ignored. As a weaker version we consider *weighted additivity*. It is in the line of additivity, but the allocation of each period (month, year, ...) is weighted by the inverse of the cost per passenger in that period.

Weighted additivity For each $(M, S, OD, C) \in \mathbb{A}$ and each $i \in M$,

$$\frac{\Omega(OD)}{C} R_i\,(M, S, OD, C) = \sum_{t=1}^{T} \frac{\Omega(OD_t)}{C_t} R_i\,(M, S, OD_t, C_t)\,,$$

where $OD = \sum_{t=1}^{T} OD_t$ and $C = \sum_{t=1}^{T} C_t$.

Weighted additivity can be seen as a compromise, since it still imposes the rule to be additive in some way and, at the same time, it allows us to exploit the flows of passengers.[2]

The next property requires some sort of independence for the rule with respect to changes in the set of municipalities. More precisely, given a rule, we consider a problem and apply the rule to the problem. Imagine now that all of the municipalities but two are excluded from the consortium, and thus these two municipalities must meet all the costs themselves. In its general formulation, the principle of *consistency* requires that in the reduced problem the contributions of these two municipalities are the same as in the original problem. This definition cannot be directly applied to our setting. Since the cost is fixed, the allocations in the reduced problem must add up to C, and therefore they cannot be the same any more.

In the line of the consistency principle, we introduce a new property that states that the solution proposed for the reduced problem must be consistent with the proposal for the

[2] Alternative versions of pure additivity have also been explored in other settings such as airport games (Dubey 1982) or claims problems (Harless 2016).

original one. Hence, *bilateral ratio consistency* requires that when the new situation is re-evaluated and the cost is divided between these two municipalities, the ratio between their allocations for the new problem is the same as the ratio between their allocations for the original problem.

The reduced problem is defined in a natural manner, where the set of municipalities still comprises the same two in the original situation (say $N = \{i, j\}$), the set of stations is formed by all the stations that belong to either i or j (and only those), the fixed cost C is the same as in the original problem, and the reduced flow matrix $OD_{\{i,j\}}$ is obtained by removing in OD all the columns and rows that correspond to stations not in N.

Bilateral ratio consistency For each $a = (M, S, OD, C) \in \mathbb{A}$ and each pair of municipalities $\{i, j\} \subseteq M$ we have that

$$\frac{R_i(a)}{R_j(a)} = \frac{R_i(a_{\{i,j\}})}{R_j(a_{\{i,j\}})},$$

where $a_{\{i,j\}} = \big(\{i, j\}, S_i \cup S_j, OD_{\{i,j\}}, C\big)$.

5 Characterizations of the uniform rule

The next theorem states that if we simply require symmetry and additivity, then the cost is split equally among all the municipalities regardless of the number of stations and their users. The independence of the properties is relegated to Appendix A.

Theorem 1 *A rule satisfies symmetry and additivity if and only if it is the uniform rule.*

Proof First, we check that the uniform rule satisfies the two properties of the statement.

(a) Symmetry. Let $\{i, j\} \subseteq M$ such that $\Omega_{ii}(OD) = \Omega_{jj}(OD)$, and $\Omega_{ik}(OD) = \Omega_{jk}(OD)$ and $\Omega_{ki}(OD) = \Omega_{kj}(OD)$ for all $k \in M \setminus \{i, j\}$. Then, it holds that $\Omega_i(OD) = \Omega_j(OD)$. By definition of the uniform rule, we immediately find that $U_i(a) = U_j(a)$.
(b) Additivity. Now, for each $i \in M$,

$$\sum_{t=1}^{T} U_i\left(M, S, OD_t, C_t\right) = \sum_{t=1}^{T} \frac{C_t}{m} = \frac{C}{m} = U_i\left(M, S, OD, C\right).$$

Let us consider the converse. Let R be a rule that is symmetric and additive. We show that $R = U$.

Notice that, in application of Proposition 1, we already know that the rule does not depend on the flow matrix. Now, let us consider that $R \neq U$, then there exists a problem $a = (M, S, OD, C) \in \mathbb{A}$ such that there are at least two municipalities i, j with $R_i(a) \neq R_j(a)$. Now we consider the problem $a' = (M, S, OD', C) \in \mathbb{A}$ such that municipalities i and j are *symmetric*. On the one hand, since R does not depend on the flow matrix because it satisfies *additivity*, we have that $R(a) = R(a')$ which implies that $R_i(a') \neq R_j(a')$, but on the other hand, since R satisfies *symmetry*, $R_i(a') = R_j(a')$. Therefore, we obtain a contradiction and R must be equal to U. $\square$

The previous theorem characterizes the uniform rule by means of just two properties. If we substitute additivity by its weaker version other rules may emerge. However, Theorem 2 shows that, if we replace additivity by the combination of weighted additivity and bilateral ratio consistency, we go back to the uniform rule.

Theorem 2 *A rule satisfies symmetry, bilateral ratio consistency, and weighted additivity if and only if it is the uniform rule.*

Proof First, we check that the uniform rule satisfies bilateral ratio consistency and weighted additivity.[3]

(a) Bilateral ratio consistency. Let us consider the problems where $a = (M, S, OD, C)$ and $a_{\{i,j\}} = \big(\{i, j\}, S_i \cup S_j, OD_{\{i,j\}}, C\big)$, and if we apply the uniform rule, we obtain the following.

$$\frac{U_i(a)}{U_j(a)} = \frac{\frac{C}{m}}{\frac{C}{m}} = 1 = \frac{\frac{C}{2}}{\frac{C}{2}} = \frac{U_i(a_{\{i,j\}})}{U_j(a_{\{i,j\}})}.$$

(b) Weighted additivity. Now, for each $i \in M$,

$$\begin{aligned}
\sum_{t=1}^{T} \frac{\Omega(OD_t)}{C_t} U_i\left(M, S, OD_t, C_t\right) &= \sum_{t=1}^{T} \frac{\Omega(OD_t)}{C_t} \frac{C_t}{m} \\
&= \frac{1}{m} \sum_{t=1}^{T} \Omega(OD_t) \\
&= \frac{C}{m} \frac{\Omega(OD)}{C} \\
&= \frac{\Omega(OD)}{C} \cdot U_i\left(M, S, OD, C\right).
\end{aligned}$$

Let us consider the converse. Let R be a rule that is symmetric, bilateral ratio consistent, and weighted additive. We show that $R = U$. Let $a^{gh} = (M, S, OD^{gh}, C) \in \mathbb{A}$ be a problem such that all of the entries in the flow matrix are null, except for the entry gh that is equal to one ($\omega_{ef} = 0$ for all $(e, f) \neq (g, h)$, and $\omega_{gh} = 1$). In this case, there are two municipalities (not necessarily different) $i, j \in M$ to which these stations belong, i.e., $s_g \in S_i$ and $s_h \in S_j$. The flow matrix for problem a^{gh} has the following form

$$OD^{gh} = \begin{array}{c c} & \begin{array}{ccccc} & & h & & \end{array} \\ g & \begin{pmatrix} 0 & \dots & 0 & \dots & 0 \\ \vdots & \ddots & 0 & \ddots & \vdots \\ 0 & \dots & 1 & \dots & 0 \\ \vdots & \ddots & 0 & \ddots & \vdots \\ 0 & \dots & 0 & \dots & 0 \end{pmatrix} \end{array}.$$

We note that for any other pair of municipalities $\{k, l\} \neq \{i, j\}$, the reduced flow matrix $OD^{gh}_{\{k,l\}}$ is null (all of the entries are equal to zero). Therefore, k and l are *symmetric* in the reduced problem $\left(\{k, l\}, S_k \cup S_l, OD^{gh}_{\{k,l\}}, C\right)$, and thus due to the *symmetry*, it must be true that

$$R_k\left(\{k, l\}, S_k \cup S_l, OD^{gh}_{\{k,l\}}, C\right) = R_l\left(\{k, l\}, S_k \cup S_l, OD^{gh}_{\{k,l\}}, C\right) = \frac{C}{2}.$$

[3] The fulfillment of symmetry is already proved in Theorem 1.

By applying the *bilateral ratio consistency*, we also know that

$$\frac{R_k\left(a^{gh}\right)}{R_l\left(a^{gh}\right)} = \frac{R_k\left(\{k,l\}, S_k \cup S_l, OD^{gh}_{\{k,l\}}, C\right)}{R_l\left(\{k,l\}, S_k \cup S_l, OD^{gh}_{\{k,l\}}, C\right)} = 1 \quad \Leftrightarrow \quad R_k\left(a^{gh}\right) = R_l\left(a^{gh}\right).$$

This fact combined with the requirement that $\sum_{k=1}^{m} R_k\left(a^{gh}\right) = C^{gh}$ imply that $R_k\left(a^{gh}\right) = \frac{C^{gh}}{m}$ for all $k \in M$.

Let $a = (M, S, OD, C) \in \mathbb{A}$ be a problem without any restriction. We can additively split problem a into other $\Omega(OD)$ problems a^{gh} such that

$$OD = \sum_{gh} OD^{gh} \quad \text{and} \quad C^{gh} = \frac{C}{\Omega(OD)}.$$

We already know that for each of those problems, $a^{gh} = (M, S, OD^{gh}, C^{gh})$, the *weighted additivity* implies that for each $i \in M$:

$$\frac{\Omega(OD)}{C} R_i(a) = \sum_{gh} \frac{\Omega(OD^{gh})}{C^{gh}} R_i(a^{gh}),$$

or equivalently,

$$R_i(a) = \frac{C}{\Omega(OD)} \sum_{gh} \frac{\Omega(OD^{gh})}{C^{gh}} R_i(a^{gh}).$$

We already know that $R_i(a^{gh}) = \frac{C^{gh}}{m}$ for any $i \in M$, so we have the following.

$$\begin{aligned} R_i(a) &= \frac{C}{\Omega(OD)} \sum_{gh} \frac{\Omega(OD^{gh})}{C^{gh}} R_i(a^{gh}) \\ &= \frac{C}{\Omega(OD)} \sum_{gh} \frac{\Omega(OD^{gh})}{C^{gh}} \frac{C^{gh}}{m} \\ &= \frac{C}{m} \\ &= U_i(a). \end{aligned}$$

□

6 Rules that use additional information

The nature of a fixed cost makes it invariant with respect to the potential use of the line. If a central authority desires to obviate the variable elements of the problem, it can be done, simply by applying solutions similar to the aforementioned uniform rule. If, alternatively, the government is interested in a method that considers all the available information, the solutions must be more elaborated. In this section we follow this alternative approach and present two new rules. The first one uses the stations, while the second uses the segments between stations as structural elements that define the network.

The next rule allocates the cost in proportion to the number of passengers who use the stations (to board or to get off) in a municipality.

Station-based proportional rule For each $a \in \mathbb{A}$ and each $i \in M$,

$$SP_i(a) = \frac{C}{2\Omega(OD)} \cdot \Omega_i(OD)$$

The second rule allocates the fixed cost in proportion to the use of segments of the line. To do that, it assumes that each passenger is divided into as many parts as the number of tracks used in her trip, and then each part of the passenger corresponding to a track is distributed equally between the two stations delimiting that track.

Track-based proportional rule For each $a \in \mathbb{A}$ and each $i \in M$,

$$TP_i(a) = \frac{C}{\Omega(OD)} \cdot \sum_{s_f \in S_i} \sum_{\substack{s_g, s_h \in S, s_g \neq s_h \\ f \in [g,h] \text{ or } f \in [h,g]}} \frac{\omega_{gh}}{\left(2 - \left\lceil \frac{|f-g| \cdot |h-f|}{(h-g)^2} \right\rceil\right) |h-g|},$$

where $\lceil z \rceil = min\,\{k \in \mathbb{Z} : k \geq z\}$.

Although the mathematical expression of the track-based proportional rule is a bit tricky, its application is very intuitive and easy to understand. In the next example we illustrate the functioning of the rules defined above.

Example 3 We consider the problem described in Example 1.

(a) Station-based proportional rule:

$$SP(a) = \left(\frac{12}{30} \cdot 12, \frac{12}{30} \cdot 18, \frac{12}{30} \cdot 0\right) = \left(\frac{24}{5}, \frac{36}{5}, 0\right).$$

(b) Track-based proportional rule: We use this example to illustrate the function of the formula in the definition of this rule. We start by counting the number of passengers between each pair of stations. As illustrated in Fig. 1, five passengers are traveling between s_1 and s_2, seven between s_1 and s_3, and so on. Second, we compute the use of each track. In the case of the track between s_1 and s_2, it is natural to assign to this track, the five travelers between s_1 and s_2, half of the travelers between s_1 and s_3 (because it is a half of their whole trip), and a third of the travelers between s_1 and s_4 (because this track represents only a third of their whole journey). The dashed line in the figure shows the use of each of the three tracks in this example.
Now, we must allocate the utilization of the tracks to the stations. The simplest approach is to assume that the use is split equally between the two stations at the extreme ends of the track (Fig. 2).
Finally, the cost is divided in proportion to the use of the tram line by each municipality, which is the sum of the uses of its stations. Hence,

$$TP(a) = \left(\frac{12}{15} \cdot \frac{17}{4}, \frac{12}{15} \cdot \frac{43}{4}, \frac{12}{15} \cdot 0\right) = \left(\frac{51}{15}, \frac{129}{15}, 0\right).$$

It is quite obvious, in application of Theorem 2, that neither the station-based proportional rule nor the track-based proportional rule satisfy symmetry, bilateral ratio consistency, and weighted additivity together. Next, we explore the properties the two new rules may fulfills and we show that they characterize them.

Null municipality, states that a municipality does not have to contribute to the fixed cost if none of its stations is used by passengers. Thus, if nobody departs from or arrives at any of the stations located in a municipality, then this municipality is exempted from payment.

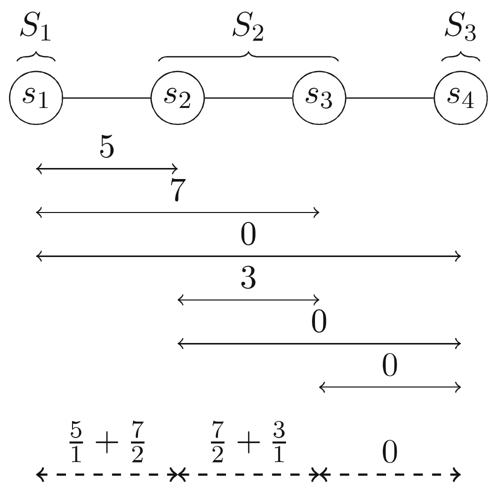


Fig. 1 Computation of the track-based proportional rule (first step)

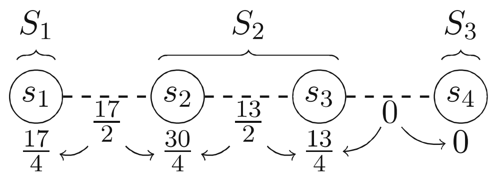


Fig. 2 Computation of the track-based proportional rule (second step)

In our context, municipalities cannot decide whether to participate or not, the line is already built and the maintenance cost has to be paid, and its distribution is decided by a superior authority. However, this central planner, may want to take into account the lack of use of the stations.

Null municipality For each $a \in \mathbb{A}$ and each $i \in M$, if $\Omega_i(OD) = 0$ then $R_i(a) = 0$.

In Example 1, Municipality 3 is null because nobody uses its single station.

The next property is a weakening of the previous one. Now, we consider that a municipality is null if no traveler uses its stations and no train passes through its stations and tracks. We note that only municipalities at the extreme ends of the line may potentially satisfy this requirement.

Weak null municipality For each $a \in \mathbb{A}$ and each $i \in M$, if one of the following two conditions holds

- $\omega_{gh} = \omega_{hg} = 0$, for all $j \leq i$, for all $s_g \in S_j$, and all $s_h \in S_i$;
- $\omega_{gh} = \omega_{hg} = 0$, for all $j \geq i$, for all $s_g \in S_j$, and all $s_h \in S_i$;

then $R_i(a) = 0$.

Clearly, null municipality implies weak null municipality but the converse is not true.

Adjacent symmetry states that if all traffic is on the line between two adjacent stations that belong to two different municipalities, then both municipalities must contribute equally. We note that this requirement is very weak because it only imposes a minimal condition of fairness in a very specific situation.

