COMMUNICATIONS ON
PURE AND APPLIED ANALYSIS
Volume **19**, Number **4**, April **2020**

doi:10.3934/cpaa.2020096

pp. **2197–2217**

CHAIN RECURRENCE AND STRUCTURE OF ω-LIMIT SETS OF MULTIVALUED SEMIFLOWS

Dedicated to Prof. Tomás Caraballo on the occasion of his 60-th birthday

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Abstract. We study properties of ω-limit sets of multivalued semiflows like chain recurrence or the existence of cyclic chains.

First, we prove that under certain conditions the ω-limit set of a trajectory is chain recurrent, applying this result to an evolution differential inclusion with upper semicontinous right-hand side.

Second, we give conditions ensuring that the ω-limit set of a trajectory contains a cyclic chain. Using this result we are able to check that the ω-limit set of every trajectory of a reaction-diffusion equation without uniqueness of solutions is an equilibrium.

1. **Introduction.** The asymptotic behavior of infinite-dimensional dynamical systems without uniqueness (that is, multivalued dynamical systems) have been intensively studied during the last three decades (see, among many others, [2, 4, 5, 6, 9, 10, 11, 12, 13, 14, 15, 18, 19, 25, 26, 27, 28, 29, 30, 31, 33, 34, 36, 39]).

One important question when studying the asymptotic behavior of solutions of partial differential equations is to know the internal structure of ω-limit sets and global attractors, which gives us an insight into the dynamics of solutions in the long term. While in the single-valued case (for differential equations with uniqueness of the Cauchy problem) such question has been widely studied, the multivalued case (for differential equations without uniqueness of the Cauchy problem) is more difficult to tackle. Nevertheless, several results in this direction have been published over the last years (see [2, 5, 9, 11, 12, 13, 18, 19, 28, 29, 33]).

2000 *Mathematics Subject Classification.* 35B40, 35B41, 35K55, 37B20, 37B25, 37D15, 58C06.

Key words and phrases. Multivalued semiflows, ω-limit set, global attractor, differential inclusions, reaction-diffusion equations, structure, asymptotic behaviour.

The first two authors were partially supported by the State Fund for Fundamental Research of Ukraine. The third author was partially supported by Spanish Ministry of Economy and Competitiveness and FEDER, project MTM2016-74921-P, and by Spanish Ministry of Science, Innovation and Universities, project PGC2018-096540-B-I00.

In this paper we are not interested in studying the structure of the whole global attractor but the dynamical properties of the ω-limit set of each particular trajectory of the dynamical system.

First, in Section 2, following the classical theory for single-valued dynamical systems [24, 35], we introduce the notion of chain recurrence for multivalued semiflows and prove that under certain conditions the ω-limit set of an individual trajectory is chain recurrent. It is worth observing that Conley proved in [17, Lemma 4.1E] that if a point is not chain recurrent for a flow, then a Lyapunov function exists along its trajectory. Therefore, the property of being chain recurrent is somehow opposite to the existence of a Lyapunov function.

Second, in Section 3 we apply the abstract theorem of Section 2 to an evolution inclusion with upper-semicontinuous right-hand side. Moreover, the converse statement saying that a given compact, connected, quasi-invariant, chain recurrent set has to be the ω-limit set of a certain differential inclusion is also established.

Third, in Sections 4-5 we study the internal structure of ω-limit sets, and in particular the existence of cyclic chains. Such results are very useful in order to determine whether a trajectory converges towards an equilibrium as time goes to infinity or not. When a Lyapunov function exists it is possible to establish, in a similar way as in the single-valued case, that each trajectory converges to the set of stationary points (see [2, 28]). If, moreover, the number of stationary points is finite (or even infinite but countable), then the ω-limit set of any trajectory is equal to one stationary point. However, in absence of a Lyapunov function such results are much harder to prove.

In Section 4 we extend first a classical result [8] about the existence of stable and unstable sets for compact, isolated, invariant sets intersecting with (but not containing) the ω-limit set of one trajectory. Using it we establish that under certain conditions the ω-limit set of a trajectory contains a cyclic chain. This theorem generalized a classical one for semigroups [37].

In Section 5 we apply these results to a reaction-diffusion equation without uniqueness of solutions. Although it was proved in [29] that inside the global attractor the ω-limit set of every trajectory belongs to the set of stationary points, whether such result is also true or not for any weak solution of the equation was an open problem so far. Using the theoretical results of Section 4 we give an answer to this question by proving that indeed any trajectory converges to an equilibrium if its number is finite. The idea behind the proof is the following: if the ω-limit set of trajectory was not an equilibrium, then a cyclic chain connecting equilibria would exist inside the ω-limit set; however, as a Lyapunov function exists in the global attractor, cyclic chains are forbidden.

2. **Chain recurrence for multivalued semiflows.** In this section we will prove that the ω-limit sets of trajectories of multivalued semiflows are chain recurrent, generalizing in this way the classical result of Conley for single-valued flows.

Let X be a complete metric space with metric ρ. As usual, the Hausdorff semidistance from the set A to the set B is given by

$$dist\,(A,B)=\sup_{a\in A}\inf_{b\in B}\ \rho\,(a,b)\,.$$

We consider a set of functions $\mathcal{K}\subset\mathcal{C}(\mathbb{R}^{+},X)$ satisfying the following conditions:

(K1) for any $x\in X$ there exists $\varphi\in\mathcal{K}$ such that $\varphi(0)=x$;

(K2) $\varphi^{\tau}(\cdot)=\varphi(\cdot+\tau)\in\mathcal{K}$, if $\varphi\in\mathcal{K}$ for any $\tau\geq 0$;

(K3) if $\varphi_1, \varphi_2 \in \mathcal{K}$ satisfy $\varphi_2(0) = \varphi_1(s)$, $s > 0$, then φ given by

$$\varphi(t) = \begin{cases} \varphi_1(t), & \text{if } t \leq s \\ \varphi_2(t-s), & \text{if } t > s, \end{cases}$$

belongs to $\mathcal{K}$;

(K4) if $\varphi_n \in \mathcal{K}$ is a sequence such that $\varphi_n(0) \to x_0$, for some $x_0 \in X$, then there is a subsequence and $\varphi_0 \in \mathcal{K}$ such that $\varphi_{n_k}(t) \to \varphi_0(t)$ uniformly on compact subsets of $[0, \infty)$.

Remark 1. Condition $(K4)$ is stronger than the usual one [5], where pointwise convergence is assumed.

Let $\mathcal{P}(X)$ be the set of all non-empty subsets of X. The multivalued map $G\colon \mathbb{R}^+ \times X \to \mathcal{P}(X)$ is said to be a multivalued semiflow (m-semiflow for short) if:

(i) $x = G(0, x)$, for all $x \in X$;

(ii) $G(t + s, x) \subset G(t, G(s, x))$ for all $t, s \geq 0$ and $x \in X$.

It is called strict if, additionally, $G(t + s, x) = G(t, G(s, x))$ for all $t, s \geq 0$ and $x \in X$.

We define the multivalued map $G\colon \mathbb{R}^+ \times X \to \mathcal{P}(X)$ associated with the family $\mathcal{K}$ as follows:

$$G(t, x) = \{y \in X : y = \varphi(t) \text{ for some } \varphi \in \mathcal{K} \text{ such that } \varphi(0) = x\}. \tag{1}$$

Conditions $(K1) - (K2)$ imply that G is a multivalued semiflow; if, additionally, $(K3)$ is true, then G is strict (see e.g. [28, Lemma 5]).

We say that a map $\phi\colon \mathbb{R} \to X$ is a complete trajectory of $\mathcal{K}$ if

$$\phi(\cdot + h)|_{[0,\infty)} \in \mathcal{K}, \text{ for any } h \in \mathbb{R}.$$

We recall several definitions of invariance for a set $A \subset X$. A is said to be positively invariant if $G(t, A) \subset A$, for all $t \geq 0$, and negatively invariant if $A \subset G(t, A)$ for all $t \geq 0$. It is invariant if it is both positively and negatively invariant, that is, $G(t, A) = A$ for all $t \geq 0$. A is quasi-invariant (or weakly invariant as well) if for all $x \in A$ there is at least one complete trajectory ϕ of $\mathcal{K}$ such that $\phi(t) \in A$, for all $t \in \mathbb{R}$.

It is obvious that any quasi-invariant set is negatively invartiant. It is also well known that under conditions $(K1) - (K4)$ any compact invariant set is quasi-invariant [18, Corollary 7].

For any trajectory $\varphi \in \mathcal{K}$ we define its ω-limit set by

$$\begin{aligned} \omega(\varphi) &= \bigcap_{\tau \geq 0} \overline{\bigcup_{t \geq \tau} \varphi(t)} \\ &= \{y \in X : \text{there exists a sequence } t_n \to +\infty \text{ such that } \varphi(t_n) \to y\}. \end{aligned}$$

The positive orbit of $\varphi \in \mathcal{K}$ is the set $\gamma^+(\varphi) = \cup_{t \geq 0} \varphi(t)$.

