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Dynamics and Wong-Zakai Approximations of Stochastic Nonlocal PDEs with Long Time Memory

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Abstract

In this paper, a combination of Galerkin's method and Dafermos' transformation is first used to prove the existence and uniqueness of solutions for a class of stochastic nonlocal PDEs with long time memory driven by additive noise. Next, the existence of tempered random attractors for such equations is established in an appropriate space for the analysis of problems with delay and memory. Eventually, the convergence of solutions of Wong-Zakai approximations and upper semicontinuity of random attractors of the approximate random system, as the step sizes of approximations approach zero, are analyzed in a detailed way.

Keywords Long time memory · Wong-Zakai approximation · Dafermos transformation · Random attractors · Upper semicontinuity

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1 Introduction

Motivated by some interesting physical problems related to thermal memory or materials with memory, several papers have been published (see [9, 10, 14–16, 24, 34, 37] and the references therein) concerning a semilinear partial differential equation to model the heat flow in a rigid, isotropic, homogeneous heat conductor with linear memory. The equation is the following,

$$\begin{cases} c_0\partial_t u - k_0\Delta u - \displaystyle\int_{-\infty}^{t} k(t-s)\Delta u(s)ds + f(u) = h, & \text{in } \mathcal{O}\times(\tau,+\infty), \\ u(x,t) = 0, & \text{on } \partial\mathcal{O}\times(\tau,+\infty), \\ u(x,\tau+t) = u_0(x,t), & \text{in } \mathcal{O}\times(-\infty,0], \end{cases} \tag{1.1}$$

where $\tau \in \mathbb{R}$, $\mathcal{O} \subset \mathbb{R}^N$ is a bounded domain with regular boundary, $u : \mathcal{O}\times\mathbb{R} \to \mathbb{R}$ is the temperature field, $k : \mathbb{R}^+ \to \mathbb{R}$ is the heat flux memory kernel, $\mathbb{R}^+$ denotes the interval $(0, +\infty)$, c_0 and k_0 denote the specific heat and the instantaneous conductivity, respectively.

To solve (1.1) successfully, one can make the past history of u, from $-\infty$ to 0, be part of the forcing term given by the causal function g as follows,

$$g(x,t) = h(x,t) + \int_{-\infty}^{\tau} k(t-s)\Delta u(x,s)ds, \qquad x \in \mathcal{O},\ \ t \geq \tau.$$

In this way, (1.1) becomes an initial value problem without delay or memory,

$$\begin{cases} c_0\partial_t u - k_0\Delta u - \displaystyle\int_{\tau}^{t} k(t-s)\Delta u(s)ds + f(u) = g, & \text{in } \mathcal{O}\times(\tau,+\infty), \\ u(x,t) = 0, & \text{on } \partial\mathcal{O}\times(\tau,+\infty), \\ u(x,\tau) = u_0(x), & \text{in } \mathcal{O}. \end{cases} \tag{1.2}$$

However, proceeding in this way, we cannot construct a dynamical system generated by the solutions of the original problem (1.1) in a correct way, since the history part of the function u is necessary to solve problem (1.2).

Therefore, two alternatives have been carried out to handle the problem in a correct mathematical way.

- Alternative 1: Based on Dafermos' idea for linear viscoelasticity problems (see, e.g., [10]), one can define the new variables,

$$u^t(x,s) = u(x,t-s), \qquad s \geq 0,$$

$$\eta^t(x,s) = \int_0^s u^t(x,r)dr = \int_{t-s}^{t} u(x,r)dr, \qquad s \geq 0. \tag{1.3}$$

Assuming $k(\infty) = 0$, thanks to a change of variable and a formal integration by parts, we obtain

$$\int_{-\infty}^{t} k(t-s)\Delta u(s)ds = -\int_0^\infty k'(s)\Delta\eta^t(s)ds.$$

Here and in the sequel, the *prime* denotes derivation with respect to the variable s. Setting

$$\mu(s) = -k'(s),$$

the original problem (1.2) becomes an autonomous one without delay,

$$\begin{cases} c_0\partial_t u - k_0\Delta u - \int_0^\infty \mu(s)\Delta\eta^t(s)ds + f(u) = h, & \text{in } \mathcal{O}\times(\tau,\infty), \\ \eta_t^t(s) = -\eta_s^t(s) + u(t), & \text{in } \mathcal{O}\times(\tau,\infty), s>0, \\ u(x,t) = \eta^t(s,x) = 0, & \text{on } \partial\mathcal{O}\times(\tau,\infty), s>0, \\ u(x,\tau) = u_0(x), & \text{in } \mathcal{O}, \\ \eta^\tau(x,s) = \eta_0(s), & \text{in } \mathcal{O}\times\mathbb{R}^+. \end{cases} \tag{1.4}$$

From the definition of $\eta^t(x,s)$ (see (1.3)), we see that

$$\eta_0(s) = \int_{\tau-s}^{\tau} u(r)dr = \int_{\tau-s}^{\tau} u_0(r-\tau)dr = \int_{-s}^{0} u_0(r)dr, \tag{1.5}$$

which is the initial integrated past history of u with vanishing boundary. Consequently, any solution to (1.2) is a solution to (1.4) for the corresponding initial values (u_0,η_0) given by (1.5).

Observe that problem (1.4) can be solved for arbitrary initial values (u_0,η_0) in a proper phase space $L^2(\mathcal{O})\times L^2_\mu(\mathbb{R}^+;H^1_0(\mathcal{O}))$, i.e., the second component η_0 does not necessarily depend on $u_0(\cdot)$, where $L^2_\mu(\mathbb{R}^+;H^1_0(\mathcal{O}))$ is a Hilbert space specified later.

Let μ satisfy the hypotheses:

(h_1) $\mu\in C^1(\mathbb{R}^+)\cap L^1(\mathbb{R}^+)$, $\mu(s)\geq 0$, $\mu'(s)\leq 0$, $\forall s\in\mathbb{R}^+$;
(h_2) $\mu'(s)+\varpi\mu(s)\leq 0$, $\forall s\in\mathbb{R}^+$, for some $\varpi>0$.

Then $L^2_\mu(\mathbb{R}^+;H^1_0(\mathcal{O}))$ is a Hilbert space of functions $w:\mathbb{R}^+\to H^1_0(\mathcal{O})$ with inner product,

$$((w_1,w_2))_\mu = \int_0^\infty \mu(s)(\nabla w_1(s),\nabla w_2(s))ds.$$

The solutions of (1.4) are proved to exist in [10] and permit to construct a dynamical system $S(t): L^2(\mathcal{O})\times L^2_\mu(\mathbb{R}^+;H^1_0(\mathcal{O}))\to L^2(\mathcal{O})\times L^2_\mu(\mathbb{R}^+;H^1_0(\mathcal{O}))$ via,

$$S(t)(u_0,\eta_0) = (u(t;0,(u_0,\eta_0)),\eta^t(\cdot;0,(u_0,\eta_0))),$$

which possesses a global attractor in this phase space. However, this global attractor does not reflect the complete asymptotic dynamics of the original problem (1.1) since the latter problem is not equivalent to (1.1). In other words, not for every $\eta_0 \in L^2_\mu(\mathbb{R}^+; H^1_0(\mathcal{O}))$, there exists $u_0 : (-\infty, 0] \to H^1_0(\mathcal{O})$ such that,

$$\eta_0(s) = \int_{-s}^{0} u_0(r)dr, \qquad \forall s \geq 0.$$

In fact, both problems are equivalent (cf. [16]) if and only if the initial value η_0 belongs to a proper subspace of $L^2_\mu(\mathbb{R}^+; H^1_0(\mathcal{O}))$. Precisely, the domain of the distributional derivative with respect to s, denoted by $D(\mathbf{T})$,

$$D(\mathbf{T}) = \left\{\eta(\cdot) \in L^2_\mu(\mathbb{R}^+; H^1_0(\mathcal{O})) \mid \eta_s(\cdot) \in L^2_\mu(\mathbb{R}^+; H^1_0(\mathcal{O})),\ \eta(0) = 0\right\},$$

and $\mathbf{T}$ is defined by $\mathbf{T}\eta = -\eta_s$, $\eta \in D(\mathbf{T})$.

Hence, it seems natural to construct a dynamical system generated by (1.4) in $L^2(\mathcal{O}) \times D(\mathbf{T})$ and to prove the existence of attractors to problem (1.1) via the above relationship. Up to our knowledge, it is not possible to prove the existence of attractors in $L^2(\mathcal{O}) \times D(\mathbf{T})$ unless the solutions admit more regularity.

- Alternative 2: The idea comes from a simpler case in [5] when the kernel is the so called non-singular one and has the expression $k(t) = e^{-d_0 t}$, $d_0 > 0$, considering the phase space $L^2_{H^1_0}$ formed by the functions $\varphi : (-\infty, 0] \to H^1_0(\mathcal{O})$ with $\int_{-\infty}^0 e^{\gamma s}\|\varphi(s)\|^2_{H^1_0} ds < +\infty$ for certain $\gamma > 0$. The authors in [5] proved that solutions of problem (1.1) with initial value u_0 generate a dynamical system which possesses a global attractor in $L^2_{H^1_0}$. However, when working with delay problems, it is natural (see e.g., [1, 3] and the references therein) to consider the phase space $L^2(\mathcal{O}) \times L^2_{H^1_0}$ and set up the problem as,

$$\begin{cases} c_0\partial_t u - k_0\Delta u - \displaystyle\int_{-\infty}^{t} k(t-s)\Delta u ds + f(u) = h, & \text{in } \mathcal{O} \times (\tau, +\infty), \\ u(x,t) = 0, & \text{on } \partial\mathcal{O} \times (\tau, +\infty), \\ u(x,\tau) = u_0(x), & \text{in } \mathcal{O}, \\ u(x, \tau + t) = \varphi(x,t), & \text{in } \mathcal{O} \times (-\infty, 0). \end{cases} \tag{1.6}$$

Thanks to the results in [5], we are able to construct a dynamical system $S(t) : L^2(\mathcal{O}) \times L^2_{H^1_0} \to L^2(\mathcal{O}) \times L^2_{H^1_0}$ via the relation,

$$S(t)(u_0, \varphi) := (u(t; 0, (u_0, \varphi)), u_t(\cdot; 0, (u_0, \varphi))), \tag{1.7}$$

where $u(\cdot; 0, (u_0, \varphi))$ denotes the solution of problem (1.6) (see [3]), and u_t the history up to time t,

$$u_t(s; 0, (u_0, \varphi)) = u(t + s; 0, (u_0, \varphi)), \quad s \leq 0.$$

The existence of global attractors in the space $L^2(\mathcal{O}) \times L^2_{H_0^1}$ is proved in [5]. In fact, it was proved for a non-autonomous version which is much more general than the one explained here. Nevertheless, as we mentioned before, the technique applied in this case (essentially the Galerkin approach) requires the kernel to be non-singular ($k(t) = e^{-d_0 t}$, $d_0 > 0$). This is a strong restriction on the kernel k (and consequently, on μ) because in applied science singularities appear very often, e.g., $k(t) = e^{-d_0 t} t^{-\alpha}$, $\alpha \in (0, 1)$. Motivated by this fact, recently we have proved in [37] the existence of global attractors in $L^2(\mathcal{O}) \times L^2_{H_0^1}$ for the general singular case, even for a more general model containing nonlocal diffusion coefficients thanks to a combination of Galerkin's method and Dafermos' transformation. More precisely, the following nonlocal PDE associated with singular memory was considered in [37],

$$\begin{cases} \partial_t u - a(l(u))\Delta u - \int_{-\infty}^{t} k(t-s)\Delta u ds + f(u) = g, & \text{in } \mathcal{O} \times (\tau, \infty), \\ u(x,t) = 0, & \text{on } \partial\mathcal{O} \times (\tau, \infty), \\ u(x,\tau) = u_0(x), & \text{in } \mathcal{O} \\ u(x, t+\tau) = \phi(x,t), & \text{in } \mathcal{O} \times (-\infty, 0), \end{cases} \tag{1.8}$$

where the function $a \in C(\mathbb{R}; \mathbb{R}^+)$ satisfies,

$$0 < m \leq a(r), \qquad \forall r \in \mathbb{R}, \tag{1.9}$$

with initial value $u_0 \in L^2(\mathcal{O})$ and initial function $\phi \in L^2_{H_0^1(\mathcal{O})}$. Then, the semigroup defined as in (1.7) for the solutions of (1.8) possesses a global attractor in the phase space $L^2(\mathcal{O}) \times L^2_{H_0^1}$. Also it is worth mentioning that the same results were proved in the phase space $L^2(\mathcal{O}) \times L^2_\mu(\mathbb{R}^+; H_0^1(\mathcal{O}))$ (see [38]) by using the classical Dafermos' method.

Our interest in this paper is to analyze the behavior of the nonlocal problem with memory when some stochastic disturbance appears in the model. Assume that this perturbation appears as an additive noise, namely, our objective is to study the following stochastic nonlocal PDEs with long time memory,

$$\begin{cases} \partial_t u - a(l(u))\Delta u - \int_{-\infty}^{t} k(t-s)\Delta u ds + f(u) = h + \phi\frac{dW(t)}{dt}, & \text{in } \mathcal{O} \times (\tau, \infty), \\ u(x,t) = 0, & \text{on } \partial\mathcal{O} \times (\tau, \infty), \\ u(x,\tau) = u_0(x), & \text{in } \mathcal{O}, \\ u(x, t+\tau) = \varphi(x,t), & \text{in } \mathcal{O} \times (-\infty, 0), \end{cases} \tag{1.10}$$

where $\tau \in \mathbb{R}$, $l \in \mathcal{L}(L^2(\mathcal{O}); \mathbb{R})$, $\mathcal{O} \subset \mathbb{R}^N$ is a fixed bounded domain with regular boundary, $h \in L^2(\mathcal{O})$, $W(t)$ is a two-sided standard Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and f is a polynomial of odd degree $2p-1$, $p \in \mathbb{N}$. Suppose that there are two constants m and M such that the function $a \in C(\mathbb{R}; \mathbb{R}^+)$ satisfies,

$$0 < m \le a(r) \le M, \qquad \forall r \in \mathbb{R}. \tag{1.11}$$

Here, $k : \mathbb{R}^+ \to \mathbb{R}$ is the memory kernel whose properties will be specified later. The initial value u_0 belongs to $L^2(\mathcal{O})$, while the initial function φ belongs to the space $L^2_{H_0^1}$, which is given by the measurable functions $\varphi : (-\infty, 0) \to H_0^1(\mathcal{O})$, such that

$$\int_{-\infty}^{0} e^{\gamma s} \|\varphi(s)\|^2_{H_0^1} ds < \infty,$$

for certain $\gamma > 0$. Furthermore, assume that $\phi \in H_0^1(\mathcal{O}) \cap H^2(\mathcal{O}) \cap L^{2p}(\mathcal{O})$ is such that $\Delta\phi \in L^{2p}(\mathcal{O})$. A local version of this problem has been analyzed in [4] (see also [6]) when the noise is a Hilbert valued Wiener process by using Dafermos' transformation, obtaining the existence of random attractors in the corresponding phase space for the Dafermos set-up. In this case, our intention now is to construct a framework to solve problems with delay and memory in an appropriate phase space, like in the deterministic model (see [37]).

In general, the Wiener process W can be chosen as a stochastic process to represent the position of the Brownian particle, but the velocity of the particle cannot be obtained from the Wiener process because of the nowhere differentiability of the sample paths of W [17]. Therefore, it is natural to approximate Brownian motion by more regular stochastic process, which is the so-called colored noise. In recent decades, the Wong-Zakai approximations to reaction-diffusion differential equations have been extensively studied in the literature, see, e.g., [2, 18, 20, 22, 23, 27, 28, 32, 33, 36, 39] and the references therein. One of the goals of this paper is to derive the relations between the solutions of problem (1.10) and the corresponding limiting problem.

To this end, in Sect. 2, we will first set-up problem (1.10) in an appropriate form and will do the transformation to obtain a random partial differential equation with delay. Then, a random partial differential system is obtained thanks to Dafermos' transformation. We will also include in Sect. 2 some necessary preliminaries and notation to tackle our problem. The well-posedness of the transformed system is proved in Sect. 3. Next, we prove in Sect. 4 the existence of random attractors to problem (1.10) in the phase space $L^2(\mathcal{O}) \times L^2_{H_0^1}$. In Sect. 5, we consider the approximation of the original problem by a parameterized family of problems containing colored noise which possesses a parameterized family of corresponding random attractors. Finally, in Sect. 6, we prove the upper-semicontinuity property of this family of parameterized random attractors with respect to the random attractors to problem (1.10).

2 Preliminaries

2.1 Set-Up of the Problem

The standard probability space $(\Omega, \mathcal{F}, \mathbb{P})$ will be used throughout this paper, where $\Omega = \{\omega \in C(\mathbb{R}; \mathbb{R}) : \omega(0) = 0\}$, $\mathcal{F}$ is the Borel σ-algebra induced by the compactopen topology of Ω and $\mathbb{P}$ is the Wiener measure on $(\Omega, \mathcal{F})$. Given $t \in \mathbb{R}$, define $\theta_t : \Omega \to \Omega$ by

$$\theta_t \omega(\cdot) = \omega(\cdot + t) - \omega(t), \quad \omega \in \Omega.$$

Then $(\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\}_{t\in\mathbb{R}})$ is a metric dynamical system and we identify $W(t, \omega) = \omega(t)$. Let $z(t, \omega)$ be the unique stationary solution of the stochastic equation $dz = -zdt + dW(t)$. This stationary solution is given by $z(t, \omega) = z_*(\theta_t\omega)$, where the random variable $z_*(\omega)$ is defined as,

$$z_*(\omega) = -\int_{-\infty}^{0} e^s W(s, \omega)ds. \tag{2.1}$$

In addition, it follows from [35] that there exists a θ_t-invariant set of full measure such that $z_*(\theta_t\omega)$ is pathwise continuous for each fixed $\omega \in \Omega$ and satisfies,

$$\lim_{t\to\pm\infty} \frac{|z_*(\theta_t\omega)|}{|t|} = 0 \quad \text{and} \quad \lim_{t\to\pm\infty} \frac{1}{t}\int_0^t z_*(\theta_s\omega)ds = 0. \tag{2.2}$$

We now transform the stochastic Eq. (1.10) into a pathwise deterministic one by using the random variable z_*. Given $\tau \in \mathbb{R}$, $t \geq \tau$ and $\omega \in \Omega$, if $u(t, \omega)$ is a solution of (1.10), then we introduce a new variable $v(t, \omega)$, by

$$v(t, \omega) = u(t, \omega) - \phi z_*(\theta_t\omega). \tag{2.3}$$

In this way, problem (1.10) can be rewritten as,

$$\begin{cases} v_t - a(l(v + \phi z_*(\theta_t\omega)))\Delta v - a(l(v + \phi z_*(\theta_t\omega)))z_*(\theta_t\omega)\Delta\phi & \\ \quad -\displaystyle\int_{-\infty}^{t} k(t-s)\Delta v ds + f(v + \phi z_*(\theta_t\omega)) = h + \phi z_*(\theta_t\omega) + z_k^{\phi}(\theta_t\omega), & \text{in } \mathcal{O}\times(\tau,\infty), \\ v(x,t) = 0, & \text{on } \partial\mathcal{O}\times(\tau,\infty), \\ v(x,\tau) = v_0(x) := u_0(x) - \phi z_*(\theta_\tau\omega), & \text{in } \mathcal{O}, \\ v(x,t+\tau) := u(x,t+\tau) - \phi z_*(\theta_{t+\tau}\omega) = \varphi(x,t) - \phi z_*(\theta_{t+\tau}\omega) := \varphi_v(x,t), & \text{in } \mathcal{O}\times(-\infty,0), \end{cases} \tag{2.4}$$

where $z_k^{\phi}(\omega)$ is a process defined by

$$z_k^{\phi}(\omega) = \Delta\phi \int_{-\infty}^{0} k(-s)z_*(\theta_s\omega)ds. \tag{2.5}$$

Notice that a change of variable yields that

$$z_k^{\phi}(\theta_t\omega) = \Delta\phi \int_{-\infty}^{0} k(-s)z_*(\theta_s\theta_t\omega)ds = \Delta\phi \int_{-\infty}^{t} k(t-s)z_*(\theta_s\omega)ds.$$

In order to use Dafermos' transform (see [10]) to establish the well-posedness of problem (2.4), let us define the new variables,

$$\begin{aligned} &v^t(x,s,\omega)=v(x,t-s,\omega), \qquad s\geq 0,\\ &\eta^t(x,s,\omega)=\int_0^s v^t(x,r,\omega)dr=\int_{t-s}^t v(x,r,\omega)dr, \qquad s\geq 0. \end{aligned}$$

Besides, assuming $k(\infty)=0$, a change of variable and a formal integration by parts (see Lemma 3.6 for a rigorous calculation) imply,

$$\int_{-\infty}^t k(t-s)\Delta v(s)ds=-\int_0^\infty k'(s)\Delta\eta^t(s)ds.$$

Setting $\mu(s)=-k'(s)$, problem (2.4) turns into the following system without delay,

$$\begin{cases} v_t-a(l(v+\phi z_*(\theta_t\omega)))\Delta v-a(l(v+\phi z_*(\theta_t\omega)))z_*(\theta_t\omega)\Delta\phi & \\ \quad -\int_0^\infty \mu(s)\Delta\eta^t(s)ds+f(v+\phi z_*(\theta_t\omega))=h+\phi z_*(\theta_t\omega)+z_k^\phi(\theta_t\omega), & \text{in } \mathcal{O}\times(\tau,\infty),\\ \eta_t^t(s)=-\eta_s^t(s)+v(t), & \text{in } \mathcal{O}\times(\tau,\infty), s>0,\\ v(x,t)=\eta^t(x,s)=0, & \text{on } \partial\mathcal{O}\times(\tau,\infty), s>0,\\ v(x,\tau)=v_0(x):=u_0(x)-\phi z_*(\theta_\tau\omega), & \text{in } \mathcal{O},\\ \eta^\tau(x,s)=\eta_0(x,s), & \text{in } \mathcal{O}\times\mathbb{R}^+, \end{cases} \tag{2.6}$$

where

$$\eta_0(s)(\omega)=\int_{\tau-s}^\tau v(x,r,\omega)dr=\int_{-s}^0(\varphi(r)-\phi z_*(\theta_{r+\tau}\omega))dr:=\int_{-s}^0\varphi_v(r)dr, \tag{2.7}$$

which contains the initial integrated past history of φ with vanishing boundary and a piece of value of $z_*(\theta_{\cdot+\tau}\omega)$ on $(-s,0]$. Moreover, η_s^t denotes the distributional derivative of $\eta^t(s)$ with respect to the internal variable s.