Adjacent symmetry For each $a \in \mathbb{A}$ and each $\{i, j\} \subseteq M$, if $\omega_{gh} + \omega_{hg} = \Omega(OD)$, such that $|g - h| = 1$, and $g \in S_i$, $h \in S_j$, then $R_i(a) = R_j(a)$.

The following requirement states that the distribution of the cost is not altered by splitting a long trip into small trips. Thus, the allocation must be the same regardless of whether an individual goes from station s_g to station s_h directly or indirectly (from s_g to an intermediate station and from there to s_h). This property is in the line of other requirements that prevent manipulations in the routing. For example, Henriet and Moulin (1996) introduced a similar concept called *no transit*, and it has been more recently applied in other papers such as Moulin (2009) or Juarez and Wu (2019) under the name *routing-proofness*.

Example 4 For Example 1, the flow matrix is as follows

$$OD = \begin{pmatrix} 0 & 4 & 2 & 0 \\ 1 & 0 & 1 & 0 \\ 5 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

According to OD, there are two travelers from s_1 to s_3. Now, the two passengers split, one that goes from s_1 to s_2 and the other goes from s_2 to s_3. In this case, the flow matrix becomes as follows

$$OD' = \begin{pmatrix} 0 & 4+1 & 2-2 & 0 \\ 1 & 0 & 1+1 & 0 \\ 5 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 5 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 5 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

If the rule satisfies trip decomposition, then the distribution of the fixed cost is the same for both OD and OD'.

Trip decomposition For each $(M, S, OD, C), (M, S, OD', C) \in \mathbb{A}$. If $s_g, s_h \in S$, are stations such that $h - g > 1$, and

either

1. $\omega'_{g(g+1)} = \omega_{g(g+1)} + \frac{\omega_{gh}}{|h-g|}$; $\omega'_{(g+1)(g+2)} = \omega_{(g+1)(g+2)} + \frac{\omega_{gh}}{|h-g|}, \ldots, \omega'_{(h-1)h} = \omega_{(h-1)h} + \frac{\omega_{gh}}{|h-g|}$; and $\omega'_{gh} = 0$;
2. $\omega'_{ef} = \omega_{ef}$, if $(ef) \neq (gh)$,

or

1. $\omega''_{h(h-1)} = \omega_{h(h-1)} + \frac{\omega_{hg}}{|h-g|}$; $\omega''_{(h-1)(h-2)} = \omega_{(h-1)(h-2)} + \frac{\omega_{hg}}{|h-g|}, \ldots, \omega'_{(g+1)g} = \omega_{(g+1)g} + \frac{\omega_{hg}}{|h-g|}$; and $\omega''_{hg} = 0$;
2. $\omega''_{ef} = \omega_{ef}$, if $(ef) \neq (hg)$,

then, $R(M, S, OD, C) = R(M, S, OD', C)$ and $R(M, S, OD, C) = R(M, S, OD'', C)$.

Now, we present the characterizations of the rules introduced in this section. Our first result states that the unique rule that fulfills null municipality, symmetry, and weighted additivity is the rule that divides the cost in proportion to the number of users in each municipality. The independence of the properties is relegated to Appendix A.

Theorem 3 *A rule satisfies null municipality, symmetry, and weighted additivity if and only if it is the station-based proportional rule.*

Proof First, we show that the station-based proportional rule satisfies the three properties in this statement.

(a) Null municipality. If $\Omega_i(OD) = 0$ then $SP_i(a) = 0$.
(b) Symmetry. Let $\{i, j\} \subseteq M$ such that $\Omega_{ii}(OD) = \Omega_{jj}(OD)$, and $\Omega_{ik}(OD) = \Omega_{jk}(OD)$ and $\Omega_{ki}(OD) = \Omega_{kj}(OD)$ for all $k \in M \setminus \{i, j\}$. Then, it holds that $\Omega_i(OD) = \Omega_j(OD)$. By definition of the station-based proportional rule, we immediately find that $SP_i(a) = SP_j(a)$.
(c) Weighted additivity. First, it is easy to prove that $\Omega_{ij}(OD) = \sum_{t=1}^{T} \Omega_{ij}(OD_t)$. Thus, we can find that $\Omega_i(OD) = \sum_{t=1}^{T} \Omega_i(OD_t)$. Now, for each $i \in M$, we have the following.

$$\sum_{t=1}^{T} \frac{\Omega(OD_t)}{C_t} SP_i(M, S, OD_t, C_t) = \sum_{t=1}^{T} \frac{\Omega(OD_t)}{C_t} \frac{C_t}{2\Omega(OD_t)} \cdot \Omega_i(OD_t)$$

$$
\begin{aligned}
&= \frac{\Omega(OD)}{C} \cdot \frac{C}{\Omega(OD)} \cdot \frac{1}{2} \sum_{t=1}^{T} \Omega_i(OD_t) \\
&= \frac{\Omega(OD)}{C} \cdot \frac{C}{\Omega(OD)} \cdot \frac{1}{2} \Omega_i(OD) \\
&= \frac{\Omega(OD)}{C} \cdot SP_i\,(M, S, OD, C).
\end{aligned}
$$

Let us prove the converse, i.e., let R be a rule that satisfies null municipality, symmetry, and weighted additivity. We show that $R = SP$. Let $a^{gh} = (M, S, OD^{gh}, C) \in \mathbb{A}$ be a problem such that all of the entries of the flow matrix are null, except for the entry gh that is equal to one ($\omega_{ef} = 0$ for all $(e, f) \neq (g, h)$, and $\omega_{gh} = 1$). In this case, there are two municipalities (not necessarily different) $i, j \in M$ to which these stations belong, i.e., $s_g \in S_i$ and $s_h \in S_j$. The flow matrix for the problem a^{gh} has the following form

$$
OD^{gh} = \begin{array}{c} \\ \\ \\ g \\ \\ \\ \end{array}
\begin{array}{c}
\begin{array}{ccccc} & & h & & \end{array} \\
\begin{pmatrix}
0 & \dots & 0 & \dots & 0 \\
\vdots & \ddots & 0 & \ddots & \vdots \\
0 & \dots & 1 & \dots & 0 \\
\vdots & \ddots & 0 & \ddots & \vdots \\
0 & \dots & 0 & \dots & 0
\end{pmatrix}
\end{array}.
$$

First, we note that, for any other municipality $k \in S \setminus \{i, j\}$, $\omega_{hf} = \omega_{fg} = 0$ for all $s_h, s_g \in S_q$ and all $s_f \in S \setminus S_k$. By applying *null municipality*, we find that $R_k(a^{gh}) = 0$. In addition, municipalities i and j are symmetric in the problem a^{gh}, and thus by *symmetry*, it must be true that $R_i(a^{gh}) = R_j(a^{gh}) = \frac{C}{2}$.

Let $a = (M, S, OD, C) \in \mathbb{A}$ be a problem without any restriction. We can additively split the problem a into other $\Omega(OD)$ problems a^{gh} such that

$$
OD = \sum_{\{g,h\} \subseteq S} OD^{gh} \quad \text{and} \quad C^{gh} = \frac{C}{\Omega(OD)}.
$$

We already know that for each of those problems, $a^{gh} = (M, S, OD^{gh}, C^{gh})$. *Weighted additivity* implies that, for each $i \in M$,

$$
\frac{\Omega(OD)}{C} R_i(a) = \sum_{\{g,h\} \subseteq S} \frac{\Omega(OD^{gh})}{C^{gh}} R_i(a^{gh}),
$$

or equivalently,

$$
R_i(a) = \frac{C}{\Omega(OD)} \sum_{\{g,h\} \subseteq S} \frac{\Omega(OD^{gh})}{C^{gh}} R_i(a^{gh}).
$$

We note the following

$$
R_i(a^{gh}) = \begin{cases} 0 & \text{if } s_g, s_h \notin S_i \\ \frac{C^{gh}}{2} & \text{otherwise} \end{cases}.
$$

Therefore,

$$
\begin{aligned}
R_i(a) &= \frac{C}{\Omega(OD)} \sum_{\{g,h\}\subseteq S} \frac{\Omega(OD^{gh})}{C^{gh}} R_i(a^{gh}) \\
&= \frac{C}{\Omega(OD)} \left[\sum_{s_g\in S_i,\, s_h\in S} \frac{\Omega(OD^{gh})}{C^{gh}} \frac{C^{gh}}{2} + \sum_{s_g\in S,\, s_h\in S_i} \frac{\Omega(OD^{gh})}{C^{gh}} \frac{C^{gh}}{2} \right] \\
&= \frac{C}{\Omega(OD)} \left[\frac{1}{2}\Omega_i^{+}(OD) + \frac{1}{2}\Omega_i^{-}(OD) \right] \\
&= \frac{C}{2\Omega(OD)} \cdot \Omega_i(OD) \\
&= SP_i(a).
\end{aligned}
$$

□

Finally, the next theorem provides an axiomatic characterization of the track-based proportional rule.

Theorem 4 *The unique rule that satisfies adjacent symmetry, weak null municipality, trip decomposition, and weighted additivity is the track-based proportional rule.*

Proof First, we prove that the track-based proportional rule satisfies the four properties in the statement above.

(a) Adjacent symmetry. Let $\{i, j\} \subseteq M$, such that $g \in S_i, h \in S_j$, $|g - h| = 1$, and $\omega_{gh} + \omega_{hg} = \Omega(OD)$. Then

$$
TP_i(a) = \frac{C}{\Omega(OD)} \frac{\omega_{gh} + \omega_{hg}}{2} = \frac{C}{2} = TP_j(a).
$$

(b) Weak null municipality. Let us assume that for $i \in M$, the condition that $\omega_{gh} = \omega_{hg} = 0$ for all $j \leq i$, for all $s_g \in S_j$, and all $s_h \in S$ holds. Then, this condition implies that for all $s_g, s_h \in S_i$, and $s_f \in S_i$, such that $g \leq f \leq h$, $g \neq h$, $\omega_{gh} = \omega_{hg} = 0$. Therefore, by the definition of the track-based proportional rule, $R_i(a) = 0$. The proof is completely analogous for the other condition.

(c) Trip decomposition. Let $a = (M, S, OD, C)$, $a' = (M, S, OD', C) \in \mathbb{A}$, such that:

(a) $h - g > 1$

(b) $\omega'_{g(g+1)} = \omega_{g(g+1)} + \frac{\omega_{gh}}{|h-g|}$; $\omega'_{(g+1)(g+2)} = \omega_{(g+1)(g+2)} + \frac{\omega_{gh}}{|h-g|}, \ldots, \omega'_{(h-1)h} = \omega_{(h-1)h} + \frac{\omega_{gh}}{|h-g|}$; and $\omega'_{gh} = 0$;

(c) $\omega'_{ef} = \omega_{ef}$, if $(ef) \neq (gh)$,

then for each $i \in M$, $TP_i(a')$ is given by

$$
TP_i(a') = \frac{C}{\Omega(OD')} \cdot \sum_{s_f\in S_i} \sum_{\substack{s_d, s_e\in S, s_d\neq s_e \\ f\in[e,d] \text{ or } f\in[d,e]}} \frac{\omega'_{de}}{(2 - \lceil \frac{|f-d|\cdot|e-f|}{(e-d)^2} \rceil)|e-d|}.
$$

We distinguish two situations, as follows:

(a) $\forall s_f \in S_i$, $f \notin [g, h]$. In this situation, we have:

$$\omega'_{de} = \omega_{de} \text{ and } \omega'_{ed} = \omega_{ed}, \forall s_d, s_e \in S, \text{ such that } f \in [e, d].$$

Therefore, $TP_i(a') = TP_i(a)$.

(b) There exists $s_f \in S_i$, such that $f \in [g, h]$. In this situation only the trips in ascending and consecutive order from g to h are different from a to a'. Therefore, we will now focus on these trips. We distinguish three cases, as follows:

i. If we assume that $g < f = g + k < h$, $k \in \mathbb{Z}_+$, then we only have to consider the trips from $s_{(g+k-1)}$ to $s_{(g+k)}$ and from $s_{(g+k)}$ to $s_{(g+k+1)}$. In this case, the corresponding two terms in $TP_i(a')$ are given by:

$$\begin{aligned}&\frac{\omega_{(g+k-1)(g+k)} + \frac{\omega_{gh}}{|h-g|}}{(2-0)1} + \frac{\omega_{(g+k)(g+k+1)} + \frac{\omega_{gh}}{|h-g|}}{(2-0)1} \\ &= \frac{\omega_{(g+k-1)(g+k)}}{2} + \frac{\omega_{(g+k)(g+k+1)}}{2} + \frac{\omega_{gh}}{|h-g|}.\end{aligned}$$

The first two terms correspond to the same terms in problem a and the third term corresponds to the term associated with station s_f in the trip from s_g to s_h in problem a.

ii. Let us consider that $f = g$. In this case, we only have to consider the trip from s_g to s_{g+1}. The corresponding term in $TP_i(a')$ is given by

$$\frac{\omega_{g(g+1)} + \frac{\omega_{gh}}{|h-g|}}{(2-0)1} = \frac{\omega_{g(g+1)}}{2} + \frac{\omega_{gh}}{2|h-g|}.$$

The first term corresponds to the same term in problem a and the second term corresponds to the term associated with station s_f in the trip from s_g to s_h in problem a.

iii. Let us consider $f = h$. This case is analogous to Case ii.

Therefore, we can conclude that $TP_i(a') = TP_i(a)$.

(c) Weighted additivity. First, it is easy to prove that $\omega_{gh} = \sum_{t=1}^{T} \omega^t_{gh}$. Now, for each $i \in M$, we have the following:

$$\begin{aligned}&\sum_{t=1}^{T} \frac{\Omega(OD_t)}{C_t} TP_i(a_t) \\ &= \sum_{t=1}^{T} \frac{\Omega(OD_t)}{C_t} \left(\frac{C_t}{\Omega(OD_t)} \cdot \sum_{s_f \in S_i} \sum_{\substack{s_g, s_h \in S, s_g \neq s_h \\ f \in [g,h] \text{ or } f \in [h,g]}} \frac{\omega^t_{gh}}{\left(2 - \left\lceil \frac{|f-g|\cdot|h-f|}{(h-g)^2} \right\rceil\right) |h-g|} \right) \\ &= \sum_{t=1}^{T} \sum_{s_f \in S_i} \sum_{\substack{s_g, s_h \in S, s_g \neq s_h \\ f \in [g,h] \text{ or } f \in [h,g]}} \frac{\omega^t_{gh}}{\left(2 - \left\lceil \frac{|f-g|\cdot|h-f|}{(h-g)^2} \right\rceil\right) |h-g|} \\ &= \sum_{s_f \in S_i} \sum_{\substack{s_g, s_h \in S, s_g \neq s_h \\ f \in [g,h] \text{ or } f \in [h,g]}} \frac{\sum_{t=1}^{T} \omega^t_{gh}}{\left(2 - \left\lceil \frac{|f-g|\cdot|h-f|}{(h-g)^2} \right\rceil\right) |h-g|}\end{aligned}$$

$$= \sum_{s_f \in S_i} \sum_{\substack{s_g, s_h \in S, s_g \neq s_h \\ f \in [g,h] \text{ or } f \in [h,g]}} \frac{\omega_{gh}}{\left(2 - \left\lceil \frac{|f-g| \cdot |h-f|}{(h-g)^2} \right\rceil\right) |h-g|} = \frac{\Omega(OD)}{C} TP_i(a).$$

Conversely, let R be a rule that satisfies adjacent symmetry, weak null municipality, trip decomposition, and weighted additivity. We now show that $R = TP$. Let $a^{gh} = (M, S, OD^{gh}, C) \in \mathbb{A}$ be a problem such that all of the entries in the flow matrix are null, except for the gh entry, which is equal to w_{gh} ($\omega_{ef} = 0$ for all $(e, f) \neq (g, h)$). In this case, there are two municipalities (not necessarily different) $i, j \in M$ to which those stations belong, i.e., $s_g \in S_i$ and $s_h \in S_j$. The flow matrix for the problem a^{gh} has the following form

$$OD^{gh} = \begin{array}{c c} & h \\ g & \begin{pmatrix} 0 & \dots & 0 & \dots & 0 \\ \vdots & \ddots & 0 & \ddots & \vdots \\ 0 & \dots & \omega_{gh} & \dots & 0 \\ \vdots & \ddots & 0 & \ddots & \vdots \\ 0 & \dots & 0 & \dots & 0 \end{pmatrix} \end{array}.$$

Now, ω_{gh} in OD^{gh} can be distributed equally among the consecutive trips from s_g to s_h. The latter matrix can be written as the sum of matrices such that the only non-null entry corresponds to two consecutive stations. Therefore, we consider $a'^{gh} = (M, S, OD'^{gh}, C) \in \mathbb{A}$, such that $|g - h| = 1$.