If ϕ is a complete trajectory of $\mathcal{K}$, the α-limit set is defined by

$$\begin{aligned} \alpha(\phi) &= \bigcap_{\tau \leq 0} \overline{\bigcup_{t \leq \tau} \phi(t)} \\ &= \{y \in X : \text{there exists a sequence } t_n \to -\infty \text{ such that } \phi(t_n) \to y\}. \end{aligned}$$

The negative orbit of ϕ is the set $\gamma^-(\phi) = \cup_{t \leq 0} \phi(t)$.

The following lemma can be proved in the same way as in Lemma 3.4 and Proposition 4.1 in [5].

Lemma 2.1. *Let $(K1)-(K4)$ be satisfied. If the closure of the positive orbit of $\varphi \in \mathcal{K}$ is compact, then $\omega(\varphi)$ is non-empty, compact, connected, quasi-invariant and*

$$\lim_{t\to+\infty} dist\left(\varphi\left(t\right),\omega\left(\varphi\right)\right)=0.$$

If ϕ is a complete trajectory such that the closure of the negative orbit is compact, then $\alpha(\phi)$ is non-empty, compact, connected, quasi-invariant and

$$\lim_{t\to-\infty} dist\left(\phi\left(t\right),\alpha\left(\phi\right)\right)=0.$$

Definition 2.2. Let $B\subset H$ be a quasi-invariant set with respect to m-semiflow G, and let $y,z\in B$. For $\varepsilon>0$, $t>0$ an (ε,t)-chain from y to z is a sequence $\{y=y_1,y_2,...,y_{n+1}=z\}\subset B$, $\{t_1,t_2,...,t_n\}\subset[t,+\infty)$ such that

$$\mathrm{dist}(y_{i+1},G(t_i,y_i))<\varepsilon,\ \ i=\overline{1,n}. \tag{2}$$

A point $y\in A$ is called chain recurrent with respect to G if for every $\varepsilon>0$, $t>0$ there exists an (ε,t)-chain from y to y. The set B is said to be chain recurrent with respect to G if every point of B is chain recurrent with respect to G.

Remark 2. If G is single-valued, then quasi-invariance implies positively invariance and, as a consequence, Definition 2.2 coincides with the classical definition of chain recurrence [35].

Let us consider the following conditions:

$(N1)$ There exist a sequence of sets of functions $\mathcal{K}_N\subset\mathcal{C}(\mathbb{R}^+,X)$ satisfying $(K1)-(K4)$ such that:

$$G(t,y)\subset G_N(t,y),\ \text{for all } t\geq0,\ N\geq1,\ y\in X, \tag{3}$$

$$\mathrm{dist}_H\left(G_N(t,y_1),G_N(t,y_2)\right)\leq e^{c_N t}\|y_1-y_2\|,\ \text{for all } y_1,y_2\in X,\ t\geq0,\ N\geq1, \tag{4}$$

where $G_N:\mathbb{R}^+\times X\to\mathcal{P}(X)$ are the strict m-semiflows corresponding to $\mathcal{K}_N$ and $\mathrm{dist}_H\left(A,B\right)=\max\{\mathrm{dist}\left(A,B\right),\mathrm{dist}(B,A)\}$ is the Hausdorff distance.

$(N2)$ If $y_N\in\mathcal{K}_N$, $y_N(0)\to y_0$, then up to subsequence

$$y_N\to y \text{ in } C([0,T];X),\ \text{for any } T>0, \tag{5}$$

where $y\in\mathcal{K}$, $y(0)=y_0$.

Theorem 2.3. *Assume that conditions $(K1)-(K4)$, $(N1)-(N2)$ hold. If the closure of the positive orbit of $\varphi\in\mathcal{K}$ is compact, then the set $\omega(\varphi)$ is chain recurrent with respect to the strict m-semiflow G.*

Proof. It is well known that $\varphi(t+s)\in G(t,\varphi(s))$ [29]. Let us fix T and let us put

$$\gamma^T(\varphi)=\bigcup_{t\geq T}\varphi(t).$$

We take arbitrary $\varepsilon>0$, $y\in\omega(\varphi)$, $t_0>0$. Because of the equality

$$y=\lim\varphi(s_n) \text{ as } s_n\to\infty,$$

and $(K4)$ we can choose n such that $s_n>T$ and

$$\mathrm{dist}\left(\varphi(s_n+t),G(t,y)\right)<\varepsilon\,,\ \text{for all } t\in[0,t_0].$$

Let us put $y_1=y$, $y_2=\varphi(s_n+t_0)$, $t_1=t_0$. Then

$$\mathrm{dist}(y_2,G(t_1,y_1))=\mathrm{dist}(\varphi(s_n+t_0),G(t_0,y))<\varepsilon.$$

Choose $m \geq n$ such that

$$s_m > s_n + 2t_0, \ \ \|\varphi(s_m) - y\| < \varepsilon.$$

Let $k \geq 1$ be such that

$$s_m - s_n - t_0 = kt_0 + r, \quad r \in [0, t_0).$$

Let us put

$$y_3 = \varphi(s_n + 2t_0), \dots, \ \ y_{k+1} = \varphi(s_n + kt_0), \ \ y_{k+2} = y,$$
$$t_1 = \ldots = t_k = t_0, \quad t_{k+1} = t_0 + r.$$

Then

$$\operatorname{dist}(y_{i+1}, G(t_i, y_i)) = \operatorname{dist}(\varphi(s_n + it_0), G(t_0, \varphi(s_n + (i-1)t_0))) = 0,$$

for any $i \in [2, k]$, and

$$\operatorname{dist}(y_{k+2}, G(t_{k+1}, y_{k+1})) = \operatorname{dist}(y, G(t_0 + r, \varphi(s_n + kt_0))) \leq \|y - \varphi(s_m)\| < \varepsilon.$$

In other words, for any $y \in \omega(\varphi)$, $\varepsilon > 0$, $t_0 > 0$ there exists an (ε, t_0)-chain $\{y = y_1, \dots, y_{l+1} = y\}$, $\{t_1, \dots, t_l\}$ such that $y_i \in \gamma^T(\varphi)$, $i = \overline{2, l}$, $t_l \in [t_0, 2t_0)$.

From (3) the same is true for G_N for every $N \geq 1$. So, putting $\varepsilon = \frac{1}{n}$, $T = n$ we obtain that for every $n \geq 1$ there exist $\{y_i^n\}_{i=1}^{l_n+1} \subset \gamma^n(\varphi)$, $y_1^n = y = y_{l_n+1}^n$, $\{t_i^n\}_{i=1}^{l_n}$, $t_i^n = t_0$, $i = 1, 2, \dots, l_n - 1$, $t_{l_n}^n \in [t_0, 2t_0)$ such that for any $N \geq 1$,

$$\operatorname{dist}(y_{i+1}^n, G_N(t_i^n, y_i^n)) < \frac{1}{n}, \ \ i = 1, 2, \dots, l_n - 1.$$

Let us denote

$$C^n = \bigcup_{i=1}^{l_n} y_i^n.$$

Then C^n is compact, $y \in C^n$, $C^n \subset \gamma(\varphi) = \bigcup_{t \geq 0} \varphi(t)$, for all $n \geq 1$. Due to the compactness of $\overline{\gamma(\varphi)}$ up to a subsequence

$$\operatorname{dist}_H(C^n, C) \to 0, \quad n \to \infty,$$

where $y \in C$, $C \subset \omega(\varphi)$. For every $N \geq 1$ we choose n_0 such that for all $n \geq n_0$,

$$\frac{1}{n} < \frac{\varepsilon}{3}, \ \ \alpha_n := \operatorname{dist}_H(C^n, C) < \frac{\varepsilon}{3} \ \text{ and } \ e^{2c_N t_0} \alpha_n < \frac{\varepsilon}{3}.$$

Also let us put $z_1 = y$, $t_1 = t_0$, $z_2^n \in C$ such that

$$\operatorname{dist}(y_2^n, C) = \operatorname{dist}(y_2^n, z_2^n).$$

Then

$$\operatorname{dist}(z_2^n, G_N(t_1, z_1)) \leq \operatorname{dist}(z_2^n, y_2^n) + \operatorname{dist}(y_2^n, G_N(t_0, y)) < \frac{2\varepsilon}{3} < \varepsilon.$$

Let us put $t_2 = t_0$, $z_3^n \in C$ such that $dist(y_3^n, C) = dist(y_3^n, z_3^n)$. Then

$$\operatorname{dist}(z_3^n, G_N(t_2, z_2^n)) \leq \operatorname{dist}(z_3^n, y_3^n) + \operatorname{dist}(y_3^n, G_N(t_0, y_2^n + \operatorname{dist}(G_N(t_0, y_2^n), G_N(t_0, z_2^n))$$
$$< \frac{\varepsilon}{3} + \frac{1}{n} + \frac{\varepsilon}{3} < \varepsilon.$$

Repeating this argument we obtain $z_i^n \in C$, $i \in \{1, ..., l_n\}$, $t_i = t_0$, for $i = 1, ..., l_n - 1$, and

$$\operatorname{dist}(z_{i+1}^n, G_N(t_i, z_i^n)) < \varepsilon.$$

Let us put $z_{l_n+1}=y$, $t^n_{l_n}\in[t_0,2t_0)$. Then
$\mathrm{dist}(z_{l_n+1},G_N(t^n_{l_n},z_{l_n})\le \mathrm{dist}(y,G_N(t^n_{l_n},y^n_{l_n}))+\mathrm{dist}(G_N(t^n_{l_n},y^n_{l_n}),G_N(t^n_{l_n},z_{l_n}))<\varepsilon.$
Due to (5) and the compactness of the set C we can choose a number N such that

$$\sup_{t\in[t_0,2t_0]}\sup_{z\in C}\mathrm{dist}(G_N(t,z),G(t,z))<\varepsilon$$

Then

$$\mathrm{dist}(z^n_{i+1},G(t_i,z^n_i))\le \mathrm{dist}(z^n_{i+1},G_N(t_i,z^n_i))+\varepsilon<2\varepsilon$$

and the theorem is proved. □

As a consequence of (2) we have the following result, which can be proved by repeating without any changes the arguments of Lemmas 1.4, 3.3 from [35].