2.2 Assumptions

We will enumerate the assumptions on the nonlinear term f and the variable μ. In our analysis, suppose that $h\in L^2(\mathcal{O})$, $f:\mathbb{R}\to\mathbb{R}$ is a polynomial of odd degree with positive leading coefficient,

$$f(u)=\sum_{k=1}^{2p} f_{2p-k}u^{k-1}, \qquad p\in\mathbb{N}. \tag{2.8}$$

The variable μ is required to verify the following hypotheses:

(h_1) $\mu \in C^1(\mathbb{R}^+) \cap L^1(\mathbb{R}^+)$, $\mu(s) \geq 0$, $\forall s \in \mathbb{R}^+$;
(h_2) $\mu'(s) + \varpi \mu(s) \leq 0$, $\forall s \in \mathbb{R}^+$, for some $\varpi > 0$.

Remark 2.1 (i) Recall that from $\mu(s) = -k'(s)$, together with condition (h_1), we infer immediately that there exists a constant $M_1 > 0$ such that,

$$k(t) = -\int_t^\infty k'(s)ds = \int_t^\infty \mu(s)ds < M_1, \qquad \forall\, 0 \leq t < \infty.$$

(ii) In terms of assumption (h_2) imposed on μ, it is easy to see that,

$$\mu(s_2) \leq \mu(s_1)e^{-\varpi(s_2-s_1)}, \qquad \forall\, 0 < s_1 < s_2.$$

(iii) Combining the results of (i) and (ii), we have

$$k(t) = \int_t^\infty \mu(s)ds \leq \mu(t)\int_t^\infty e^{-\varpi(s-t)}ds := \frac{\mu(t)}{\varpi}, \qquad \forall\, 0 < t < \infty.$$

2.3 Notation

Let $\mathcal{O}$ be a fixed bounded domain in $\mathbb{R}^N$ with regular boundary. On this set, we introduce the Lebesgue space $L^p(\mathcal{O})$ with the natural norm $\|\cdot\|_p$, where $1 \leq p \leq \infty$. Besides, $W^{1,p}(\mathcal{O})$ is the subspace of $L^p(\mathcal{O})$ consisting of functions such that the first order weak derivative belongs to $L^p(\mathcal{O})$. For convenience, $L^2(\mathcal{O})$ is denoted by H, $H_0^1(\mathcal{O})$ is denoted by V and $H^{-1}(\mathcal{O})$, the dual space of $H_0^1(\mathcal{O})$, is denoted by V^*. We will use the norms and inner products of H and V as $|\cdot|$, $\|\cdot\|$, and $(\cdot,\cdot)$, $((\cdot,\cdot))$, respectively. Moreover, $<\cdot,\cdot>$ will denote the duality pairing between V and V^*.

Taking into account system (2.6) and (h_1), we need to modify slightly the notation before showing main results. Let $L^2_\mu(\mathbb{R}^+; H)$ be a Hilbert space of functions $w : \mathbb{R}^+ \to H$ endowed with the inner product,

$$(w_1, w_2)_\mu = \int_0^\infty \mu(s)(w_1(s), w_2(s))ds,$$

and let $|\cdot|_\mu$ denote the corresponding norm. In a similar way, we introduce the inner products $((\cdot,\cdot))_\mu$, $(((\cdot,\cdot)))_\mu$ and relative norms $\|\cdot\|_\mu$, $|||\cdot|||_\mu$ on $L^2_\mu(\mathbb{R}^+; V)$, $L^2_\mu(\mathbb{R}^+; V \cap H^2(\mathcal{O}))$, respectively. It follows then that

$$((\cdot,\cdot))_\mu = (\nabla\cdot, \nabla\cdot)_\mu, \text{ and } (((\cdot,\cdot)))_\mu = (\Delta\cdot, \Delta\cdot)_\mu.$$

We also define the Hilbert spaces,

$$\mathcal{H} = H \times L^2_\mu(\mathbb{R}^+; V),$$

and

$$\mathcal{V} = V \times L^2_\mu(\mathbb{R}^+; V \cap H^2(\mathcal{O})),$$

which are respectively endowed with the inner products,

$$(w_1, w_2)_{\mathcal{H}} = (w_1, w_2) + ((w_1, w_2))_\mu,$$

and

$$(w_1, w_2)_{\mathcal{V}} = ((w_1, w_2)) + (((w_1, w_2)))_\mu,$$

where $w_i \in \mathcal{H}$ or $\mathcal{V}$ $(i = 1, 2)$. The norms induced on $\mathcal{H}$ and $\mathcal{V}$ are the so-called energy ones, which read

$$\|(w_1, w_2)\|^2_{\mathcal{H}} = |w_1|^2 + \int_0^\infty \mu(s)\|w_2(s)\|^2 ds,$$

and

$$\|(w_1, w_2)\|^2_{\mathcal{V}} = \|w_1\|^2 + \int_0^\infty \mu(s)\|\nabla w_2(s)\|^2 ds.$$

At last, with standard notation, $\mathcal{D}(I; X)$ is the space of infinitely differentiable X-valued functions with compact support in $I \subset \mathbb{R}$, whose dual space is the distribution one on I with values in X^* (dual of X), denoted by $\mathcal{D}'(I; X^*)$. For convenience, we define L^2_V as the space of functions $u\,(\cdot)$ such that,

$$\int_{-\infty}^0 e^{\gamma s}\, \|u\,(s)\|^2\, ds < \infty,$$

where $0 < \gamma < \min\{\frac{m\lambda_1}{2}, \varpi\}$, λ_1 is the first eigenvalue of $-\Delta$ with zero Dirichlet boundary condition and ϖ comes from (h_2).

3 Well-Posedness of Problem (1.10)

In this section, before presenting the well-posedness of problem (1.10), we first state an auxiliary result for the regularity of initial value η_0 (cf. (2.7)), which is the essential point to prove the existence and uniqueness of solutions to problem (2.6).

3.1 An Auxiliary Result

Let us first recall a crucial technical lemma in [37, Lemma 3.1].

Lemma 3.1 *Assume* (h_1)-(h_2) *hold. Then, the operator* $\mathcal{J} : L^2_V \to L^2_\mu(\mathbb{R}^+; V)$ *defined by,*

$$(\mathcal{J}\varphi)(s) = \int_{-s}^0 \varphi(r)dr, \qquad s \in \mathbb{R}^+,$$

is a linear and continuous operator. In particular, there exists a positive constant $K_\mu = e^\gamma \int_0^1 \mu(s)ds + \mu(1)e^{\varpi}(\gamma - \varpi)^{-2}$ *such that for any* $\varphi \in L_V^2$*, it holds*

$$\|\mathcal{J}\varphi\|^2_{L^2_\mu(\mathbb{R}^+;V)} \leq K_\mu \|\varphi\|^2_{L^2_V}.$$

For the sake of simplicity, define $(\mathcal{J}_{\omega,\tau}\varphi)(s) := \mathcal{J}(\varphi - \phi z_*(\theta_{\cdot+\tau}\omega))(s)$. By slightly modifying the proof of Lemma 3.1 in [37], we have the following corollary.

Corollary 3.2 *Assume* (h_1)*-*(h_2) *hold and* $\phi \in V \cap H^2(\mathcal{O}) \cap L^{2p}(\mathcal{O})$*. Then, for every* $\omega \in \Omega$ *and* $\tau \in \mathbb{R}$*, the operator* $\mathcal{J}_{\omega,\tau} : L_V^2 \to L_\mu^2(\mathbb{R}^+; V)$ *defined by*

$$(\mathcal{J}_{\omega,\tau}\varphi)(s) := \int_{-s}^0 \varphi(r)dr - \int_{-s}^0 z_*(\theta_{r+\tau}\omega)\phi dr = \mathcal{J}(\varphi - \phi z_*(\theta_{\cdot+\tau}\omega))(s), \quad (3.1)$$

is continuous. Additionally, there exists a positive constant K_μ *which is the same as in Lemma 3.1, such that for any* $\varphi \in L_V^2$*, we have*

$$\|\mathcal{J}_{\omega,\tau}\varphi\|^2_{L^2_\mu(\mathbb{R}^+;V)} \leq K_\mu \left(\|\varphi - z_*(\theta_{\cdot+\tau}\omega)\phi\|^2_{L^2_V}\right) \leq 2K_\mu \left(\|\varphi\|^2_{L^2_V} + \|z_*(\theta_{\cdot+\tau}\omega)\phi\|^2_{L^2_V}\right).$$

Proof First of all, we show that the operator $\mathcal{J}_{\omega,\tau}$ is well-defined. As $\phi \in V \cap H^2(\mathcal{O}) \cap L^{2p}(\mathcal{O})$, it is easy to check that $z_*(\theta_{\cdot+\tau}\omega)\phi \in L_V^2$. Indeed, for every $\omega \in \Omega$,

$$\begin{aligned}\|z_*(\theta_{\cdot+\tau}\omega)\phi\|^2_{L^2_V} &= \int_{-\infty}^0 e^{\gamma t}|z_*(\theta_{t+\tau}\omega)|^2\|\phi\|^2 dt = \|\phi\|^2 \int_{-\infty}^{\tau} e^{\gamma(t-\tau)}|z_*(\theta_t\omega)|^2 dt \\ &\leq e^{-\gamma\tau}\|\phi\|^2 \left(\int_{-\infty}^0 e^{\gamma t}|z_*(\theta_t\omega)|^2 dt + \int_0^\tau e^{\gamma t}|z_*(\theta_t\omega)|^2 dt\right) < +\infty,\end{aligned}$$

where we have used the continuity property of $z_*(\theta_t\omega)$ with respect to t and (2.2). Notice that, by Remark 2.1(ii), we find

$$\begin{aligned}&\left\|\int_{-s}^0 \varphi - z_*(\theta_{r+\tau}\omega)\phi dr\right\|^2_{L^2_\mu(\mathbb{R}^+;V)} \\ &\leq \int_0^\infty \mu(s)\left(\int_{-s}^0 \|\varphi(r) - z_*(\theta_{r+\tau}\omega)\phi\|dr\right)^2 ds \\ &\leq \int_0^1 \mu(s)s\int_{-s}^0 \|\varphi(r) - z_*(\theta_{r+\tau}\omega)\phi\|^2 drds \\ &\quad + \int_1^\infty \mu(s)\left(\int_{-s}^0 \|\varphi(r) - z_*(\theta_{r+\tau}\omega)\phi\|dr\right)^2 ds \\ &\leq \int_{-1}^0 \|\varphi(r) - z_*(\theta_{r+\tau}\omega)\phi\|^2 e^{\gamma r}e^{-\gamma r}\int_0^1 s\mu(s)dsdr \\ &\quad + \mu(1)e^{\varpi}\int_{-\infty}^0 e^{\gamma r}\|\varphi(r) - z_*(\theta_{r+\tau}\omega)\phi\|^2 \int_{-r}^\infty se^{-\varpi s}e^{\gamma s}dsdr\end{aligned}$$

$$\begin{aligned}
&\le \|\varphi - z_*(\theta_{\cdot+\tau}\omega)\phi\|^2_{L^2_V} e^{\gamma} \int_0^1 s\mu(s)ds \\
&\quad + \mu(1)e^{\varpi}(\varpi-\gamma)^{-2}\|\varphi - z_*(\theta_{\cdot+\tau}\omega)\phi\|^2_{L^2_V} \\
&\le \left(e^{\gamma}\int_0^1 \mu(s)ds + \mu(1)e^{\varpi}(\varpi-\gamma)^{-2}\right)\|\varphi - z_*(\theta_{\cdot+\tau}\omega)\phi\|^2_{L^2_V} \\
&= K_\mu \|\varphi - z_*(\theta_{\cdot+\tau}\omega)\phi\|^2_{L^2_V}.
\end{aligned}$$

The proof of this corollary is complete. ◻

Remark 3.3 Once we fix an initial function $\varphi \in L^2_V$ to problem (1.10), then for every $\omega \in \Omega$, the initial function $\varphi_v := \varphi - \phi z_*(\theta_{\cdot+\tau}\omega)$ of (2.4) belongs to L^2_V as $\phi \in V \cap H^2(\mathcal{O}) \cap L^{2p}(\mathcal{O})$. Also, the initial value for the second component $\eta_0(s,\omega) = (\mathcal{J}_{\omega,\tau}\varphi)(s)$ of problem (2.6) belongs to $L^2_\mu(\mathbb{R}^+; V)$ thanks to Corollary 3.2. Analogously, if we pick up the initial function $\varphi \in L^2_{V\cap H^2(\mathcal{O})}$ to problem (1.10), then the initial function $\varphi_v := \varphi - \phi z_*(\theta_{\cdot+\tau}\omega)$ of (2.4) also belongs to $L^2_{V\cap H^2(\mathcal{O})}$. Thus, making use of a similar proof as in Corollary 3.2, it is easy to check that $\eta_0(s,\omega)$ defined by (3.1) belongs to $L^2_\mu(\mathbb{R}^+; V\cap H^2(\mathcal{O}))$.

3.2 Well-Posedness of Problem (2.6)

Theorem 3.4 *Assume that (1.11), (2.8) and* (h_1) *-*(h_2) *hold. Let* $\phi \in V \cap H^2(\mathcal{O}) \cap L^{2p}(\mathcal{O})$ *be such that* $\Delta\phi \in L^{2p}(\mathcal{O})$*, let* $h \in H$ *and a be a locally Lipschitz function. Then:*

(i) *For every* $\omega \in \Omega$*, any initial value* $v_0 \in H$ *and initial function* $\varphi \in L^2_V$*, there exists a unique solution* (v,η) *to problem (2.6) in the weak sense with initial value* (v_0,η_0)*, where* $\eta_0(s,\omega) = (\mathcal{J}_{\omega,\tau}\varphi)(s)$*, fulfilling*

$$\begin{aligned}
&v \in L^\infty(\tau,T;H) \cap L^2(\tau,T;V) \cap L^{2p}(\tau,T;L^{2p}(\mathcal{O})), \qquad \forall T > \tau, \\
&\eta \in L^\infty(\tau,T;L^2_\mu(\mathbb{R}^+;V)), \qquad \forall T > \tau.
\end{aligned}$$

Furthermore, the solution (v,η) *of (2.6) is continuous with respect to the initial value* (v_0,η_0) *for all* $t \in [\tau,T]$ *in* $\mathcal{H}$*;*

(ii) *For any initial value* $(v_0,\eta_0) \in \mathcal{V}$*, the unique solution* (v,η) *to problem (2.6) satisfies,*

$$\begin{aligned}
&v \in L^\infty(\tau,T;V) \cap L^2(\tau,T;V\cap H^2(\mathcal{O})), \qquad \forall T > \tau, \\
&\eta \in L^\infty(\tau,T;L^2_\mu(\mathbb{R}^+;V\cap H^2(\mathcal{O}))), \qquad \forall T > \tau.
\end{aligned}$$

In addition, the solution (v,η) *of (2.6) is continuous with respect to the initial value* (v_0,η_0) *for all* $t \in [\tau,T]$ *in* $\mathcal{V}$*.*

Remark 3.5 By a similar argument as in [37, p.443] one can prove that

$$v \in C([\tau, T], H), \ \eta \in C([\tau, T], L^2_\mu(\mathbb{R}^+; V)).$$

Proof On the one hand, for every $\omega \in \Omega$, it follows from Corollary 3.2 that $(\mathcal{J}_{\omega,\tau}\varphi)(s) \in L^2_\mu(\mathbb{R}^+; V)$ thanks to the facts $\varphi \in L^2_V$ and $\phi \in V \cap H^2(\mathcal{O}) \cap L^{2p}(\mathcal{O})$. We will prove (i) in five steps.

Step 1. (Faedo-Galerkin scheme) Let $\{w_j\}_{j=1}^\infty$ be the eigenfunctions of the operator $-\Delta$, which is an orthonormal basis in H. For regular domains these functions belong to $V \cap L^{2p}(\mathcal{O})$ [25, Chapter 8]. We select also an orthonormal basis $\{\zeta_j\}_{j=1}^\infty$ of $L^2_\mu(\mathbb{R}^+; V)$ which belongs to $\mathcal{D}(\mathbb{R}^+; V)$ as well. Fix $T > \tau$. For a given integer n and each ω, denote by P_n and Q_n the projectors on the subspaces,

$$\text{span}\{w_1, \cdots, w_n\} \subset V, \qquad \text{and} \qquad \text{span}\{\zeta_1, \cdots, \zeta_n\} \subset L^2_\mu(\mathbb{R}^+; V),$$

respectively. We will look for a function (v_n, η_n) of the form,

$$v_n(t) = \sum_{j=1}^n b_j(t) w_j \qquad \text{and} \qquad \eta_n^t(s) = \sum_{j=1}^n c_j(t)\zeta_j(s),$$

satisfying for all $t \geq \tau$,

$$\begin{cases} \dfrac{d}{dt} b_k(t) + \lambda_k a \left(l\left(\sum_{j=1}^n b_j(t) w_j\right) + l(\phi) z_*(\theta_t\omega)\right) b_k(t) + a\left(l\left(\sum_{j=1}^n b_j(t) w_j\right)\right. \\ \quad \left. + l(\phi) z_*(\theta_t\omega)\right) z_*(\theta_t\omega)\lambda_k < \phi, w_k > + \sum_{j=1}^n c_j(t)((\zeta_j, w_k))_\mu \\ \quad + \left(f\left(\sum_{j=1}^n b_j(t) w_j + \phi z_*(\theta_t\omega)\right), w_k\right) \\ \quad = z_*(\theta_t\omega)(\phi, w_k) + (z_k^\phi(\theta_t\omega), w_k) + (h, w_k), \\ \dfrac{d}{dt} c_k(t) = -\sum_{j=1}^n c_j(t)((\zeta_j', \zeta_k))_\mu + \sum_{j=1}^n b_j(t)((w_j, \zeta_k))_\mu, \quad k = 1, 2, \cdots, n, \end{cases} \tag{3.2}$$

where λ_j is the eigenvalue associated to the eigenfunction w_j, subject to the initial conditions

$$b_k(\tau) = (v_0, w_k), \qquad c_k(\tau) = ((\eta_0, \zeta_k))_\mu. \tag{3.3}$$

According to the standard existence theory for ordinary differential systems, there exists a continuous solution of (3.2)–(3.3) on some interval (τ, t_n). Proceeding as usual, by establishing some a priori estimates below, we can ensure that $t_n = \infty$.