R satisfies *adjacent symmetry* and *weak null municipality*, so $R_i(a'^{gh}) = R_j(a'^{gh}) = \frac{C}{2}$, which matches with the track-based proportional rule.

Let $a^{gh} = (M, S, OD^{gh}, C) \in \mathbb{A}$. Let us consider that $g < h$, and the other case is completely analogous. Let $a'^{gh} = (M, S, OD'^{gh}, C) \in \mathbb{A}$, such that $\omega'_{(g+i)(g+i+1)} = \frac{\omega_{gh}}{|h-g|}$, for all $i = 0, 1, 2, \dots, h - g - 1$ and the remaining entries are zero.

Now, R satisfies *trip decomposition*, so we find that $R(a^{gh}) = R(a'^{gh})$. We can additively split the problem a'^{gh} into $|h - g|$ problems $a''^{(g+i)(g+i+1)} = (M, S, OD''^{(g+i)(g+i+1)}, C^{(g+i)(g+i+1)}) \in \mathbb{A}$, $i = 0, 1, 2, \dots, h-g-1$, such that $\omega''_{(g+i)(g+i+1)} = \frac{\omega_{gh}}{|h-g|}$ and the remaining entries are zero, and

$$C^{(g+i)(g+i+1)} = \frac{C}{|h-g|}.$$

Therefore, we have:

$$OD'^{gh} = \sum_{i=0}^{h-g-1} OD''^{(g+i)(g+i+1)} \quad \text{and} \quad C = \sum_{i=0}^{h-g-1} C^{(g+i)(g+i+1)}.$$

R satisfies *weighted additivity*, for each $i \in M$, so the following holds:

$$\frac{\Omega(OD'^{gh})}{C} R_i(a'^{gh}) = \sum_{i=0}^{h-g-1} \frac{\Omega(OD''^{(g+i)(g+i+1)})}{C^{(g+i)(g+i+1)}} R_i(a''^{(g+i)(g+i+1)}),$$

or equivalently,

$$R_i(a'^{gh}) = \frac{C}{\Omega(OD'^{gh})} \sum_{i=0}^{h-g-1} \frac{\Omega(OD''^{(g+i)(g+i+1)})}{C^{(g+i)(g+i+1)}} R_i(a''^{(g+i)(g+i+1)}).$$

We have proved that R and TP coincide for problems with only traffic of passengers between two consecutive stations, so we find that:

$$R_i(a'^{gh}) = \frac{C}{\Omega(OD'^{gh})} \sum_{i=0}^{h-g-1} \frac{\Omega(OD''^{(g+i)(g+i+1)})}{C^{(g+i)(g+i+1)}} TP_i(a''^{(g+i)(g+i+1)}).$$

Now, the track-based proportional rule satisfies *weighted additivity*, so the following holds

$$R_i(a'^{gh}) = \frac{C}{\Omega(OD'^{gh})} \cdot \frac{\Omega(OD'^{gh})}{C} TP_i(a'^{gh}) = TP_i(a'^{gh}).$$

Therefore, $R_i(a^{gh}) = TP_i(a^{gh})$, for all $i \in M$.

Finally, let $a = (M, S, OD, C) \in \mathbb{A}$ be a problem without any restriction. We can additively split the problem a into other $\Omega(OD)$ problems a^{gh} such that

$$OD = \sum_{\{g,h\}\subseteq S} OD^{gh} \quad \text{and} \quad C^{gh} = \frac{C \cdot \omega_{gh}}{\Omega(OD)}.$$

We already know that for each of those problems, $a^{gh} = (M, S, OD^{gh}, C^{gh})$, the *weighted additivity* implies that, for each $i \in M$,

$$\frac{\Omega(OD)}{C} R_i(a) = \sum_{\{g,h\}\subseteq S} \frac{\Omega(OD^{gh})}{C^{gh}} R_i(a^{gh}),$$

or equivalently,

$$R_i(a) = \frac{C}{\Omega(OD)} \sum_{\{g,h\}\subseteq S} \frac{\Omega(OD^{gh})}{C^{gh}} R_i(a^{gh}).$$

Therefore, $R(a^{gh}) = TP(a^{gh})$ and the track-based proportional rule satisfies *weighted additivity*, so for each $i \in M$, we obtain:

$$R_i(a) = \frac{C}{\Omega(OD)} \sum_{\{g,h\}\subseteq S} \frac{\Omega(OD^{gh})}{C^{gh}} TP_i(a^{gh}) = TP_i(a).$$

□

7 Final comments

In this study, we propose a model for addressing the problem of dividing the fixed cost of a tram line among the municipalities along that line. The infrastructure is already built and the municipalities cannot decide whether to participate or not. A central planner must choose a method to allocate the maintenance cost. As in many situations, the total cost is decomposed into two parts: the variable cost (that depends on the use of the line, among other elements) and the fixed cost. In the paper we have focused on the latter, and we have introduced and characterized solutions for this particular setting.

We have analyzed two types of rules. First, the uniform rule, which does not take into account the intensity in the use of the line and it simply distributes the cost uniformly. And second, rules that exploit the available information more intensively. Even though we are allocating the part of the cost that is not due to the use, the central planner may be interested in considering the information on flows to determine the allotment because municipalities are obliged to participate. In this sense we have defined the station-based proportional and the track-based proportional rules.

The uniform rule is the unique solution that satisfies two combinations of properties: symmetry and additivity, or symmetry, weighted additivity and bilateral ratio consistency. We have also shown that the station-based proportional rule is characterized by means of symmetry, weighted additivity, and null municipality. Interestingly, both the uniform and the station-based proportional rule share the common brach of symmetry and weighted additivity, and their characterizations only differ by one axiom. We also prove that the track-based proportional rule is the unique rule that satisfies adjacent symmetry, weak null municipality, trip decomposition, and weighted additivity.

In Sect. 3 we have studied a cost game that naturally arises from the situation we deal with. As shown by Shapley (1953), the unique solution for this cost game that satisfies symmetry, null player, and additivity is the Shapley value, which matches with the uniform rule in our setting. Interestingly, Theorem 3 proves that if these same principles are translated into our model, then we characterize the station-based proportional rule instead.

There are still some open questions that we have not addressed. One is the structure of the network. Here we have focused on line because we believe it is the canonical problem that helps to understand how the rules work and the relations among the properties. A more general setting would consider other network structures with overlaps and loops. Some other natural rules, such as the station-based uniform rule in Appendix A also deserve a deeper analysis.

Appendix A. Logical independence of the properties

In this section we show that all of the properties used in the characterization of each solution are necessary.

Proposition 2 *Symmetry, and additivity are necessary to characterize the uniform rule.*

Proof (a) Let us consider the station-based uniform rule given by:

$$SU_i(a) = \frac{C}{n} \cdot |S_i|.$$

It is clear that this rule does not satisfy symmetry because it depends only on the number of stations in each municipality but not on the traffic through the network. It is easy to check that this rule satisfies additivity.

(b) The station-based proportional rule satisfies symmetry, but not additivity because it depends on the flow matrix.

□

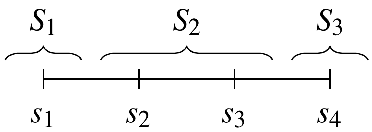
Proposition 3 *Symmetry, bilateral ratio consistency, and weighted additivity are necessary to characterize the uniform rule.*

Proof (a) Let us consider the station-based uniform rule given by:

$$SU_i(a) = \frac{C}{n} \cdot |S_i|.$$

As stated above, it does not satisfy symmetry. It is easy to check that this rule satisfies bilateral ratio consistency and weighted additivity.

(b) The station-based proportional rule satisfies symmetry and weighted additivity but not bilateral ratio consistency. We consider the case of a trolley line that passes across three municipalities $M = \{1, 2, 3\}$ with four stations $S = \{s_1, s_2, s_3, s_4\}$, which are distributed as follows:



The fixed cost is $C = 4$ and the OD matrix is

$$OD = \begin{pmatrix} 0 & 2 & 3 & 1 \\ 2 & 0 & 4 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 2 & 0 & 0 \end{pmatrix}.$$

In this case, $SP(a) = \left(1, \frac{19}{8}, \frac{5}{8}\right)$ Now, if we suppose that Municipality 3 leaves the consortium, then the new (reduced) problem is: $a_{\{1,2\}} = \left(\{1, 2\}, S_1 \cup S_2, OD_{\{1,2\}}, C\right)$, where

$$OD_{\{1,2\}} = \begin{pmatrix} 0 & 2 & 3 \\ 2 & 0 & 4 \\ 0 & 0 & 0 \end{pmatrix},$$

For this reduced problem, we obtain:

$$SP(a_{\{1,2\}}) = \left(\frac{14}{11}, \frac{30}{11}\right).$$

Now we have:

$$\frac{1}{\frac{19}{8}} \neq \frac{\frac{14}{11}}{\frac{30}{11}}.$$

Therefore, the station-based proportional rule does not satisfy bilateral ratio consistency.

(c) Example of a rule that satisfies symmetry, bilateral ratio consistency but does not satisfy weighted additivity. Let $a = (M, S, OD, C) \in \mathbb{A}$ be a problem. We define the following rule for each $i \in M$:

$$R_i(a) = \begin{cases} \frac{C}{\sum_{j \in M} \Omega_{jj}(OD)} \cdot \Omega_{ii}(OD) & \text{if } \Omega_{kk}(OD) \neq 0 \text{ for all } k \in M \\ U_i(a) & \text{otherwise.} \end{cases}$$

By definition, this rule satisfies symmetry and bilateral ratio consistency. However, it does not satisfy weighted additivity because we can transform a problem with $\Omega_{kk}(OD) \neq 0$ for all $k \in M$ into problems where this condition does not hold, and we can then apply the uniform rule instead of the proportional distribution to the inner traffic in each municipality.

$\square$

Proposition 4 *Null municipality, symmetry and weighted additivity are necessary in the characterization of the station-based proportional rule.*

Proof (a) The uniform rule satisfies symmetry and weighted additivity but it does not satisfy null municipality by definition, because all municipalities are allocated with part of the fixed cost independently of the traffic in the transport system.

(b) Let $a = (M, S, OD, C) \in \mathbb{A}$ be a problem and for each $i \in M$ we define the following rule:

$$R_i(a) = \begin{cases} 0 & \text{if } \Omega_i(OD) = 0 \\ \frac{C}{|K|} & \text{otherwise} \end{cases},$$

where $K = \{i \in M : \Omega_i(OD) \neq 0\}$.

By definition, this rule satisfies null municipality and symmetry, but not weighted additivity. Indeed, let us consider the case of a trolley line that passes across three municipalities $M = \{1, 2, 3\}$ with three stations $S = \{s_1, s_2, s_3\}$, which are distributed as follows: $S_1 = \{s_1\}$, $S_2 = \{s_2\}$ and $S_3 = \{s_3\}$. The fixed cost is $C = 6$ and the OD matrix is given by

$$OD = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix},$$

and

i	$\Omega_i^+(OD)$	$\Omega_i^-(OD)$	$\Omega_i(OD)$
1	2	2	4
2	2	2	4
3	2	2	4

Therefore, all of the municipalities are symmetric so they must pay the same $\frac{C}{3}$, and thus:

$$R(a) = (2, 2, 2)\,.$$

Now we divide the cost C into $C_1 + C_2 = 2 + 4 = 6$ and the OD matrix into $OD_1 + OD_2$ in the following manner:

$$OD = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

For example, for Municipality 1, we obtain:

$$\frac{6}{6} \cdot 2 \neq \frac{2}{2} \cdot 0 + \frac{4}{4} \cdot \frac{4}{3}.$$

Therefore, this rule does not satisfy weighted additivity.

(c) Let $a = (M, S, OD, C) \in \mathbb{A}$ be a problem. For each $i \in M$, we define the following rule:

$$R_i(a) = \begin{cases} \frac{i}{3} \cdot C & \text{if } |M| = 2 \text{ and } |S| = 2 \\ SP_i(a) & \text{otherwise.} \end{cases}$$

For $|M| = 2$ and $|S| = 2$, it is clear that this rule satisfies weighted additivity because it does not depend on the OD matrix. Furthermore, in this case, null municipality is meaningless because it implies that there is no traffic at all in the network. Finally, in this case, this rule does not satisfy symmetry because the allocation of the fixed cost depends on the names of the agents. For the remaining cases, this rule satisfies null municipality, symmetry and weighted additivity. Therefore, this rule satisfies null municipality, weighted additivity but no symmetry.

□

Proposition 5 *Adjacent symmetry, weak null municipality, trip decomposition, and weighted additivity are necessary to characterize the track-based proportional rule.*

Proof (a) By the definition of the uniform rule, it is straightforward to check whether it satisfies adjacent symmetry, trip decomposition, and weighted additivity but not weak null municipality.

(b) Before giving a rule that satisfies weak null municipality, trip decomposition, and weighted additivity but not adjacent symmetry, we introduce the following

$$\omega_{[g,g+1]} = \sum_{\substack{k,h \\ k \le g < g+1 \le h}} \frac{\omega_{kh} + \omega_{hk}}{|h-k|},$$

where $\omega_{[g,g+1]}$ is the number of passengers between two consecutive stations when all passengers are distributed equally among all tracks that they use in their trips. Now, we define the following rule:

$$R_i(a) = \frac{C}{\Omega(OD)} \sum_{s_g \in S_i} \left(\frac{g}{2g-1} \omega_{[g-1,g]} + \frac{g}{2g+1} \omega_{[g,g+1]} \right), \text{ for all } i \in M,$$

where $\omega_{[0,1]} = \omega_{[n,n+1]} = 0$.

By definition, this rule satisfies weak null municipality and trip decomposition. Analogous to the track-based proportional rule, we can prove that this rule satisfies weighted additivity. However, this rule does not satisfies adjacent symmetry because it depends on the name of the stations.

(c) It is easy to check that the station-based proportional rule satisfies weak null municipality, adjacent symmetry, and weighted additivity. However, it does not satisfy trip decomposition as shown by the following example. Let us consider a problem with two municipalities and three stations, $S_1 = \{s_1\}$ and $S_2 = \{s_2, s_3\}$, where the fixed cost that needs to be distributed is 1 and the OD matrix is given by

$$OD = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

The station-based proportional rule is $SP(a) = \left(\frac{1}{3}, \frac{2}{3}\right)$. Now, if we distribute the passengers such that there are only trips between consecutive stations, we obtain the following OD' matrix:

$$OD' = \begin{pmatrix} 0 & 1\frac{1}{2} & 0 \\ 1\frac{1}{2} & 0 & 1\frac{1}{2} \\ 0 & 1\frac{1}{2} & 0 \end{pmatrix},$$

and the station-based proportional rule is $SP(a') = \left(\frac{1}{4}, \frac{3}{4}\right)$. Therefore, the station-based proportional rule does not satisfies trip decomposition.

(d) Given an OD matrix, we define the following $[OD]$ matrix:

$$[\omega_{gh}] = \begin{cases} 0 & \text{if } |g-h| > 1 \\ \omega_{[g,h]} & \text{otherwise,} \end{cases}$$

where $\omega_{[g,h]}$ is defined as in (b).
Now, we define the following rule for each $i \in M$:

$$R_i(a) = \begin{cases} 0 & \text{if } \Omega_i([OD]) = 0 \\ \frac{C}{|K|} & \text{otherwise} \end{cases}$$

where $K = \{i \in M : \Omega_i([OD]) \neq 0\}$.
By definition, we can prove that this rule satisfies weak null municipality, adjacent symmetry, and trip decomposition but not weighted additivity.

□

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Appendix B

Manipulability in the cost allocation of transport systems.

MANIPULABILITY IN THE COST ALLOCATION OF TRANSPORT SYSTEMS*

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Abstract

In this work we study the allocation of the maintenance cost of a tram line that goes across several cities. Each city may have one or several stations. Information about the flow of passengers between any pair of stations is available. We particularly focus on the distribution of the fixed part of the cost (salaries of the executive staff, repair facilities, or fixed taxes). As our main finding, we obtain that the cost must be allocated proportionally to the number of stations, as long as we require some conditions on fairness and non-manipulability.