Corollary 1. *Assume that conditions $(K1)-(K4)$ hold. Let B be a compact connected chain recurrent set with respect to the m-semiflow G. Then for any $T>0$, $\varepsilon>0$, $y_0\in B$ there exist sequences $\{y_i\}_{i\ge1}\subset B$, $\{t_i\}_{i\ge0}\subset[T,+\infty)$ such that*

$$\mathrm{dist}(y_{i+1},G(t_i,y_i))<\varepsilon \text{ for all } i\ge 0, \tag{6}$$

$$\mathrm{dist}(y_{i+1},G(t_i,y_i))\to 0, \text{ as } i\to\infty, \tag{7}$$

$$B=\overline{\{y_i\}}_{i\ge n} \text{ for all } n\ge 1. \tag{8}$$

Let us denote

$$D(y_0)=\{y(\cdot)\in\mathcal{K}\mid y(0)=y_0\}.$$

Then for any $i\ge0$ there exists $\varphi_i\in D(y_i)$ such that

$$\mathrm{dist}(y_{i+1},G(t_i,y_i))\le\|y_{i+1}-\varphi_i(t_i)\|=:\varepsilon_i<\varepsilon,\ \varepsilon_i\searrow 0,\ i\to\infty.$$

Corollary 1, and mappings $\{\varphi_i\}_{i\ge0}$ allow us to construct a pseudo-trajectory, the ω-limit set of which contains B.

Corollary 2. *Assume the conditions of Corollary 1. Then for any $\varepsilon>0$, $T>0$, $y_0\in B$ there exists a mapping (an (ε,T)-pseudo-trajectory) $\varphi^*(\cdot)$ starting at y_0 such that*

$$B\subset\omega(\varphi^*). \tag{9}$$

This mapping is defined by the following formula

$$\varphi^*(t)=\begin{cases}\varphi_0(t)\in G(t,y_0),\ t\in[0,t_0),\\ \varphi_i(t-s_{i-1})\in G(t-s_{i-1},y_i),\ s_{i-1}\le t<s_i.\end{cases} \tag{10}$$

where $s_i=\sum_{k=0}^{i}t_k$ and

$$\varepsilon_i=\|\varphi^*(s_i)-\varphi^*(s_i-0)\|\to 0. \tag{11}$$

3. **Chain recurrence for differential inclusions.** Let $V\subset H\subset V'$ be a Gelfand triple with compact dense embeddings, $\|\cdot\|$ and $(\cdot,\cdot)$ be the norm and the scalar product in H, $\langle\cdot,\cdot\rangle$ be pairing between V and V'. We are interested in the limit behavior of trajectories of the following evolution inclusion

$$\begin{cases}\dfrac{dy}{dt}+Ay\in F(y),\\ y|_{t=0}=y_0\in H,\end{cases} \tag{12}$$

where $A:V\to V'$ is a linear, continuous, self-adjoint operator satisfying

$$\lambda_1\|u\|_V^2\le\langle Au,u\rangle\le\lambda_2\|u\|_V^2,\ 0<\lambda_1<\lambda_2. \tag{13}$$

and the multi-valued term $F : H \to \mathcal{P}(H)$ satisfies the following assumptions:

$$\begin{gathered} F \text{ has closed, convex, bounded values,} \\ F \text{ is w-upper semicontinuous,} \\ \|F(y)\|_+ := \sup_{a\in F(y)} \|a\| \leq C_1 + C_2 \|y\|, \text{ for all } y \in H, \end{gathered} \tag{14}$$

for some constants $C_1, C_2 > 0$.

We note that the continuous embedding $V \subset H$ and (13) imply that

$$\langle Au, u\rangle \geq \gamma \|u\|^2, \text{ for some } \gamma > 0.$$

The assumptions on A imply that it is in fact a densely defined maximal monotone operator in H, so it is well known [7, p.60] that A is equal to the subdifferential $\partial\psi$ of the proper, convex lower semicontinuous function

$$\psi(u) = \begin{cases} \frac{1}{2}\left\|A^{\frac{1}{2}}u\right\|, & \text{if } u \in D(A^{\frac{1}{2}}), \\ +\infty, & \text{otherwise.} \end{cases}$$

As $\left\|A^{\frac{1}{2}}u\right\|$ is an equivalent norm in V, the compact embedding $V \subset H$ implies also that the sets

$$M_R = \{u \in H \mid \|u\| \leq R,\ \psi(u) \leq R\} \tag{15}$$

are compact for any $R > 0$.

We recall that F is called w-upper semicontinuous at y_0 if for any $\varepsilon > 0$ there exists $\delta > 0$ such that $F(y) \subset O_\varepsilon(F(y_0))$ as soon as $\|y - y_0\| < \delta$, where for a set $A \subset H$ we define its ε-neighborhood by

$$O_\varepsilon(A) = \{z \in H : dist\,(z, A) < \varepsilon\}.$$

F is w-upper semicontinuous if it is w-upper semicontinuous at any $y_0 \in H$. F is called upper semicontinuous if in the above definition we replace the ε-neighborhood $O_\varepsilon(F(y_0))$ by an arbitrary neighborhood $O(F(y_0))$. It is obvious that any upper semicontinuous map is w-upper semicontinuous, the converse being true as well when F possesses compact values.

It is known [20] that under conditions (13), (14) for every $y_0 \in H$ problem (12) has at least one (mild) solution $y = y(t)$, $t \geq 0$, i.e., for any $T > 0$ $y \in L^2(0, T; V)$, $\frac{dy}{dt} \in L^2(0, T; V')$ and there exists $f \in L^2(0, T; H)$ such that

$$\begin{cases} \dfrac{dy}{dt} + Ay = f(t), \ \ f(t) \in F(y(t)) \text{ for a.a. } t \in (0, T), \\ y|_{t=0} = y_0. \end{cases} \tag{16}$$

Let $\mathcal{K} \subset C([0, +\infty); H)$ be the collection of all mild solutions of (12). It is known [27] that properties $(K1) - (K4)$ are satisfied. Then we define the strict m-semiflow $G : \mathbb{R}^+ \times H \to \mathcal{P}(H)$ by (1).

We will use the following condition: there exists a sequence of mappings $\{F_N : H \to \mathcal{P}(H)\}$ such that F_N have closed, convex, bounded values and

$$F(y) \subset F_N(y), \text{ for all } y \in H,\ N \geq 1, \tag{17}$$

$$\text{dist}(F_N(y), F(y)) \to 0, \text{ as } N \to \infty, \text{ for all } y \in H, \tag{18}$$

$$\text{dist}_H(F_N(y_1), F_N(y_2)) \leq c_N \|y_1 - y_2\|, \text{ for all } y_1, y_2 \in H, \tag{19}$$

for some $c_N > 0$. Let us consider the evolution inclusion

$$\begin{cases} \dfrac{dy}{dt} + Ay \in F_N(y), \\ y|_{t=0} = y_0 \in H. \end{cases} \tag{20}$$

All mild solutions K_N of (20) generate the strict m-semiflows $G_N : \mathbb{R}^+ \times H \to \mathcal{P}(H)$ (see [34]) and (3)-(5) hold true. Property (3) is obvious, whereas (4) was proved in [34, Lemma 8]. Finally, for (5) see the proof of Theorem 3.1 in [27].

Therefore, conditions $(K1) - (K4)$, $(N1) - (N2)$ are satisfied and from Theorem 2.3 we obtain the following result.

Theorem 3.1. *Asume that (17)-(19) are satisfied. If the closure of the positive orbit of $\varphi \in \mathcal{K}$ is compact, then the set $\omega(\varphi)$ is chain recurrent with respect to the strict m-semiflow G.*

Let us give an additional condition ensuring that the closure of every positive orbit of $\varphi \in \mathcal{K}$ is compact. For this aim we need to apply a result on existence of global attractors given in [27]. We recall that the set Θ is a global attractor for G if $\Theta \subset G(t, \Theta)$ for all $t \geq 0$ (negatively semi-invariance), $dist(G(t, B), \Theta) \to 0$, as $t \to +\infty$, for any bounded ser B (attraction property) and it is minimal (that is, it is contained in any closed attracting set). It is called invariant if, moreover, $\Theta = G(t, \Theta)$ for all $t \geq 0$.