Step 2. (Energy Estimates) Multiplying the first equation of (3.2) by b_k and the second one by c_k, respectively, summing over k ($k = 1, 2, \cdots, n$) and adding the results, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|v_n(t)|^2 + a(l(v_n(t)) + l(\phi)z_*(\theta_t\omega))\|v_n(t)\|^2 + a(l(v_n(t))\\
&\quad + l(\phi)z_*(\theta_t\omega))z_*(\theta_t\omega)((\phi, v_n)) + ((\eta_n^t(s), v_n(t)))_\mu + (f(v_n(t) + \phi z_*(\theta_t\omega)), v_n(t))\\
&\quad = z_*(\theta_t\omega)(\phi, v_n(t)) + (z_k^\phi(\theta_t\omega), v_n(t)) + (h, v_n(t)),\\
&\frac{1}{2}\frac{d}{dt}\|\eta_n^t(s)\|_\mu^2 = -((\eta_n^t(s), (\eta_n^t(s))'))_\mu + ((\eta_n^t(s), v_n(t)))_\mu.
\end{aligned}$$

Combining the two equations above, for every fixed $\omega \in \Omega$ and $t \in [\tau, T]$, by condition (1.11), we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|v_n(t)|^2 + \frac{1}{2}\frac{d}{dt}\|\eta_n^t(s)\|_\mu^2 + m\|v_n(t)\|^2 + a(l(v_n(t))\\
&\quad + l(\phi)z_*(\theta_t\omega))z_*(\theta_t\omega)((\phi, v_n(t))) + (f(v_n(t) + \phi z_*(\theta_t\omega)), v_n(t))\\
&\quad \le -((\eta_n^t(s), (\eta_n^t(s))'))_\mu + |z_*(\theta_t\omega)||\phi||v_n(t)| + (z_k^\phi(\theta_t\omega), v_n(t)) + (h, v_n(t)),
\end{aligned}$$

which is equivalent to

$$\begin{aligned}
&\frac{d}{dt}|v_n(t)|^2 + \frac{d}{dt}\|\eta_n^t(s)\|_\mu^2 + 2m\|v_n(t)\|^2 + 2a(l(v_n(t))\\
&\quad + l(\phi)z_*(\theta_t\omega))z_*(\theta_t\omega)((\phi, v_n(t)))\\
&\quad + 2(f(v_n(t) + \phi z_*(\theta_t\omega)), v_n(t))\\
&\le -2((\eta_n^t(s), (\eta_n^t(s))'))_\mu + 2|z_*(\theta_t\omega)||\phi||v_n(t)|\\
&\quad + 2(z_k^\phi(\theta_t\omega), v_n(t)) + 2(h, v_n(t)).
\end{aligned} \tag{3.4}$$

Let us do estimates for (3.4) one by one. First of all, by integration by parts, we have

$$-2((\eta_n^t(s), (\eta_n^t(s))'))_\mu = \int_0^\infty \mu'(s)|\nabla\eta_n^t(s)|^2 ds \le 0. \tag{3.5}$$

Second, by (1.11) and the Young inequality, we obtain

$$\begin{aligned}
&-2a(l(v_n(t)) + l(\phi)z_*(\theta_t\omega))z_*(\theta_t\omega)((\phi, v_n(t))) \le 2M|z_*(\theta_t\omega)|\|\phi\|\|v_n(t)\|\\
&\quad \le \frac{m}{4}\|v_n(t)\|^2 + \frac{4M^2}{m}|z_*(\theta_t\omega)|^2\|\phi\|^2.
\end{aligned} \tag{3.6}$$

Third, by the Poincaré and Young inequalities, we derive

$$\begin{aligned}
2|z_*(\theta_t\omega)||\phi||v_n(t)| &\le 2|z_*(\theta_t\omega)||\phi|\frac{\|v_n(t)\|}{\sqrt{\lambda_1}}\\
&\le \frac{m}{4}\|v_n(t)\|^2 + \frac{4}{m\lambda_1}|z_*(\theta_t\omega)|^2|\phi|^2.
\end{aligned} \tag{3.7}$$

Fourth, for the nonlinear term f, making use of the Young inequality, we infer that there exist constants $\alpha, \beta > 0$, such that

$$f(u)u \geq \frac{1}{2} f_0 u^{2p} - \alpha, \qquad \text{and} \qquad |f(u)| \leq \beta(1 + |u|^{2p-1}). \tag{3.8}$$

Therefore, by the continuity of $z_*(\theta_t\omega)$, (3.8) and the Young inequality, we deduce there exists a constant $\widetilde{C}_1$ such that

$$\begin{aligned}
&2(f(v_n(t) + \phi z_*(\theta_t\omega)), v_n(t)) \\
&\quad = 2\int_{\mathcal{O}} f(v_n(t) + \phi z_*(\theta_t\omega))\,(v_n(t) + \phi z_*(\theta_t\omega))\,dx \\
&\qquad -2\int_{\mathcal{O}} f(v_n(t) + \phi z_*(\theta_t\omega))\phi z_*(\theta_t\omega)dx \\
&\quad \geq f_0 \int_{\mathcal{O}} |v_n(t) + \phi z_*(\theta_t\omega)|_{2p}^{2p}\,dx \\
&\qquad -\widetilde{C}_1 \int_{\mathcal{O}} \left(1 + |v_n(t)|^{2p-1} + |z_*(\theta_t\omega)|^{2p-1}|\phi|^{2p-1}\right)|z_*(\theta_t\omega)||\phi|dx \\
&\qquad -2\alpha|\mathcal{O}|.
\end{aligned} \tag{3.9}$$

Since

$$|v|^{2p} = |v + r - r|^{2p} \leq D\left(|v + r|^{2p} + |r|^{2p}\right),$$

for some $D = D(p) > 0$, we obtain

$$\begin{aligned}
&2(f(v_n(t) + \phi z_*(\theta_t\omega)), v_n(t)) \\
&\quad \geq \frac{f_0}{D}\int_{\mathcal{O}} |v_n(t)|^{2p}\,dx - \frac{f_0}{D}|z_*(\theta_t\omega)|^{2p}\|\phi\|_{2p}^{2p} \\
&\qquad - \widetilde{C}_1 \max\{|z_*(\theta_t\omega)|, |z_*(\theta_t\omega)|^{2p}\}\int_{\mathcal{O}}\left(1 + |v_n(t)|^{2p-1} + |\phi|^{2p-1}\right)|\phi|dx - 2\alpha|\mathcal{O}| \\
&\quad \geq \frac{f_0}{2D}\|v_n(t)\|_{2p}^{2p} - C_1(\theta_t\omega)(1+\|\phi\|_{2p}^{2p}) - 2\alpha|\mathcal{O}|.
\end{aligned}$$

Here, $C_1(\omega) := C_1(|z_*(\omega)|, p, |\mathcal{O}|) = \widetilde{C}_2(1+|z_*(\omega)|^{4p^2})$, for some $\widetilde{C}_2 = \widetilde{C}_2(p, |\mathcal{O}|) > 0$. As for the last term, by the Young inequality, the properties of $z_*(\theta_t\omega)$ (cf. (2.2)) and Remark 2.1, we deduce that there exists a random variable $C_2(\omega)$, such that

$$\begin{aligned}
&2(z_k^{\phi}(\theta_t\omega), v_n(t)) \\
&\quad \leq 2\left(\int_0^{\infty} k(s)|z_*(\theta_{t-s}\omega)|ds\right)\|\phi\|\|v_n(t)\| \\
&\quad \leq 2\left(\int_0^{1} k(s)|z_*(\theta_{t-s}\omega)|ds + \frac{1}{\varpi}\int_1^{\infty} \mu(s)|z_*(\theta_{t-s}\omega)|ds\right)\|\phi\|\|v_n(t)\|
\end{aligned}$$

$$
\begin{aligned}
&\leq 2\left(M_1\int_0^1 |z_*(\theta_{t-s}\omega)|ds+\frac{\mu(1)e^{\varpi}}{\varpi}\int_1^\infty e^{-\varpi s}|z_*(\theta_{t-s}\omega)|ds\right)\|\phi\|\|v_n(t)\|\\
&= C_2(\theta_t\omega)\|\phi\|\|v_n(t)\| \leq \frac{C_2(\theta_t\omega)^2}{m}\|\phi\|^2+\frac{m}{4}\|v_n(t)\|^2. \qquad (3.10)
\end{aligned}
$$

Also, by the Young inequality, we have

$$
2\,(h, v_n) \leq \frac{4}{m\lambda_1}|h|^2+\frac{m}{4}\|v_n(t)\|^2. \qquad (3.11)
$$

Substituting (3.5)–(3.11) into (3.4), we obtain

$$
\begin{aligned}
&\frac{d}{dt}|v_n(t)|^2+\frac{d}{dt}\|\eta_n^t(s)\|_\mu^2+m\|v_n(t)\|^2+\frac{f_0}{2D}\|v_n(t)\|_{2p}^{2p} \leq \frac{4M^2}{m}|z_*(\theta_t\omega)|^2\|\phi\|^2\\
&+\frac{4}{m\lambda_1}|z_*(\theta_t\omega)|^2|\phi|^2+2\alpha|\mathcal{O}|+C_1(\theta_t\omega)(1+\|\phi\|_{2p}^{2p})+\frac{C_2(\theta_t\omega)^2}{m}\|\phi\|^2+\frac{4}{m\lambda_1}|h|^2.
\end{aligned} \qquad (3.12)
$$

Denote

$$
\begin{aligned}
\Theta_1(\omega) =\ &\frac{4M^2}{m}|z_*(\omega)|^2\|\phi\|^2+\frac{4}{m\lambda_1}|z_*(\omega)|^2|\phi|^2+2\alpha|\mathcal{O}|+C_1(\omega)(1+\|\phi\|_{2p}^{2p})\\
&+\frac{C_2(\omega)^2}{m}\|\phi\|^2+\frac{4}{m\lambda_1}|h|^2. \qquad (3.13)
\end{aligned}
$$

Subsequently, it follows from (3.12) that

$$
\frac{d}{dt}|v_n(t)|^2+\frac{d}{dt}\|\eta_n^t(s)\|_\mu^2+m\|v_n(t)\|^2+\frac{f_0}{2D}\|v_n(t)\|_{2p}^{2p} \leq \Theta_1(\theta_t\omega).
$$

Denote $y_n(t) := (v_n(t), \eta_n^t(s))$, then $y_n(\tau) := y_0^n = (P_n v_0, Q_n\eta_0)$. Integrating the above inequality over (τ, T) for every $T > \tau$ and $\omega \in \Omega$, we have

$$
\|y_n(T)\|_{\mathcal{H}}^2+m\int_\tau^T \|v_n(t)\|^2dt+\frac{f_0}{2D}\int_\tau^T \|v_n(t)\|_{2p}^{2p}dt \leq \|y_0^n\|_{\mathcal{H}}^2+\int_\tau^T \Theta_1(\theta_t\omega)dt. \qquad (3.14)
$$

Thanks to $\phi \in V \cap H^2(\mathcal{O}) \cap L^{2p}(\mathcal{O})$, together with the fact that for every $\omega \in \Omega$, $z_*(\theta_t\omega)$ is continuous with respect to t, we infer that for every $T > \tau$, $\Theta_1(\theta.\omega) \in L^1(\tau, T)$. Hence, for each $\omega \in \Omega$, there exists a constant $C_3(\omega, T) > 0$, such that

$$
\|y_n(T)\|_{\mathcal{H}}^2+m\int_\tau^T \|v_n(t)\|^2dt+\frac{f_0}{2D}\int_\tau^T \|v_n(t)\|_{2p}^{2p}dt \leq C_3(\omega, T). \qquad (3.15)
$$

Making use of (3.8), a compactness argument and the Aubin-Lions lemma, for every $\omega \in \Omega$, there exist subsequences $\{v_n\}$ and $\{\eta_n\}$ (relabeled the same), $v \in$

$L^\infty(\tau, T; H) \cap L^2(\tau, T; V) \cap L^{2p}(\tau, T; L^{2p}(\mathcal{O}))$ and $\eta \in L^\infty(\tau, T; L^2_\mu(\mathbb{R}^+; V))$, such that

$$\begin{aligned}
v_n &\to v \ \text{ weak-}* \ \text{ in } \ L^\infty(\tau, T; H);\\
v_n &\to v \ \text{ weakly in } \ L^2(\tau, T; V);\\
v_n &\to v \ \text{ weakly in } \ L^{2p}(\tau, T; L^{2p}(\mathcal{O}));\\
\eta_n &\to \eta \ \text{ weak-}* \ \text{ in } \ L^\infty(\tau, T; L^2_\mu(\mathbb{R}^+; V));\\
\frac{dv_n}{dt} &\to \frac{dv}{dt} \ \text{ weakly in } \ L^2(\tau, T; V^*) + L^q(\tau, T; L^q(\mathcal{O}));\\
v_n &\to v \ \text{ strongly in } \ L^2(\tau, T; H);\\
v_n(t,\omega) &\to v(t,\omega) \ \text{ strongly in } \ H, \ \text{ a.e. } t \in (\tau, T];\\
v_n(x,t,\omega) &\to v(x,t,\omega) \ \text{ a.e. } (x,t) \in \mathcal{O} \times (\tau, T];\\
f(v_n + \phi z_*(\theta.\omega)) &\to f(v + \phi z_*(\theta.\omega)) \text{ weakly in } \ L^q(\tau, T; L^q(\mathcal{O})),
\end{aligned} \tag{3.16}$$

for all $T > \tau$, where $q = \frac{2p}{2p-1}$ is the conjugate number of $2p$.

Step 3. (Passage to Limit) For a fixed integer m and each $\omega \in \Omega$, choose a function

$$l = (\sigma, \pi) \in \mathcal{D}((\tau, T); V \cap L^{2p}(\mathcal{O})) \times \mathcal{D}((\tau, T); \mathcal{D}(\mathbb{R}^+; V)),$$

of the form

$$\sigma(t,\omega) = \sum_{j=1}^m \tilde{b}_j(t) w_j \qquad \text{and} \qquad \pi^t(s,\omega) = \sum_{j=1}^m \tilde{c}_j(t)\zeta_j(s),$$

where $\{\tilde{b}_j\}_{j=1}^m$ and $\{\tilde{c}_j\}_{j=1}^m$ are given functions in $\mathcal{D}((\tau, T))$.

Our main target is to prove that problem (2.6) has a solution in the weak sense, i.e., for arbitrary $l \in \mathcal{D}((\tau, T); V \cap L^{2p}(\mathcal{O})) \times \mathcal{D}((\tau, T); \mathcal{D}(\mathbb{R}^+; V))$, the following equality

$$\begin{aligned}
&\int_\tau^T (\partial_t v_n, \sigma) dt + \int_\tau^T ((\partial_t \eta_n^t, \pi))_\mu dt\\
&= -\int_\tau^T \big(a(l(v_n) + l(\phi) z_*(\theta_t\omega))((v_n, \sigma)) + a(l(v_n)\\
&\quad + l(\phi) z_*(\theta_t\omega)) z_*(\theta_t\omega)((\phi, \sigma))\big) dt - \int_\tau^T ((\eta_n^t, \sigma))_\mu dt\\
&\quad - \int_\tau^T (f(v_n + \phi z_*(\theta_t\omega)), \sigma) dt + \int_\tau^T z_*(\theta_t\omega)(\phi, \sigma) dt\\
&\quad + \int_\tau^T (z_k^\phi(\theta_t\omega), \sigma) dt + \int_\tau^T (h, \sigma) dt - \int_\tau^T \langle\!\langle (\eta_n^t)', \pi \rangle\!\rangle dt + \int_\tau^T ((v_n, \pi))_\mu dt,
\end{aligned}$$

holds in the space $\mathcal{D}'((\tau, T))$. Here, we denote by $\langle\langle \cdot, \cdot \rangle\rangle$ the duality map between $H^1_\mu(\mathbb{R}^+; V)$ and its dual space. With the help of (3.16) and the continuity property of function a, we proceed likewise as in the proof of [15, p.344] (see also [37, Step 3. Appendix A]) to finish these arguments.

Step 4. (Continuity of Solution) By means of similar arguments as in [37, Step 4. Appendix A], it is immediate to see that for every $\omega \in \Omega$, $(\frac{dv}{dt}, \frac{d\eta}{dt})$ fulfills

$$\begin{aligned} \tfrac{dv}{dt} &\in L^2(\tau, T; V^*) + L^q(\tau, T; L^q(\mathcal{O})); \\ \tfrac{d\eta}{dt} &\in L^2(\tau, T; H^{-1}_\mu(\mathbb{R}^+; V)). \end{aligned}$$

Using a slightly modified version of [26, Lemma III.1.2], together with (3.16), we deduce that $v \in C([\tau, T]; H)$. As for the second component, by applying the same arguments as for the theorem in [15, Sect. 2], we obtain that $\eta \in C([\tau, T]; L^2_\mu(\mathbb{R}^+; V))$. Thus, $(v(\tau), \eta^\tau)$ makes sense and the equality $(v(\tau), \eta^\tau) = (v_0, \eta_0)$ follows from the fact that $(P_n v_0, Q_n \eta_0)$ converges to (v_0, η_0) strongly for each $\omega \in \Omega$.

Step 5. (Continuity with Respect to Initial Value and Uniqueness) Let $y_1 = (v_1, \eta_1)$ and $y_2 = (v_2, \eta_2)$ be two solutions of (2.6) with initial data y_{10} and y_{20}, respectively. Due to the fact that $v \in C([\tau, T]; H)$, for every $\omega \in \Omega$, there exists a bounded set $S \subset H$ such that $v_i(t) \in S$ for all $t \in [\tau, T]$ and $i = 1, 2$. In addition, taking into account that $l \in \mathcal{L}(H; \mathbb{R})$, $\phi \in V \cap H^2(\mathcal{O}) \cap L^{2p}(\mathcal{O})$ and $z_*(\theta_t\omega)$ is uniformly bounded for each $\omega \in \Omega$ and all $t \in [\tau, T]$, we have $\{l(v_i(t) + l(\phi) z_*(\theta_t\omega))\}_{t \in [\tau, T]} \subset [-R, R]$ for $i = 1, 2$ and some $R > 0$. Hence, let $\bar{y} = y_1 - y_2 = (\bar{v}, \bar{\eta}) = (v_1 - v_2, \eta_1 - \eta_2)$ and $\bar{y}_0 = y_{10} - y_{20}$, we have

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}|\bar{v}(t)|^2 + \frac{1}{2}\frac{d}{dt}\|\bar{\eta}^t\|^2_\mu \le\ & |a(l(v_1) + l(\phi)z_*(\theta_t\omega)) - a(l(v_2) \\ & + l(\phi)z_*(\theta_t\omega))||z_*(\theta_t\omega)|\|\phi\|\|\bar{v}\| \\ & - (((\bar{\eta}^t)', \bar{\eta}^t))_\mu + |a(l(v_1) + l(\phi)z_*(\theta_t\omega)) - a(l(v_2) \\ & + l(\phi)z_*(\theta_t\omega))|\|v_2\|\|\bar{v}\| \\ & - a(l(v_1) + l(\phi)z_*(\theta_t\omega))\|\bar{v}\|^2 \\ & - (f(v_1 + \phi z_*(\theta_t\omega)) - f(v_2 + \phi z_*(\theta_t\omega)), \bar{v})_{L^{p,q}}, \end{aligned}$$

where $(\cdot, \cdot)_{L^{p,q}}$ is the duality between L^{2p} and L^q. Observe that it follows from integration by parts and the fact $\mu' \le 0$ that,

$$2(((\bar{\eta}^t)', \bar{\eta}^t))_\mu = -\lim_{s \to 0} \mu(s)|\nabla\bar{\eta}^t(s)|^2 - \int_0^\infty \mu'(s)|\nabla\bar{\eta}^t(s)|^2 ds \ge 0.$$

All these operations are formal but can be justified using mollifiers (see [15, Sect. 2]). Applying the Poincaré and Young inequalities, together with (1.11) and the above results, we obtain

$$\frac{d}{dt}\|\bar{y}\|^2_{\mathcal{H}} \le -2m\|\bar{v}\|^2 + 2L_a(R)|l||\bar{v}||z_*(\theta_t\omega)|\|\phi\|\|\bar{v}\| + 2L_a(R)|l||\bar{v}|\|v_2\|\|\bar{v}\|$$

$$
\begin{aligned}
&-2(f(v_1+\phi z_*(\theta_t\omega))-f(v_2+\phi z_*(\theta_t\omega)),\bar{v})_{L^{p,q}}\\
\le\ &-2m\|\bar{v}\|^2+2m\|\bar{v}\|^2+\frac{1}{m}L_a^2(R)|l|^2|\bar{v}|^2\left(\|\phi\|^2|z_*(\theta_t\omega)|^2+\|v_2\|^2\right)\\
&-2(f(v_1+\phi z_*(\theta_t\omega))-f(v_2+\phi z_*(\theta_t\omega)),\bar{v})_{L^{p,q}}. \qquad (3.17)
\end{aligned}
$$

Since f is a polynomial of odd degree with positive leading coefficient, we find that there exists a positive constant σ, such that

$$
f'(s)\ge-\frac{\sigma}{2},\qquad \forall s\in\mathbb{R}. \qquad (3.18)
$$

With help of the mean value theorem, we deduce

$$
\begin{aligned}
&-2(f(v_1+\phi z_*(\theta_t\omega))-f(v_2+\phi z_*(\theta_t\omega)),\bar{v})_{L^{p,q}}\\
&=-2\int_{\mathcal{O}}(f(v_1+\phi z_*(\theta_t\omega))-f(v_2+\phi z_*(\theta_t\omega)))\bar{v}dx\\
&=-2\int_{\mathcal{O}}f'(s_x)|\bar{v}|^2dx\le\sigma|\bar{v}|^2\le\sigma\|\bar{y}\|_{\mathcal{H}}^2, \qquad (3.19)
\end{aligned}
$$

where s_x is an intermediate point between $v_1(x)+\phi(x)z_*(\theta_t\omega)$ and $v_2(x)+\phi(x)z_*(\theta_t\omega)$.

Subsequently, (3.17)–(3.19) imply that

$$
\frac{d}{dt}\|\bar{y}\|_{\mathcal{H}}^2\le\left(\frac{1}{m}L_a^2(R)|l|^2\big(\|\phi\|^2|z_*(\theta_t\omega)|^2+\|v_2\|^2\big)+\sigma\right)\|\bar{y}\|_{\mathcal{H}}^2.
$$

The uniqueness and continuous dependence on initial data of solutions to problem (2.6) follow from the Gronwall lemma. Till now, the proof of the first assertion is finished.