Keywords: axiom, cost sharing, fairness, non-manipulability, proportionality.

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1 Introduction

The problem of how to divide the cost of maintaining infrastructures has become increasingly relevant since regions or countries cooperate to manage those constructions. That is the case, for example, of the Interstate Highway System in the United States, or the railroads that are planned at European level and cross more than one country. Imagine that the Federal Government builds a high-speed line from Miami to Boston with stations in the main cities. The further maintenance cost is left to the states, which cannot exclude themselves from participation. How much should each state contribute? The maintaining cost has two main components: the variable cost (which depends on the intensity in use, flows of passengers, length of the line,...) and the fixed cost (the salaries of the executive staff, the maintenance of the railway yard, the payment of some fixed local taxes, and other expenses that do not depend on the usage). The problem of how to distribute the variable cost has already been addressed by several authors (see Algaba et al. (2019), Sudhölter and Zarzuelo (2017), Bergantiños and Martínez (2014), Kuipers et al. (2013), Bergantiños et al. (2012), Mallozzi (2011), Sánchez-Soriano et al. (2002), Henriet and Moulin (1996), and Littlechild and Owen (1973), among others). However, the fixed cost has not been deeply analyzed. In this work, following Estañ et al. (2020), we study how to allocate the invariant part of the cost.

In our setting a *problem* has four elements. One, the set of states or cities (if we think, for instance, on a tram line across cities instead of a high-speed railway) located along the line. Two the sets of stations that belong to each city. Three, the flow matrix that indicates the number of users between any pair of stations. And four, the cost to be split (which is not a function of the previous elements, unlike the problems considered in the other aforementioned models). A *rule* is simply a way to distribute the cost among the cities.

We follow the axiomatic approach, in which the rules are justified in terms of the axioms or properties that they satisfy. The goal is to identify the rules that satisfy these properties. Thus, we introduce a collection of axioms that are suitable for the studied model: *symmetry*, *non-manipulability via merging* and *non-manipulability via splitting. Symmetry* says that two cities with the same number of stations contribute equally. *Non-manipulability via merging* imposes that no group of cities should benefit by merging their stations and acting as a single city. Dually, *non-manipulability via splitting* avoids manipulations by splitting the stations and acting as different cities.

In our main result we prove that, if the three aforementioned axioms are required, then the cost of the transport network must be allocated proportionally to the number of stations in each city.

The remainder of the paper is organized as follows. In Section 2, we present the mathematical model. In Section 3, we define a collection of rules. In Section 4, we introduce the axioms we consider in this study. Section 5 is devoted to the characterization. We conclude by giving some final remarks in Section 7.

2 The model

Let $M = \{1, \ldots, m\}$ $(m \geq 3)$ be the set of **municipalities** and let S_i the **set of stations of municipality** i , $S_i = \{1, \ldots, s_i\}$. where if two stations belong to i, $s_h, s_{h+l} \in S_i$ then $s_g \in S_i$ for all $g \in [h, h+l]$. Therefore, $S = S_1 \cup \ldots \cup S_m$ is the ordered set of **stations**, which are located on a line. For a given station s_h we assume that s_{h-1} and s_{h+1} are located on the left and on the right of s_h, respectively. Each station belongs to one (and only one) municipality, so $|S| = \sum_{i \in M} |S_i| = n$.

The flows of passengers are described by a **flow matrix** (denoted by OD), which specifies the number of people that use the line between each pair of stations.

$$OD = \begin{pmatrix} 0 & \pi_{12} & \ldots & \pi_{1n} \\ \pi_{21} & 0 & \ldots & \pi_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \pi_{n1} & \pi_{n2} & \ldots & 0 \end{pmatrix} \in \mathbb{R}_+^{n \times n},$$

where π_{gh} is a measure of the number of passengers whose trip starts in station s_g and ends in station s_h. We assume that at least one entry of OD is different from zero. Finally, the network has a **fixed cost**, $C \in \mathbb{R}_+$, which must be distributed among the municipalities in M

The **allocation problem**, is defined by the 4-tuple $a = (M, S, OD, C)$. The class of all these allocation problems is denoted by $\mathbb{A}$.

An **allocation vector** for $a \in \mathbb{A}$ is an efficient distribution of the cost among the municipalities, that is, it is a vector $x \in \mathbb{R}_+^M$ such that $\sum_{i \in M} x_i = C$. Let $X(a)$ be the set of all allocation vectors for $a \in \mathbb{A}$. A **rule** is a way of selecting allocation vectors, that is, it is a function, $R : \mathbb{A} \longrightarrow \bigcup_{a \in \mathbb{A}} X(a)$, that selects, for each problem $a \in \mathbb{A}$, a unique allocation vector $R(a) \in X(a)$.

3 Rules

In this section we present some different rules that allocate a fixed cost of a tram transport network. All rules are related to the principle of proportionality but each of them takes into account a different concept on what the fixed cost distribution has to be proportional to.

The *uniform rule* is straightforward. It splits the cost uniformly among municipalities, regardless other elements of the problem.

Uniform rule. For each $a \in \mathbb{A}$ and each $i \in M$,

$$R_i^U(a) = \frac{C}{m}.$$

The next rule allocates the cost in proportion to the number of passengers who use the stations (to board or to get off) in a municipality.

Flow proportional rule. For each $a \in \mathbb{A}$ and each $i \in M$,

$$R_i^{FP}(a) = \frac{C}{2\Pi(OD)} \cdot \Pi_i(OD),$$

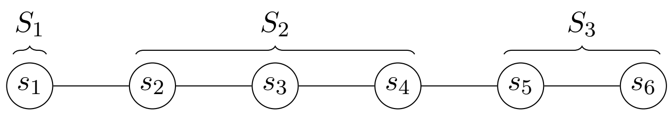
where $\Pi_i(OD) = \sum_{s_g \in S_i} \sum_{s_h \in S} (\pi_{gh} + \pi_{hg})$ and $\Pi(OD) = \|OD\|_1 = \frac{1}{2} \sum_{i=1}^{m} \Pi_i(OD)$.

In the line of the previous one, our last rule also divides the cost on a proportional basis, but with respect to the number of stations in the municipalities.

Station proportional rule. For each $a \in \mathbb{A}$ and each $i \in M$,

$$R_i^{SP}(a) = \frac{C}{n} \cdot |S_i|$$

Example 1. *Consider the case of a trolley line that passes across three municipalities* $M = \{1, 2, 3\}$ *with six stations* $S = \{s_1, s_2, s_3, s_4, s_5, s_6\}$ *that are distributed as follows:*



Thus, Municipality 1 only has one station, Municipality 2 has three stations, and Municipality 3 has two stations. Now, suppose that the line has a fixed cost of maintenance $C = 21$*. The flow matrix is*

$$OD = \begin{pmatrix} 0 & 4 & 2 & 0 & 0 & 3 \\ 1 & 0 & 1 & 0 & 4 & 2 \\ 5 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Here, $\pi_{52} = 1$ *means that one person is traveling from* s_1 *to* s_2*, and* $\pi_{31} = 5$ *indicates that five people are traveling from* s_3 *to* s_1*. For this particular situation, the three rules presented in this section allocate the cost as follows:*

i	R^U	R^{FP}	R^{SP}
1	7	5.08	3.5
2	7	10.16	10.5
3	7	5.76	7.0

4 Properties

We now list some axioms whose fulfillment may be desirable for the problem of distributing the fixed cost of a transport network.

The first requirement imposes a minimal criterion of equity, *symmetry* states that if two municipalities has the same number of stations then they must contribute equally, so this municipalities are *symmetric*.

Symmetry. For each $a \in \mathbb{A}$ and each $\{i,j\} \subseteq M$, if $|S_i| = |S_j|$. Then $R_i(a) = R_j(a)$.

The next property guarantees that the rule is inmune to a special type of manipulations.[1] In particular, it states that the municipalities can not alter their contributions by merging and pretending to be just one municipality.

Non-manipulability via merging: For each pair M, M' such that $M' \subset M$, each $(M, S, OD, C) \in \mathbb{A}$, and each $(M', S', OD, C) \in \mathbb{A}$. If there is $i \in M'$ such that $S'_i = S_i \cup \bigcup_{j \in M \setminus M'} S_j$, and for each $j \in M' \setminus \{i\}$, $S'_j = S_j$, then $R_i(M', S', OD, C) \geq R_i(M, S, OD, C) + \sum_{j \in M \setminus M'} R_j(M, S, OD, C)$.

Our last axiom is the dual of the previous one. It states that a municipality can not alter its contribution by splitting itself into several municipalities.

Non-manipulability via splitting: For each pair M, M' such that $M' \subset M$, each $(M, S, OD, C) \in \mathbb{A}$, and each $(M', S', OD, C) \in \mathbb{A}$. If there is $i \in M'$ such that $S'_i = S_i \cup \bigcup_{j \in M \setminus M'} S_j$, and for each $j \in M' \setminus \{i\}$, $S'_j = S_j$, then $R_i(M', S', OD, C) \leq R_i(M, S, OD, C) + \sum_{j \in M \setminus M'} R_j(M, S, OD, C)$

5 A characterization result

Our main result states that symmetry, non-manipulability via merging, and non-manipulability via splitting leads to a distribution of the cost that is proportional to the number of stations in each municipality.[2]

Theorem 1. *A rule satisfies* symmetry*,* non-manipulability via merging*, and* non-manipulability via splitting *if and only if it is the station proportional rule.*

Proof. It is not difficult to check that the station proportional rule satisfies the axioms of the statement. In order to prove the converse, we first need to consider the segregation of a municipality into as many municipalities as stations it has, i.e., one new municipality for each station. We denote by $MS_i = \{m_{i1}, \ldots, m_{is_i}\}$ the set of all new municipalities obtained from municipality i, where m_{ij} is the new municipality obtained from the j-th station of municipality i, and by $M^M = \bigcup_{i \in M} MS_i = \bigcup_{i \in M} \{m_{i1}, \ldots, m_{is_i}\}$. Thus, it is obvious that $|MS_i| = |S_i|$ for all $i \in M$, and $|M^M| = \sum_{i=1}^{m} |MS_i| = \sum_{i=1}^{m} |S_i| = n$.

We now denote by M^R the set of municipalities where municipalities in R are segregated into as many municipalities as stations they have and the rest of municipalities remain the same. In particular, we have the following:

- For one municipality, $i \in M$, we have the problem $a^i = (M^i, S, OD, C)$ and $M^i = \{1, \ldots, i-1\} \cup MS_i \cup \{i+1, \ldots, m\} = \{1, \ldots, i-1\} \cup \{m_{i1}, \ldots, m_{is_i}\} \cup \{i+1, \ldots, m\}$.

[1] The requirement has been widely used in other models (O'Neill (1982), Moulin (1985), Chun (1988), de Frutos (1999), Ju (2003), Ju et al. (2007), Moulin (2008), Knudsen and Østerdal (2012), Valencia-Toledo and Vidal-Puga (2019), among others).

[2] See Estañ et al. (2020) for characterizations of the uniform and flow proportional rules.

- For two municipalities, $\{i,k\} \in M$, we have the problem $a^{\{i,k\}} = (M^{\{i,k\}}, S, OD, C)$ and $M^{\{i,k\}} = \{1,\ldots,i-1\} \cup MS_i \cup \{i+1,\ldots,k-1\} \cup MS_k \cup \{k+1,\ldots,m\} = \{1,\ldots,i-1\} \cup \{m_{i1},\ldots,m_{is_i}\} \cup \{i+1,\ldots,k-1\} \cup \{m_{k1},\ldots,m_{ks_k}\} \cup \{k+1,\ldots,m\}$.
- In general, $a^K = (M^K, S, OD, C)$, where $M^K = \{i \in M \backslash K\} \cup \bigcup_{i \in K} MS_i$, and $a^M = (M^M, S, OD, C)$.

Note that all problems a^R, $R \subset M$, have the same number of stations, matrix OD and fixed cost C to be distributed. The only change is the number of municipalities.

Let R be a rule that satisfies *symmetry*, *non-manipulable via merging* and *non-manipulable via splitting*. We will prove that it coincides with the station proportional rule by starting with the most segregate problem and then reconstructing municipality-by-municipality the original distribution of stations among municipalities.

Let $a = (M, S, OD, C)$ be a problem where $M = \{1,\ldots,m\}$ is the set of municipalities, $|M| = m$, and S is the set of stations where $|S| = n$, we should have into account that the number of stations is always the same throughout the proof.

We start with problem $a^M = (M^M, S, OD, C)$, we know $|M^M| = n$ coincides with the number of total stations $|S| = n$.

For this problem, by *symmetry* all the allocations in problem a^M are the same, i.e., $R_i(a^M) = \alpha, \forall i \in M^M$. Now, by definition we have that

$$\sum_{i=1}^{n} R_i(a^M) = C \Rightarrow n \cdot \alpha = C \Rightarrow \alpha = \frac{C}{n} \Leftrightarrow \alpha = \frac{C}{|S|}, \forall i \in M^M.$$

In the second step, we consider all problems in which all municipalities are segregated into their stations but one. For all $i \in M$, we consider the problem $a^{M\backslash\{i\}} = (M^{M\backslash\{i\}}, S, OD, C)$. By *non-manipulability via merging*, the following holds:

$$R_i(a^{M\backslash\{i\}}) \geq \sum_{j \in MS_i} R_j(a^M) = \alpha \cdot |S_i| = \frac{C}{n} \cdot |S_i| = \frac{C}{|S|} \cdot |S_i|.$$

However, by *non-manipulability via splitting*, we have that

$$R_i(a^{M\backslash\{i\}}) \leq \sum_{j \in MS_i} R_j(a^M) = \alpha \cdot |S_i| = \frac{C}{n} \cdot |S_i| = \frac{C}{|S|} \cdot |S_i|.$$

Therefore, $R_i(a^{M\backslash\{i\}}) = \frac{C}{|S|} \cdot |S_i|$.

Now, by definition all municipalities except municipality i must pay

$$\sum_{l \in M^{M\backslash\{i\}} \backslash \{i\}} R_l(a^{M\backslash\{i\}}) = C - R_i(a^{M\backslash\{i\}}) = C - \frac{C}{|S|} \cdot |S_i| = \frac{C}{|S|} \cdot (|S| - |S_i|).$$

By *symmetry*, the previous amount should be equally distributed among all remaining $|S| - |S_i|$ municipalities in problem $a^{M\setminus\{i\}}$, i.e.,

$$R_l(a^{M\setminus\{i\}}) = \frac{C}{|S|}, \quad \forall l \in M^{M\setminus\{i\}} \setminus \{i\}.$$

In the third step, we consider all problems in which all municipalities are segregated into their stations but two. For every $\{i,j\} \in M$, we have the problem $a^{M\setminus\{i,j\}} = (M^{M\setminus\{i,j\}}, S, OD, C)$. By *non-manipulability via merging* and *non-manipulability via splitting*, and reasoning as in the previous step with respect to problems $a^{M\setminus\{j\}}$ and $a^{M\setminus\{i\}}$, respectively, we have that

- the allocation for municipality $i \in M$ is given by

$$R_i(a^{M\setminus\{i,j\}}) = \sum_{l \in MS_i} R_l(a^{M\setminus\{j\}}) = \frac{C}{|S|} \cdot |S_i|.$$

- and the allocation for municipality $j \in M$ is given by

$$R_j(a^{M\setminus\{i,j\}}) = \sum_{l \in MS_j} R_l(a^{M\setminus\{i\}}) = \frac{C}{|S|} \cdot |S_j|.$$

By definition, all municipalities except municipalities $i, j \in M$ have to contribute with

$$\sum_{l \in M^{M\setminus\{i,j\}}\setminus\{i,j\}} R_l(a^{M\setminus\{i,j\}}) = \frac{C}{|S|} \cdot (|S| - |S_i| - |S_j|).$$

By *symmetry*, the previous amount must be equally distributed between the remaining $|S| - |S_i| - |S_j|$, so

$$R_l(a^{M\setminus\{i,j\}}) = \frac{C}{|S|}, \quad \forall l \in M^{M\setminus\{i,j\}} \setminus \{i,j\}.$$

Now, we assume that for all $K \subset M$, such that $|K| \leq r < m$, $m \geq 3$, we have that

$$R_i(a^{M\setminus K}) = \frac{C}{|S|} \cdot |S_i|, \quad \forall i \in K,$$
$$R_l(a^{M\setminus K}) = \frac{C}{|S|}, \quad \forall l \in M^{M\setminus K} \setminus K.$$

Let $H \subset M$, such that $|H| = r + 1 \leq m$. By *non-manipulability via merging* and *non-manipulability via splitting*, and reasoning as in the previous steps, we have that

$$R_i(a^{M\setminus H}) = \sum_{l \in MS_i} R_l(a^{M\setminus(H\setminus\{i\})}) = \frac{C}{|S|} \cdot |S_i|, \quad \forall i \in H.$$

By definition, all municipalities except municipalities $H \subset M$ have to contribute with

$$\sum_{l \in M^{M\setminus H}\setminus H} R_l(a^{M\setminus H}) = \frac{C}{|S|} \cdot (|S| - \sum_{i \in H} |S_i|).$$

By *symmetry*, the previous amount must be equally distributed between the remaining $|S| - \sum_{i\in H}|S_i|$, so

$$R_l(a^{M\setminus H}) = \frac{C}{|S|}, \quad \forall l \in M^{M\setminus H} \setminus H.$$

Therefore, we have that $R_i(a^{M\setminus M}) = R_i(a) = \frac{C}{|S|} \cdot |S_i| = R_i^{SP}(a), \ \forall i \in M.$ $\square$

Next examples show that the properties in Theorem 1 are independent.