If we suppose also that for some $\delta > 0$,

$$(z, u) \leq (\lambda_1 - \delta) \|u\|^2, \text{ for any } u \in H, \, z \in F(u), \tag{21}$$

then G has the invariant compact global attractor Θ [27]. In particular, every positive orbit $\gamma^+(\varphi)$, $\varphi \in \mathcal{K}$, has compact closure in H and by Lemma 2.1 its ω-limit set $\omega(\varphi)$ is nonempty, compact, quasi-invariant, connected and $\omega(\varphi) \subset \Theta$.

Corollary 3. *Asume that (17)-(19) are satisfied. If (21) holds, then for any $\varphi \in \mathcal{K}$ the set $\omega(\varphi)$ is chain recurrent with respect to the strict m-semiflow G.*

Now let us prove in some sense the converse statement to the previous theorem: under some additional restrictions on problem (12) every chain recurrent quasi-invariant compact connected set of the m-semiflow G is the ω-limit set of some trajectory of a perturbed inclusion.

Theorem 3.2. *Asume that (17)-(19) are satisfied. Let B be a compact, connected, chain recurrent, quasi-invariant set with respect to the m-semiflow G. Assume that $A \in \mathcal{L}(H)$. Then there exist maps $F^N : H \to \mathcal{P}(H)$ satisfying (14) and*

$$\mathrm{dist}_H(F^N(y), F(y)) \to 0, \text{ as } N \to \infty, \text{ for all } y \in H,$$

such that for any $y_0 \in B$, $N \geq 1$ there is a solution $y_N(\cdot)$ of the problem

$$\begin{cases} \dfrac{dy}{dt} + Ay \in F^N(y), \\ y(0) = y_0, \end{cases} \tag{22}$$

for which $B \subset \omega(y_N)$.

Proof. Let us fix $N \geq 1$, $\varphi(\cdot) \in K$, $\varphi_0 \in H$ and $\bar{t} > 1$. Denote by $\|A\|_{\mathcal{L}}$ the norm of A in $\mathcal{L}(H)$.

Let us show that for any $\varepsilon \in (0, \frac{1}{N(c_N+\|A\|_{\mathcal{L}}+1)})$, $b \in O_\varepsilon(\varphi(\bar{t}))$ there exists a solution $y_\varepsilon(\cdot)$ of the problem

$$\begin{cases} \dfrac{dy}{dt} + Ay \in F^N(y), \ t \in (0, \bar{t}) \\ y(0) = \varphi_0, \end{cases} \tag{23}$$

with $y_\varepsilon(\bar{t}) = b$, where

$$F^N(y) = \overline{O_{\frac{1}{N}}(F_N(y))}.$$

Indeed, $\varphi(\cdot) \in K_N$. We put $\xi = b - \varphi(\bar{t})$ and consider

$$y_\varepsilon(t) = \varphi(t) + \xi \frac{t}{\bar{t}}, \ t \in [0, \bar{t}].$$

Then $y_\varepsilon(0) = \varphi_0, \ y_\varepsilon(\bar{t}) = b$

$$\dot{y}_\varepsilon(t) + Ay_\varepsilon \in A\xi\frac{t}{\bar{t}} + \xi\frac{1}{\bar{t}} + F_N(y_\varepsilon(t) - \xi\frac{t}{\bar{t}}).$$

Due to (19) we deduce that for a.a. $t \in (0, \bar{t})$

$$\dot{y}_\varepsilon(t) + Ay_\varepsilon \in O_{(c_N+\|A\|_{\mathcal{L}}+1)\varepsilon}(F_N(y_\varepsilon(t)) \subset F^N(y_\varepsilon(t)).$$

Let us denote this solution by $y_\varepsilon(t, \varphi_0)$.

We observe that the maps F^N satisfy the conditions in (14). It is clear that F^N has closed, convex, bounded values. Also, if we prove that

$$dist_H(F^N(x), F^N(y)) \le c_N \|x - y\|, \text{ for all } x, y \in H, \tag{24}$$

then, in particular, F^N are w-upper semicontinous. Indeed, first by (19) we have

$$dist(F_N(x), F^N(y)) \le dist(F_N(x), F_N(y)) \le c_N \|x - y\|.$$

If $z \in F^N(x) \backslash F_N(x)$, then we choose $v \in F_N(x)$ satisfying $z = v + w$, where $\|w\| < \frac{1}{N}$. Take an arbitrary $\delta > 0$. There is some $u \in F_N(y)$ such that

$$\|v - u\| \le dist\,(v, F_N(y))) + \delta \le c_N \|x - y\| + \delta.$$

Then $\widetilde{u} = u + w \in F^N(y)$ and

$$\|z - \widetilde{u}\| = \|v - u\| \le c_N \|x - y\| + \delta.$$

Since δ and z are arbitrary, we obtain that

$$dist\left(F^N(x), F^N(y)\right) \le c_N \|x - y\|.$$

Arguing the other way round we obtain

$$dist\left(F^N(y), F^N(x)\right) \le c_N \|x - y\|$$

as well and so (24). Finally, from (24) the existence of constants $C_1^N, C_2^N > 0$ such that

$$\left\|F^N(y)\right\|_+ \le C_1^N + C_2^N \|y\|$$

follows easily.

Now let $T > \bar{t}, y_0 \in B$ and $\varphi^*(\cdot)$ be the corresponding (ε, T)-pseudo-trajectory from Corollary 2. Let us set

$$y(t) = \begin{cases} \varphi^*(t), \text{ if } t \notin \bigcup\limits_{j=0}^{\infty} (s_j - \bar{t}, s_j), \\ y_{\varepsilon_j}(t - s_j + \bar{t}, \varphi_j(s_j - \bar{t})), \text{ if } t \in (s_j - \bar{t}, s_j). \end{cases}$$

Then $y(\cdot)$ is absolutely continuous function which satisfies (22) and $B \subset \omega(\varphi^*) \subset \omega(y)$. The theorem is proved. □

We observe that we cannot expect that F^N could be replaced by F in Theorem 3.2. The reason is that there are examples in the literature of connected, compact, chain-recurrent, quasi-invariant sets B which cannot be included in the ω-limit set of any trajectory of a given differential equation. Indeed, this fact was shown in [35]. More precisely, let us consider the Duffing equation

$$\begin{cases} \dot{x} = y, \\ \dot{y} = x - x^3 \end{cases} \tag{25}$$

The function $V(x,y) = \frac{y^2}{2} - \frac{x^2}{2} + \frac{x^4}{4}$ is constant along trajectories of (25) and any connected component of

$$V^{-1}([a,b]), \quad -\frac{1}{4} \leq a < b \leq \infty,$$

is compact, invariant, chain recurrent set for the semigroup generated by (25). In particular, it is true for

$$B = V^{-1}([-\frac{1}{4}, 0]) \cap \{x \geq 0\},$$

and there is no trajectory φ of (25) such that $B \subset \omega(\varphi)$.

We give a simple sufficient condition for chain-recurrence.

Lemma 3.3. *Let G be a strict m-semiflow and for arbitrary $x \in B$ there exists $t_x > 0$ such that $x \in G(t_x, x)$. Then the set B is chain-recurrent.*

Proof. Inclusion $x \in G(t_x, x)$ implies that for all $n \geq 1$,

$$x \in G(nt_x, x).$$

Then for any $t > 0$ there exists $n \geq 1$ such that $nt_x > t$ and $x \in G(nt_x, x)$. Thus, $\{x_1 = x, x_2 = x\}$ with $t_1 = nt_x$ is the required (ε, t)-chain. The lemma is proved. □

Further, we will apply Theorem 3.1 to the following partial differential inclusion:

$$\begin{cases} \dfrac{\partial y}{\partial t} - \Delta y \in f(y), \\ y\mid_{\partial\Omega} = 0, \\ y(x,0) = y_0(x), \ x \in \Omega, \end{cases} \tag{26}$$

where $\Omega \subset \mathbb{R}^n$ is an open, bounded subset with smooth boundary $\partial\Omega$ and the multivalued map $f : \mathbb{R} \to \mathcal{P}(\mathbb{R})$ satisfies the following assumptions:

$(f1)$ f has non-empty, closed, convex, bounded values.

$(f2)$ f is upper semicontinuous.

$(f3)$ There are $C_1, C_2 > 0$ such that $\sup_{z \in f(y)} |z| \leq C_1 + C_2 |y|$.

$(f4)$ There exist $C_3, \varepsilon > 0$ for which

$$zy \leq (\lambda_1 - \varepsilon) y^2 + C_3, \text{ for any } z \in f(y),$$

being $\lambda_1 > 0$ the first eigenvalue of the operator $-\Delta$ in $H_0^1(\Omega)$.