(ii) **(Further regularity)** We are going to study further regularity of (v,η). To this end, for every $\omega\in\Omega$ and $\tau\in\mathbb{R}$, let us first consider the operator $\mathcal{I}_{\tau,\omega}: L^2_{V\cap H^2(\mathcal{O})}\to L^2_\mu(\mathbb{R}^+;V\cap H^2(\mathcal{O}))$ defined by

$$
(\mathcal{I}_{\tau,\omega}\varphi)(s)=\int_{-s}^0\varphi(r)dr-\int_{-s}^0 z_*(\theta_{r+\tau}\omega)\phi dr.
$$

Thus, similar to [37], we know that the operator $\mathcal{I}_{\tau,\omega}$ introduced above is a continuous mapping. Particularly, there exists a positive constant K_μ, which is the same as in Corollary 3.2, such that for any $\varphi\in L^2_{V\cap H^2(\mathcal{O})}$ and $\phi\in V\cap H^2(\mathcal{O})\cap L^{2p}(\mathcal{O})$, it holds

$$
\|\mathcal{I}_{\tau,\omega}\varphi\|^2_{L^2_\mu(\mathbb{R}^+;V\cap H^2(\mathcal{O}))}\le 2K_\mu\left(\|\varphi\|^2_{L^2_{V\cap H^2(\mathcal{O})}}+\|z_*(\theta_{\cdot+\tau}\omega)\phi\|^2_{L^2_{V\cap H^2(\mathcal{O})}}\right).
$$

Next, multiplying (2.6_1) by $-\Delta v$ with respect to the inner product of H, the Laplacian of (2.6_2) by $\Delta\eta^t$ with respect to the inner product of $L^2_\mu(\mathbb{R}^+;H)$ and adding the two terms, for every $\omega \in \Omega$, it follows from (1.11) that,

$$\frac{d}{dt}\|v\|^2 + \frac{d}{dt}\|\Delta\eta^t\|_\mu^2 + 2m|\Delta v|^2 + 2((((\eta^t)', \eta^t)))_\mu \le 2M|z_*(\theta_t\omega)||\Delta\phi||\Delta v| \\ +2(f(v+\phi z_*(\theta_t\omega)), \Delta v) + 2|z_*(\theta_t\omega)|\|\phi\|\|v\| + 2(z_k^\phi(t,\omega), -\Delta v) + 2(h, -\Delta v). \tag{3.20}$$

Applying the same arguments as in [37, Appendix A], we know that

$$2((((\eta^t)', \eta^t)))_\mu = -2\int_0^\infty \mu'(s)|\Delta\eta^t(s)|^2 ds > 0. \tag{3.21}$$

By means of the Young inequality, we infer that

$$2M|z_*(\theta_t\omega)||\Delta\phi||\Delta v| \le \frac{2M^2}{m}|z_*(\theta_t\omega)|^2|\Delta\phi|^2 + \frac{m}{2}|\Delta v|^2, \tag{3.22}$$

and

$$2|z_*(\theta_t\omega)|\|\phi\|\|v\| \le |z_*(\theta_t\omega)|^2\|\phi\|^2 + \|v\|^2. \tag{3.23}$$

Next, taking into account of (3.18) and assumption $\Delta\phi \in L^{2p}(\mathcal{O})$, together with the Young inequality and the Green formula, it yields

$$\begin{aligned} &2(f(v+\phi z_*(\theta_t\omega)), \Delta v) \\ &= 2(f(v+\phi z_*(\theta_t\omega)), \Delta(v+\phi z_*(\theta_t\omega))) - 2(f(v+\phi z_*(\theta_t\omega)), z_*(\theta_t\omega)\Delta\phi) \\ &\le 2\int_{\mathcal{O}} f_{2p-1}(\Delta v + z_*(\theta_t\omega)\Delta\phi)dx + 2|z_*(\theta_t\omega)|\int_{\mathcal{O}} |f(v+\phi z_*(\theta_t\omega))||\Delta\phi|dx \\ &\quad - 2\int_{\mathcal{O}} \nabla(v+\phi z_*(\theta_t\omega)) \cdot f'(v+z_*(\theta_t\omega)\phi)\nabla(v+\phi z_*(\theta_t\omega))dx \\ &\le \frac{4}{m}f_{2p-1}^2|\mathcal{O}| + \frac{m}{2}|\Delta v|^2 + \frac{m}{2}|z_*(\theta_t\omega)|^2|\Delta\phi|^2 + 2\sigma\|v\|^2 + 2\sigma|z_*(\theta_t\omega)|^2\|\phi\|^2 \\ &\quad + \frac{2|z_*(\theta_t\omega)|}{q}\|f(v+\phi z_*(\theta_t\omega))\|_q^q + \frac{|z_*(\theta_t\omega)|}{p}\|\Delta\phi\|_{2p}^{2p}, \end{aligned} \tag{3.24}$$

where $f(v+\phi z_*(\theta_t\omega)) \in L^q(\mathcal{O})$ since $v \in L^{2p}(\mathcal{O})$ and $\phi \in L^{2p}(\mathcal{O})$. In the end, it follows from the Young inequality that,

$$2(z_k^\phi(\theta_t\omega), -\Delta v) \le \frac{C_2(\theta_t\omega)^2}{m}|\Delta\phi|^2 + \frac{m}{4}|\Delta v|^2, \tag{3.25}$$

$$2(h, -\Delta v) \le \frac{4}{m}|h|^2 + \frac{m}{4}|\Delta v|^2,$$

where C_2 is the same as in (3.10).

Substituting (3.21)–(3.25) into (3.20), we have

$$\begin{aligned}\frac{d}{dt}\|v\|^2+\frac{d}{dt}\|\Delta\eta^t\|_\mu^2+\frac{m}{2}|\Delta v|^2 &\le \left(\frac{2M^2}{m}+\frac{m}{2}\right)|z_*(\theta_t\omega)|^2|\Delta\phi|^2\\ &+(2\sigma+1)|z_*(\theta_t\omega)|^2\|\phi\|^2\\ &+\frac{4}{m}f_{2p-1}^2|\mathcal{O}|+(2\sigma+1)\;\|v\|^2+\frac{C_2(\theta_t\omega)^2}{m}|\Delta\phi|^2\\ &+\frac{2|z_*(\theta_t\omega)|}{q}\|f(v+\phi z_*(\theta_t\omega))\|_q^q+\frac{|z_*(\theta_t\omega)|}{p}\|\Delta\phi\|_{2p}^{2p}+\frac{4}{m}\,|h|^2\,.\end{aligned}$$

Denote

$$\begin{aligned}\Theta_2(t,\omega):=&\left(\frac{2M^2}{m}+\frac{m}{2}\right)|z_*(\omega)|^2|\Delta\phi|^2+(2\sigma+1)|z_*(\omega)|^2\|\phi\|^2+\frac{4}{m}f_{2p-1}^2|\mathcal{O}|\\ &+\frac{C_2(\omega)^2}{m}|\Delta\phi|^2+\frac{2|z_*(\omega)|}{q}\|f(v(t)+\phi z_*(\omega))\|_q^q\\ &+\frac{|z_*(\omega)|}{p}\|\Delta\phi\|_{2p}^{2p}+\frac{4}{m}|h|^2\in L^1(\tau,T).\end{aligned}$$

Then, for every $\omega\in\Omega$ and $t\in(\tau,T]$, we obtain

$$\frac{d}{dt}\|y\|_{\mathcal{V}}^2+\frac{m}{2}|\Delta v|^2\le\Theta_2(t,\theta_t\omega)+(2\sigma+1)\;\|v\|^2. \tag{3.26}$$

By the continuity of $z_*(\theta_t\omega)$ on $(\tau,T]$ and integrating the above inequality between τ and t with $\tau\le t\le T$, we have

$$\|y(t)\|_{\mathcal{V}}^2+\frac{m}{2}\int_\tau^t|\Delta v(s)|^2ds\le\|y_0\|_{\mathcal{V}}^2+\int_\tau^t\Theta_2(s,\theta_s\omega)ds+(2\sigma+1)\int_\tau^t\|v(s)\|^2ds.$$

Thus, we conclude that

$$\begin{aligned}&v\in L^\infty(\tau,T;V)\cap L^2(\tau,T;V\cap H^2(\mathcal{O}));\\ &\eta\in L^\infty(\tau,T;L_\mu^2(\mathbb{R}^+;V\cap H^2(\mathcal{O}))).\end{aligned}$$

Furthermore, the continuity of v follows again using a slightly modified version of [26, Lemma III.1.2.] and the continuity of η can be proved mimicking the idea of the proof of Step 4 of (i), with $V\cap H^2(\mathcal{O})$ in place of V. The proof of this theorem is complete. □

Lemma 3.6 *Let conditions (h1)–(h2) hold. If* $u\in L_V^2$*, then* $\eta(s)=\int_{-s}^0u(r)\,dr$ *belongs to* $L_\mu^2(\mathbb{R}^+;V)$ *and*

$$\int_0^\infty\mu(s)\Delta\eta(s)ds=\int_{-\infty}^0k(-s)\Delta u(s)ds. \tag{3.27}$$

Proof The fact that $\eta \in L^2_\mu(\mathbb{R}^+; V)$ is given by Lemma 3.1. From the arguments in [13, pp-174-175], it follows the existence of a sequence of functions $u_n(\cdot) \in C^1((-\infty, 0], V) \cap L^2_V$ such that

$$u_n \to u \text{ in } L^2_V.$$

First, we will show that u_n, η_n, where $\eta_n(s) = \int_{-s}^0 u_n(r)\,dr$, satisfy (3.27). For any $w \in V$, we have

$$\begin{aligned}\left\langle \int_0^\infty \mu(s)\Delta\eta_n(s)ds, w\right\rangle &= \int_0^\infty \mu(s)\left\langle \Delta\eta_n(s), w\right\rangle ds = \int_0^\infty k'(s)\left(\nabla\eta_n(s), \nabla w\right) ds\\ &= \int_0^\infty k'(s)\left(\nabla\int_{-s}^0 u_n(r)dr, \nabla w\right) ds\\ &= \int_0^\infty k'(s)\int_{-s}^0 \left(\nabla u_n(r), \nabla w\right) drds\\ &= -\int_0^\infty k(s)\left(\nabla u_n(-s), \nabla w\right) ds\\ &\quad + \lim_{s\to\infty} k(s)\int_{-s}^0 \left(\nabla u_n(r), \nabla w\right) dr\\ &\quad - \lim_{s\to 0} k(s)\int_{-s}^0 \left(\nabla u_n(r), \nabla w\right) dr.\end{aligned}$$

Let us check that the last two limits of the above equality are equal to 0. By Remark 2.1, we derive

$$k(s)e^{\gamma s} \le \frac{\mu(1)}{\varpi}e^{\varpi}e^{(\gamma-\varpi)s}, \quad \text{for any } s \ge 1.$$

Hence, $\gamma < \varpi$ implies

$$\begin{aligned}\left|k(s)\int_{-s}^0 \left(\nabla u_n(r), \nabla w\right) dr\right| &\le k(s)e^{\gamma s}\left\|w\right\|\int_{-s}^0 e^{\gamma r}\left\|u_n(r)\right\| dr\\ &\le \frac{k(s)e^{\gamma s}\left\|w\right\|}{2}\left(\int_{-\infty}^0 e^{\gamma r}\left\|u_n(r)\right\|^2 dr + \frac{1}{\gamma}\right)\\ &\le C_1 e^{(\gamma-\varpi)s} \underset{s\to\infty}{\to} 0.\end{aligned}$$

Also, from $k(s) \underset{s\to 0}{\to} \int_0^\infty \mu(r)\,dr$ and $u_n \in L^2_V$, it follows that the second limit is 0 as well. Hence,

$$\begin{aligned}\left\langle \int_0^\infty \mu(s)\Delta\eta_n(s)ds, w\right\rangle &= -\int_0^\infty k(s)\left(\nabla u_n(-s), \nabla w\right) ds\\ &= \left\langle \int_{-\infty}^0 k(-s)\Delta u_n(s)ds, w\right\rangle,\end{aligned}$$

this proves (3.27) for u_n.

Furthermore, for any $w \in V$, we infer

$$\begin{aligned}
&\left|\left\langle \int_{-\infty}^{0} k(-s)(\Delta u_n(s) - \Delta u(s))ds,\, w\right\rangle\right| \\
&\quad = \left|\int_{-\infty}^{0} k(-s)\, \langle \Delta u_n(s) - \Delta u(s), w\rangle\, ds\right| \\
&\quad \leq \|w\| \left(C_2 \int_{-1}^{0} \|u_n(s) - u(s)\|\, ds + \frac{\mu(1)}{\varpi} e^{\varpi} \int_{-\infty}^{-1} e^{\varpi s} \|u_n(s) - u(s)\|\, ds \right) \\
&\quad \leq C_3 \left(\left(\int_{-1}^{0} \|u_n(s) - u(s)\|^2\, ds \right)^{\frac{1}{2}} + \left(\int_{-\infty}^{-1} e^{\gamma s} \|u_n(s) - u(s)\|^2\, ds \right)^{\frac{1}{2}} \right) \underset{n\to\infty}{\to} 0,
\end{aligned}$$

and Lemma 3.1 implies

$$\begin{aligned}
&\left|\left\langle \int_{0}^{\infty} \mu(s)(\Delta \eta_n(s) - \Delta \eta(s))ds,\, w\right\rangle\right| \\
&= \left|\int_{0}^{\infty} \mu(s)\, \langle \Delta \eta_n(s) - \Delta \eta(s), w\rangle\, ds\right| \leq \|w\| \int_{0}^{\infty} \mu(s)\, \|\eta_n(s) - \eta(s)\|\, ds \\
&\leq \|w\| \left(\int_{0}^{\infty} \mu(s) ds \right)^{\frac{1}{2}} \left(\int_{0}^{\infty} \mu(s)\, \|\eta_n(s) - \eta(s)\|^2\, ds \right)^{\frac{1}{2}} \leq C_4 \|u_n - u\|_{L_V^2} \underset{n\to\infty}{\to} 0.
\end{aligned}$$

By these convergences we deduce (3.27). The proof of this lemma is complete. □

Lemma 3.6 implies that the solution given in Theorem 3.4 is in fact the unique weak solution to problem (2.4).

Corollary 3.7 *Assume that (1.11), (2.8) and* (h_1)-(h_2) *hold, and that* $\phi \in V \cap H^2(\mathcal{O}) \cap L^{2p}(\mathcal{O})$ *is such that* $\Delta\phi \in L^{2p}(\mathcal{O})$. *Let* $h \in H$ *and a be a locally Lipschitz function. If for fixed* $\tau \in \mathbb{R}$ *and* $\omega \in \Omega$, *the function* (v, η) *is the unique weak solution to problem (2.6) corresponding to the initial values* $v_0 \in H$ *and* $\varphi \in L_V^2$, *then v is the unique weak solution to problem (2.4).*

Now by the transform (2.3), we derive the well-posedness of problem (1.10).

Theorem 3.8 *Assume that (1.11), (2.8) and* (h_1) -(h_2) *hold, and that* $\phi \in V \cap H^2(\mathcal{O}) \cap L^{2p}(\mathcal{O})$ *is such that* $\Delta\phi \in L^{2p}(\mathcal{O})$. *Let* $h \in H$ *and a be a locally Lipschitz function. Then, for every* $\tau \in \mathbb{R}$ *and* $\omega \in \Omega$, *it holds:*

(i) For any initial vale $u_0 \in H$ *and initial function* $\varphi \in L_V^2$, *there exists a unique solution u to problem (1.10) in the weak sense, fulfilling*

$$u \in L^\infty(\tau, T; H) \cap L^2(\tau, T; V) \cap L^{2p}(\tau, T; L^{2p}(\mathcal{O})), \qquad \forall T > \tau.$$

Furthermore, the solution u of (1.10) is continuous with respect to the initial values (u_0, φ) *for all* $t \in [\tau, T]$ *in H;*

(ii) For any initial value $u_0 \in V$ *and initial function* $\varphi \in L^2_{V\cap H^2(\mathcal{O})}$*, the unique solution* u *to problem (1.10) satisfies,*

$$u \in L^\infty(\tau, T; V) \cap L^2(\tau, T; V \cap H^2(\mathcal{O})), \qquad \forall T > \tau.$$

In addition, the solution u *of (1.10) is continuous with respect to the initial values* (u_0, φ) *for all* $t \in [\tau, T]$ *in* V*.*

Remark 3.9 The proof of Theorem 3.4 is correct for a general function $f \in C^1(\mathbb{R})$ satisfying (3.8) and (3.18). The same applies to the results in Sects. 4–5.

4 Existence of Random Attractors

This section is devoted to studying the long time behavior of (1.10) in the natural phase space,

$$X = H \times L^2_V,$$

endowed with the norm

$$\|(w_1, w_2)\|^2_X = |w_1|^2 + \|w_2\|^2_{L^2_V}.$$

It is worth emphasizing that we will take $\tau = 0$ in this section since problem (1.10) is autonomous. Taking into account the results in the previous section, problem (1.10) generates a random dynamical system in X. Let us denote by $u(\cdot; 0, \omega, (u_0, \varphi))$ the unique solution to (1.10). Then, the random dynamical system (RDS) generated by (1.10), denoted by $\Xi : \mathbb{R}^+ \times \Omega \times X \to X$, is defined, for every $t \in \mathbb{R}^+$, $\omega \in \Omega$ and $(u_0, \varphi) \in X$, as

$$\Xi(t, \omega, (u_0, \varphi)) = (u(t; 0, \omega, (u_0, \varphi)), u_t(\cdot; 0, \omega, (u_0, \varphi))).$$

Moreover, problem (2.6) also generates a random dynamical system Φ on the phase space $H \times L^2_\mu(\mathbb{R}^+; V)$, which is defined, for every $t \in \mathbb{R}^+$, $\omega \in \Omega$ and $(v_0, \eta_0) \in H \times L^2_\mu(\mathbb{R}^+; V)$, by

$$\Phi(t, \omega, (v_0, \eta_0)) = (v(\cdot; 0, \omega, (v_0, \eta_0)), \eta^{\cdot}(\cdot; 0, \omega, (v_0, \eta_0))),$$

where the right-hand side of the above equality denotes the solution to (2.6) for $\tau = 0$, the initial values $(v_0, \eta_0) \in H \times L^2_\mu(\mathbb{R}^+; V)$ and η_0 is given in (2.7). Thanks to Dafermos' transformation, we can obtain a random dynamical system $\Psi : \mathbb{R}^+ \times \Omega \times X \to X$ generated by (2.4) which is given, for every $t \in \mathbb{R}^+$, $\omega \in \Omega$ and $(v_0, \psi) \in X$, by

$$\Psi(t, \omega, (v_0, \psi)) = (v(t; 0, \omega, (v_0, (\mathcal{J}\psi))), v_t(\cdot; 0, \omega, (v_0, (\mathcal{J}\psi)))).$$

Then, on account of the random transformation (2.3), for $(u_0, \varphi) \in X$, we deduce

$$\begin{aligned}
&\Xi(t, \omega, (u_0, \varphi)) \\
&\quad = (u(t; 0, \omega, (u_0, \varphi)), u_t(\cdot; 0, \omega, (u_0, \varphi))) \\
&\quad = (v(t; 0, \omega, (u_0 - \phi z_*(\omega), (\mathcal{J}_{\omega,0}\varphi))) \\
&\qquad + \phi z_*(\theta_t\omega), v_t(\cdot; 0, \omega, (u_0 - \phi z_*(\omega), (\mathcal{J}_{\omega,0}\varphi))) + \phi z_*(\theta_{t+\cdot}\omega)) \\
&\quad = \Psi(t, \omega, (u_0 - \phi z_*(\omega), \varphi_v)) + (\phi z_*(\theta_t\omega), \phi z_*(\theta_{t+\cdot}\omega)).
\end{aligned} \tag{4.1}$$

It is straightforward to check that the cocycles Ξ and Ψ are conjugated. Indeed, consider the mapping $T : \Omega \times X \to X$ defined by,

$$T(\omega, (u_0, \varphi)) = (u_0 - \phi z_*(\omega), \varphi - \phi z_*(\theta_\cdot\omega)).$$

Then, it holds that

$$T^{-1}(\omega, (u_0, \varphi)) = (u_0 + \phi z_*(\omega), \varphi + \phi z_*(\theta_\cdot\omega)).$$

In addition, by (4.1), it is clear that

$$\begin{aligned}
\Xi(t, \omega, (u_0, \varphi)) &= \Psi(t, \omega, (u_0 - \phi z_*(\omega), \varphi - \phi z_*(\theta_\cdot\omega))) + (\phi z_*(\theta_t\omega), \phi z_*(\theta_{t+\cdot}\omega)) \\
&= T^{-1}(\theta_t\omega, \Psi(t, \omega, T(\omega, (u_0, \varphi)))).
\end{aligned} \tag{4.2}$$

Let $D = \{D(\omega) : \omega \in \Omega\}$ be a family of bounded nonempty subsets of X. Such a family D is called tempered if for every $c > 0$ and $\omega \in \Omega$,

$$\lim_{t\to\infty} e^{-ct}\|D(\theta_{-t}\omega)\| = 0,$$

where the norm $\|D\|$ of a set D in X is defined by $\|D\| = \sup_{u\in D} \|u\|_X$. From now on, we will use $\mathcal{D}$ to denote the collection of all tempered families of bounded nonempty subsets of X:

$$\mathcal{D} = \{D = \{D(\omega) : \omega \in \Omega\} : D \text{ is tempered in } X\}.$$

This family will be adopted to prove the existence of random pullback attractors for the RDS Ξ. Notice that, for $D \in \mathcal{D}$, the set $\widetilde{D}$ whose fibers are given by,

$$\widetilde{D}(\omega) = \{(u_0 - \phi z_*(\omega), \varphi - \phi z_*(\theta_\cdot\omega)) : (u_0, \varphi) \in D(\omega)\},$$

also belongs to $\mathcal{D}$ thanks to the arguments in the proof of Corollary 3.2 and the properties of the random variable $z_*(\omega)$ (cf. (2.2)).

Lemma 4.1 *Under assumptions of Theorem 3.8, there exists $B \in \mathcal{D}$ which is $\mathcal{D}$-pullback absorbing for the RDS Ξ. In other words, for any given $\omega \in \Omega$ and $D \in \mathcal{D}$, there exists $t_0 := t_0(\omega, D) \geq 0$, such that*

$$\Xi(t, \theta_{-t}\omega, D(\theta_{-t}\omega)) \subset B(\omega), \quad \textit{for all } t \geq t_0(\omega, D),$$

where $B(\omega)$ is the ball in X centered at 0 with radius $\rho(\omega)$ and

$$\rho^2(\omega) = 1 + 2K_2 \int_{-\infty}^{0} e^{\gamma s}\Theta_1(\theta_s\omega)ds + 2|\phi|^2|z_*(\omega)|^2 + 2\|\phi z_*(\theta.\omega)\|^2_{L^2_V},$$

where $\Theta_1(\omega)$ is given in (3.13) and $K_2 > 0$ is a constant.