Example 2. *A rule that satisfies symmetry, non-manipulability via merging but violates non-manipulability via splitting. For each $a \in \mathbb{A}$ and each $i \in M$,*

$$R_i(a) = \frac{C}{\sum_{i\in M}\sqrt{|S_i|}} \cdot \sqrt{|S_i|}$$

Example 3. *A rule that satisfies symmetry, non-manipulability via splitting but violates non-manipulability via merging. For each $a \in \mathbb{A}$ and each $i \in M$,*

$$R_i(a) = \frac{C}{\sum_{i\in M}|S_i|^2} \cdot |S_i|^2$$

Example 4. *The flow proportional rule satisfies non-manipulability via merging and splitting but violates symmetry.*

6 Conclusions

We have showed that symmetry and non-manipulability (via merging and splitting) characterize the station proportional rule. This result is in the line of some others in the literature for different models (see, for example, Chun (1988), de Frutos (1999), Moreno-Ternero (2006) and Ju et al. (2007)), in which the absence of manipulability is closely related with proportional mechanisms. However, in our setting, proportionality is not unambiguously determined, since it can refer to the stations, flows, or even a combination of both. Theorem 1 states that the properties we consider are only compatible with one of those.

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Appendix C

On the difficulty of budget allocation in claims problems with indivisible items and prices.

On the Difficulty of Budget Allocation in Claims Problems with Indivisible Items and Prices

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Abstract
In this paper we study the class of claims problems where the amount to be divided is perfectly divisible and claims are made on indivisible units of several items. Each item has a price, and the available amount falls short to be able to cover all the claims at the given prices. We propose several properties that may be of interest in this particular framework. These properties represent the common principles of fairness, efficiency, and non-manipulability by merging or splitting. Efficiency is our focal principle, which is formalized by means of two axioms: *non-wastefulness* and *Pareto efficiency*. We show that some combinations of the properties we consider are compatible, others are not.

Keywords Claims problems · Indivisible items · Equal treatment of equals · Non-wastefulness · Manipulability

1 Introduction

Resource allocation problems have been extensively studied in the literature for their relevance, particularly when there is a shortage of resources. Decisions made about the allocation problem can lead to grievances and tough negotiations. For this

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reason, it is interesting to know what difficulties are faced by those who have to make decisions about the allocation of resources, particularly when they are scarce. Claims problems or bankruptcy problems are one type of these resource allocation problems. In the classical claims problem, introduced by O'Neill (1982), a central authority has to decide how to divide among the creditors the liquidation value of a firm that goes into bankruptcy. Obviously, this liquidation value does not exceed the debt to the creditors.[1] Usually, both the creditors' claims and the value to be divided are perfectly divisible. In this work we present a novel model where the amount to be divided is in money (and therefore divisible) but the demands are made on indivisible items,[2] each with an associated price. Imagine a health authority responsible for several hospitals that, as in the case of the COVID-19 pandemic, has to allocate to all of them the scarce resources. Each of those hospitals will ask for several medical items (beds, ventilators, defibrillators...). Each of those items has a market price and the health authority has a budget with which to buy them. How many items of each type should be allocated to each hospital taking into consideration their claims, the prices and the available budget? The fact that the demands are expressed in indivisible units of different items, while the amount to divide is continuous, presents several decision difficulties that we analyze in this work.

To be more precise, in our setting, a claims problem can be condensed into five elements. Namely: a set of *agents* of the claimants, a set of possible *items* demanded, a vector of *prices* of those items, a matrix of *claims* that specifies the number of units each agent claims on each item, and the available amount (called *estate*). In addition, it happens that the estate falls short to be able to cover the whole claim at the given prices. A rule is a way in which to solve claims problems. In particular, we consider multi-valued functions, which may be more convenient in order to deal with the indivisibilities in the model.

There are other authors that have studied allocations problems with indivisibilities. In some cases both the budget and the demands are integers (Chen 2015; Herrero and Martínez 2011, 2008a, 2008b), while in other papers the estate is indivisible but the claims are continuous (Fragnelli et al. 2016, 2014). With respect to those works, the novelty we present is twofold. One, the claims are on multiple items. And two, and more significant, the existence of prices, which allow us to consider and combine a continuous estate with indivisible claims.

Following the axiomatic methodology, we wonder if rules exist that satisfy suitable combinations of properties (called *axioms*) that represent criteria on efficiency, fairness, and stability. A claims problem represents a situation where there is a shortage, and therefore the first requirement that comes to mind is to use the limited budget efficiently, trying to satisfy most of the claims with the least amount of money. We implement this principle by means of the *non-wastefulness* condition,

[1] Thomson (2003, 2015) are two excellent surveys on this literature.

[2] We should point out that if we consider situations in which the claims are also perfectly divisible, the so-called multi-issue situations introduced by Calleja et al. (2005) arise.

which simply says that we would waste as little estate as possible.[3] As an alternative to non-wastefulness, we also analyze the *Pareto efficiency* condition. While the former represents efficiency from the point of view of a central authority/government (wasting the least of the budget), the latter takes the perspective of the agents (their allocation cannot be improved at the cost of worsening other individual). We also study other properties that implement several principles of fairness and stability. For the former we consider *weak equal treatment of equals* (agents with equal claims should get equal allotments), while for the latter we impose *non-manipulability by merging or splitting* (agents cannot manipulate their assignments either by splitting or merging their claims).

Interestingly, the finding of non-wasteful rules is closely related with a well-known programming problem, the so-called bounded knapsack problem.[4] Since the seminal paper by Dantzig (1957), several extensions have been widely studied due to their practical applications Kellerer et al. (2010), including choice theory Feuerman and Weiss (1973). As examples of interest which relate to our situation, Darmann and Klamler (2014) study how to share the estate in a continuous setting by means of optimal solutions, and Arribillaga and Bergantiños (2019) analyze two rules related to the Shapley value of an optimistic game.

In the context of claims problems with indivisibilities, several papers have proposed different type of solutions. Moulin (2000), Herrero and Martínez (2008a), Chen (2015) use priority methods, while Giménez-Gómez and Vilella Bach (2012) adopt a P-rights recursive process, described in Giménez-Gómez and Marco-Gil (2014), to ensure weak order preservation.[5] Discrete claim models have been widely used to deal with scarce resources in technological problems such as mobile radio networks (Lucas-Estañ et al. 2012; Gozálvez et al. 2012) or social problems such as apportionment problems Sánchez-Soriano et al. (2016). On the other hand, in claims problems with multi-dimensional and perfectly divisible claims Calleja et al. (2005) introduce the run to the bank rule, while Bergantiños et al. (2011) present several characterizations of the constrained equal awards rule, and Moreno-Ternero (2009) studies the proportional rule.

With respect to our findings, we show that if we require non-wastefulness together with properties such as weak equal treatment of equals Young (1988; 1994) and non-manipulability (O'Neill 1982; de Frutos 1999; Ju et al. 2007), exemption Herrero and Villar (2001), conditional full compensation (Herrero and Villar 2002; Herrero and Martínez 2008b; securement (Moreno-Ternero and Villar 2004), or self-duality Aumann and Maschler (1985), we end up with an impossibility. As for Pareto efficiency, it is also incompatible with weak equal treatment of equal and

[3] Because of the particular nature of the kind of situations we are dealing with (continuous estate and discrete claims) we cannot impose that the rule must exhaust the estate completely. And this is a particularity of our model that differs from other works, since in the vast majority of the papers in the literature, both in continuous and discrete settings, the estate is completely allocated, and nothing remains.

[4] Bounded knapsack problems are knapsack problems in which the variables are bounded from above.

[5] This property states that if the claim of agent i is larger than the claim of agent j, she should obtain (lose) at least (at most) as much as agent j, and if two agents have equal claims, their amounts should differ at most by one unit.

non-manipulability. However, this notion of efficiency is compatible with exemption, conditional full compensation, and weak securement.

The rest of the paper is structured as follows: In Sect. 2 we set the model. In Sect. 3 we present the three core properties we analyze, including non-wastefulness. In Sect. 4 we explore the compatibility of the axioms. In Sect. 5 we study several protective and duality properties, and we illustrate their incompatibilities with non-wastefulness. In Sect. 6 we analyze an alternative formulation of the efficiency principle: Pareto efficiency. Finally, Sect. 7 concludes with a final discussion.

2 The mathematical Model

Let $N = \{1, \dots, n\}$ be the set of agents and let $H = \{1, \dots, h\}$ be the set of possible items, whose prices are given by $p = (p_1, \dots, p_h) \in \mathbb{R}^h_+$.

A claims problem with indivisible items and different prices, or simply a **problem**, represents a situation in which a perfectly divisible quantity, $E \in \mathbb{R}_{++}$ (called **estate**) must be distributed among agents in N according to their demands. Those demands are described by means of a matrix of claims $c \in \mathbb{Z}_+$ that has as many rows as agents, and as many columns as items

$$c = \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1h} \\ c_{21} & c_{22} & \cdots & c_{2h} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{nh} \end{pmatrix},$$

where $c_{ig} \in \mathbb{Z}_+$ indicates the amount of item g claimed by agent i. In any claims problem, the estate falls short to fully cover all the demands, that is, $\sum_{i=1}^{n} \sum_{g=1}^{h} c_{ig} p_g \geq E$.

Therefore, a problem is given by a tuple $a = (N, H, p, c, E)$, where N is the set of agents, H is the set of items, p is the vector of prices, c is the matrix of claims, and E is the estate. Since the elements N, H, and p are fixed throughout the paper, when no confusion arises we simply write the claims problem as $a = (c, E)$. Let $\mathbb{A}$ be the set of all problems:

$$\mathbb{A} = \left\{ a = (c, E) \in \mathbb{Z}^{n\times h}_+ \times \mathbb{R}_{++} : \|c \cdot p\| = \sum_{i=1}^{n} \sum_{g=1}^{h} c_{ig} p_g \geq E \right\}.$$

An **allocation** for $a \in \mathbb{A}$ is a distribution of the estate among the agents that specifies how many items of each price are awarded to each agent. Thus, it is a matrix $x \in \mathbb{Z}^{n\times h}_+$ that satisfies the following two conditions:

(a) Each agent receives a non-negative amount of each type of item, which is not larger than her claim:

$$0 \leq x_{ig} \leq c_{ig} \quad \text{for all } i \in N \text{ and all } g \in H.$$

(b) The overall cost does not exceed the available estate:

$$\|x \cdot p\| = \sum_{i=1}^{n} \sum_{g=1}^{h} x_{ig} p_g \leq E.$$

We denote by $X(a)$ the set of all feasible allocations for the problem $a \in \mathbb{A}$.

All standard models on claims problems impose that the estate must be exhausted and nothing remains without being allocated. Notice that, Condition (b) relaxes this requirement and part of the estate may be unassigned. Otherwise, if equality is imposed, the set of allocations will be empty for some problems.

Example 1 Let $N = \{1, 2, 3, 4\}$, $H = \{1, 2, 3\}$, and $p = (2, 4, 5)$. For the problem $a = (N, H, p, c, E)$, where $E = 16$ and

$$c = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 2 \end{pmatrix}.$$

The overall claimed amount is[6]

$$\|c \cdot p\| = \sum_{i=1}^{4} \sum_{g=1}^{3} c_{ig} p_g = 38 \geq 16 = E.$$

Three possible allocations for this problem are

$$x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad x' = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad x'' = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}.$$

All the three previous matrices satisfy the two conditions to be allocations of the problem, but they differ in that part of the estate that is wasted or non-used $(E - \|x \cdot p\|)$. Hence, for the null allocation x we have that $\|x \cdot p\| = 0$, so all the estate is left, for x', $\|x' \cdot p\| = 15$, so only one unit is left, while in x'', $\|x'' \cdot p\| = 16$, hence the estate is exhausted.

A **rule** is a way of selecting allocations. In our setting, it is a correspondence, $R : \mathbb{A} \rightrightarrows \mathbb{Z}_+^{n \times h}$, that selects, for each problem $a \in \mathbb{A}$, a non-empty subset of allocations $R(a) \subseteq X(a)$.

We next present some examples of rules than can be used to solve a claims problem with indivisible items and prices.

The first rule is straightforward, it simply stipulates that no agent receives anything. Obviously, from a practical perspective this rule is pointless, but it is useful from a

[6] For a given matrix A we denote by $\|A\|$ the norm 1 of A, that is, the sum of all the entries of the matrix.

theoretical point of view, since it can be used to illustrate certain problems with the properties of the rules.

Null rule, R^N. For each $a \in \mathbb{A}$ and each $x \in X(a)$,

$$x \in R^N(a) \iff x_{ig} = 0 \ \forall i \in N \text{ and } \forall g \in H$$

A rule is a multi-valued mapping, so it may select more than one allocation. The next proposal is an extreme case, since it selects the whole set of allocations $X(a)$. It is the counterpart of the null rule.

Greedy rule, R^G. For each $a \in \mathbb{A}$,

$$R^G(a) = X(a).$$

Let $\succ_N$ be an ordering on the set of claimants N, where $i \succ_N j$ means i has priority over j. Let $\succ_H$ be an ordering on the set of items H, where $f \succ_H g$ means f has priority over g. Consider now a rule as the following *modus operandi*: the agents arrive one at a time in the ordering $\succ_N$, and try to fully satisfy them, starting with the items with the highest priority in $\succ_H$. This process continues until, eventually, the estate runs out. The formal definition of this rule is given below.

Agent-item priority arrival rule, R^{AIPA}. For each $a \in \mathbb{A}$ and each $x \in X(a)$,

$$x \in R^{AIPA}(a) \iff \left[x_{ig} > 0 \Rightarrow x_{if} = c_{if} \ \forall f \succ_H g \text{ and } x_{jf} = c_{jf} \ \forall j \succ_N i \ \forall f \in H\right].$$

As a result of the application of this rule, agents with higher priority are satisfied before those with lower priority. Besides, for each agent, more relevant items are fully served first. Consider, for example, the impact of the COVID-19 pandemic in a country whose regions (agents) are significantly heterogenous with respect to the pressure levels of their ICUs. It is natural to prioritize those regions with more pressing needs. In addition, some items (ventilators, for instance) are more critical than others for the patients survival. The agent-item priority arrival rule could be appropriate for such a kind of situations.

Several generalizations of the previous rule can be defined by, for example, considering different orderings of items for different claimants. Alternatively, instead of applying $\succ_N$ and then $\succ_H$, it is also possible to do the converse.

Another rule based on priority which better captures the idea behind rules with similar name in other settings (see Thomson 2019) is the following. Given an ordering $\succ_N$ on the set of claimants, agents arrive one at a time in the ordering. The first agent in the ordering selects the set of items so that she maximizes the value of her choice subject to the budget constrained given by E. Let E^1 be the remaining estate. Now, the second agent in the ordering selects the set of items so that she maximizes the value of her choice subject to the budget constrained given by E^1. We continue the process until the estate, eventually, runs out.

Agent priority arrival rule, R^{APA}. For each $a \in \mathbb{A}$ and each $x \in X(a)$,

$$x \in R^{APA}(a) \iff \left[x_{jg} > 0 \Rightarrow x_{ig} = c_{ig} \ \forall i \succ_N j\right].$$

Notice that, in comparison with the agent-item priority arrival rule, this rule consumes more budget in each step of the process, since there is no ordering on the items that restricts the agent's choice. Therefore, these two rules are suitable for similar situations, depending on whether some item must or must no be prioritized.