Let $V = H_0^1(\Omega)$, $V' = H^{-1}(\Omega)$, $H = L^2(\Omega)$. The operator $A : V \to V'$, defined by

$$\langle Au, v \rangle = \int_\Omega \nabla u \cdot \nabla v dx,$$

is continuous and self-adjoint. Also, the operator $-\Delta : D(-\Delta) \subset H \to H$ is the subdifferential of the proper, convex lower semicontinuous function

$$\psi(u) = \begin{cases} \frac{1}{2}\int_\Omega |\nabla u|^2 \, dx, & \text{if } u \in V, \\ +\infty, & \text{otherwise,} \end{cases}$$

and the level set given in (15) are compact in H. Moreover, it is easy to see from $(f4)$ that condition (21) is satisfied.

We define the Nemytskii operator $F : H \to \mathcal{P}(H)$ given by

$$F(y) = \{\xi \in H \mid \xi(x) \in f(y(x)) \text{ for a.a. } x \in \Omega\}.$$

In view of Lemma 6.28 in [32] F satisfies (14).

The map f can be written as follows:

$$f(y) = [\underline{f}(y), \bar{f}(y)], \ \forall y \in \mathbb{R}, \tag{27}$$

where $\underline{f} : \mathbb{R} \to \mathbb{R}$ is a lower semicontinuous single-valued function and $\bar{f} : \mathbb{R} \to \mathbb{R}$ is an upper semicontinuous single-valued function. Indeed, due to $(f1)$ the map f can be written as (27) with single-valued functions $\underline{f}$, $\bar{f}$. Since f is upper semicontinuous by $(f2)$, for any $\epsilon > 0$ and $y_0 \in \mathbb{R}$ there exists $\delta > 0$ such that

$$f(y) \subset O_\epsilon(f(y_0)), \text{ for all } y \in O_\delta(y_0).$$

Therefore, for any $y \in O_\delta(y_0)$ one has

$$\underline{f}(y) > \underline{f}(y_0) - \epsilon, \quad \bar{f}(y) < \bar{f}(y_0) + \epsilon$$

and we obtain the required semicontinuity properties.

We recall that the multivalued map $h : \Omega \to \mathcal{P}(\mathbb{R})$ is called measurable if for any open set $O \subset \mathbb{R}$ the inverse

$$h^{-1}(O) = \{z \in \Omega : h(z) \cap O \neq \varnothing\}$$

is measurable.

Lemma 3.4. *If $u : \Omega \to \mathbb{R}$ is measurable, then the multivalued map $h : \Omega \to \mathcal{P}(\mathbb{R})$ given by the composition $h(x) = f(u(x))$ is measurable.*

Proof. Since it is known that any open set of $\mathbb{R}$ is the union of a sequence of intervals (a_n, b_n), where $a_n, b_n \in \overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$, and $h^{-1}(A \cup B) = h^{-1}(A) \cup h^{-1}(B)$, it is enough to show that the inverse of any set (a, b) is Lebesgue measurable. This follows from the equalities

$$\begin{aligned} h^{-1}((a,b)) &= \{x \in \Omega \mid f(u(x)) \cap (a,b) \neq \varnothing\} \\ &= \{x \in \Omega \mid \exists \xi \in f(u(x)) \text{ such that } a < \xi < b\} \\ &= \{x \in \Omega \mid \underline{f}(u(x)) < b, \ \bar{f}(u(x)) > a\} \\ &= \{x \in \Omega \mid \underline{f}(u(x)) < b\} \cap \{x \in \Omega \mid \ \bar{f}(u(x)) > a\} \end{aligned}$$

and the fact that the single-valued maps $\underline{g}(x) = \underline{f}(u(x))$, $\overline{g}(x) = \bar{f}(u(x))$ are measurable. □

Lemma 3.5. *There exists an approximative sequence $F_N : H \to \mathcal{P}(H)$ having closed, convex, bounded values and satisfying (17)-(19).*

Proof. Taking into account (27), we define the Moreau-Yosida regularization:

$$f_N(x) = \sup_{y \in \mathbb{R}} (\bar{f}(y) - \frac{N}{2}|y - x|^2).$$

Let us prove that this sequence of functions possesses the following properties:

1. $f_N(x) < +\infty$, for all $N \geq 1,\ x \in \mathbb{R}$,
2. for all $x \in \mathbb{R}$ there exist x_N such that $f_N(x) = \bar{f}(x_N) - \frac{N}{2}|x_N - x|^2$,
3. $\bar{f}(x) \leq f_N(x) \leq \bar{f}(x_N)$, for all $N \geq 1,\ x \in \mathbb{R}$,
4. $x_N \to x$, $\bar{f}(x_N) \to \bar{f}(x)$, $\quad f_N(x) \to \bar{f}(x)$, as $N \to \infty$,
5. $|f_N(x)| \leq D(1 + |x|)$, for any $x \in \mathbb{R}$,
6. $|f_N(x) - f_N(y)| \leq DN(1 + |x| + |y|)|x - y|$, for any $x, y \in \mathbb{R}$ and $N \geq 1$,

where $D > 0$ does not depend on N, x, y. Indeed, points 1,2 are a consequence of the Weierstrass theorem and the sublinear growth of $\bar{f}$. Point 3 is obvious. From the sublinear growth of $\bar{f}$ and the inequality

$$\bar{f}(x_N) - \frac{N}{2}|x_N - x|^2 \geq \bar{f}(x)$$

we deduce that

$$|x_N| \leq \bar{C}(1 + |x|), \tag{28}$$

where $\bar{C}$ does not depend on x, N. These inequalities imply the convergence $x_N \to x$, and the first inequality and the upper semicontinuity of $\bar{f}$ imply the convergence $\bar{f}(x_N) \to \bar{f}(x)$. Hence, using point 3 we get that $f_N(x) \to \bar{f}(x)$. The sublinear growth of $\bar{f}$, (28) and point 3 imply point 5. For proving point 6 we will use that for any $x, y \in \mathbb{R}$ and their corresponding sequences x_N, y_N one has

$$f_N(x) = \bar{f}(x_N) - \frac{N}{2}|x_N - x|^2 \geq \bar{f}(y_N) - \frac{N}{2}|y_N - x|^2,$$
$$f_N(y) = \bar{f}(y_N) - \frac{N}{2}|y_N - y|^2 \geq \bar{f}(x_N) - \frac{N}{2}|x_N - y|^2.$$

Combining these inequalities we obtain

$$\begin{aligned} f_N(x) - f_N(y) &= \bar{f}(x_N) - \bar{f}(y_N) - \frac{N}{2}|x_N - x|^2 + \frac{N}{2}|y_N - y|^2 \\ &\leq \frac{N}{2}|x_N - y|^2 - \frac{N}{2}|x_N - x|^2, \\ f_N(y) - f_N(x) &= \bar{f}(y_N) - \bar{f}(x_N) + \frac{N}{2}|x_N - x|^2 - \frac{N}{2}|y_N - y|^2 \\ &\leq \frac{N}{2}|y_N - x|^2 - \frac{N}{2}|y_N - y|^2. \end{aligned}$$

From the last inequalities and (28) we deduce point 6.

For $N \geq 1$ let D_{N+1} be the Lipschitz constant of f_N in $[-N-1, N+1]$. Then

$$f_N(x) \leq f_N(N) + D_{N+1}\,(x - N) \text{ if } x \in [N, N+1],$$
$$f_N(x) \leq f_N(-N) - D_{N+1}(x + N) \text{ if } x \in [-N-1, -N].$$

On the other hand, we know that $f_N(x) \leq D + D\,|x|$ for any x. We choose $K_N^+ \geq D_{N+1},\ K_N^- \geq D_{N+1}$ such that the point of intersection x_N^+ of $f_N(N) + K_N^+\,(x - N)$ with $D + Dx$ (respectively, x_N^- of $f_N(-N) - K_N^-(x + N)$ with $D - Dx$) belongs to $[N, N+1]$ (respectively, to $[-N-1, -N]$). We put

$$f^{(N)}(x) = \begin{cases} D - Dx, \text{ if } x \leq x_N^-, \\ f_N(-N) - K_N^-(x + N), \text{ if } x_N^- \leq x \leq -N, \\ f_N(x), \text{ if } |x| \leq N, \\ f_N(N) + K_N^+(x - N), \text{ if } N \leq x \leq x_N^+, \\ D\,(1 + x)\,, \text{ if } x \geq x_N^+. \end{cases}$$

Then:

1. $\bar{f}(x) \leq f^{(N)}(x)$, for any $N \geq 1,\ x \in \mathbb{R}$,
2. $|f^{(N)}(x)| \leq D(1+|x|)$, for any $N \geq 1,\ x \in \mathbb{R}$,
3. $f^{(N)}(x) \to \bar{f}(x)$, as $N \to \infty$, for any $x \in \mathbb{R}$,
4. for any $N \geq 1$ there exists $C(N) > 0$ such that $\left|f^{(N)}(x) - f^{(N)}(y)\right| \leq C(N)\,|x-y|$, for all $x, y \in \mathbb{R}$.