Proof Let us first pick $(u_0, \varphi) \in D$. Thanks to (4.1), we have

$$\begin{aligned}
&\Xi(t, \omega, (u_0, \varphi)) \\
&\quad = (v(t; 0, \omega, (u_0 - \phi z_*(\omega), (\mathcal{J}_{\omega,0}\varphi))), v_t(\cdot; 0, \omega, (u_0 - \phi z_*(\omega), (\mathcal{J}_{\omega,0}\varphi))) \\
&\qquad + (\phi z_*(\theta_t\omega), \phi z_*(\theta_{t+\cdot}\omega)) \\
&\quad = \Psi(t, \omega, (u_0 - \phi z_*(\omega), \varphi - \phi z_*(\theta.\omega))) + (\phi z_*(\theta_t\omega), \phi z_*(\theta_{t+\cdot}\omega)).
\end{aligned}$$

For the sake of simplicity, denote by $y(t,\omega) := (v(t,\omega), \eta^t(s,\omega))$ the solution to (2.6) with initial value $(v_0, \eta_0) = (u_0 - \phi z_*(\omega), \mathcal{J}_{\omega,0}\varphi)$. Now, for every $\omega \in \Omega$, we multiply the first equation of (2.6) by $v(t)$ in H and the second equation of (2.6) by η^t in $L^2_\mu(\mathbb{R}^+; V)$, respectively. Then, by means of the same estimates as in the proof of Theorem 3.4 (cf. (3.12)) and the Poincaré inequality, we obtain

$$\begin{aligned}
&\frac{d}{dt}\|y(t)\|^2_{\mathcal{H}} + \frac{m\lambda_1}{2}|v(t)|^2 + \frac{m}{2}\|v(t)\|^2 + 2((\eta^t(s), (\eta^t(s))'))_\mu \\
&\quad + \frac{f_0}{2D}\|v(t)\|^{2p}_{2p} \le \Theta_1(\theta_t\omega),
\end{aligned} \tag{4.3}$$

(see (3.13) for the expression of $\Theta_1(\cdot)$). With the help of condition (h_2), we infer

$$\begin{aligned}
2(((\eta^t)', \eta^t))_\mu &= -\int_0^\infty \mu'(s)|\nabla\eta^t(s)|^2 ds \\
&\ge \varpi \int_0^\infty \mu(s)|\nabla\eta^t(s)|^2 ds := \varpi\|\eta^t\|^2_\mu.
\end{aligned} \tag{4.4}$$

Recall that $0 < \gamma < \min\{\frac{m\lambda_1}{2}, \varpi\}$, which, together with (4.3) and (4.4), implies that

$$\frac{d}{dt}\|y(t)\|^2_{\mathcal{H}} + \gamma\|y(t)\|^2_{\mathcal{H}} + \frac{m}{2}\|v(t)\|^2 + \frac{f_0}{2D}\|v(t)\|^{2p}_{2p} \le \Theta_1(\theta_t\omega). \tag{4.5}$$

Next, multiplying the above inequality by $e^{\gamma t}$ and integrating over $(0, t)$, neglecting the last term on the left hand side of (4.5), we find

$$\|y(t)\|^2_{\mathcal{H}} + \frac{m}{2}\int_0^t e^{-\gamma(t-s)}\|v(s)\|^2 ds \le \|y_0\|^2_{\mathcal{H}} e^{-\gamma t} + \int_0^t e^{-\gamma(t-s)}\Theta_1(\theta_s\omega)ds. \tag{4.6}$$

Then,

$$\begin{aligned}\frac{m}{2}\|v_t\|^2_{L^2_V} &= \frac{m}{2}\int_{-\infty}^{0} e^{-\gamma(t-s)}\|\varphi(s)-\phi z_*(\theta_s\omega)\|^2 ds + \frac{m}{2}\int_0^t e^{-\gamma(t-s)}\|v(s)\|^2 ds\\ &\leq \frac{m}{2}e^{-\gamma t}\left(\|\varphi-\phi z_*(\theta.\omega)\|^2_{L^2_V}\right)+\|y_0\|^2_{\mathcal{H}}e^{-\gamma t}+\int_0^t e^{-\gamma(t-s)}\Theta_1(\theta_s\omega)ds.\end{aligned} \tag{4.7}$$

On account of Corollary 3.2, we infer

$$\|y_0\|^2_{\mathcal{H}} = |v_0|^2+\|\mathcal{J}_{\omega,0}\varphi\|^2_{L^2_\mu(\mathbb{R}^+;V)} \leq |u_0-\phi z_*(\omega)|^2+2K_\mu\left(\|\varphi-\phi z_*(\theta.\omega)\|^2_{L^2_V}\right). \tag{4.8}$$

Hence, collecting (4.6)–(4.8), we arrive at

$$\begin{aligned}&\|\Psi(t,\omega,(u_0-\phi z_*(\omega),\varphi_v))\|^2_X = |v(t)|^2+\|v_t\|^2_{L^2_V} \leq \|y(t)\|^2_{\mathcal{H}}+\|v_t\|^2_{L^2_V}\\ &\leq K_1 e^{-\gamma t}\left(|u_0-\phi z_*(\omega)|^2+\|\varphi-\phi z_*(\theta.\omega)\|^2_{L^2_V}\right)+K_2\int_0^t e^{-\gamma(t-s)}\Theta_1(\theta_s\omega)ds,\end{aligned} \tag{4.9}$$

where $K_1, K_2 > 0$ are constants which neither depend on ω nor on the initial functions. Now, replacing ω by $\theta_{-t}\omega$ in (4.9), we obtain

$$\begin{aligned}&\|\Psi(t,\theta_{-t}\omega,(u_0-\phi z_*(\theta_{-t}\omega),\varphi-\phi z_*(\theta_{-t+\cdot}\omega)))\|^2_X\\ &\quad\leq K_1 e^{-\gamma t}\left(|u_0-\phi z_*(\theta_{-t}\omega)|^2+\|\varphi-\phi z_*(\theta_{-t+\cdot}\omega)\|^2_{L^2_V}\right)\\ &\qquad+K_2\int_0^t e^{-\gamma(t-s)}\Theta_1(\theta_{-t+s}\omega)ds\\ &\quad\leq K_1 e^{-\gamma t}\left(|u_0-\phi z_*(\theta_{-t}\omega)|^2+\|\varphi-\phi z_*(\theta_{-t+\cdot}\omega)\|^2_{L^2_V}\right)\\ &\qquad+K_2\int_{-\infty}^0 e^{\gamma s}\Theta_1(\theta_s\omega)ds.\end{aligned} \tag{4.10}$$

Therefore, for any $(u_0,\varphi)\in D(\theta_{-t}\omega)$, we have

$$\begin{aligned}&\|\Xi(t,\theta_{-t}\omega,(u_0,\varphi))\|^2_X\\ &\quad= \|\Psi(t,\theta_{-t}\omega,(u_0-\phi z_*(\theta_{-t}\omega),\varphi-\phi z_*(\theta_{-t+\cdot}\omega)))\\ &\qquad+(\phi z_*(\omega),\phi z_*(\theta.\omega))\|^2_X\\ &\quad\leq 2K_1 e^{-\gamma t}\left(|u_0-\phi z_*(\theta_{-t}\omega)|^2+\|\varphi-\phi z_*(\theta_{-t+\cdot}\omega)\|^2_{L^2_V}\right)\\ &\qquad+2K_2\int_{-\infty}^0 e^{\gamma s}\Theta_1(\omega)ds+2|\phi|^2|z_*(\omega)|^2+2\|\phi z_*(\theta.\omega)\|^2_{L^2_V}\\ &\quad\leq 2K_1 e^{-\gamma t}\|\widetilde{D}(\theta_{-t}\omega)\|^2+2K_2\int_{-\infty}^0 e^{\gamma s}\Theta_1(\theta_s\omega)ds\end{aligned}$$

$$+2|\phi|^2|z_*(\omega)|^2+2\|\phi z_*(\theta.\omega)\|^2_{L^2_V}. \tag{4.11}$$

Consequently, let

$$\rho^2(\omega)=1+2K_2\int_{-\infty}^0 e^{\gamma s}\Theta_1(\theta_s\omega)ds+2|\phi|^2|z_*(\omega)|^2+2\|\phi z_*(\theta.\omega)\|^2_{L^2_V}.$$

Taking into account the temperedness of $\widetilde{D}$ and the property of Ornstein-Uhlenbeck process, it is straightforward to check that the set,

$$B_\rho(\omega)=\{(u_0,\varphi)\in X:\|(u_0,\varphi)\|_X\le\rho(\omega)\},$$

is tempered, i.e., $B_\rho(\omega)$ belongs to $\mathcal{D}$ and is pullback absorbing for the universe $\mathcal{D}$. The proof of this lemma is finished. □

Remark 4.2 It follows from the proof of Lemma 4.1 that, under the assumptions of Theorem 3.4, there exists $\tilde{B}\in\mathcal{D}$ which is $\mathcal{D}$-pullback absorbing for the RDS Ψ. In other words, for any given $\omega\in\Omega$ and $D\in\mathcal{D}$, there exists $\tilde{t}_0:=\tilde{t}_0(\omega,D)\ge 0$, such that

$$\Psi(t,\theta_{-t}\omega,D(\theta_{-t}\omega))\subset\tilde{B}(\omega),\quad \text{for all } t\ge\tilde{t}_0(\omega,D).$$

Next, we will prove the asymptotic compactness of the cocycle Ξ. Namely, we will show that for any $\omega\in\Omega$, $D\in\mathcal{D}$ and for any sequence $t_n\to+\infty$, $(u_0^n,\varphi^n)\in D(\theta_{-t_n}\omega)$, the sequence $\{\Xi(t_n,\theta_{-t_n}\omega,(u_0^n,\varphi^n))\}$ possesses a convergent subsequence in X. To this end, let us first prove an auxiliary result.

Lemma 4.3 *Assume the hypotheses in Theorem 3.4 hold. Let $\{v_0^n,\varphi_v^n\}$ be a sequence such that $(v_0^n,\varphi_v^n)\to(v_0,\varphi_v)$ weakly in X as $n\to\infty$. Then, for every $\omega\in\Omega$, $\Psi(t,\omega,(v_0^n,\varphi_v^n))=(v^n(t),v_t^n(\cdot))$ fulfills:*

$$v^n\to v\ \ \text{in}\ \ C([r,T];H)\ \text{for all}\ 0<r<T; \tag{4.12}$$

$$v^n\to v\ \ \text{weakly in}\ \ L^2(0,T;V)\ \text{for all}\ T>0; \tag{4.13}$$

$$v^n\to v\ \ \text{in}\ \ L^2(0,T;H)\ \text{for all}\ T>0; \tag{4.14}$$

$$\limsup_{n\to\infty}\|v_t^n-v_t\|^2_{L^2_V}\le Ke^{-\gamma t}\limsup_{n\to\infty}\left(|v_0^n-v_0|^2+\|\varphi_v^n-\varphi_v\|^2_{L^2_V}\right)\ \text{for all}\ t\ge 0, \tag{4.15}$$

where $K=(1+\frac{2K_\mu}{m}+\frac{1}{m})$ and $(v(t),v_t(\cdot))=\Psi(t,\omega,(v_0,\varphi_v))$. Moreover, if $(v_0^n,\varphi_v^n)\to(v_0,\varphi_v)$ strongly in X as $n\to\infty$, then

$$v^n\to v\ \ \text{in}\ \ L^2(0,T;V)\ \text{for all}\ T>0; \tag{4.16}$$

$$v_t^n\to v_t\ \ \text{in}\ \ L^2_V\ \text{for all}\ t\ge 0. \tag{4.17}$$

Proof Let $T > 0$ be arbitrary. Integrating in (4.5), we deduce that v^n is bounded in $L^\infty(0, T; H)$, $L^2(0, T; V)$ and $L^{2p}(0, T; L^{2p}(\mathcal{O}))$, η_n is bounded in $L^\infty(0, T; L^2_\mu(\mathbb{R}^+; V))$. Hence, passing to a subsequence, for every $\omega \in \Omega$, we have

$$\begin{cases} v^n \to v \;\text{ weak-}* \;\text{ in }\; L^\infty(0, T; H); \\ v^n \to v \;\text{ weakly in }\; L^2(0, T; V); \\ v^n \to v \;\text{ weakly in }\; L^{2p}(0, T; L^{2p}(\mathcal{O})); \\ \eta_n \to \eta \;\text{ weak-star in }\; L^\infty(0, T; L^2_\mu(\mathbb{R}^+; V)), \end{cases}$$

thus (4.13) holds. By the same arguments in the proof of Theorem 3.4, we deduce

$$\begin{cases} \dfrac{dv^n}{dt} \to \dfrac{dv}{dt} \text{ weakly in }\; L^2(0, T; V^*) + L^q(0, T; L^q(\mathcal{O})); \\ f(v^n + \phi z_*(\theta.\omega)) \to \chi \;\text{ weakly in }\; L^q(0, T; L^q(\mathcal{O})). \end{cases} \tag{4.18}$$

In view of (4.13) and the above results, making use of the Compactness Theorem [25], we infer that (4.14) is true. Thus, $v^n(t, x, \omega) \to v(t, x, \omega)$, $f(v^n(t, x, \omega) + \phi z_*(\theta_t\omega)) \to f(v(t, x, \omega) + \phi z_*(\theta_t\omega))$ for a.a. $(t, x) \in (0, T) \times \mathcal{O}$. Also it follows from [21, Lemma 1.3] that $\chi = f(v + \phi z_*(\theta.\omega))$.

By proceeding as in the proof of Theorem 3.4, we obtain that $y = (v, \eta)$ is a solution to problem (2.6) with initial value $y(0) = (v_0, \eta_0) := (v_0, \mathcal{J}_{\omega,0}\varphi)$. Thanks to the uniqueness of solution, a standard argument implies that the above convergences are true for the whole sequence. Furthermore, we know that $v^n \in C([0, T]; H)$ and $v \in C([0, T]; H)$ for each $\omega \in \Omega$.

Since $\{(v^n)'\}$ is bounded in $L^q(0, T; V^* + L^q(\mathcal{O}))$, we have that $\{v^n\}$ is equicontinuous in $V^* + L^q(\mathcal{O})$ on $[0, T]$. Indeed,

$$\begin{aligned} \|v^n(s_2) - v^n(s_1)\|_{V^*+L^q} &\le \int_{s_1}^{s_2} \left\|(v^n)'\right\|_{V^*+L^q} ds \\ &\le |s_2 - s_1|^{\frac{1}{2p}} \left\|(v^n)'\right\|_{L^q(0,T;V^*+L^q(\mathcal{O}))}. \end{aligned} \tag{4.19}$$

In addition, as $\{v^n\}$ is bounded in $C([0, T]; H)$ and the embedding $H \subset V^*$ is compact, by the Arzelà-Ascoli theorem, we obtain (relabeled the same) that

$$v^n \to v \;\text{ strongly in }\; C([0, T]; V^* + L^q(\mathcal{O})). \tag{4.20}$$

Now, consider a sequence $\{s_n\} \in [0, T]$ which converges to $s_* \in (0, T]$. Since $\{v^n\}$ is bounded in $C([0, T]; H)$, there exist a subsequence of $\{v^n(s_n)\}$ and $\hat{u} \in H$ such that

$$v^n(s_n) \rightharpoonup \hat{u} \;\text{ weakly in }\; H.$$

Using (4.20) we deduce that $\hat{u} = u(s_*)$ and that the whole sequence converges. Therefore,

$$|v(s_*)| \le \liminf_{n\to\infty} |v^n(s_n)|. \tag{4.21}$$

We will prove that $v^n(s_n) \to v(s_*)$ strongly in H, which implies (4.12). By Corollary 3.7, v^n and v are weak solutions to problem (2.4), so multiplying the equation by v^n, we obtain that

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|v^n(t)|^2 + m\|v^n\|^2 + (f(v^n + \phi z_*(\theta_t\omega)), v^n) \\
&\quad \le M|z_*(\theta_t\omega)|\|\phi\|\|v^n\| + \left(\int_{-\infty}^{t} k(t-s)\Delta v^n(s)ds, v^n(t)\right) + (h, v^n(t)) \\
&\qquad + |z_*(\theta_t\omega)||\phi||v^n| + (z_k^{\phi}(\theta_t\omega), v^n).
\end{aligned}
$$

By using similar arguments as in Theorem 3.4 and the Young inequality, we deduce

$$
\begin{aligned}
&\frac{d}{dt}|v^n|^2 + m\|v^n\|^2 + \frac{f_0}{2D}\|v^n\|_{2p}^{2p} \le 2\alpha|\mathcal{O}| + C_1(\theta_t\omega)(1 + \|\phi\|_{2p}^{2p}) \\
&\quad + \frac{4M^2}{m}|z_*(\theta_t\omega)|^2\|\phi\|^2 + \frac{4}{m\lambda_1}|h|^2 + \frac{4}{m\lambda_1}|z_*(\theta_t\omega)|^2|\phi|^2 \\
&\quad + \frac{C_2(\theta_t\omega)}{m}\|\phi\|^2 + 2\int_{-\infty}^{t} k(t-s)\|v^n(s)\|ds\|v^n(t)\|.
\end{aligned} \tag{4.22}
$$

Now we estimate the last term in the above inequality. Notice that

$$
\int_{-\infty}^{t} k(t-s)\|v^n(s)\|ds = \int_{-\infty}^{0} k(t-s)\|v^n(s)\|ds + \int_{0}^{t} k(t-s)\|v^n(s)\|ds := I_1 + I_2.
$$

For I_1, by the fact that $\varphi_v \in L_V^2$, $\gamma < \min\{\frac{m\lambda_1}{2}, \varpi\}$ and $k(t) \le M_1$, $\forall t \in [0, \infty)$ (see Remark 2.1), we find

$$
\begin{aligned}
I_1 &= \int_{-\infty}^{0} k(t-s)e^{-\frac{\gamma s}{2}}e^{\frac{\gamma s}{2}}\|\varphi_v(s)\|ds \le \left(\int_{-\infty}^{0} k^2(t-s)e^{-\gamma s}ds\right)^{\frac{1}{2}} \\
&\quad \times \left(\int_{-\infty}^{0} e^{\gamma s}\|\varphi_v\|^2 ds\right)^{\frac{1}{2}} \le \|\varphi_v\|_{L_V^2}\left(\int_{t}^{\infty} k(s)e^{-\gamma(t-s)}ds\right)^{\frac{1}{2}} M_1^{\frac{1}{2}} \\
&\quad \le \frac{\|\varphi_v\|_{L_V^2} M_1^{\frac{1}{2}}}{\varpi^{\frac{1}{2}}}\left(\int_{t}^{\infty} \mu(s)e^{-\gamma(t-s)}ds\right)^{\frac{1}{2}} \\
&\quad \le \frac{|\varphi_v\|_{L_V^2} M_1^{\frac{1}{2}}}{\varpi^{\frac{1}{2}}}\left(\int_{t}^{\infty} \mu(t)e^{-\varpi(s-t)}e^{-\gamma(t-s)}ds\right)^{\frac{1}{2}} \\
&\quad \le \frac{M_1^{\frac{1}{2}}\mu^{\frac{1}{2}}(t)\|\varphi_v\|_{L_V^2}}{\varpi^{\frac{1}{2}}(\varpi-\gamma)^{\frac{1}{2}}}.
\end{aligned}
$$

For I_2, by means of the property $k(t) \leq M_1$ and the boundedness of v^n in $L^2(0, T; V)$, there exists a constant M'' such that

$$I_2 \leq M_1 \int_0^t \|v^n(s)\| ds \leq M_1 M'' \sqrt{t}.$$

Therefore, it follows from the above inequalities and (4.22) that for every $t < T$,

$$\begin{aligned}\frac{d}{dt}|v^n(t)|^2 + m\|v^n(t)\|^2 + \frac{f_0}{2D}\|v^n(t)\|_{2p}^{2p} &\leq 2\alpha|\mathcal{O}| + C_1(\theta_t\omega)(1 + \|\phi\|_{2p}^{2p}) \\ &+ \frac{4M^2}{m}|z_*(\theta_t\omega)|^2\|\phi\|^2 + \frac{4}{m\lambda_1}|h|^2 + \frac{4}{m\lambda_1}|z_*(\theta_t\omega)|^2|\phi|^2 \\ &+ \frac{C_2(\theta_t\omega)}{m}\|\phi\|^2 + 2\left(\frac{M_1^{\frac{1}{2}}\mu^{\frac{1}{2}}(t)\|\varphi_v\|_{L_V^2}}{\varpi^{\frac{1}{2}}(\varpi-\gamma)^{\frac{1}{2}}} + M_1M''\sqrt{T}\right)\|v^n(t)\|. \qquad (4.23)\end{aligned}$$

We will estimate the last term of the above inequality. By the Young inequality, we have

$$2\frac{M_1^{\frac{1}{2}}\mu^{\frac{1}{2}}(t)\|\varphi_v\|_{L_V^2}}{\varpi^{\frac{1}{2}}(\varpi-\gamma)^{\frac{1}{2}}}\|v(t)\| \leq \frac{4M_1\mu(t)\|\varphi_v\|_{L_V^2}^2}{\varpi m(\varpi-\gamma)} + \frac{m}{4}\|v(t)\|^2, \qquad (4.24)$$

and

$$2M_1M''\sqrt{T}\|v(t)\| \leq \frac{4(M_1)^2(M'')^2T}{m} + \frac{m}{4}\|v(t)\|^2. \qquad (4.25)$$