The next rule is a two-step process. First, the estate is equally divided among the items ($\frac{E}{h}$ for each one). And second, for each item, amounts as equal as possible are assigned to all claimants subject to no-one receiving more than her claim.[7] This rule can be used when the central planner is interested in allocating the budget as equally as possible not only among the agents but also among the different items.

Equal-by-item rule, R^{EI}. For each $a \in \mathbb{A}$ and each $x \in X(a)$,

$$x \in R^{EI}(a) \;\Leftrightarrow\; \begin{cases} |x_{ig} - x_{jg}| \leq 1 \text{ for all } i,j \in N \\ p_g\left(\sum_{i=1}^{n} x_{ig}\right) \leq \frac{E}{h} \\ p_g\left(1 + \sum_{i=1}^{n} x_{ig}\right) > \frac{E}{h}. \end{cases}$$

In the previous definition, for each possible item, the first condition states that the difference between the awards of two agents is no larger than 1. The second condition imposes that the overall cost of all assigned units does not exceed the share of the estate for this item. Finally, the third one says that the part of the estate corresponding to this item is efficiently distributed, wasting as little as possible.

Several variations of the previous rule are possible. For example, we can consider the distribution of the estate among the items different from the uniform split. We can also restrict the set of allocations by introducing an ordering on N as a tie breaking scheme. Besides, we can obviate the third condition on the efficient usage of the estate.

Another interesting rule could be obtained by applying the two step process of the equal by item rule but to agents. Namely, some kind of equal by agent rule. For each agent we select a set of items whose price is smaller or equal than $\frac{E}{n}$ and such that adding a new item the price is larger than $\frac{E}{n}$. Later the remaining budget is assigned to any set of agents spending as much as possible.

Equal-by-agent rule, R^{EA}. For each $a \in \mathbb{A}$ and each $x \in X(a)$,

$$x = z + y \in R^{EA}(a) \;\Leftrightarrow\; \begin{cases} z, y \in \mathbb{Z}_+^{n\times h} \\ \sum_{g=1}^{h} p_g z_{ig} \leq \frac{E}{n}, \forall i \in N \\ \sum_{g=1}^{h} p_g z'_{ig} > \min\{\sum_{g=1}^{h} p_g c_{ig}, \frac{E}{n}\}, \forall z' > z, \forall i \in N \\ \sum_{i=1}^{n} \sum_{g=1}^{h} p_g y_{ig} \leq E - \|z \cdot p\| \\ \sum_{i=1}^{n} \sum_{g=1}^{h} p_g y_{ig} \geq \sum_{i=1}^{n} \sum_{g=1}^{h} p_g y'_{ig}, \forall y' \in X(a'), \end{cases}$$

where $z' > z$ means that there is at least one cell ig such that $z'_{ig} > z_{ig}$, and the others are greater or equal; and $a' = (N, H, p, c - z, E - \|x \cdot p\|)$.

[7] The second step of this procedure is closely related to some other extensions in settings with indivisibilities of the so-called *constrained equal awards rule* (Herrero and Martínez 2008a, Chen 2015).

Example 2 Continuing with the claims problem of Example 1, these are the allocations selected by each of the rules described above.

(a) Null rule.

$$R^N(a)=\left\{\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\\0&0&0\end{pmatrix}\right\}.$$

(b) Greedy rule.

$$R^G(a)=X(a).$$

(c) Agent-item priority arrival rule. Let us suppose that the ordering $\succ_N$ and $\succ_H$ are such that $1\succ_N 2\succ_N 3\succ_N 4$ and $3\succ_H 2\succ_H 1$. Given $\succ_N$, we start with Agent 1. When this agent is fully honored, the remaining estate is $16-(2\cdot 5+1\cdot 2)=4$. Claimant 2 is the next in line. According to $\succ_H$ we should start by awarding her demand on item 3. However, the unit cost of item 3 she is claiming is 5, which exceeds the available estate. Then, the process stops and the allocation is the following:

$$R^{\text{AIPA}}(a)=\left\{\begin{pmatrix}1&0&2\\0&0&0\\0&0&0\\0&0&0\end{pmatrix}\right\}.$$

(d) Agent priority arrival rule. Let us suppose again that the ordering $\succ_N$ is such that $1\succ_N 2\succ_N 3\succ_N 4$. We start with Agent 1. He is fully honored obtaining $x_{11}=1$ and $x_{13}=2$, and the remaining budget is 4. Agents 2 and 3 cannot obtain anything since the items they demand have a price of 5. Finally, Agent 4 can choose 1 unit of item 1 or 1 unit of item 2, but if we consider agents are maximizers of the budget allocated to them, then the only alternative is $x_{42}=1$. Then, the process ends and the rule gives the following allocation:

$$R^{\text{APA}}(a)=\left\{\begin{pmatrix}1&0&2\\0&0&0\\0&0&0\\0&1&0\end{pmatrix}\right\}.$$

Note that if c_{41} had been 2 instead of 1, then $R^{\text{APA}}(a)$ would have two possible allocations.

(e) Equal-by-item rule. First, we equally divide the estate among the items ($E_1=E_2=E_3=\frac{16}{3}$). We start with item 1. Since $p_1=2$, E_1 is enough to fully cover the demands of Agents 1 and 4. The same argument applies to item 2. Finally, item 3 must be rationed because the cost of honoring all the demands exceeds the share of the estate devoted to this item. With $E_3=\frac{16}{3}$ we can only distribute 1 unit at a price $p_3=5$. Following the definition of the rule, this unit may be assigned to any agent. Therefore, the equal-by-item rule selects the following allocations:

$$R^{\mathrm{EI}}(a) = \left\{ \begin{pmatrix} 1&0&1\\0&0&0\\0&0&0\\1&1&0 \end{pmatrix}, \begin{pmatrix} 1&0&0\\0&0&1\\0&0&0\\1&1&0 \end{pmatrix}, \begin{pmatrix} 1&0&0\\0&0&0\\0&0&1\\1&1&0 \end{pmatrix}, \begin{pmatrix} 1&0&0\\0&0&0\\0&0&0\\1&1&1 \end{pmatrix} \right\}.$$

(f) Equal-by-agent rule. First, we equally divide the estate among the agents, $\frac{16}{4} = 4$. With this distribution of the budget there are only two alternatives

$$\left\{ \begin{pmatrix} 1&0&0\\0&0&0\\0&0&0\\1&0&0 \end{pmatrix}, \begin{pmatrix} 1&0&0\\0&0&0\\0&0&0\\0&1&0 \end{pmatrix} \right\}.$$

In the first case 12 units of the budget are left and in the second case 10 units are left. In both cases, the allocations spending as much as possible of the remaining budget are

$$Y = \left\{ y \in \mathbb{Z}_+^{4\times 3} : \sum_{i=1}^{4} y_{i3} = 2, y_{i1} = y_{i2} = 0, \forall i \in N \right\}.$$

Therefore,

$$R^{\mathrm{EA}}(a) = \left\{ \begin{pmatrix} 1&0&0\\0&0&0\\0&0&0\\1&0&0 \end{pmatrix} + y, \begin{pmatrix} 1&0&0\\0&0&0\\0&0&0\\0&1&0 \end{pmatrix} + y : y \in Y \right\}$$

All the previous rules (and their possible generalizations) arise as natural ways to solve a claims problem with indivisible items of different prices. Some of them may be more appealing than others. There are rules (like the null rule) that make very inefficient usage of the available estate, which is quite undesirable in a situation of shortage. Other proposals may result in very „unfair" allocations. Thus, the priority arrival rules, for instance, do not take into account any minimal principle of justice. Some claimants are fully satisfied while others get nothing. Besides, solutions in the spirit of the equal-by-item or the equal-by-agent rules are easily manipulated by the agents merging or splitting their claims. In the next sections we deliberate over the existence of rules with good properties that can be applied to our claims problems.

3 Three Core Properties

In this section we present three minimal requirements a rule should satisfy, which are quite standard in the literature on claims problems. The restrictions they impose are so slight that they are usually compatible (O'Neill 1982; de Frutos 1999; Ju et al. 2007; Estañ et al. 2020). The first property stipulates that in a rationing situation we should waste as little as possible. The second property is a minimal criterion on fairness, and states that agents with equal claims should be equally treated. Finally, the last property makes the rule immune to certain manipulations by the agents. To summarize, efficiency, fairness, and non-manipulability will be the core requirements we impose as starting point.

As we have seen in Example 2, it may happen that not all the allocations completely exhaust the estate. Given the nature of the problem, it is natural to require that the rule chooses an allocation that misuses the estate as little as possible. This is the counterpart of the *efficiency* requirement in claims problems with continuous claims and estate, which states that the entire amount available should be allocated (see, Thomson (2003) and Thomson (2015), for example).

Non-wastefulness For each $a \in \mathbb{A}$, if $x \in R(a)$, then there is no other allocation $x' \in X(a)$ such that $E - \|x' \cdot p\| < E - \|x \cdot p\|$.

The next property introduces a minimal condition of equality, imposing that individuals with the same claims should be treated equally. Obviously, because of the nature of the problem and the existence of indivisible items, complete equality is difficult to achieve (if not impossible in most cases). So, we modify the condition to require that agents with equal claims must obtain allocations as equal as the indivisibility allows, and equal agents must have the same opportunities. That is, if two individuals demand the same units of all items, then (i) their allocations differ, at most, by one unit per item (in each allocation their awards are as equal as the indivisibility permits), and (ii) the set of selected allocations is symmetric with respect to these two agents (both have the same opportunities to receive one unit more than the other).

Weak equal treatment of equals For each $a \in \mathbb{A}$ and each $\{i,j\} \subseteq N$, if $c_{ig} = c_{jg} \forall g \in H$, then for all $x \in R(a)$ it holds that

- for all $g \in H$, $|x_{ig} - x_{jg}| \leq 1$, and
- for each $g \in H$, there is $x' \in R(a)$, such that $x'_{ig} = x_{jg}$, $x'_{jg} = x_{ig}$ and the rest of cells of x' are the same as in x.

Finally, the next principle says that the rule is immune to manipulation. More precisely, it states that agents cannot manipulate the allocation by either merging or splitting their demands. If a group of individuals merge into a single claimant whose demand (for each item) is the sum of the demands of all the members of such a group, then the allocation of this phantom claimant should coincide with the aggregate allocation the group would have obtained if they had concurred separately. Dually, if an agent splits into a group of different individuals, the aggregate allocation should coincide with the allotment this single agent would have received.[8]

Before formalizing the axiom, we must point out that we are working with correspondences, which means that the outcome of a rule is a set of allocations. Therefore, comparing two outcomes requires comparing two sets, and several possibilities arise: $S = T$, $S \subseteq T$, or $T \subseteq S$. From those alternatives, *non-manipulability by merging or splitting* states that (1) for each allocation in the shrunk problem (the problem with the phantom agent) there must exist a corresponding allocation in the expanded problem (without the phantom agent), and (2), for each allocation in the expanded problem there must exist a corresponding allocation in the shrunk problem.

Non-manipulability by merging or splitting For each $(N, c, E), (N', c', E) \in \mathbb{A}$ with $N' \subset N$, if there is $i \in N'$ such that the following two conditions hold

1. $c'_{ig} = c_{ig} + \sum_{j \in N \setminus N'} c_{jg}$ for all $g \in H$
2. $c'_{jg} = c_{jg}$ for all $j \in N' \setminus \{i\}$ and for all $g \in H$,

then

(a) $\forall x' \in R(N', c', E)$ there exists $x \in R(N, c, E)$ such that $x'_{ig} = x_{ig} + \sum_{j \in N \setminus N'} x_{jg}$ $\forall g \in H$.
(b) $\forall x \in R(N, c, E)$ there exists $x' \in R(N', c', E)$ such that $x'_{ig} = x_{ig} + \sum_{j \in N \setminus N'} x_{jg}$ $\forall g \in H$.

Note that the definition of non-manipulability by merging or splitting states that any allocation that a group of agents could receive through their merger could also have been obtained by remaining separate, and conversely, any allocation that a group of agents could receive by remaining separate could also have obtained through their merger.

4 Compatibility Results

In the previous section, we have formalized three basic properties that implement principles of efficiency, fairness, and non-manipulability. Now, we explore their compatibility, that is, we analyze if rules exist that satisfy all or some of those properties.

Given a problem $a \in \mathbb{A}$. Consider the following integer linear programming problem (ILP, for short):

[8] It was first studied in O'Neill (1982) with the name *strategy-proofness*, and analyzed in works as de Frutos (1999) or Ju et al. (2007).

$$\left.\begin{array}{l} \min\limits_{x \in \mathbb{Z}_+^{n\times h}} E - \sum\limits_{i=1}^{n}\sum\limits_{g=1}^{h} p_g x_{ig} \\ \text{s.t.:} \quad \sum\limits_{i=1}^{n}\sum\limits_{g=1}^{h} p_g x_{ig} \leq E \\ \qquad\quad 0 \leq x_{ig} \leq c_{ig}, \ \forall i \in N, \forall g \in H \end{array}\right\}$$

or equivalently,

$$\left.\begin{array}{l} \max\limits_{x \in \mathbb{Z}_+^{n\times h}} \sum\limits_{i=1}^{n}\sum\limits_{g=1}^{h} p_g x_{ig} \\ \text{s.t.:} \quad \sum\limits_{i=1}^{n}\sum\limits_{g=1}^{h} p_g x_{ig} \leq E \\ \qquad\quad 0 \leq x_{ig} \leq c_{ig}, \ \forall i \in N, \forall g \in H \end{array}\right\} \tag{1}$$

Let us denote by $ILP(a)$ the set of all optimal solutions for the program in (1). It is easy to observe that a rule R satisfies non-wastefulness if it is a selection of solutions of the previous optimization problem, i.e., $R(a) \subseteq ILP(a)$ for all $a \in \mathbb{A}$. The integer linear program defined by (1) belongs to the class of bounded knapsack problems.[9] Since the seminal paper by Dantzig (1957) several extensions have been widely studied due to their practical applications (Kellerer et al. 2010). In general, the solutions of a bounded knapsack problem cannot be obtained in polynomial time. Besides, most of the algorithms are heuristic, and they are usually unable to find all the possible allocations. In other words, finding the set of non-wasteful allocations of the claims problem (and therefore the goal of efficiency) is a very hard task, if not impossible. However, despite these technical difficulties, the following results shows that some interesting conclusions can be derived.

Theorem 1 *There are rules that satisfy non-wastefulness, weak equal treatment of equals and non-manipulability by merging or splitting if and only if there are rules that satisfy those properties for the subclass of problems with* $|H| = 1$.

Proof It is obvious that if there are rules that satisfy non-wastefulness, weak equal treatment of equals and non-manipulability by merging or splitting then those rules satisfy those properties for the subclass of problems with $|H| = 1$.

Conversely, let us suppose that there are rules that satisfy non-wastefulness, weak equal treatment of equals and non-manipulability by merging or splitting for the subclass of problems with $|H| = 1$. Let R be one of those rules, then we define the following procedure for all problems $a = (N, H, p, c, E) \in \mathbb{A}$:

[9] Notice that the constrains $0 \leq x_{ig} \leq c_{ig}, \ \forall i \in N, \forall g \in H$ restrict the possible values of the optimization variables, and therefore the knapsack problem is bounded.

First, we consider the bankruptcy problem $b(a) = (\mathcal{N}, d, u, E)$ given by

- The set of claimants $\mathcal{N} = H$, i.e the claimants are the different types of items.
- $d_g = p_g \sum_{i \in N} c_{ig}$, for all $g \in H$, i.e. the claim of item g is exactly the total amount claimed by all agents for the item g.
- u is a vector of utility functions defined for each $g \in \mathcal{N} = H$ as follows:

$$u_g(y) = \left\lfloor \frac{y}{p_g} \right\rfloor \times p_g,$$

  where $\lfloor r \rfloor$ is the integer part of $r \in \mathbb{R}$.
- Finally, the estate E is exactly the same as in $a = (N, H, p, c, E) \in \mathbb{A}$.