In the same way, for the function $\underline{f}\,(x)$ we define a sequence of functions $g^{(N)}\,(x)$ satisfying:

1. $\underline{f}\,(x) \geq g^{(N)}(x)$, for any $N \geq 1$, $x \in \mathbb{R}$,
2. $|g^{(N)}(x)| \leq D(1+|x|)$, for any $N \geq 1,\ x \in \mathbb{R}$,
3. $g^{(N)}(x) \to \underline{f}\,(x)\,,$ as $N \to \infty$, for any $x \in \mathbb{R}$,
4. for any $N \geq 1$ there exists $C(N) > 0$ such that $\left|g^{(N)}(x) - g^{(N)}(y)\right| \leq C(N)\,|x-y|$, for all $x, y \in \mathbb{R}$.

We define now the maps $F_N : H \to \mathcal{P}(H)$ by

$$F_N(y) = \{\xi \in H : \xi\,(x) \in [g^{(N)}\,(y(x))\,, f^{(N)}(y(x))], \text{ for a.a. } x \in \Omega\}.$$

It follows from [34, Lemmas 11, 12] that F_N has non-empty, closed, convex, bounded values and that there exists $c_N > 0$ such that (19) holds true. Also, it is obvious that (17) is satisfied. Finally, let us prove (18). Assume the opposite, that is, that there exists $y \in H$, $\varepsilon > 0$ and a sequence $\xi^N \in F_N(y)$ such that

$$dist(\xi^N, F(y)) > \varepsilon. \tag{29}$$

The multivalued function $x \mapsto f(y(x))$ is measurable by Lemma 3.4. As the functions $\underline{f}\,(y(x))\,,\ \bar{f}(y(x)),\ g^{(N)}\,(y(x))\,,\ f^{(N)}(y(x))$ are measurable, the map

$$\rho_N(x) = \left|f^{(N)}(y(x)) - \bar{f}(y(x))\right| + \left|g^{(N)}\,(y(x)) - \underline{f}\,(y(x))\right| + \frac{1}{N}$$

is measurable as well. Let $B(a,c) = [a-c, a+c]$. We define the multivalued map $\mathcal{P}_N : \Omega \to \mathcal{P}(\mathbb{R})$ by $\mathcal{P}_N(x) = B(\xi^N(x), \rho_N(x))$. By [3, p.316] this map is measurable. Then the multivalued map $\mathcal{D}_N : \Omega \to \mathcal{P}(\mathbb{R})$ given by $\mathcal{D}_N(x) = \mathcal{P}_N(x) \cap f(y(x))$ has non-empty values and is measurable as the intersection of measurable maps [3, p.312]. Therefore, there exists a measurable selection $z^N(x) \in \mathcal{D}_N(x)$ [3, p.308]. Since $g^{(N)}(x) \to \underline{f}\,(x)\,,\ \ f^{(N)}(x) \to \bar{f}(x)$, we obtain that $\rho_N(x) \to 0$ as $N \to \infty$, so

$$\left|\xi^N\,(x) - z^N(x)\right| \to 0, \text{ for a.a. } x \in \Omega.$$

The uniform sublinear growth of the functions $g^{(N)},\ f^{(N)}$ implies that

$$\left|\xi^N\,(x) - z^N(x)\right| \leq K\,(1+|y(x)|)\,, \text{ for a.a. } x \in \Omega, \text{ for any } N,$$

for some constant $K > 0$. Hence, Lebesgue's dominated convergence theorem yields

$$v^N = \xi^N - z^N \to 0 \text{ in } L^2\,(\Omega)\,.$$

Hence,

$$dist\left(\xi^N, F(y)\right) \leq \left\|\xi^N - z^N\right\| \to 0,$$

which contradicts (29). □

In view of Lemma 3.5 and Corollary 3 we obtain the following result.

Theorem 3.6. *For any solution $\varphi \in \mathcal{K}$ of problem (26) the set $\omega(\varphi)$ is chain recurrent.*

4. **Stable and unstable sets of compact quasi-invariant isolated sets.** In this section we generalize to multivalued semiflows a well known result about the existence of stable and unstable sets for compact isolated invariant sets of flows (see [8, 23]).

Definition 4.1. A quasi-invariant set M is said to be isolated if there exists a neighborhood O of M such that M is the maximal quasi-invariant set in O. We call O an isolating neighborhood.

If the quasi-invariant set M is compact, then in the previous definition we can replace O by a δ-neighborhood $O_\delta(M)$.

We denote the set of all complete trajectories of $\mathcal{K}$ by $\mathcal{F}$.

Definition 4.2. Let M be an isolated quasi-invariant set. The weakly stable set $W_w^s(M)$ of M is defined by

$$W_w^s(M) = \{x \in X \mid \exists \phi \in \mathcal{K} \text{ with } \phi(0) = x \text{ such that } \omega(\phi) \neq \varnothing,\ \omega(\phi) \subset M\}. \tag{30}$$

The unstable set $W^u(M)$ of M is given by

$$W^u(M) = \{x \in X \mid \exists \varphi \in \mathcal{F} \text{ with } \varphi(0) = x \text{ such that } \alpha(\varphi) \neq \varnothing,\ \alpha(\varphi) \subset M\}. \tag{31}$$

Remark 3. The strong stable set $W_{str}^s(M)$ of M would be defined by

$$W_s^+(M) = \{x \in X \mid \omega(x) \subset M\},$$

where

$$\omega(x) = \{y \in X \mid \exists t_n \to +\infty,\ y_n \in G(t_n, x) \text{ such that } y_n \to y\}.$$

Of course, in the single-valued case the weak and strong stable sets are the same.

Theorem 4.3. *Let $(K1)-(K4)$ hold and let the closure of the positive orbit of $\varphi \in \mathcal{K}$ be compact. We consider the compact, isolated, quasi-invariant set M. If $\omega(\varphi) \cap M \neq \varnothing$ but $\omega(\varphi) \not\subset M$, then*

$$\omega(\varphi) \cap \{W_w^s(M) \backslash M\} \neq \varnothing, \tag{32}$$

$$\omega(\varphi) \cap \{W^u(M) \backslash M\} \neq \varnothing. \tag{33}$$

Moreover, we obtain a function $\phi \in \mathcal{K}$ in the Definition (30) such that $\phi(t) \in \omega(\varphi) \cup M$ for all $t \geq 0$, and a complete trajectory ϕ in the Definition (31) such that $\phi(t) \in \omega(\varphi) \cup M$ for all $t \in \mathbb{R}$.

Proof. We prove first (32), that is, the existence of the weakly stable set.

The assumptions $\omega(\varphi) \cap M \neq \varnothing$ but $\omega(\varphi) \not\subset M$ imply the existence of a δ-neighborhood $O_\delta(M)$ such that the map $\varphi(t)$ enters and leaves it infinitely often as $t \to +\infty$. Hence, there exist $0 < \tau_k < t_k$ such that $\tau_k, t_k \to +\infty$ and

$$\begin{aligned} dist(\varphi(\tau_k), M) &= \delta, \\ dist(\varphi(t), M) &< \delta, \text{ for all } t \in (\tau_k, t_k], \\ dist(\varphi(t_k), M) &< \frac{1}{k}. \end{aligned}$$

We choose $O_\delta(M)$ in such a way that $\overline{O_\delta(M)} \subset O_\varepsilon(M)$, being $O_\varepsilon(M)$ an isolating neighborhood of M.

First, let $t_k - \tau_k$ be bounded. Then up to a subsequence $t_k - \tau_k \to T > 0$. We put $\varphi_k(t) = \varphi(t + \tau_k)$. Then, as $\tau_k \to +\infty$, we obtain that

$$\varphi_k(0) = \varphi(\tau_k) \to \varphi_0 \in \omega(\varphi),$$
$$dist(\varphi_0, M) = \delta,$$

so $\varphi_0 \notin M$. By $(K4)$ there exists $\overline{\varphi} \in \mathcal{K}$ such that

$$\varphi_k(s_k) \to \overline{\varphi}(s) \text{ as } s_k \to s,\ s_k, s \geq 0.$$

In particular,

$$\varphi_k(0) \to \overline{\varphi}(0) = \varphi_0,$$
$$\varphi_k(t_k - \tau_k) \to \overline{\varphi}(T) = y \in M.$$

Since M is quasi-invariant, there exists $\psi \in \mathcal{K}$ such that $\psi(0) = y$, $\psi(t) \in M$ for any $t \geq 0$. Using $(K3)$ we can concatenate $\overline{\varphi}$ and ψ by

$$\phi(t) = \begin{cases} \overline{\varphi}(t) & \text{if } 0 \leq t \leq T, \\ \psi(t - T) & \text{if } t \geq T, \end{cases}$$

so that

$$\varphi_0 \in \omega(\varphi) \cap \{W_w^+(M) \backslash M\}.$$

It is clear that $\phi(t) \in \omega(\varphi) \cup M$ for any $t \geq 0$.

Second, let $t_k - \tau_k \to +\infty$. Since $\tau_k \to +\infty$, up to a subsequence $\varphi(\tau_k) \to y \in \omega(\varphi)$. We put $\varphi_k(t) = \varphi(t+\tau_k)$. By $(K4)$ there exists $\psi \in \mathcal{K}$ for which $\varphi_k(t) \to \psi(t)$ uniformly in compact sets of $[0, +\infty)$. Moreover,

$$\psi(0) = y \in \omega(\varphi),$$
$$dist(y, M) = \delta,$$
$$\psi(t) \in O_\delta(M) \text{ for all } t > 0.$$

The last result is a consequence of $dist(\varphi_k(t), M) < \delta$, for all $t \in (0, t_k - \tau_k]$.