Collecting (4.23)–(4.25), we obtain

$$\begin{aligned}\frac{d}{dt}|v^n(t)|^2 + \frac{m}{2}\|v^n(t)\|^2 + \frac{f_0}{2D}\|v^n(t)\|_{2p}^{2p} &\leq 2\alpha|\mathcal{O}| + C_1(\theta_t\omega)(1 + \|\phi\|_{2p}^{2p}) \\ &+ \frac{4M^2}{m}|z_*(\theta_t\omega)|^2\|\phi\|^2 + \frac{4}{m\lambda_1}|h|^2 + \frac{4}{m\lambda_1}|z_*(\theta_t\omega)|^2|\phi|^2 \\ &+ \frac{4M_1\mu(t)\|\varphi_v\|_{L_V^2}^2}{\varpi m(\varpi-\gamma)} + \frac{4(M_1)^2(M'')^2T}{m} + \frac{C_2(\theta_t\omega)}{m}\|\phi\|^2, \qquad (4.26)\end{aligned}$$

and the same is true for the function v. Hence, we define the functions

$$\begin{aligned}J_n(t) = |v^n(t)|^2 - 2\alpha|\mathcal{O}|t - \frac{4(M_1)^2(M'')^2T}{m}t - \int_0^t C_1(\theta_r\omega)(1 + \|\phi\|_{2p}^{2p})dr \\ - \int_0^t \left(\frac{4M^2}{m}|z_*(\theta_r\omega)|^2\|\phi\|^2 + \frac{4}{m\lambda_1}|z_*(\theta_r\omega)|^2|\phi|^2\right)dr\end{aligned}$$

$$
\begin{aligned}
&-\frac{4\,|h|^2}{m\lambda_1}t-\frac{4M_1\|\varphi_v\|_{L_V^2}^2}{\varpi m(\varpi-\gamma)}\int_0^t\mu(r)dr-\int_0^t\frac{C_2(\theta_r\omega)}{m}\,\|\phi\|^2\,dr,\\
J(t)=|v(t)|^2-2\alpha|\mathcal{O}|t-\frac{4(M_1)^2(M'')^2T}{m}t-\int_0^t C_1(\theta_r\omega)(1+\|\phi\|_{2p}^{2p})dr\\
&-\int_0^t\left(\frac{4M^2}{m}|z_*(\theta_r\omega)|^2\|\phi\|^2+\frac{4}{m\lambda_1}|z_*(\theta_r\omega)|^2|\phi|^2\right)dr\\
&-\frac{4\,|h|^2}{m\lambda_1}t-\frac{4M_1\|\varphi_v\|_{L_V^2}^2}{\varpi m(\varpi-\gamma)}\int_0^t\mu(r)dr-\int_0^t\frac{C_2(\theta_r\omega)}{m}\,\|\phi\|^2\,dr.
\end{aligned}
$$

From the regularity of v and all v^n, together with (4.26), it holds that these functions J and J_n are continuous and non-increasing on $[0, T]$, and

$$J_n(s)\to J(s)\ \text{ a.e. }\ s\in[0,T]\text{ as }n\to\infty.$$

Hence, there exists a sequence $\{\tilde{t}_k\}\in(0,s_*)$ such that $\tilde{t}_k\to s_*$ when $k\to\infty$, and

$$\lim_{n\to\infty}J_n(\tilde{t}_k)=J(\tilde{t}_k),\qquad\forall k\geq 1.$$

Fix an arbitrary value $\epsilon>0$. From the continuity of J on $[0,T]$, there exists $k(\epsilon)\geq 1$ such that

$$|J(\tilde{t}_k)-J(s_*)|\leq\frac{\epsilon}{2},\quad\forall k\geq k(\epsilon).$$

Now consider $n(\epsilon)\geq 1$ such that

$$t_n\geq\tilde{t}_{k(\epsilon)}\ \text{ and }\ |J_n(\tilde{t}_{k(\epsilon)})-J(\tilde{t}_{k(\epsilon)})|\leq\frac{\epsilon}{2},\quad\forall n\geq n(\epsilon).$$

Then, since all J_n are non-increasing, we deduce that

$$
\begin{aligned}
J_n(t_n)-J(s_*)&\leq J_n(\tilde{t}_{k(\epsilon)})-J(s_*)\leq|J_n(\tilde{t}_{k(\epsilon)})-J(s_*)|\\
&\leq|J_n(\tilde{t}_{k(\epsilon)})-J(\tilde{t}_{k(\epsilon)})|+|J(\tilde{t}_{k(\epsilon)})-J(s_*)|\leq\epsilon,\qquad\forall n\geq n(\epsilon).
\end{aligned}
$$

As $\epsilon>0$ is arbitrary, we obtain

$$\limsup_{n\to\infty}J_n(t_n)\leq J(s_*).$$

Thus,

$$\limsup_{n\to\infty}\ |u^n(t_n)|\leq|u(s_*)|.\tag{4.27}$$

Therefore, (4.21) and (4.27) imply that $v^n(s_n)\to v(s_*)$ strongly in H, and (4.12) holds true.

Define the functions $\bar{y}^n = y^n - y$ and $\bar{\eta}_n^t = \eta_n^t - \eta^t$ with $\bar{y}_0^n = \bar{y}_0^n - y_0$, where $y_0 = (v_0, \eta_0)$. Similar to the uniqueness part in the proof of Theorem 3.4, for every $\omega \in \Omega$, we have

$$\begin{aligned}
&\frac{d}{dt}\|\bar{y}^n\|_{\mathcal{H}}^2 + 2(((\bar{\eta}_n^t)', \eta_n^t))_\mu \\
&\leq -2\int_{\mathcal{O}} (f(v^n + \phi z_*(\theta_t\omega)) - f(v + \phi z_*(\theta_t\omega)))(v^n - v)dx \\
&\quad -2\int_{\mathcal{O}} \big(a(l(v^n) + l(\phi)z_*(\theta_t\omega))\nabla v^n - a(l(v) + l(\phi)z_*(\theta_t\omega))\nabla v\big) \cdot \nabla(v^n - v)dx \\
&\quad +2\int_{\mathcal{O}} (a(l(v^n) + l(\phi)z_*(\theta_t\omega)) - a(l(v) + l(\phi)z_*(\theta_t\omega)))z_*(\theta_t\omega)\Delta\phi(v^n - v)dx.
\end{aligned}$$

Since a is a locally Lipschitz function, by (1.11) and the Young inequality, we find

$$\begin{aligned}
&-2\int_{\mathcal{O}} \big(a(l(v^n) + l(\phi)z_*(\theta_t\omega))\nabla v^n - a(l(v) + l(\phi)z_*(\theta_t\omega))\nabla v\big) \cdot \nabla(v^n - v)dx \\
&\quad \leq -2m\|v^n - v\|^2 + 2L_a(R)|l||v^n - v|\|v\|\|v^n - v\| \\
&\quad \leq (\alpha - 2m)\|v^n - v\|^2 + \frac{L_a^2(R)|l|^2}{\alpha}|v^n - v|^2\|v\|^2,
\end{aligned}$$

where $\alpha \leq (m\lambda_1 - \gamma)/\lambda_1$ and for all $n \geq 1$, $t \geq 0$, and we have chosen $R > 0$ such that $\{l(v^n(t) + l(\phi)z_*(\theta_t\omega))\}_{t\in[\tau,T]} \subset [-R, R]$, $\{l(v(t) + l(\phi)z_*(\theta_t\omega))\}_{t\in[\tau,T]} \subset [-R, R]$, which can be done because $|v^n(t)|$ are uniformly bounded in $[\tau, T]$. Then, by the above estimates, we deduce

$$\begin{aligned}
&\frac{d}{dt}\|\bar{y}^n\|_{\mathcal{H}}^2 + \gamma\|\bar{y}^n\|_{\mathcal{H}}^2 + m\|v^n - v\|^2 \\
&\quad \leq \frac{d}{dt}\|\bar{y}^n\|_{\mathcal{H}}^2 + (2m - \alpha)\|v^n - v\|^2 + \varpi\int_0^\infty \mu(s)|\nabla\bar{\eta}_n^t(s)|^2 ds \\
&\quad \leq \frac{L_a^2(R)|l|^2}{\alpha}|v^n - v|^2\|v\|^2 \\
&\qquad -2\int_{\mathcal{O}} (f(v^n + \phi z_*(\theta_t\omega)) - f(v + \phi z_*(\theta_t\omega)))(v^n - v)dx \\
&\qquad +2\int_{\mathcal{O}} (a(l(v^n) + l(\phi)z_*(\theta_t\omega)) - a(l(v) \\
&\qquad +l(\phi)z_*(\theta_t\omega)))z_*(\theta_t\omega)\Delta\phi(v^n - v)dx,
\end{aligned}$$

where we have used that $0 < \gamma \leq \min\{(m-\alpha)\lambda_1, \varpi\}$ by the choice of α. Multiplying by $e^{\gamma t}$ on both sides of the above inequality and integrating over $(0, t)$, we obtain

$$\begin{aligned}
&\|\bar{y}^n(t)\|^2_{\mathcal{H}} + m\int_0^t e^{-\gamma(t-s)}\|v^n(s) - v(s)\|^2 ds \\
&\leq e^{-\gamma t}\|\bar{y}^n_0\|^2_{\mathcal{H}} + \frac{L_a^2(R)|l|^2}{\alpha}\int_0^t e^{-\gamma(t-s)}|v^n - v|^2\|v\|^2 ds \\
&\quad - 2\int_0^t e^{-\gamma(t-s)}\int_{\mathcal{O}}(f(v^n + \phi z_*(\theta_s\omega)) \\
&\quad - f(v + \phi z_*(\theta_s\omega)))(v^n - v)dxds \\
&\quad + 2\int_0^t e^{-\gamma(t-s)}\int_{\mathcal{O}}(a(l(v^n) + l(\phi)z_*(\theta_s\omega)) - a(l(v) \\
&\quad + l(\phi)z_*(\theta_s\omega)))|z_*(\theta_s\omega)||\Delta\phi||v^n - v|dxds.
\end{aligned}$$

On the one hand, by (4.12), we know that $|v^n(s) - v(s)|^2\|v(s)\|^2 \to 0$ and $|v^n(s) - v(s)| \to 0$ for a.e. $s \in (0, t)$. On the other hand, $e^{-\gamma(t-s)}|v^n(s) - v(s)|^2\|v(s)\|^2$ and $e^{-\gamma(t-s)}(a(l(v^n)+l(\phi)z_*(\theta_s\omega)) - a(l(v)+l(\phi)z_*(\theta_s\omega)))|z_*(\theta_s\omega)||\Delta\phi||v^n - v|$ can be bounded by $4R^2e^{-\gamma(t-s)}\|v(s)\|^2$ and $4MRe^{-\gamma(t-s)}\sup_{s\in[0,t]}|z_*(\theta_s\omega)||\Delta\phi|$, respectively. Hence, the Lebesgue theorem implies that

$$\int_0^t e^{-\gamma(t-s)}|v^n(s) - v(s)|^2\|v(s)\|^2 ds \to 0 \quad \text{as} \quad n \to 0,$$

and

$$\begin{aligned}
&\int_0^t e^{-\gamma(t-s)}\int_{\mathcal{O}}(a(l(v^n) + l(\phi)z_*(\theta_s\omega)) - a(l(v) \\
&\quad + l(\phi)z_*(\theta_s\omega)))|z_*(\theta_s\omega)||\Delta\phi||v^n - v|dxds \longrightarrow 0 \quad \text{as} \quad n \to 0,
\end{aligned}$$

respectively. Moreover, it follows from the argument after (4.18) that $f(v^n + \phi z_*(\theta_s\omega)) \to f(v + \phi z_*(\theta_s\omega))$ weakly in $L^q(0, T; L^q(\mathcal{O}))$ as $n \to \infty$, therefore

$$\int_0^t e^{-\gamma(t-s)}\int_{\mathcal{O}}\big(f(v^n + \phi z_*(\theta_s\omega)) - f(v + \phi z_*(\theta_s\omega))\big)\, v dxds \to 0 \quad \text{as} \quad n \to \infty.$$

By (3.8) and the Young inequality, we deduce that there are two positive constants $\kappa_1(\omega, \phi, T) > 0$ and $\kappa_2 > 0$ such that

$$\begin{aligned}
& f(v^n(s,x,\omega) + \phi z_*(\theta_s\omega))v^n(s,x,\omega) \\
&\quad = f(v^n(s,x,\omega) + \phi z_*(\theta_s\omega)) \left(v^n(s,x,\omega\right) + \phi z_*(\theta_s\omega)) \\
&\qquad - f(v^n(s,x,\omega) + \phi z_*(\theta_s\omega))\phi z_*(\theta_s\omega) \\
&\quad \geq \frac{1}{2} f_0 \left|v^n(s,x,\omega) + \phi z_*(\theta_s\omega)\right|^{2p} - \alpha \\
&\qquad - \beta \left(1 + \left|v^n(s,x,\omega) + \phi z_*(\theta_s\omega)\right|^{2p-1}\right) |\phi|\, |z_*(\theta_s\omega)| \\
&\quad \geq -\kappa_1 + \kappa_2 \left|v^n(s,x,\omega) + \phi z_*(\theta_s\omega)\right|^{2p}.
\end{aligned}$$

Thus, the Fatou-Lebesgue theorem implies that

$$\begin{aligned}
& \limsup_{n\to\infty} \left(-2\int_0^t e^{-\gamma(t-s)} \int_{\mathcal{O}} f(v^n + \phi z_*(\theta_s\omega))v^n dxds\right) \\
&\quad \leq -2 \liminf_{n\to\infty} \int_0^t e^{-\gamma(t-s)} \int_{\mathcal{O}} f(v^n + \phi z_*(\theta_s\omega))v^n dxds \\
&\quad \leq -2 \int_0^t e^{-\gamma(t-s)} \int_{\mathcal{O}} \liminf_{n\to\infty} f(v^n + \phi z_*(\theta_s\omega))v^n dxds \\
&\quad = -2 \int_0^t e^{-\gamma(t-s)} \int_{\mathcal{O}} f(v + \phi z_*(\theta_s\omega))v dxds.
\end{aligned}$$

This inequality, together with

$$\int_0^t e^{-\gamma(t-s)} \int_{\mathcal{O}} f(v + \phi z_*(\theta_s\omega))(v^n - v)dxds \to 0 \quad \text{as} \quad n \to \infty, \tag{4.28}$$

shows that

$$\begin{aligned}
& \limsup_{n\to\infty} \left(-2\int_0^t e^{-\gamma(t-s)} \int_{\mathcal{O}} (f(v^n + \phi z_*(\theta_s\omega)) - f(v + \phi z_*(\theta_s\omega)))(v^n - v)dxds\right) \\
&\quad \leq 0.
\end{aligned}$$

Notice that (4.28) follows from the facts that $f(v + \phi z_*(\theta.\omega)) \in L^q(0,T; L^q(\mathcal{O}))$ and $v^n \to v$ weakly in $L^{2p}(0,T; L^{2p}(\mathcal{O}))$ for every $\omega \in \Omega$.

Collecting all inequalities derived above and using Corollary 3.2, we find

$$\begin{aligned}
& \limsup_{n\to\infty} \int_0^t e^{-\gamma(t-s)} \|v^n(s) - v(s)\|^2 ds \\
&\quad \leq \frac{1}{m} e^{-\gamma t} \limsup_{n\to\infty} \|\bar{y}_0^n\|_{\mathcal{H}}^2 \\
&\quad \leq \frac{1}{m} e^{-\gamma t} \limsup_{n\to\infty} \left(|v^n(0) - v(0)|^2 + \int_0^\infty \mu(s) \|\bar{\eta}_n^0(s)\|^2 ds\right)
\end{aligned}$$

$$\leq \frac{1}{m} e^{-\gamma t} \limsup_{n\to\infty} \left(|v_0^n - v_0|^2 + 2K_\mu \int_{-\infty}^0 e^{\gamma s} \|\varphi_v^n(s) - \varphi_v(s)\|^2 ds \right).$$

Finally, (4.15) follows from

$$\begin{aligned} \|v_t^n - v_t\|_{L_V^2}^2 &= \int_{-t}^0 e^{\gamma s} \|v^n(t+s) - v(t+s)\|^2 ds \\ &\quad + \int_{-\infty}^{-t} e^{\gamma s} \|v^n(t+s) - v(t+s)\|^2 ds \\ &= \int_0^t e^{-\gamma(t-s)} \|v^n(s) - v(s)\|^2 ds \\ &\quad + e^{-\gamma t} \int_{-\infty}^0 e^{\gamma s} \|\varphi_v^n(s) - \varphi_v(s)\|^2 ds. \end{aligned}$$

If $(v_0^n, \varphi_v^n) \to (v_0, \varphi_v)$ in X, then (4.15) implies (4.16)–(4.17). The proof is complete. □

Remark 4.4 The proof of (4.12) also works in the deterministic case (see the Appendix).

Lemma 4.5 *Suppose that the conditions of Theorem 3.4 hold. Then the cocycle Ξ is asymptotically compact.*

Proof Let $D \in \mathcal{D}$. It is sufficient to prove that for any sequence $\{(u_0^n, \varphi^n)\}_{n\in\mathbb{N}} \subset D(\theta_{-t_n}\omega)$, the sequence $\{\Xi(t_n, \theta_{-t_n}\omega, (u_0^n, \varphi^n))\}_{n\in\mathbb{N}}$ is relatively compact in X as $t_n \to \infty$. Recall that

$$\begin{aligned} \Xi(t_n, \theta_{-t_n}\omega, (u_0^n, \varphi^n)) = \Psi(t_n, \theta_{-t_n}\omega, (u_0^n - \phi z_*(\theta_{-t_n}\omega), \mathcal{J}(\varphi^n - \phi z_*(\theta_{-t_n+\cdot}\omega)))) \\ + (\phi z_*(\omega), \phi z_*(\theta_\cdot\omega)). \end{aligned}$$

Hence, we only need to prove that the sequence $\{\Psi(t_n, \theta_{-t_n}\omega, (u_0^n - \phi z_*(\theta_{-t_n}\omega), \mathcal{J}(\varphi^n - \phi z_*(\theta_{-t_n+\cdot}\omega))))\}$ possesses a convergent subsequence in X. Observe that for $T > 0$,

$$\begin{aligned} &\Psi(t_n, \theta_{-t_n}\omega, (u_0^n - \phi z_*(\theta_{-t_n}\omega), \varphi^n - \phi z_*(\theta_{-t_n+\cdot}\omega))) \\ &\quad = \Psi(T, \theta_{-T}\omega, \Psi(t_n - T, \theta_{-t_n}\omega, (u_0^n - \phi z_*(\theta_{-t_n}\omega), \varphi^n - \phi z_*(\theta_{-t_n+\cdot}\omega)))) \\ &\quad = \Psi(T, \theta_{-T}\omega, \Psi(t_n - T, \theta_{-t_n+T}\theta_{-T}\omega, (u_0^n - \phi z_*(\theta_{-t_n+T}\theta_{-T}\omega), \varphi^n \\ &\qquad - \phi z_*(\theta_{-t_n+T+\cdot}\theta_{-T}\omega)))) \subset \Psi(T, \theta_{-T}\omega, \tilde{B}(\theta_{-T}\omega)), \end{aligned}$$

for $t_n - T \geq \tilde{t}_0(\omega, D)$, where $\tilde{B}$ is the absorbing ball of Ψ. Let

$$\mathcal{Y}_n := (\alpha^n, \beta^n) = \Psi(t_n, \theta_{-t_n}\omega, (u_0^n - \phi z_*(\theta_{-t_n}\omega), \varphi^n - \phi z_*(\theta_{-t_n+\cdot}\omega))).$$

Then $(\alpha^n, \beta^n) = \Psi(T, \theta_{-T}\omega, \xi_n^T)$, where $\xi_n^T \in \tilde{B}(\theta_{-T}\omega)$. Let $(V^n(\cdot), V_\cdot^n)$ be a sequence of solutions to problem (2.6) with initial condition ξ_n^T and $(V^n(T), V_T^n) =$

(α^n, β^n). Since $\tilde{B}(\omega)$, $\tilde{B}(\theta_{-T}\omega)$ are bounded in X, by Lemma 4.1, we can assume (up to a subsequence) that $\mathcal{Y}_n \to \mathcal{Y} = (\vartheta, \pi)$, $\xi_n^T \to \xi^T$ weakly in X for every ω.

It follows from Lemma 4.3 that $(V^n(T), V_T^n(\cdot)) = \Psi(T, \theta_{-T}\omega, \xi_n^T)$ satisfies (4.12)–(4.14). We deduce from the above convergence that $\vartheta = V(T)$ in H and $\pi = V_T$ in L_V^2, $\pi(s) = V_T(s)$ for almost all $s \in (-\infty, 0)$ and $\omega \in \Omega$. Also, in view of (4.12), we find that

$$\alpha^n = V^n(T) \to V(T) = \vartheta \quad \text{in} \quad H.$$

Hence, in order to prove that $\mathcal{Y}_n \to \mathcal{Y}$ in X, it remains to show that $\beta^n \to \pi$ in L_V^2 (up to a subsequence). Notice that $\beta^n = V_T^n$ for all $T > 0$ and $V_T = \pi$. Since the family $\tilde{B}$ is tempered, we have that

$$\lim_{T\to\infty} e^{-cT} \sup_n \left\| \xi_n^T \right\|_X = 0, \tag{4.29}$$

for any $c > 0$. Thanks to (4.15), we have, for each $T \in \mathbb{N}$,

$$\begin{aligned}\limsup_{n\to\infty} \|\beta^n - \pi\|_{L_V^2}^2 &= \limsup_{n\to\infty} \|V_T^n - V_T\|_{L_V^2}^2 \\ &\le K e^{-(\gamma-c)T} e^{-cT} \limsup_{n\to\infty} \left(\|\xi_n^T - \xi^T\|_X^2 \right) \\ &\le \tilde{K} e^{-(\gamma-c)T},\end{aligned}$$

where $0 < c < \gamma$ and the last inequality follows from (4.29). For every $k > 0$, there exists $T(k)$ such that for all $T \ge T(k)$,

$$\limsup_{n\to\infty} \|\beta^n - \pi\|_{L_V^2}^2 = \limsup_{n\to\infty} \|V_T^n - V_T\|_{L_V^2}^2 \le \frac{1}{k}.$$

Taking $k \to \infty$ and using a diagonal argument, we obtain that there exists a subsequence $\{\beta^{n_k}\}$ such that $\beta^{n_k} \to \pi$ in L_V^2 for all $\omega \in \Omega$. The proof of this lemma is complete. □

A family of sets $K(\omega)$ is said to be measurable with respect to $\mathcal{F}$, if for any $x \in X$ the mapping $\omega \mapsto dist(x, K(\omega))$ is $(\mathcal{F}, \mathcal{B}(\mathbb{R}))$-measurable.