This bankruptcy problem is closely related to those studied in Gozálvez et al. (2012), Lucas-Estañ et al. (2012) and Carpente et al. (2013), in which the agents are considered to have different utility functions in the sense that the same part of the estate has different degrees of satisfaction for the claimants. In our particular case, for example, Claimant 1 must receive at least p_1 units of the estate to obtain one level of satisfaction, i.e. one item of type $1 \in H$, and this can be different for each claimant $g \in \mathcal{N} = H$.

Second, we distribute the estate E amongst the claimants $\mathcal{N} = H$ by solving the following linear program:

$$\left.\begin{aligned} &\max_{y \in \mathbb{R}^h_+} \sum_{g=1}^{h} u_g(x_g) \\ &\text{s.t.:} \quad \sum_{g=1}^{h} y_g \leq E \\ &\qquad 0 \leq y_g \leq p_g \sum_{i=1}^{n} c_{ig}, \ \forall g \in \mathcal{N} = H \end{aligned}\right\} \tag{2}$$

Note that every optimal solution of the integer linear program defined by (1) results in a feasible solution of the linear program defined by (2), and this must be optimal because, otherwise, we would be able to find a better solution for the problem given in (1) from an optimal solution of the problem given in (2). We denote by $LP(b(a))$ the set of all optimal solutions $\bar{x}$ of the linear program given by (2) such that $\frac{\bar{y}_g}{p_g} \in \mathbb{Z}_+, \forall g \in \mathcal{N}$.

Third, for every optimal solution $\bar{y} \in LP(b(a))$, we consider the family of problems $\bar{a}_g = (N, H = \{g\}, p_g, c_g, \bar{y}_g)$, where $c_g = (c_{1g}, \ldots, c_{ng})$.

Finally, for each problem $\bar{a}_g$ we consider $R(\bar{a}_g)$. The set of allocations given by this procedure is exactly

$$\bigcup_{\bar{y} \in LP(b(a))} \bigotimes_{g=1}^{h} R(\bar{a}_g),$$

where $\bigotimes$ denotes the Cartesian product.

This procedure provides a rule $\bar{R}$ which satisfies non-wastefulness, weak equal treatment of equals and non-manipulability by merging or splitting. Indeed, by definition and taking into account that R satisfies non-wastefulness and weak equal treatment of equals, $\bar{R}$ satisfies non-wastefulness and weak equal treatment of equals too.

Let $a = (N, H, p, c, E), a' = (N', H, p, c', E) \in \mathbb{A}$ with $N' \subset N$ such that for $i \in N'$ the following holds

1. $c'_{ig} = c_{ig} + \sum_{j \in N \setminus N'} c_{jg}$ for all $g \in H$
2. $c'_{jg} = c_{jg}$ for all $j \in N' \setminus \{i\}$ and for all $g \in H$

First of all, note that $LP(b(a)) = LP(b(a'))$. Now let $x' \in \bar{R}(a')$, then there is an optimal solution $\bar{y}$ such that $x'_g \in R(\bar{a}'_g)$ for all $g \in H$. Since $LP(b(a)) = LP(b(a'))$ and R satisfies non-manipulability by merging or splitting, then for each $g \in H$ there is an $x_g \in R(\bar{a}_g)$ such that $x'_{ig} = x_{ig} + \sum_{j \in N \setminus N'} x_{jg}$. The proof of the second condition is analogous. Therefore, $\bar{R}$ satisfies non-manipulability by merging or splitting. □

The previous result states that, if we are able to obtain rules that satisfy the three conditions in a reduced domain of problems (with just one item), then they can be extended to the general domain. And conversely, if the three properties are not compatible when $|H| = 1$, then they are not compatible in general. Theorem 2 exploits this relation to conclude that, in this setting, it is not possible to find a rule that fulfills non-wastefulness, weak equal treatment of equals and non-manipulability by merging or splitting.

Theorem 2 *There is no rule that satisfies non-wastefulness, weak equal treatment of equals and non-manipulability by merging or splitting.*

Proof Let us consider $a = (N = \{1, 2, 3, 4, 5, 6\}, H = \{1\}, p = (1), c = (1, 1, 1, 1, 1, 5), E = 4)$, and a rule R that satisfies non-wastefulness, non-manipulability by merging or splitting, and weak equal treatment of equals. Let us consider $a' = \left(N' = \{1, 6\}, H = \{1\}, p = (1), c' = (5, 5), E = 4\right)$, in this case by non-wastefulness and weak equal treatment of equals $R(a') = \{(2, 2)\}$.

Now, let us consider

$$\begin{aligned} a'' = (N = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}, H = \{1\}, p = (1), \\ c = (1, 1, 1, 1, 1, 1, 1, 1, 1, 1), E = 4), \end{aligned}$$

in this case, by non-wastefulness and weak equal treatment of equals $R(a'')$ is the set of all possible 0-1 vectors, such that the sum of their coordinates is 4. Therefore, by non-manipulability by splitting the allocation (0, 0, 0, 0, 0, 4) belongs to $R(a)$. Again, by non-manipulability by splitting $R(a') = \{(2, 2)\}$ is the unique result and the impossibility of $x'_1 = 0$ is more evident. □

Theorem 2 provides a surprising result, since it states an incompatibility among some principles that are compatible in the classical claims problem (see Thomson (2019)). Notice that none of the properties in the previous result is very demanding by itself. Indeed, the next propositions show that any pairwise combination of non-wastefulness, weak equal treatment of equals and non-manipulability by merging or splitting is feasible. Besides, the set of rules that satisfy each pairwise combination of properties is so wide that it does not seem to have a clear structure.

Proposition 1 *There are rules that satisfy non-wastefulness and weak equal treatment of equals together.*

Proof To prove the result is sufficient to show that there is at least one rule satisfying these two properties for each $a \in \mathbb{A}$. We define the following rule $WE(a) = ILP(a) \cap E(a)$ for each $a \in \mathbb{A}$, where $E(a)$ is the set of all weak equal treatment of equals allocations in $X(a)$. Now we only need to prove that $WE(a)$ is always nonempty for each $a \in \mathbb{A}$.

Let $a = (N, H, p, c, E) \in \mathbb{A}$ and let $i, j \in N$ be such that $c_{ig} = c_{jg}$ for all $g \in H$. Given an allocation $x^* \in ILP(a)$, let us suppose that there is some item $g \in H$, such that $|x^*_{ig} - x^*_{jg}| > 1$. Now we consider the following allocation:

$$x'_{kf} = \begin{cases} x^*_{kf} & \text{if } k \neq i,j, \\ x^*_{kf} & \text{if } f \neq g, \\ min\{x^*_{ig}, x^*_{jg}\} + \left\lfloor \frac{|x^*_{ig} - x^*_{jg}|}{2} \right\rfloor & \text{if } k = i \text{ and } f = g, \\ min\{x^*_{ig}, x^*_{jg}\} + \left\lfloor \frac{|x^*_{ig} - x^*_{jg}|}{2} \right\rfloor + \left\lceil m\{\frac{|x^*_{ig} - x^*_{jg}|}{2}\} \right\rceil & \text{if } k = j \text{ and } f = g, \end{cases}$$

where given $r \in \mathbb{R}$, $\lfloor r \rfloor$ is the integer part of r, $\lceil r \rceil$ is the lowest integer larger than or equal to r, and $m\{r\}$ is the fractional part of r.

Finally, it is easy to check that $x' \in ILP(a)$ and, obviously, also $x' \in E(a)$, therefore $x' \in WE(a)$. Now, since $ILP(a) \neq \emptyset$ and $E(a) \neq \emptyset$ for each $a \in \mathbb{A}$, $WE(a) \neq \emptyset$ for each $a \in \mathbb{A}$. □

Proposition 2 *There are rules that satisfy non-wastefulness and non-manipulability by merging or splitting together.*

Proof To prove the result is sufficient to show that for each $a \in \mathbb{A}$, the set $ILP(a)$ satisfies non-manipulability by merging or splitting.

Let $a = (N, H, p, c, E), a' = (N', H, p, c', E) \in \mathbb{A}$ with $N' \subset N$, and let $i \in N'$ such that the following two conditions hold

1. $c'_{ig} = c_{ig} + \sum_{j \in N \setminus N'} c_{jg}$ for all $g \in H$
2. $c'_{jg} = c_{jg}$ for all $g \in H$ and for all $j \in N' \setminus \{i\}$.

Let $x' \in ILP(a')$, then we define for each $j \in N$ and for each $g \in H$, the following allocation:

$$x_{jg} = \begin{cases} x'_{jg} & \text{if } j \in N' \text{ and } j \neq i, \\ GR((N \backslash N') \cup i, \left(c_{kg}\right)_{k \in (N \backslash N') \cup i}, x'_{ig}) & \text{if } j \in (N \backslash N') \cup i, \end{cases}$$

where $GR((N \backslash N') \cup i, \left(c_{kg}\right)_{k \in (N \backslash N') \cup i}, x'_{ig})$ is the application of the greatest remainder method[10] to distribute x'_{ig} according to the vector $\left(c_{kg}\right)_{k \in (N \backslash N') \cup i}$. Now, it is easy to check that this allocation x belongs to $ILP(a)$.

Conversely, let $x \in ILP(a)$, then we define for each $j \in N'$ and for each $g \in H$, the following allocation:

$$x'_{jg} = \begin{cases} x_{jg} & \text{if } j \in N' \text{ and } j \neq i, \\ x_{ig} + \sum_{k \in N \backslash N'} x_{kg} & \text{if } j = i. \end{cases}$$

Again, it is easy to check that this allocation x' belongs to $ILP(a')$. Therefore, the rule $R^{ILP}(a) = ILP(a)$ for each $a \in \mathbb{A}$ satisfies non-wastefulness and non-manipulability by merging or splitting. □

Proposition 3 *There are rules that satisfy weak equal treatment of equals and non-manipulability by merging or splitting together.*

Proof The null rule satisfies both non-manipulability by merging or splitting and weak equal treatment of equals. □

5 Protective Properties and Duality

In this section we study the compatibility between the non-wastefulness condition and other standard properties required when solving claims problems. In particular, we focus on requirements that protect small claimants. In some cases these properties establish the conditions under which an agent has such a small claim that she should be excluded from rationing. In other cases they guarantee a minimum amount of resources to each individual.

Consider an agent i, and replace any other agent's claim (for all the items) by the claim of agent i. Imagine that in the new problem resulting from this replacement the overall demand does not exceed the available estate. *Exemption* states that, in such a case, the claim of i is so small that she is not responsible for the shortage, and she should be excluded from rationing. This is, i should receive her claim.[11]

Exemption For each $a \in \mathbb{A}$ and each $i \in N$, if

[10] This is a well-known method in apportionment problems also known as the method of largest fractions or the Hare Quota method (see, for instance, Lucas (1982)). The integer budget is distributed proportionally to the integer claims, if the allocation is integer for all claimants that is the final allocation, if not, each claimant receives the integer part of his allocation, and an extra unit is allocated to the claimants with the highest fractional parts until the estate is exhausted.

[11] This property was introduced by Herrero and Villar (2001).

$$n \cdot \left(\sum_{g=1}^{h} p_g c_{ig} \right) \leq E,$$

then, for any $x \in R(a)$, $x_{ig} = c_{ig}\ \forall g \in H$.

The next property applies a different criterion to determine when an agent has a small claim. Consider an agent i, and replace any other agent's claim (for all the items) by the minimum between her claim and the claim of agent i. Imagine that in the new problem resulting from this replacement the overall demand does not exceed the available estate. *Conditional full compensation* states that, in such a case, agent i should be excluded from rationing and receive her whole claim.[12]

Conditional full compensation For each $a \in \mathbb{A}$ and each $i \in N$, if

$$\sum_{j \in N_i^-} \sum_{g=1}^{h} p_g c_{jg} + (n - |N_i^-|) \sum_{g=1}^{h} p_g c_{ig} \leq E,$$

then, for any $x \in R(a)$, $x_{ig} = c_{ig}\ \forall g \in H$, where $N_i^- = \left\{ j \in N : \sum_{g=1}^{h} p_g c_{jg} < \sum_{g=1}^{h} p_g c_{ig} \right\}$.

Notice that exemption implies conditional full compensation, and both properties coincide when $|N| = 2$.

Theorem 3 *There is no rule that satisfies non-wastefulness and conditional full compensation together.*

Proof Let R be a rule that satisfies both properties in the statement of the theorem. Let us consider the problem where $N = \{1, 2, 3\}$, $H = \{1, 2\}$, $p = (3, 7)$, $E = 35$, and

$$c = \begin{pmatrix} 2 & 3 \\ 1 & 2 \\ 1 & 1 \end{pmatrix}.$$

For Claimant 3 we have that $N_3^- = \phi$ and therefore $\sum_{j \in N_3^-} \sum_{g=1}^{2} p_g c_{jg} + (3 - |N_3^-|) \sum_{g=1}^{2} p_g c_{3g} = 30 \leq 35$. Since the rule satisfies *conditional full compensation*, it must happen that $x_{31} = x_{32} = 1$, which is not compatible with non-wastefulness. Indeed, if $x_{31} = x_{32} = 1$ then 25 units of estate remains. But this remaining estate cannot be allocated to Agents 1 and 2 fulfilling non-wastefulness, because the unique positive and integer linear combination of the numbers $p_1 = 3$ and $p_2 = 7$ is $6p_1 + 1p_2 = 25$. However, this would imply to assign 6 units of the first item, which exceeds the joint claim of Agents 1 and 2. □

As a consequence of the previous result, neither exemption and non-wastefulness are compatible. Theorem 3 illustrates that, for the problem of adjudicating conflicting indivisible claims with different prices, efficiency (non-wastefulness) and some protective conditions (exemption or conditional full compensation)

[12] This requirement is called *sustainability* in Herrero and Villar (2002).

cannot be conciliated. It is worth noting that this impossibility is a particularity of the model with several items and prices. Both when claims and estate are divisible, and when they are expressed in indivisible units these two properties are compatible (Herrero and Martínez (2008a)).

The next property, called *securement*, was introduced by Moreno-Ternero and Villar (2004) and guarantees a minimal share to every agent. More precisely, it imposes two conditions. One, an individual holding a feasible claim (the value of her demand at the prices of the items is below the estate) should receive allocations whose value is, at least, one nth of the value of her claim. And two, an individual holding an unfeasible claim (the value of her demand at the prices of the items is above the estate) should receive allocations whose value is, at least, one nth of the estate.

Securement For each $a \in \mathbb{A}$, each $x \in R(a)$, and each $i \in N$

$$\sum_{g=1}^{h} p_g x_{ig} \geq \frac{1}{n} \min \left\{ \sum_{g=1}^{h} p_g c_{ig}, E \right\}, \quad \forall i \in N.$$

Example 3 Let us consider the problem where $N = \{1, 2, 3\}$, $H = \{1, 2\}$, $p = (2, 4)$, $E = 10$, and

$$c = \begin{pmatrix} 1 & 2 \\ 1 & 2 \\ 1 & 2 \end{pmatrix}$$

For any agent $i \in N$, securement implies that $2x_{i1} + 4x_{i2} \geq \frac{10}{3}$. Besides, by the definition of a rule, it must happen that $2(x_{11} + x_{21} + x_{31}) + 4(x_{12} + x_{22} + x_{32}) \leq 10$ with $x_{ig} \in \mathbb{Z}_+$, but this is impossible.

The previous example illustrates that securement, as it is defined above, cannot be directly applied to this model because it is incompatible with the definition of a rule itself. The main reason is that the lower bound securement imposed is too high. Therefore, it must be definitively discarded.

In the spirit of securement, the next criterion guarantees to each agent a minimum amount that, at the same time, is also compatible with the existence of feasible allocations. To this end, we look for the largest lower bound of the value of the allocation of any agent $i \in N$. That is, we are looking for a value α_i such that (i) we can impose that $\sum_{g=1}^{h} p_g x_{ig} \geq \alpha_i$ (in the line of securement), (ii) α_i is compatible with the existence of feasible allocations, and (iii) if α_i increases by an infinitely small amount $\varepsilon \in \mathbb{R}_{++}$ then the impossibility emerges again.