Hence, $\omega(\psi) \subset \overline{O_\delta(M)} \subset O_\varepsilon(M)$. As $\psi(t) \in \omega(\varphi)$ for all $t \geq 0$, $\omega(\psi)$ is non-empty and quasi-invariant, so $\omega(\psi) \subset M$. Thus,

$$y \in \omega(\varphi) \cap \{W_w^s(M) \backslash M\}.$$

Second, we prove (33), that is, the existence of the unstable set.

There exist $0 < t_k < \tau_k$ such that $t_k, \tau_k \to +\infty$ and

$$dist(\varphi(\tau_k), M) = \delta,$$
$$dist(\varphi(t), M) < \delta, \text{ for all } t \in [t_k, \tau_k),$$
$$dist(\varphi(t_k), M) < \frac{1}{k}.$$

As before, first let $\tau_k - t_k$ be bounded, so passing to a subsequence $\tau_k - t_k \to T$. We define $\varphi_k(t) = \varphi(t + t_k)$. We know that

$$\varphi_k(0) = \varphi(t_k) \to x \in M$$

and by $(K4)$ there exists $\overline{\varphi} \in \mathcal{K}$ such that $\overline{\varphi}(0) = x$ and

$$\varphi_k(s_k) \to \overline{\varphi}(s) \text{ if } s_k \to s,\ s_k, s \geq 0.$$

Hence,

$$\varphi_k(\tau_k - t_k) = \varphi(\tau_k) \to \overline{\varphi}(T) = \varphi_0 \in \omega(\varphi),$$
$$dist(\varphi_0, M) = \delta,$$

so $\varphi_0 \notin M$ and $T > 0$. As M is quasi-invariant, there is a complete trajectory ψ such that

$$\psi(0) = x,$$
$$\psi(t) \in M \text{ for all } t \leq 0.$$

Using $(K3)$ we can concatanate ψ and $\overline{\varphi}$ by

$$\phi(t) = \begin{cases} \psi(t+T) & \text{if } t \leq -T, \\ \overline{\varphi}(t+T) & \text{if } t \geq -T, \end{cases}$$

and obtain that $\varphi_0 \in \omega(\varphi) \cap \{W^-(M) \backslash M\}$. It is obvious that $\phi(t) \in \omega(\varphi) \cup M$ for all $t \in \mathbb{R}$.

Assume now that $\tau_k - t_k \to +\infty$. As $\tau_k \to +\infty$, up to a subsequence $y_k = \varphi(\tau_k) \to y \in \omega(\varphi)$. Let $\varphi_k(t) = \varphi(t + \tau_k)$. From [18, Lemma 13] there exists a complete trajectory ψ satisfying $\varphi_k(t) \to \psi(t)$ uniformly in bounded sets. Moreover,

$$\psi(0) = y \in \omega(\varphi),$$
$$dist(y, M) = \delta,$$
$$\psi(t) \in O_\delta(M) \text{ for all } t < 0.$$

The last inclusion is a consequence of the fact that $\varphi_k(t) \in O_\delta(M)$ for all $t \in [-\tau_k + t_k, 0)$. Thus, $\alpha(\psi) \subset \overline{O_\delta(M)} \subset O_\varepsilon(M)$. But $\psi(t) \in \omega(\varphi)$, for all t, implies that $\alpha(\psi)$ is non-empty and quasi-invariant, so $\alpha(\psi) \subset M$. Therefore, we have

$$y \in \omega(\varphi) \cap \{W^u(M) \backslash M\}.$$

□

With a similar proof the same result is obtained for the α-limit set of a complete trajectory.

Theorem 4.4. *Let $(K1) - (K4)$ hold and let the closure of the negative orbit of a complete trajectory $\phi \in \mathcal{F}$ be compact. We consider the compact, isolated, quasi-invariant set M. If $\alpha(\phi) \cap M \neq \varnothing$ but $\alpha(\phi) \not\subset M$, then*

$$\alpha(\phi) \cap \{W^s_w(M) \backslash M\} \neq \varnothing,$$
$$\alpha(\phi) \cap \{W^u(M) \backslash M\} \neq \varnothing.$$

The existence of stable and unstable sets given in Theorem 4.3 allows us to prove the existence of homoclinic structures inside ω-limit sets of trajectories. This is the opposite situation to that considered in [1] (see also [18]), in which Morse decompositions, characterized by the absense of homoclinic trajectories, are constructed.

We say that there exists a connection from the set M to the set N if there are $x \notin M \cup N$ and a bounded complete trajectory $\phi \in \mathcal{F}$ satisfying $\phi(0) = x$ and $\omega(\phi) \subset N$, $\alpha(\phi) \subset M$, being $\omega(\phi)$, $\alpha(\phi)$ non-empty. When M and N are disjoint the connection is called heteroclinic, whereas if $M = N$, it is called homoclinic. It is also said that M is chained to N.

A finite number of pairwise disjoint sets $\{M_1, ..., M_k\}$, $k \geq 1$, is said to be cyclically chained to each other (or a cyclical chain) if for any $j \in \{1, ..., k\}$ there exists a connection from M_j to M_{j+1}, where $M_{k+1} = M_1$. When $k = 1$, M_1 is just chained to itself. A cyclical chain is called also an homoclinic structure.

Let us consider the ω-limit set $\omega(\varphi)$ of a trajectory $\varphi \in \mathcal{K}$ having a compact positive orbit. Since this set is quasi-invariant, there exists a set of bounded complete trajectories $\mathbb{K}_\varphi$ such that

$$\omega(\varphi) = \{\phi(0) \mid \phi \in \mathbb{K}_\varphi\}.$$

We define the set Ω_φ by

$$\Omega_\varphi = \cup_{\phi \in \mathbb{K}_\varphi} \omega(\phi).$$

The following lemma extends to the multivalued case Proposition 3.3 in [37].

Lemma 4.5. *Let $(K1) - (K4)$ hold and let the closure of the positive orbit of $\varphi \in \mathcal{K}$ be compact. Let $\Omega_\varphi \subset M = \cup_{j=1}^{m} M_j$, $m \geq 1$, where $M_j \subset \omega(\varphi)$ are compact, pairwise disjoint, isolated, quasi-invariant subsets.*

Then either $\omega(\varphi) = M_1$ (so $m = 1$) or there exists a cyclical chain $\{\widetilde{M}_1, ..., \widetilde{M}_k\}$, $1 \leq k \leq m$, where each $\widetilde{M}_i$ is equal to some M_j. Moreover, the connections in this chain belong entirely to $\omega(\varphi)$.

Proof. If $\omega(\varphi) \neq M_1$, we choose some M_i and name it $\widetilde{M}_1$. Since $\omega(\phi) \setminus \widetilde{M}_1 \neq \varnothing$, Theorem 4.3 and $M_i \subset \omega(\varphi)$ imply the existence of a complete trajectory ϕ such that $\phi(\mathbb{R}) \subset \omega(\varphi)$, $\phi(0) \notin \widetilde{M}_1$ and $dist\left(\phi(t), \widetilde{M}_1\right) \to 0$ as $t \to -\infty$. By Lemma 2.1 the set $\omega(\phi)$ is connected, so it belong to one of the sets M_j, renamed $\widetilde{M}_2$. If $\widetilde{M}_2 = \widetilde{M}_1$, we are done. Otherwise, we have obtained a connection from $\widetilde{M}_1$ to $\widetilde{M}_2$. Further, arguing in the same way we get a connection from $\widetilde{M}_2$ to some $\widetilde{M}_3$. If either $\widetilde{M}_3 = \widetilde{M}_1$ or $\widetilde{M}_3 = \widetilde{M}_2$, we have finished. In other case we continue in the same way until in a finite number of steps we obtain the desired chain. □

5. Convergence to equilibria for reaction-diffusion equations without uniqueness.

In a bounded domain $\Omega \subset \mathbb{R}^n$, $1 \leq n \leq 3$, with sufficiently smooth boundary $\partial\Omega$ we consider the problem

$$\left\{ \begin{array}{l} u_t - \Delta u + f(u) = h, \quad x \in \Omega, \ t > 0, \\ u|_{\partial\Omega} = 0, \ t > 0, \\ u(0,x) = u_0(x), \ x \in \Omega, \end{array} \right. \tag{34}$$

where

$$\begin{gathered} h \in L^\infty(\Omega), \quad f \in C(\mathbb{R}), \\ |f(u)| \leq \alpha(1 + |u|^{p-1}), \quad \forall u \in \mathbb{R}, \\ f(u)u \geq \beta u^p + \gamma, \ \forall u \in \mathbb{R}, \end{gathered} \tag{35}$$

with $2 \leq p \leq 4$, $\alpha, \beta, \gamma > 0$.