We recall that the family of non-empty compact sets $\mathcal{A} = \mathcal{A}(\omega) \in \mathcal{D}$ is called a random attractor for the cocycle Ξ, if $\mathcal{A}(\omega)$ is measurable with respect to $\mathcal{F}$, it is invariant, that is,

$$\Xi(t, \omega, \mathcal{A}(\omega)) = \mathcal{A}(\theta_t\omega) \text{ for all } \omega \in \Omega,\ t \ge 0,$$

and it is pullback $\mathcal{D}$-attracting, that is, for any $D \in \mathcal{D}$, it holds that

$$\lim_{t\to+\infty} dist(\Xi(t, \theta_{-t}\omega, D(\theta_{-t}\omega)), \mathcal{A}(\omega)) = 0,$$

where $dist\,(C_1, C_2) = \sup_{x\in C_1} \inf_{y\in C_2} \|x-y\|_X$ is the Hausdorff semidistance between sets from X.

The existence and uniqueness of the random attractor $\mathcal{A}$ follow from [30, Proposition 2.10] (see also [29, 31] for related results) immediately based on Lemmas 4.1, 4.3 and 4.5. We observe that the radius $\rho\,(\omega)$ is measurable, so it is easy to see that the family of closed balls $B(\omega)$ is measurable with respect to $\mathcal{F}$.

Theorem 4.6 *Assume that (1.11), (2.8) and* (h_1)-(h_2) *hold, and that* $\phi \in V\cap H^2(\mathcal{O})\cap L^{2p}(\mathcal{O})$ *is such that* $\Delta\phi \in L^{2p}(\mathcal{O})$. *Let* $h \in H$ *and* $a(\cdot)$ *be a locally Lipschitz function. Then the cocycle* Ξ *of problem (1.10) has a unique random attractor* $\mathcal{A} = \{\mathcal{A}(\omega) : \omega \in \Omega\}$ *in* H.

5 Stochastic Nonlocal PDEs with Long Time Memory Driven by Colored Noise

This section is devoted to discuss the approximations of stochastic nonlocal PDEs with long time memory, namely, the following pathwise Wong-Zakai approximated problem,

$$\begin{cases} \partial_t u_\delta - a(l(u_\delta))\Delta u_\delta - \displaystyle\int_{-\infty}^{t} k(t-s)\Delta u_\delta(s)ds + f(u_\delta) = h + \phi\zeta_\delta(\theta_t\omega), & \text{in } \mathcal{O}\times(\tau,\infty),\\ u_\delta(t,x) = 0, & \text{on } \partial\mathcal{O}\times(\tau,\infty),\\ u_\delta(\tau,x) = u_{0,\delta}(x), & \text{in } \mathcal{O},\\ u_\delta(t+\tau,x) = \varphi_\delta(t,x), & \text{in } \mathcal{O}\times(-\infty,0), \end{cases} \tag{5.1}$$

where $\zeta_\delta(\theta_t\omega)$ is the colored noise with correlation time $\delta > 0$, which is a stationary solution of the stochastic differential equation

$$d\zeta_\delta + \frac{1}{\delta}\zeta_\delta = \frac{1}{\delta}dW.$$

This process satisfies

$$\lim_{t\to\pm\infty} \frac{|\zeta_\delta(\theta_t\omega)|}{t} = 0 \text{ for all } 0 < \delta \le 1,$$
$$\lim_{\delta\to 0^+} \sup_{t\in[\tau,\tau+T]} \left| \int_0^t \zeta_\delta(\theta_s\omega)ds - \omega(t) \right| = 0,\ \forall\tau\in\mathbb{R},\ T > 0.$$

For more details about colored noise see [7, 8, 19] and the references therein. For applications to stochastic Hamiltonian flows see [11, 12]. The functions a, f and ϕ fulfill the same assumptions as in Sect. 2. Define a random variable,

$$v_\delta(t,\omega) = u_\delta(t,\omega) - \phi y_\delta(\theta_t\omega). \tag{5.2}$$

Recall that y_δ satisfies,

$$\frac{dy_\delta}{dt} = -\sigma y_\delta + \zeta_\delta(\theta_t\omega). \tag{5.3}$$

For almost all $\omega \in \Omega$, one special solution of (5.3) can be represented by,

$$Y_\delta(t,\omega) = e^{-\sigma t}\int_{-\infty}^{t} e^{\sigma s}\zeta(\theta_s\omega)ds,$$

which, in fact, can be rewritten as $Y_\delta(t,\omega) = y_\delta(\theta_t\omega)$. Here $y_\delta : \Omega \to \mathbb{R}$ is a well-defined random variable given by $y_\delta(\omega) := \int_{-\infty}^{0} e^{\sigma s}\zeta_\delta(\theta_s\omega)ds$ and has the following properties.

Lemma 5.1 [19, Lemma 3.2] *Let y_δ be the random variable defined as above. Then the mapping*

$$(t,\omega) \to y_\delta(\theta_t\omega) = e^{-\sigma t}\int_{-\infty}^{t} e^{\sigma s}\zeta_\delta(\theta_s\omega)ds, \tag{5.4}$$

is a stationary solution of (5.3) with continuous trajectories. In addition, $\mathbb{E}(y_\delta) = 0$ and for every ω,

$$\lim_{\delta\to 0} y_\delta(\theta_t\omega) = z_*(\theta_t\omega), \quad \textit{uniformly on } [\tau,\tau+T] \textit{ with } \tau\in\mathbb{R},\ T>0; \tag{5.5}$$

$$\lim_{t\to\pm\infty} \frac{|y_\delta(\theta_t\omega)|}{|t|} = 0, \quad \textit{uniformly for } 0<\delta\le\tilde\sigma; \tag{5.6}$$

$$\lim_{t\to\pm\infty} \frac{1}{t}\int_0^t y_\delta(\theta_s\omega)ds = 0, \quad \textit{uniformly for } 0<\delta\le\tilde\sigma; \tag{5.7}$$

where $\tilde\sigma = \min\{1, \frac{1}{2\sigma}\}$ and $z_*(\theta_t\omega)$ is given in Sect. 2.

Remark 5.2 It follows from (5.5)–(5.6) that $y_\delta(\theta.\omega) \to z_*(\theta.\omega)$ in L^2_V.

Remark 5.3 Throughout this paper, to simplify the computations, we take $\sigma = 1$ in (5.3). Then the results of Lemma 5.1 are true for $\sigma = 1$.

Thus, it follows from (5.1)–(5.3) that, for $t > \tau$,

$$\begin{cases} \partial_t v_\delta - a(l(v_\delta) + y_\delta(\theta_t\omega)l(\phi))\Delta v_\delta - a(l(v_\delta) + y_\delta(\theta_t\omega)l(\phi))y_\delta(\theta_t\omega)\Delta\phi & \\ \quad -\displaystyle\int_{-\infty}^{t} k(t-s)\Delta v_\delta(s)ds + f(v_\delta + \phi y_\delta(\theta_t\omega)) = h + \phi y_\delta(\theta_t\omega) + z^{\phi}_{k,\delta}(t,\omega), & \text{in } \mathcal{O}\times(\tau,\infty), \\ v_\delta(t,x) = 0, & \text{on } \partial\mathcal{O}\times(\tau,\infty), \\ v_\delta(\tau,x) = v_{0,\delta}(x) := u_{0,\delta}(x) - \phi y_\delta(\theta_\tau\omega), & \text{in } \mathcal{O}, \\ v_\delta(t+\tau,x) := u_\delta(t+\tau,x) - \phi y_\delta(\theta_{t+\tau}\omega) = \varphi_\delta(t,x) - \phi y_\delta(\theta_{t+\tau}\omega) := \varphi_{v,\delta}(t,x), & \text{in } \mathcal{O}\times(-\infty,0), \end{cases} \tag{5.8}$$

where $z_{k,\delta}^{\phi}(t)$ is a process defined by

$$z_{k,\delta}^{\phi}(t,\omega)=\int_{-\infty}^{t}k(t-s)y_\delta(\theta_s\omega)\Delta\phi ds. \tag{5.9}$$

To use Dafermos' transformation obtaining the well-posedness of problem (5.8), let us define the new variables,

$$\begin{aligned}
v_\delta^t(s,x,\omega)&=v_\delta(t-s,x,\omega), \qquad s\geq 0,\\
\eta_\delta^t(s,x,\omega)&=\int_0^s v_\delta^t(r,x,\omega)dr=\int_{t-s}^{t}v_\delta(r,x,\omega)dr, \qquad s\geq 0.
\end{aligned}$$

Besides, assuming $k(\infty)=0$, a change of variable and a formal integration by parts imply,

$$\int_{-\infty}^{t}k(t-s)\nabla v_\delta(s)ds=-\int_0^\infty k'(s)\nabla\eta_\delta^t(s)ds.$$

Setting $\mu(s)=-k'(s)$, problem (5.8) turns into the following system without delay,

$$\begin{cases}
\partial_t v_\delta-a(l(v_\delta+\phi y_\delta(\theta_t\omega)))\Delta v_\delta-a(l(v_\delta+\phi y_\delta(\theta_t\omega)))y_\delta(\theta_t\omega)\Delta\phi & \\
\quad-\displaystyle\int_0^\infty\mu(s)\Delta\eta_\delta^t(s)ds+f(v_\delta+\phi y_\delta(\theta_t\omega))=\phi y_\delta(\theta_t\omega)+z_{k,\delta}^{\phi}(t)+h, & \text{in } \mathcal{O}\times(\tau,\infty),\\
\partial_t\eta_\delta^t(s)=-\partial_s\eta_\delta^t(s)+v_\delta(t), & \text{in } \mathcal{O}\times(\tau,\infty)\times\mathbb{R}^+,\\
v_\delta(t,x)=\eta_\delta^t(x,s)=0, & \text{on } \partial\mathcal{O}\times(\tau,\infty)\times\mathbb{R}^+,\\
v_\delta(\tau,x)=v_{0,\delta}(x):=u_{0,\delta}(x)-\phi y_\delta(\theta_\tau\omega), & \text{in } \mathcal{O},\\
\eta_\delta^\tau(s,x)=\eta_{0,\delta}(s,x), & \text{in } \mathcal{O}\times\mathbb{R}^+,
\end{cases} \tag{5.10}$$

where

$$\eta_{0,\delta}(s,x)(\omega)=\int_{\tau-s}^{\tau}v_\delta(r,x)dr=\int_{-s}^{0}(\varphi_\delta(r)-\phi y_\delta(\theta_{r+\tau}\omega))dr=\int_{-s}^{0}\varphi_{v,\delta}(r)dr.$$

The following result is proved exactly as Corollary 3.2.

Corollary 5.4 *Assume that (h_1)-(h_2) hold and $\phi\in V\cap H^2(\mathcal{O})\cap L^{2p}(\mathcal{O})$. Then, for every $\omega\in\Omega$ and $\tau\in\mathbb{R}$, the operator $\mathcal{J}_{\omega,\tau}^{\delta}:L_V^2\to L_\mu^2(\mathbb{R}^+;V)$ defined by*

$$(\mathcal{J}_{\omega,\tau}^{\delta}\varphi)(s):=\int_{-s}^{0}\varphi(r,\omega)dr-\int_{-s}^{0}y_\delta(\theta_{r+\tau}\omega)\phi dr=\mathcal{J}(\varphi-\phi y_\delta(\theta_{\tau+\cdot}\omega))(s), \tag{5.11}$$

is continuous. Additionally, there exists a positive constant K_μ which is the same as in Lemma 3.1 (which is also independent of δ), such that for any $\varphi\in L_V^2$, we have

$$\|\mathcal{J}_{\omega,\tau}^{\delta}\varphi\|_{L_\mu^2(\mathbb{R}^+;V)}^2\leq K_\mu\,\|\varphi-y_\delta(\theta_{\cdot+\tau}\omega)\phi\|_{L_V^2}^2\leq 2K_\mu\left(\|\varphi\|_{L_V^2}^2+\|y_\delta(\theta_{\cdot+\tau}\omega)\phi\|_{L_V^2}^2\right).$$

Since (5.10) can be viewed as a deterministic equation parameterized by $\omega \in \Omega$, by the same procedures as in Theorems 3.4 and 3.8, we are able to prove the following results.

Theorem 5.5 *Assume that (1.11), (2.8) and* (h_1)–(h_2) *hold. Let* $\phi \in V \cap H^2(\mathcal{O}) \cap L^{2p}(\mathcal{O})$ *be such that* $\Delta\phi \in L^{2p}(\mathcal{O})$*, let* $h \in H$ *and* a *be a locally Lipschitz function. Then, for every* $\tau \in \mathbb{R}$ *and* $\omega \in \Omega$*, it holds:*

(i) For any initial value $v_{0,\delta} \in H$ *and initial function* $\varphi_\delta \in L^2_V$*, there exists a unique solution* (v_δ, η_δ) *to problem (5.10) in the weak sense with initial value* $(v_{0,\delta}, \eta_{0,\delta})$*, where* $\eta_{0,\delta} = \mathcal{J}^\delta_{\omega,\tau}\varphi_\delta$*, fulfilling*

$$
\begin{aligned}
&v_\delta \in L^\infty(\tau, T; H) \cap L^2(\tau, T; V) \cap L^{2p}(\tau, T; L^{2p}(\mathcal{O})), \qquad \forall T > \tau;\\
&\eta_\delta \in L^\infty(\tau, T; L^2_\mu(\mathbb{R}^+; V)), \qquad \forall T > \tau.
\end{aligned}
$$

Furthermore, the solution (v_δ, η_δ) *of (5.10) is continuous with respect to the initial value* $(v_{0,\delta}, \eta_{0,\delta})$ *for all* $t \in [\tau, T]$ *in* $\mathcal{H}$*;*

(ii) For any initial value $(v_{0,\delta}, \eta_{0,\delta}) \in \mathcal{V}$*, the unique solution* (v_δ, η_δ) *to problem (5.10) satisfies,*

$$
\begin{aligned}
&v_\delta \in L^\infty(\tau, T; V) \cap L^2(\tau, T; V \cap H^2(\mathcal{O})), \qquad \forall T > \tau;\\
&\eta_\delta \in L^\infty(\tau, T; L^2_\mu(\mathbb{R}^+; V \cap H^2(\mathcal{O}))), \qquad \forall T > \tau.
\end{aligned}
$$

In addition, the solution (v_δ, η_δ) *of (5.10) is continuous with respect to the initial value* $(v_{0,\delta}, \eta_{0,\delta})$ *for all* $t \in [\tau, T]$ *in* $\mathcal{V}$*.*

Now, thanks to transformation (5.2), we obtain the well-posedness of problem (5.1).

Theorem 5.6 *Assume (1.11), (2.8) and* (h_1)-(h_2) *hold,* $\phi \in V \cap H^2(\mathcal{O}) \cap L^{2p}(\mathcal{O})$ *such that* $\Delta\phi \in L^{2p}(\mathcal{O})$*. Let* $h \in H$ *and* a *be a locally Lipschitz function. Then, for every* $\tau \in \mathbb{R}$ *and* $\omega \in \Omega$*, it holds:*

(i) For any initial value $u_{0,\delta} \in H$ *and initial function* $\varphi_\delta \in L^2_V$*, there exists a unique solution* u_δ *to problem (5.1) in the weak sense, fulfilling*

$$
u_\delta \in L^\infty(\tau, T; H) \cap L^2(\tau, T; V) \cap L^{2p}(\tau, T; L^{2p}(\mathcal{O})), \qquad \forall T > \tau.
$$

Furthermore, the solution u_δ *of (5.1) is continuous with respect to the initial values* $(u_{0,\delta}, \varphi_\delta)$ *for all* $t \in [\tau, T]$ *in* H*;*

(ii) For any initial value $u_{0,\delta} \in V$ *and initial function* $\varphi_\delta \in L^2_{V\cap H^2(\mathcal{O})}$*, the unique solution* u_δ *to problem (5.1) satisfies,*

$$
u_\delta \in L^\infty(\tau, T; V) \cap L^2(\tau, T; V \cap H^2(\mathcal{O})), \qquad \forall T > \tau.
$$

In addition, the solution u_δ *of (5.1) is continuous with respect to the initial values* $(u_{0,\delta}, \varphi_\delta)$ *for all* $t \in [\tau, T]$ *in* V*.*

Next, we can define a continuous cocycle in X associated to the solutions of problem (5.1). Let $\tau = 0$, $\Xi_\delta : \mathbb{R}^+ \times \Omega \times X \to X$ be a mapping defined, for every $t \in \mathbb{R}^+$, $\omega \in \Omega$ and $(u_{0,\delta}, \varphi_\delta) \in X$, by

$$\Xi_\delta(t, \omega, (u_{0,\delta}, \varphi_\delta)) = (u_\delta(t; 0, \omega, (u_{0,\delta}, \varphi_\delta)), u_{\delta,t}(\cdot; 0, \omega, (u_{0,\delta}, \varphi_\delta))).$$

Here and in the sequel, we denote $u_{\delta,t}(s) = u_\delta(t+s)$ for $s \le 0$. Moreover, problem (5.10) also generates a random dynamical system Φ_δ on the phase space $H \times L^2_\mu(\mathbb{R}^+; V)$, which is defined, for every $t \in \mathbb{R}^+$, $\omega \in \Omega$ and $(v_{0,\delta}, \eta_{0,\delta}) \in H \times L^2_\mu(\mathbb{R}^+; V)$, by

$$\Phi_\delta(t, \omega, (v_{0,\delta}, \eta_{0,\delta})) = (v_\delta(t; 0, \omega, (v_{0,\delta}, \eta_{0,\delta})), \eta^t_\delta(\cdot; 0, \omega, (v_{0,\delta}, \eta_{0,\delta}))),$$

where the right-hand side of the above equality denotes the solution of (5.10) for $\tau = 0$, the initial value $(v_{0,\delta}, \eta_{0,\delta}) \in H \times L^2_\mu(\mathbb{R}^+; V)$ and $\eta_{0,\delta}$ is given in (5.10). Thanks to Dafermos' transformation, we can obtain a random dynamical system $\Psi_\delta : \mathbb{R}^+ \times \Omega \times X \to X$ generated by (5.8) which is given, for every $t \in \mathbb{R}^+$, $\omega \in \Omega$ and $(v_{0,\delta}, \psi_\delta) \in X$, by

$$\Psi_\delta(t, \omega, (v_{0,\delta}, \psi_\delta)) = (v_\delta(t; 0, \omega, (v_{0,\delta}, (\mathcal{J}\psi_\delta))), v_{\delta,t}(\cdot; 0, \omega, (v_{0,\delta}, (\mathcal{J}\psi_\delta)))).$$

Then, on account of the random transformation (5.2), for $(u_{0,\delta}, \varphi_\delta) \in X$, we deduce

$$\begin{aligned}
&\Xi_\delta(t, \omega, (u_{0,\delta}, \varphi_\delta)) \\
&\quad = (u_\delta(t; 0, \omega, (u_{0,\delta}, \varphi_\delta)), u_{\delta,t}(\cdot; 0, \omega, (u_{0,\delta}, \varphi_\delta))) \\
&\quad = \big(v_\delta(t; 0, \omega, (u_{0,\delta} - \phi y_\delta(\omega), (\mathcal{J}^\delta_{\omega,0}\varphi_\delta))) + \phi y_\delta(\theta_t\omega), \\
&\quad v_{\delta,t}(\cdot; 0, \omega, (u_{0,\delta} - \phi y_\delta(\omega), (\mathcal{J}^\delta_{\omega,0}\varphi_\delta))) + \phi y_\delta(\theta_{t+\cdot}\omega)\big) \\
&\quad = \Psi_\delta(t, \omega, (u_{0,\delta} - \phi y_\delta(\omega), \varphi_{v,\delta})) + (\phi y_\delta(\theta_t\omega), \phi y_\delta(\theta_{t+\cdot}\omega)).
\end{aligned} \tag{5.12}$$

Therefore, similar to Sect. 4, it is easy to check that Ξ_δ and Ψ_δ are conjugated.

We will prove now the existence of absorbing sets for the cocycle Ξ_δ.