Weak securement For each $a \in \mathbb{A}$, each $x \in R(a)$, and each $i \in N$

$$\sum_{g=1}^{h} p_g x_{ig} \geq \max_{y \in X(a)} \left\{ \sum_{g=1}^{h} p_g y_{ig} \middle| \sum_{g=1}^{h} p_g y_{ig} \leq \frac{1}{n} \min \left\{ \sum_{g=1}^{h} p_g c_{ig}, E \right\} \right\}, \quad \forall i \in N.$$

This is the explanation in detail of the previous alternative definition of securement. We know, by Example 3, that it must happen that $\sum_{g=1}^{h} p_g y_{ig} \leq \frac{1}{n} \min\left\{\sum_{g=1}^{h} p_g c_{ig}, E\right\}$ because otherwise we have impossibility. So, among those allocations that generate feasibility, we take those in which the claimant obtains the largest possible value. Thus, weak securement can be applied to this model without the issues originated by the standard definition of securement.

The next result shows that, however, when non-wastefulness is required in conjunction with weak securement, an impossibility emerges.

Theorem 4 *There is no rule that satisfies non-wastefulness and weak securement together.*

Proof Let R be a rule that satisfies both properties in the statement of the theorem. Let us consider the problem where $N = \{1, 2, 3\}$, $H = \{1, 2\}$, $p = (3, 7)$, $E = 14$, and

$$c = \begin{pmatrix} 1 & 4 \\ 2 & 3 \\ 6 & 0 \end{pmatrix}.$$

Since R satifies *non-wastefulness* we have that

$$R(a) \subseteq \left\{ \begin{pmatrix} 0 & 1 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 2 \\ 0 & 0 \end{pmatrix} \right\}.$$

Notice that, in any allocation $x \in R(a)$ the third agent does not receive any unit of any item. And hence, $3x_{31} + 7x_{32} = 0$. This implies, because of *weak securement*, that $3y_{31} + 7y_{32} = 0$ for all $y \in X(a)$ such that $3y_{31} + 7y_{32} \leq \frac{1}{3} \min\{3 \cdot 6 + 7 \cdot 0, 14\} = \frac{14}{3}$. Which is not true. □

The last of the criteria, called *self-duality*, was formulated by Aumann and Maschler (1985). It states that the problem of dividing profits should be solved symmetrically to the problem of dividing losses. Before defining the property we introduce the dual problem of a claim problem. Given $a = (c, E) \in \mathbb{A}$, the *associated dual problem* of a is given by $a^d = (c, L) \in \mathbb{A}$, where $L = \|c \cdot p\| - E$.

Self-duality For each $a \in \mathbb{A}$ it holds that $R(a) = c - R(a^d)$.

Proposition 4 *If a rule satisfies the self-duality property then it exhausts the estate.*

Proof Let R be a rule that is self-dual but does not exhaust the estate. Then, for a given problem $a \in \mathbb{A}$ there exists an allocation $x \in R(a)$ such that

$$\sum_{i=1}^{n} \sum_{g=1}^{h} p_g x_{ig} < E$$

In application of *self-duality*, $x^d = c - x \in R(a^d)$. However,

$$\sum_{i=1}^{n}\sum_{g=1}^{h} p_g x_{ig}^d = \sum_{i=1}^{n}\sum_{g=1}^{h} p_g c_{ig} - \sum_{i=1}^{n}\sum_{g=1}^{h} p_g x_{ig} > \|cp\| - E = L$$

In other words, since $\|x^d \cdot p\| > L$, the allocation x^d is not feasible and hence we have a contradiction. Therefore, if R is self-dual then the estate must be completely used. $\square$

The converse of Proposition 4 is not true in general. For example, if we consider the class of problems with $H = \{1\}$ and $p_1 = 1$ and E a positive integer number, then the discrete constrained equal awards rule (see, for example, Herrero and Martínez (2008a)) always exhaust the estate but does not satisfies self-duality.

An immediate consequence of Proposition 4 is that there can be no rules that satisfy the property of self-duality, since no rule can always exhaust the estate, in general. Notice that, unlike the other results in this section, the lack of self-dual rules is absolute, and the principle of non-wastefulness plays no role in that.

6 An Alternative to Non-Wastefulness: Pareto Efficiency

As we have already mentioned, most of the models on claims problems assume that the estate must be fully distributed and nothing must remain. This requirement is called *balance* by Thomson (2019), but it is also known as *efficiency* (Thomson (2003), de Mesnard (2015)). It is quite obvious that this principle cannot be directly applied to our setting. In the previous sections we have interpreted this requirement from the point of view of a central authority whose goal is to do the most with the least, focusing on the use of the budget and trying to minimize the wasted estate. However, we have found this requirement to be very demanding and not compatible with many reasonable properties.

In this section we explore an alternative formulation of the efficiency principle: *Pareto efficiency*. In contrast with non-wastefulness, this property focuses on the agents' allocations rather than on the expenditure of the budget. An allocation is Pareto efficient if there is no other allocation in which some individual is better off and no individual is worse off.

Definition 1 For $a \in \mathbb{A}$, $x \in X(a)$ is *Pareto efficient* if there is no other allocation $x' \in X(a)$ such that $\sum_{g \in H} p_g x'_{ig} \geq \sum_{g \in H} p_g x_{ig}, \forall i \in N$, with at least one strict inequality.

Given $a \in \mathbb{A}$, we denote by $P(a) \subset X(a)$ the set of all allocations which are Pareto efficient.

Pareto efficiency For $a \in \mathbb{A}$, $R(a) \subseteq P(a)$.

Notice that it is glaringly obvious that non-wastefulness implies Pareto efficiency, but the converse is not true. Even though these two properties are not equivalent in general, it is not difficult to prove that they coincide when $|H| = 1$. As a consequence, we can replace non-wastefulness by Pareto efficiency in Theorem 1, which implies that weak equal treatment of equals and non-manipulability by merging or splitting together are incompatible with Pareto efficiency. This result is the analogous to Theorem 2.

Theorem 5 *There is no rule that satisfies Pareto efficiency, weak equal treatment of equals and non-manipulability by merging or splitting.*

Since Pareto efficiency is milder than non-wastefulness, we obtain the counterparts of Propositions 1 and 2.

Proposition 5 *There are rules that satisfy Pareto efficiency and weak equal treatment of equals together.*

Proposition 6 *There are rules that satisfy Pareto efficiency and non-manipulability by merging or splitting together.*

With regard to self-duality, it is evident that it will not be compatible with Pareto efficiency, since the latter does not guarantee that the estate is fully exhausted. Theorems 3 and 4 state that conditional full compensation and weak securement are incompatible with non-wastefulness. However, the next two results show that, if the latter requirement is weakened to Pareto efficiency, then the possibility emerges.

Theorem 6 *There are rules that satisfy Pareto efficiency and conditional full compensation together.*

Proof In order to prove this result is sufficient to show that there is at least one rule satisfying both properties. Given a problem $a = (N, H, p, c, E)$, we proceed as follows:

- The agents are ordered according to their claims on the budget in this way,

$$i \preceq j \Leftrightarrow \sum_{g=1}^{h} p_g c_{ig} \leq \sum_{g=1}^{h} p_g c_{jg}$$

  For each $i \in N$, we denote by $N_i^- = \left\{ j \in N : \sum_{g=1}^{h} p_g c_{jg} < \sum_{g=1}^{h} p_g c_{ig} \right\}$.
- Let i_0 be an agent in the previous order such that the following inequalities hold

$$\sum_{j \in N_i^-} \sum_{g=1}^{h} p_g c_{jg} + (n - |N_{i_0}^-|) \sum_{g=1}^{h} p_g c_{i_0 g} \leq E,$$

  and, for each $k \in N$ such that $i_0 \in N_k^-$,

$$\sum_{j\in N_k^-}\sum_{g=1}^{h} p_g c_{jg} + (n - |N_k^-|)\sum_{g=1}^{h} p_g c_{kg} > E.$$

Moreover, we denote by $N^0 = \{i \in N : i \preceq i_0\}$. Note that N^0 is independent of the chosen agent i_0.

- Let x^0 be the following allocation, $x^0_{ig} = c_{ig}, \forall g \in H$, if $i \in N^0$, and $x^0_{ig} = 0, \forall g \in H$, otherwise.

We define $X^0(a) = \{x \in X(a) | x - x^0 \in X(a)\}$. Now, we define the following rule

$$R^C(a) = x^0 + ILP(a'),$$

where $a' = (N, H, p, c - x^0, E - \|x^0 \cdot p\|)$.

By definition this rule satisfies Pareto efficiency and conditional full compensation. □

Theorem 7 *There are rules that satisfy Pareto efficiency and weak securement together.*

Proof Again, to prove this result is sufficient to show that there is at least one rule satisfying both properties. Given a problem $a = (N, H, p, c, E)$, we proceed as follows:

First, we consider the set $X^{WS}(a) \subset X(a)$ given by

$$x \in X^{WS}(a) \Leftrightarrow \sum_{g=1}^{h} p_g x_{ig} \ge \max_{y\in X(a)}\left\{\sum_{g=1}^{h} p_g y_{ig} \,\middle|\, \sum_{g=1}^{h} p_g y_{ig} \le \frac{1}{n}\min\left\{\sum_{g=1}^{h} p_g c_{ig}, E\right\}\right\},$$
$$\forall i \in N.$$

Now, we define the following rule

$$R^S(a) = X^{WS}(a) \cap P(a).$$

By definition R^S satisfies weak securement and Pareto efficiency. Furthermore, this rule is nonempty. Indeed, consider $R^{EA}(a)$ that we know it is nonempty. If there is an allocation $x \in R^{EA}(a)$ that is Pareto efficient, then $x \in R^S(a)$. Otherwise, for each $x \in R^{EA}(a)$ there exists $x' \in X(a)$ such that $\sum_{g\in H} p_g x'_{ig} \ge \sum_{g\in H} p_g x_{ig}, \forall i \in N$, with at least one strict inequality, but this $x' \in X^{WS}(a)$. If x' is not Pareto efficient, we can find another $x'' \in X(a)$ such that $\sum_{g\in H} p_g x''_{ig} \ge \sum_{g\in H} p_g x'_{ig}, \forall i \in N$, with at least one strict inequality. Furthermore,

$$\sum_{i\in N}\sum_{g\in H} p_g x_{ig} < \sum_{i\in N}\sum_{g\in H} p_g x'_{ig} < \sum_{i\in N}\sum_{g\in H} p_g x''_{ig} \le E.$$

Now, since $X(a)$ is a finite set there exists $K > 0$ such that any positive improvement of an agent from one allocation to another is larger than or equal to K, i.e., K is the minimal positive improvement that an agent can obtain from one allocation to another. Therefore, since E is finite the above chain of allocations cannot be continued indefinitely, so that there will be an allocation in $X^{WS}(a)$ that it is Pareto efficient. $\square$

Therefore, Pareto efficiency is a sufficiently less demanding property to be compatible with other reasonable properties. Furthermore, we can define rules that satisfy several of the properties introduced in this paper. For example, the following rule

$$R^{CS}(a) = x^0 + R^S(a'), \forall a \in \mathbb{A},$$

where $a' = \left(N, H, p, c - x^0, E - \|x^0 \cdot p\|\right)$, satisfies Pareto efficiency, conditional full compensation, and weak securement.

Consider the rule R^{CES} defined as follows. For each, $a \in \mathbb{A}$,

$$R^{CES}(a) = R^{CS}(a) \cap E(a)$$

This rule satisfies Pareto efficiency, weak equal treatment of equals, conditional full compensation, and weak securement. The converse is not true, there are rules different from R^{CES} that also fulfill these four properties. However, any rule that satisfies Pareto efficiency, weak equal treatment of equals, conditional full compensation, and weak securement must be a subselection of R^{CES}.

Theorem 8 *If a rule R satisfies Pareto efficiency, weak equal treatment of equals, conditional full compensation, and weak securement, then $R(a) \subseteq R^{CES}(a)$, $\forall a \in \mathbb{A}$.*

Proof Let R be a rule satisfying Pareto efficiency, equal treatment of equals, conditional full compensation, and weak securement. Let $a \in \mathbb{A}$ and $x \in R(a)$. Since R satisfies conditional full compensation, x can be written as $x^0 + (x - x^0)$ so that $(x - x^0) \in X^0(a)$.

Since R satisfies weak securement, we have that for each i such that $x^0_{ig} = 0, \forall g \in H$,

$$\sum_{g=1}^{h} p_g x_{ig} \geq \max_{y \in X(a)} \left\{ \sum_{g=1}^{h} p_g y_{ig} \middle| \sum_{g=1}^{h} p_g y_{ig} \leq \min \left\{ \frac{\sum_{g=1}^{h} p_g c_{ig}}{n}, \frac{E}{n} \right\} \right\},$$

otherwise, by conditional full compensation $x^0_{ig} = c_{ig}, \forall g \in H$. Moreover, R satisfies Pareto efficiency. Therefore, $x \in R^{CS}(a)$.

Finally, since R satisfies weak equal treatment of equals, $x \in E(a)$. Therefore, $x \in R^{CS}(a) \cap E(a) = R^{CES}(a)$. $\square$

We finish with a table that summarizes the properties each rule in Sect. 2 satisfies (Table 1).

Table 1 Y means that the rule fulfills the property and N means that it does not

	Non-wastefulness	Pareto efficiency	weak equal treatment of equals	Non-manipulability	Exemption	Conditional full compensation	Weak securement	Self-duality
Null rule	N	N	Y	Y	N	N	N	N
Greedy rule	N	N	N	Y	Y	N	N	N
Agent-item priority arrival rule	N	N	N	.	N	N	N	N
Agent priority arrival rule	N	Y	N	.	N	N	N	N
Equal-by-item rule	N	N	Y	N	N	N	N	N
Equal-by-agent rule	N	N	Y	N	Y	Y	Y	N

We do not analyze the non-manipulability of the agent-item priority arrival and the agent priority arrival rules because it is unclear the effect on the ordering of merging or splitting agents

7 Discussion

In this paper we have studied a particular class of claims problems. In our model a group of agents demand several units of different items, each of which has a price. The available estate is not sufficient to satisfy the aggregate claim. A rule is a multi-valued function that selects a set of allocations, which indicate the amount of units of each item that is assigned to each claimant.

In contrast with other models involving claims problems, efficiency cannot be guaranteed. The closest requirement is non-wastefulness, which states that the rule should waste as little estate as possible, and is closely related to the so-called bounded knapsack problem, whose solutions, in general, are difficult to obtain. Even though, with this milder condition of efficiency, we find that there is no rule that satisfies non-wastefulness together with other criteria that protect small agents or ensure claimants receive a minimum allocation.

In view of all the impossibility results obtained in this work, we can observe that it is not easy to reconcile efficiency (via non-wastefulness) with fairness. At the point we can follow to different paths. First, we contemplate an alternative notion of efficiency that weakens non-wastefulness. Or second, we reconsider the absolute necessity of the non-wastefulness property and simply guarantee that the maximum amount of estate is distributed, while respecting certain properties of fairness in the distribution. With regard to the first possibility, in Sect. 6 we analyze the implications of Pareto efficiency as a milder requirement of efficiency. Even though some impossibilities persist, we find out that Pareto efficiency is compatible with protective properties, in contrast with non-wastefulness. As for the second possibility, it is a promising research line which is beyond the objectives of this paper.

Finally, we should acknowledge there are several extensions of the model that are not addressed in this work. For example, Carpente et al. (2013) consider that, in addition to the claims, each agent is endowed with an utility function. In our model we obviate the latter element, which may be relevant in some situations. However, we do not expect that the addition of utilities as in Carpente et al. (2013) alters the main conclusions significantly, which rely on the discreteness of the claims (or, eventually, the utility of those claims) and on the multi-valued rules. For instance, with respect to non-wastefulness, the solution to the optimization problem in Equation (1) would take different values, but the structure (and implications) of the optimization program itself will still be the same. It is also worth mentioning that we do not provide any characterization in this paper. This is a very natural and convenient extension of this work, which we leave for further research. In contrast with other models in the literature, we define a rule as a correspondence, instead of a single-valued function. This change adds an extra layer of complexity both to the rules and to the axioms. When we try to extend the axioms from functions to correspondences many alternative arise. Think, for example, in properties that compares the outcome of two or more different problems: *additivity*, *composition*, *monotonicity*, *consistency*, etc. When rules are single-valued functions the comparison between two outcomes ($x = y$

or $x \leq y$) is straightforward. When rules are multi-valued functions, as they are in our model, the comparison between two outcomes is not so evident ($R(a) = R(a')$, $R(a) \subset R(a')$, $R(a') \subset R(a)$, etc). That is, each axiom may have different and natural extensions. In this paper we have focused on the implications of one primary requirement: *efficiency*. Other principles and their consequences deserve a deeper analysis, which exceeds the purposes of this paper.

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