Let $H = L^2(\Omega)$, $V = H_0^1(\Omega)$, whereas $\|\cdot\|$, $(\cdot,\cdot)$ will be the norm and the scalar product in $L^2(\Omega)$.

A function $u \in L^2_{loc}(0,+\infty; V) \bigcap L^p_{loc}(0,+\infty; L^p(\Omega))$ is called a weak solution of (34) on $(0,+\infty)$ if for all $T > 0$, $v \in V$, $\eta \in C_0^\infty(0,T)$,

$$-\int_0^T (u,v)\eta_t dt + \int_0^T ((u,v)_V + (f(u),v) - (h,v))\,\eta dt = 0.$$

A weak solution is called a strong one if, moreover, $u \in L^\infty(0,T;V)$, $\frac{du}{dt} \in L^2(0,T;H)$, for any $T > 0$. Any strong solution u satisfies $u \in L^2(0,T;D(A)) \cap C([0,+\infty);V)$.

It is well known [16, p.284] that for any $u_0 \in H$ there exists at least one weak solution of (34) with $u(0) = u_0$ (which might be non unique) and that any weak solution of (34) belongs to $C([0,+\infty);H)$.

The aim of this section is to prove, using Lemma 4.5, that the ω-limit set of every weak solution is a stationary point. For this purpose we need to recall some results about the properties of weak solutions and the structure of the global attractor for (34) given in [29].

Let $\mathcal{K} \subset C(\mathbb{R}^+, H)$ be the set of all weak solutions of problem (34) with initial condition in H. This set satisfies properties $(K1) - (K4)$, so it generates the strict m-semiflow $G : \mathbb{R}^+ \times H \to \mathcal{P}(H)$ by (1). Moreover, G possesses the global compact invariant attractor Θ (see the definition in Section 3). Therefore, every positive orbit $\gamma^+(\varphi)$, $\varphi \in \mathcal{K}$, has compact closure in H and by Lemma 2.1 its ω-limit set $\omega(\varphi)$ is nonempty, compact, quasi-invariant, connected and $\omega(\varphi) \subset \Theta$.

Further, we give an insight into the structure of the global attractor in terms of bounded complete trajectories. A complete trajectory ϕ is said to be bounded if $\cup_{t\in\mathbb{R}}\phi(t)$ is a bounded set. We denote by $\mathbb{K}$ the set of all bounded complete trajectories. Then the global attractor is characterized by

$$\Theta = \{\phi(0) \,:\, \phi(\cdot) \in \mathbb{K}\}.$$

We can give a more detail description of Θ in terms of the unstable and weakly stable sets of the stationary points. We denote by $\mathfrak{R}$ the set of all stationary points of (34), i.e., the points $u \in V$ such that

$$-\Delta u + f(u) = h \text{ in } H^{-1}(\Omega). \tag{36}$$

It is known [28, Lemmas 12, 14 and Theorem 13] that $\mathfrak{R} \neq \varnothing$ and that the following properties are equivalent:

1. $u_0 \in \mathfrak{R}$;
2. $u_0 \in G(t, u_0)$ for all $t \geq 0$;
3. The function $u(t) \equiv u_0$ belongs to $\mathcal{K}$.

We define now the sets:

$$\begin{aligned} M^s(\mathfrak{R}) &= \{z \,:\, \exists\phi(\cdot) \in \mathbb{K},\ \phi(0) = z,\ dist(\phi(t), \mathfrak{R}) \to 0,\ \ t \to +\infty\}, \\ M^u(\mathfrak{R}) &= \{z \,:\, \exists\phi(\cdot) \in \mathcal{F},\ \phi(0) = z,\ dist(\phi(t), \mathfrak{R}) \to 0,\ \ t \to -\infty\}. \end{aligned} \tag{37}$$

$M^u(\mathfrak{R})$ is the unstable set of $\mathfrak{R}$. $M^s(\mathfrak{R})$ is the weakly stable set of $\mathfrak{R}$ but considering only bounded complete trajectories. In the definition of $M^u(\mathfrak{R})$ we can replace $\mathcal{F}$ by $\mathbb{K}$, because the positive orbit of every complete trajectory ϕ is bounded.

Lemma 5.1. [29, Theorems 4, 5] *The global attractor Θ is bounded in $L^\infty(\Omega)$, compact in V and*

$$\Theta = M^u(\mathfrak{R}) = M^s(\mathfrak{R}).$$

Moreover, every weak solution $u(\cdot)$ with $u(0) \in \Theta$ is a strong solution.

In fact, any bounded complete trajectory ϕ satisfies the convergences given in (37), that is,

$$dist(\phi(t), \mathfrak{R}) \to 0,\ \ t \to +\infty,$$
$$dist(\phi(t), \mathfrak{R}) \to 0,\ \ t \to -\infty.$$

This follows from the fact that every solution inside the global attractor is strong, and then a Lyapunov function exists (see the proof of Theorem 37 in [28]). Hence, $\omega(\phi) \subset \mathfrak{R}$, $\alpha(\phi) \subset \mathfrak{R}$. In the particular, case when there exists a finite number of stationary points, as the sets $\omega(\phi)$, $\alpha(\phi)$ are connected, they have to be equal to one of the stationary points. In such a case the global attractor consists of the stationary points and all bounded complete trajectories connecting them.

We recall that a Lyapunov function $t \mapsto E(u(t))$ is strictly decreasing if $u(\cdot)$ is not a stationary point. Therefore, if there exists a connection from a stationary point e_1 to the stationary point e_2, then necessarily $E(e_2) < E(e_1)$.

We are now in position to prove the main result of this section.

Theorem 5.2. *Let the number of stationary points be finite. Then any $\varphi \in \mathcal{K}$ satisfies $\omega(\varphi) = e \in \mathfrak{R}$.*

Proof. As we have seen in Section 4 there exists a set of bounded complete trajectories $\mathbb{K}_\varphi$ such that

$$\omega(\varphi) = \{\phi(0) \mid \phi \in \mathbb{K}_\varphi\}.$$

In view of the previous arguments, the set $\Omega_\varphi = \cup_{\phi\in\mathbb{K}_\varphi}\omega(\phi)$ belongs to $\mathfrak{R}$. Put then $M = \Omega_\varphi = \cup_{j=1}^{m} e_j \subset \mathfrak{R}$ in Lemma 4.5. Therefore, the sets $M_j = e_j$, $j = 1, ..., m$, are the stationary points in $\omega(\varphi)$. Since there is a finite number of them, the sets M_j are compact, pairwise disjoint and quasi-invariant. In order to show that they are also isolated, we choose $\delta > 0$ such that $O_\delta(M_j) \cap O_\delta(M_j) = \varnothing$ if $i \neq j$. Let us assume that there exists a quasi-invariant set N inside of one $O_\delta(e_j)$ but $N \neq e_j$. Therefore, there is a bounded complete trajectory ϕ satisfying $\phi(0) \neq e_j$ and $\phi(\mathbb{R}) \subset N$. We know that $\phi(t)$ has to converge to a stationary point if either $t \to +\infty$ or $t \to -\infty$. Since the only possible point is e_j, there is a connection from e_j to itself. This would lead to the contradiction $E(e_j) < E(e_j)$. Thus, the sets M_j are isolated.

Finally, if $\omega(\varphi)$ was not equal to a stationary point, there would exist by Lemma 4.5 a cyclic chain in M. This is not possible again by the decreasing property of the Lyapunov function $E(\phi(t))$ along the connecting complete trajectories ϕ. We deduce that $\omega(\varphi) = e \in \mathfrak{R}$. □

Remark 4. The condition $h \in L^\infty(\Omega)$ is crucial in this theorem. If $h \in L^2(\Omega)$, then the question about the structure of $\omega(\varphi)$ for $\varphi \in \mathcal{K}$ remains open. Nevertheless, if $2 \leq p \leq 3$, then in [29] it is proved that every weak solution is regular and for such solutions a Lyapunov function also exists. Therefore, under this stronger condition on p the result $\omega(\varphi) = e \in \mathfrak{R}$ is also true.

We will consider finally a Chafee-Infante problem in which the number of stationary points is known to be finite:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \dfrac{\partial^2 u}{\partial x^2} = f(u),\ t > 0,\ x \in (0,1), \\ u(t,0) = 0,\ u(t,1) = 0, \\ u(0,x) = u_0(x). \end{cases} \tag{38}$$

The function f satisfies the conditions in (35) and also the following ones:

1. $f(0) = 0$;
2. $f'(0) > 0$ exists and is finite;
3. f is strictly concave if $u > 0$ and strictly convex if $u < 0$.

It was proved in [9] that if $n^2\pi^2 < f'(0) \leq (n+1)^2\pi^2$, where $n \geq 0$, $n \in \mathbb{Z}$, then there are exactly $2n+1$ stationary points. In this case $h \equiv 0$. Therefore, Theorem 5.2 implies the following result.

Theorem 5.3. *For any weak solution $\varphi \in \mathcal{K}$ of problem (38) we have that $\omega(\varphi) = e \in \mathfrak{R}$.*

Acknowledgments. We would like to thank the anonymous referees for their useful remarks.

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Received July 2019; revised September 2019.

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