Lemma 5.7 *Under the assumptions of Theorem 5.6, there exists $B_\delta \in \mathcal{D}$ which is $\mathcal{D}$-pullback absorbing for the RDS Ξ_δ. In other words, for any given $\omega \in \Omega$ and $D_\delta \in \mathcal{D}$, there exists $t_{0,\delta} := t_{0,\delta}(\omega, D_\delta) \ge 0$, such that*

$$\Xi_\delta(t, \theta_{-t}\omega, D_\delta(\theta_{-t}\omega)) \subset B_\delta(\omega), \quad \textit{for all } t \ge t_{0,\delta}(\omega, D_\delta),$$

where $B_\delta(\omega)$ is the ball in X centered 0 with radius $\rho_\delta(\omega)$,

$$\rho^2_\delta(\omega) = 1 + 2K_2 \int_{-\infty}^0 e^{\gamma s}\Theta_{1,\delta}(\theta_s\omega)ds + 2|\phi|^2|y_\delta(\omega)|^2 + 2\|\phi y_\delta(\theta.\omega)\|^2_{L^2_V}, \tag{5.13}$$

where $\Phi_{1,\delta}(\omega)$ is defined in (5.16) below. In addition, for every $\omega \in \Omega$,

$$\lim_{\delta \to 0} \rho_\delta^2(\omega) = 1 + 2K_2 \int_{-\infty}^0 e^{\gamma s} \Theta_1(\theta_s \omega) ds + 2|\phi|^2 |z_*(\omega)|^2 \\ + 2\|\phi z_*(\theta.\omega)\|_{L_V^2}^2 := \rho^2(\omega). \tag{5.14}$$

Proof Let us first pick $(u_{0,\delta}, \varphi_\delta) \in D_\delta$ and thanks to (5.12), we have

$$\begin{aligned}
&\Xi_\delta(t, \omega, (u_{0,\delta}, \varphi_\delta)) \\
&\quad = (v_\delta(t; 0, \omega, (u_{0,\delta} - \phi y_\delta(\omega), (\mathcal{J}_{\omega,0}\varphi_\delta))), v_{\delta,t}(\cdot; 0, \omega, (u_{0,\delta} - \phi y_\delta(\omega), (\mathcal{J}_{\omega,0}\varphi_\delta))) \\
&\quad + (\phi y_\delta(\theta_t \omega), \phi y_\delta(\theta_{t+\cdot}\omega)) \\
&\quad = \Psi_\delta(t, \omega, (u_{0,\delta} - \phi y_\delta(\omega), \varphi_{v,\delta})) + (\phi y_\delta(\theta_t \omega), \phi y_\delta(\theta_{t+\cdot}\omega)).
\end{aligned}$$

For the sake of simplicity, denote by $z_\delta(t) := (v_\delta(t), \eta_\delta^t(s))$ the solution to (5.10) with initial value $(v_{0,\delta}, \eta_{0,\delta}) = (u_{0,\delta} - \phi y_\delta(\omega), (\mathcal{J}_{\omega,0}\varphi_\delta))$. Now, for every $\omega \in \Omega$, we multiply the first equation of (5.10) by $v_\delta(t)$ in H and the second equation of (5.10) by η_δ^t in $L_\mu^2(\mathbb{R}^+; V)$, respectively. Then, by means of the same estimates as in the proof of Theorem 3.4 (cf. (3.12)) and the Poincaré inequality, we obtain

$$\frac{d}{dt}\|z_\delta(t)\|_{\mathcal{H}}^2 + \frac{m\lambda_1}{2}|v_\delta(t)|^2 + \frac{m}{2}\|v_\delta(t)\|^2 + 2((\eta_\delta^t(s), (\eta_\delta^t(s))'))_\mu \\ + \frac{f_0}{2D}\|v_\delta(t)\|_{2p}^{2p} \le \Theta_{1,\delta}(\theta_t \omega), \tag{5.15}$$

where we have used the notation

$$\Theta_{1,\delta}(\omega) = \frac{4M^2}{m}|y_\delta(\omega)|^2\|\phi\|^2 + \frac{4}{m\lambda_1}|y_\delta(\omega)|^2|\phi|^2 + 2\alpha|\mathcal{O}| + C_{1,\delta}(\omega)(1+\|\phi\|_{2p}^{2p}) \\ + \frac{C_{2,\delta}^2(\omega)}{m}\|\phi\|^2 + \frac{4}{m\lambda_1}|h|^2, \tag{5.16}$$

and $C_{1,\delta}(\omega)$, $C_{2,\delta}(\omega)$, D are the same as the ones in (3.12)–(3.13) but replacing $z_*(\omega)$ by $y_\delta(\omega)$. Taking into account condition (h_2) and recalling that $0 < \gamma < \min\{\frac{m\lambda_1}{2}, \varpi\}$, the above inequality, together with (5.15), implies that

$$\frac{d}{dt}\|z_\delta(t)\|_{\mathcal{H}}^2 + \gamma\|z_\delta(t)\|_{\mathcal{H}}^2 + \frac{m}{2}\|v_\delta(t)\|^2 + \frac{f_0}{2D}\|v_\delta(t)\|_{2p}^{2p} \le \Theta_{1,\delta}(\theta_t \omega). \tag{5.17}$$

Next, multiplying the above inequality by $e^{\gamma t}$ and integrating over $(0, t)$, neglecting the last term on the left hand side of (5.17), by doing similar computations as in

(4.6)–(4.7), we find

$$\begin{aligned}\frac{m}{2}\|v_{\delta,t}\|^2_{L^2_V} &= \frac{m}{2}\int_{-\infty}^{0} e^{-\gamma(t-s)}\|\varphi_{v,\delta}(s)\|^2 ds + \frac{m}{2}\int_0^t e^{-\gamma(t-s)}\|v_\delta(s)\|^2 ds\\ &\leq \frac{m}{2}e^{-\gamma t}\|\varphi_{v,\delta}\|^2_{L^2_V} + \|z_{0,\delta}\|^2_{\mathcal{H}}e^{-\gamma t} + \int_0^t e^{-\gamma(t-s)}\Theta_{1,\delta}(\theta_s\omega)ds.\end{aligned} \tag{5.18}$$

On account of Corollary 5.4, we have

$$\begin{aligned}\|z_{0,\delta}\|^2_{\mathcal{H}} &= |v_{0,\delta}|^2 + \|\mathcal{J}(\varphi_\delta - \phi y_\delta(\theta.\omega))\|^2_{L^2_\mu(\mathbb{R}^+;V)}\\ &\leq |u_{0,\delta} - \phi y_\delta(\omega)|^2 + 2K_\mu\left(\|\varphi_\delta - \phi y_\delta(\theta.\omega)\|^2_{L^2_V}\right).\end{aligned} \tag{5.19}$$

Hence, collecting (5.18)–(5.19), we arrive at

$$\begin{aligned}&\|\Psi_\delta(t,\omega,(u_{0,\delta} - \phi y_\delta(\omega), \mathcal{J}(\varphi_\delta - \phi y_\delta(\theta.\omega))))\|^2_X = |v_\delta(t)|^2 + \|v_{\delta,t}\|^2_{L^2_V}\\ &\leq \|z_\delta(t)\|^2_{\mathcal{H}} + \|v_{\delta,t}\|^2_{L^2_V}\\ &\leq K_1 e^{-\gamma t}\left(|u_{0,\delta} - \phi y_\delta(\omega)|^2 + \|\varphi_\delta - \phi y_\delta(\theta.\omega)\|^2_{L^2_V}\right)\\ &\quad + K_2\int_0^t e^{-\gamma(t-s)}\Theta_{1,\delta}(\theta_s\omega)ds,\end{aligned}$$

where K_1, $K_2 > 0$ are the same constants as in (4.9). Replacing ω by $\theta_{-t}\omega$ in the above inequality, we obtain

$$\begin{aligned}&\|\Psi_\delta(t,\theta_{-t}\omega,(u_{0,\delta} - \phi y_\delta(\theta_{-t}\omega), \mathcal{J}(\varphi_\delta - \phi y_\delta(\theta_{-t+\cdot}\omega))))\|^2_X\\ &\quad\leq K_1 e^{-\gamma t}\left(|u_{0,\delta} - \phi y_\delta(\theta_{-t}\omega)|^2 + \|\varphi_\delta - \phi y_\delta(\theta_{-t+\cdot}\omega)\|^2_{L^2_V}\right)\\ &\qquad + K_2\int_{-\infty}^0 e^{\gamma s}\Theta_{1,\delta}(\theta_s\omega)ds.\end{aligned}$$

Therefore, for any $(u_{0,\delta}, \varphi_\delta) \in D_\delta(\theta_{-t}\omega)$, we have

$$\begin{aligned}&\|\Xi_\delta(t,\theta_{-t}\omega,(u_{0,\delta},\varphi_\delta))\|^2_X\\ &\quad = \|\Psi_\delta(t,\theta_{-t}\omega,(u_{0,\delta} - \phi y_\delta(\theta_{-t}\omega), \mathcal{J}(\varphi_\delta - \phi y_\delta(\theta_{-t+\cdot}\omega))))\\ &\qquad + (\phi y_\delta(\omega), \phi y_\delta(\theta.\omega))\|^2_X\\ &\quad\leq 2K_1 e^{-\gamma t}\|\widetilde{D}_\delta(\theta_{-t}\omega)\|^2 + 2K_2\int_{-\infty}^0 e^{\gamma s}\Theta_{1,\delta}(\theta_s\omega)ds + 2|\phi|^2|y_\delta(\omega)|^2\\ &\qquad + 2\|\phi y_\delta(\theta.\omega)\|^2_{L^2_V}.\end{aligned}$$

Consequently, let

$$\rho_\delta^2(\omega) = 1 + 2K_2 \int_{-\infty}^0 e^{\gamma s}\Theta_{1,\delta}(\theta_s\omega)ds + 2|\phi|^2|y_\delta(\omega)|^2 + 2\|\phi y_\delta(\theta.\omega)\|_{L_V^2}^2.$$

Taking into account the temperedness of $\widetilde{D}_\delta$ and the properties of the Ornstein-Uhlenbeck process, it is straightforward to check that the set,

$$B_{\delta,\rho}(\omega) = \{(u_{0,\delta}, \varphi_\delta) \in X : \|(u_{0,\delta}, \varphi_\delta)\|_X \le \rho_\delta(\omega)\},$$

is tempered, i.e., $B_{\delta,\rho}(\omega)$ belongs to $\mathcal{D}$, and that it is pullback absorbing for this universe $\mathcal{D}$.

We now prove (5.14). To this end, suitable estimates of the functions y_δ, $C_{1,\delta}$, $C_{2,\delta}$ are needed. We see first that (5.6) implies that there exist $r < 0$ and $\delta_0 > 0$, such that for all $0 < \delta < \delta_0$,

$$|y_\delta(\theta_t\omega)| \le |t|, \quad \forall t \le r. \tag{5.20}$$

Let us analyze the functions $C_{1,\delta}(\omega)$ and $C_{2,\delta}(\omega)$. It is clear from (5.5) that when $\delta \to 0$,

$$\begin{aligned} C_{1,\delta}(\theta_t\omega) &= \widetilde{C}_2(1+|y_\delta(\theta_t\omega)|^{4p^2}) \to \widetilde{C}_2(1+|z_*(\theta_t\omega)|^{4p^2}) \\ &= C_1(\theta_t\omega) \text{ uniformly in } [\tau, T],\ \tau < T. \end{aligned} \tag{5.21}$$

Also, (5.20) gives that

$$\left|C_{1,\delta}(\theta_t\omega)\right| \le \widetilde{C}_2\left(1 + |t|^{4p^2}\right),\ \forall t \le r,\ 0 < \delta < \delta_0. \tag{5.22}$$

Furthermore, let us consider the function,

$$C_{2,\delta}(\theta_t\omega) = 2\left(M_1\int_0^1 |y_\delta(\theta_{t-s}\omega)|ds + \frac{\mu(1)e^{\varpi}}{\varpi}\int_1^\infty e^{-\varpi s}|y_\delta(\theta_{t-s}\omega)|ds\right).$$

From (5.5)–(5.6), it is easy to see that

$$C_{2,\delta}(\theta_t\omega) \to C_2(\theta_t\omega) \text{ uniformly in } [\tau, T],\ \tau < T, \ \text{ as } \delta \to 0. \tag{5.23}$$

Finally, (5.20) implies that for $t \le r,\ 0 < \delta < \delta_0$,

$$\begin{aligned} \left|C_{2,\delta}(\theta_t\omega)\right| &\le 2\left(M_1\int_0^1 |t-s|\,ds + \frac{\mu(1)e^{\varpi}}{\varpi}\int_1^\infty e^{-\varpi s}|t-s|ds\right) \\ &\le 2\left(M_1(|t|+1) + \frac{\mu(1)e^{\varpi}}{\varpi}\left(\frac{1}{\varpi}|t| + \int_1^\infty e^{-\varpi s}sds\right)\right). \end{aligned} \tag{5.24}$$

On the one hand, notice that

$$\begin{aligned}
&K_2\int_{-\infty}^{0} e^{\gamma s}\Theta_{1,\delta}(\theta_s\omega)ds+|\phi|^2|y_\delta(\omega)|^2+\|\phi y_\delta(\theta.\omega)\|^2_{L^2_V}\\
&\quad=K_2\int_{-\infty}^{r} e^{\gamma s}\Theta_{1,\delta}(\theta_s\omega)ds+K_2\int_{r}^{0} e^{\gamma s}\Theta_{1,\delta}(\theta_s\omega)ds\\
&\qquad+|\phi|^2|y_\delta(\omega)|^2+\|\phi y_\delta(\theta.\omega)\|^2_{L^2_V}.
\end{aligned}$$

For all $0<\delta<\delta_0$, we obtain by (5.20) and (5.24) that

$$\begin{aligned}
&e^{\gamma s}\Theta_{1,\delta}(\theta_s\omega)ds\\
&\quad=e^{\gamma s}\left(\frac{4M^2}{m}|y_\delta(\theta_s\omega)|^2\|\phi\|^2+\frac{4}{m\lambda_1}|y_\delta(\theta_s\omega)|^2|\phi|^2+2\alpha|\mathcal{O}\,|\right.\\
&\qquad\left.+C_{1,\delta}(\theta_s\omega)(1+\|\phi\|^{2p}_{2p})+\frac{C^2_{2,\delta}(\theta_s\omega)}{m}\|\phi\|^2+\frac{4}{m\lambda_1}\,|h|^2\right)\\
&\quad\le e^{\gamma s}\left(\frac{4M^2}{m}|s|^2\|\phi\|^2+\frac{4}{m\lambda_1}|s|^2|\phi|^2+2\alpha|\mathcal{O}|+\widetilde{C}_2(1+|s|^{4p^2})(1+\|\phi\|^{2p}_{2p})\right.\\
&\qquad\left.+\frac{2}{m}\left(M_1(|s|+1)+\frac{\mu(1)e^{\varpi}}{\varpi}\left(\frac{1}{\varpi}\,|s|+\int_1^{\infty}e^{-\varpi l}l dl\right)\right)\|\phi\|^2+\frac{4}{m\lambda_1}\,|h|^2\right)\\
&\quad:=g(s),\ \text{ for } s\le r,
\end{aligned}\tag{5.25}$$

where $g\in L^1(-\infty,r)$. By means of the above estimates, (5.5), (5.21), (5.23) and the dominated convergence theorem, we find that

$$\lim_{\delta\to0}\int_{-\infty}^{r} e^{\gamma s}\Theta_{1,\delta}(\theta_s\omega)ds=\int_{-\infty}^{r} e^{\gamma s}\Theta_{1}(\theta_s\omega)ds.\tag{5.26}$$

On the other hand, by (5.5), (5.21) and (5.23), we infer that

$$\begin{aligned}
&\lim_{\delta\to0}\int_{r}^{0} e^{\gamma s}\Theta_{1,\delta}(\theta_s\omega)ds\\
&\quad=\lim_{\delta\to0}\int_{r}^{0} e^{\gamma s}\left(\frac{4M^2}{m}|y_\delta(\theta_s\omega)|^2\|\phi\|^2+\frac{4}{m\lambda_1}|y_\delta(\theta_s\omega)|^2|\phi|^2+2\alpha|\mathcal{O}|\right.\\
&\qquad\left.+C_{1,\delta}(\theta_s\omega)(1+\|\phi\|^{2p}_{2p})+\frac{C^2_{2,\delta}(\theta_s\omega)}{m}\|\phi\|^2+\frac{4}{m\lambda_1}\,|h|^2\right)ds\\
&\quad=\int_{r}^{0} e^{\gamma s}\left(\frac{4M^2}{m}|z_*(\theta_s\omega)|^2\|\phi\|^2+\frac{4}{m\lambda_1}|z_*(\theta_s\omega)|^2|\phi|^2+2\alpha|\mathcal{O}|\right.\\
&\qquad\left.+C_{1}(\theta_s\omega)(1+\|\phi\|^{2p}_{2p})+\frac{C^2_{2}(\theta_s\omega)}{m}\|\phi\|^2+\frac{4}{m\lambda_1}\,|h|^2\right)ds
\end{aligned}$$

$$= \int_r^0 e^{\gamma s} \Theta_1(\theta_s \omega) ds.$$

Combining the above inequality with (5.26), we deduce

$$\lim_{\delta \to 0} \int_{-\infty}^0 e^{\gamma s} \Theta_{1,\delta}(\theta_s \omega) ds = \int_{-\infty}^0 e^{\gamma s} \Theta_1(\theta_s \omega) ds. \tag{5.27}$$

Moreover, (5.5)–(5.6) and the dominated convergence theorem imply that,

$$\lim_{\delta \to 0} \left(|\phi|^2 |y_\delta(\omega)|^2 + \|\phi y_\delta(\theta.\omega)\|_{L_V^2}^2 \right) = |\phi|^2 |z_*(\omega)|^2 + \|\phi z_*(\theta.\omega)\|_{L_V^2}^2. \tag{5.28}$$

Therefore, (5.27)–(5.28) finish the proof of this lemma. □

Remark 5.8 It follows from the proof of Lemma 5.7 that, under the assumptions of Theorem 5.5, there exists $\tilde{B}_\delta \in \mathcal{D}$ which is $\mathcal{D}$-pullback absorbing for the RDS Ψ_δ. In other words, for any given $\omega \in \Omega$ and $D_\delta \in \mathcal{D}$, there exists $\tilde{t}_{0,\delta} := \tilde{t}_{0,\delta}(\omega, D_\delta) \geq 0$, such that

$$\Psi_\delta(t, \theta_{-t}\omega, D(\theta_{-t}\omega)) \subset \tilde{B}_\delta(\omega), \quad \text{for all } t \geq \tilde{t}_{0,\delta}(\omega, D_\delta).$$

Next, by means of the same procedure and estimates as in the proof of Lemma 4.3, before stating the asymptotic compactness of the cocycle Ξ_δ, we first need the following auxiliary lemma. Since the details are similar to those in Lemma 4.3, we omit the proof here.

Lemma 5.9 *Assume the hypotheses in Theorem 5.5 hold. Let $\{v_{0,\delta}^n, \varphi_{v,\delta}^n\}$ be a sequence such that $(v_{0,\delta}^n, \varphi_{v,\delta}^n) \to (v_{0,\delta}, \varphi_{v,\delta})$ weakly in X as $n \to \infty$. Then, for every $\omega \in \Omega$, $\Psi_\delta(t, \omega, (v_{0,\delta}^n, \varphi_{v,\delta}^n)) = (v_\delta^n(t), v_{\delta,t}^n(\cdot))$ fulfills:*

$$v_\delta^n \to v_\delta \quad in \quad C([r, T]; H) \ \ for\ all\ \ 0 < r < T; \tag{5.29}$$

$$v_\delta^n \to v_\delta \quad weakly\ in \quad L^2(0, T; V) \ \ for\ all\ \ T > 0; \tag{5.30}$$

$$v_\delta^n \to v_\delta \quad in \quad L^2(0, T; H) \ \ for\ all\ \ T > 0; \tag{5.31}$$

$$\limsup_{n\to\infty} \|v_{\delta,t}^n - v_{\delta,t}\|_{L_V^2}^2 \leq K e^{-\gamma t} \limsup_{n\to\infty} \left(|v_{0,\delta}^n - v_{0,\delta}|^2 + \|\varphi_{v,\delta}^n - \varphi_{v,\delta}\|_{L_V^2}^2 \right)$$
$$for\ all\ \ t \geq 0, \tag{5.32}$$

where $K = (1 + \frac{2K_\mu}{m} + \frac{1}{m})$ and $(v_\delta(t), v_{\delta,t}(\cdot)) = \Psi_\delta(t, \omega, (v_{0,\delta}, \varphi_{v,\delta}))$. Moreover, if $(v_{0,\delta}^n, \varphi_{v,\delta}^n) \to (v_{0,\delta}, \varphi_{v,\delta})$ strongly in X as $n \to \infty$, then

$$v_\delta^n \to v_\delta \quad in \quad L^2(0, T; V)\ for\ all\ \ T > 0; \tag{5.33}$$

$$v_{\delta,t}^n \to v_{\delta,t} \quad in \quad L_V^2 \ \ for\ all\ \ t \geq 0. \tag{5.34}$$

Lemma 5.10 *Suppose that the conditions of Theorem 5.5 hold, then the cocycle Ξ_δ is asymptotically compact.*

As a consequence of Lemmas 5.7, 5.9 and 5.10, we can ensure the existence of a random attractor to problem (5.1).

Theorem 5.11 *Assume that (1.11), (2.8) and* (h_1)-(h_2) *hold,* $\phi \in V \cap H^2(\mathcal{O}) \cap L^{2p}(\mathcal{O})$ *is such that* $\Delta\phi \in L^{2p}(\mathcal{O})$. *Let* $h \in H$ *and* a *be a locally Lipschitz function. Then the cocycle* Ξ_δ *of problem (5.1) has a unique random attractor* $\mathcal{A}_\delta = \{\mathcal{A}_\delta(\omega) : \omega \in \Omega\}$ *in* H.

Let us define the family $\overline{B}$ by

$$\overline{B}(\omega) = \cup_{0<\delta\le\delta_0} B_\delta(\omega),$$

where $\delta_0 \le \widetilde{\sigma}$. Then, the following lemma holds true.

Lemma 5.12 $\overline{B}$ *is tempered for some* $\delta_0 > 0$.

Proof It is enough to check that for any $c > 0$ and $\omega \in \Omega$, we have

$$\lim_{t\to\infty} e^{-ct} \sup_{0<\delta\le\delta_0} \rho_\delta^2(\theta_{-t}\omega) = 0, \tag{5.35}$$

where $\rho_\delta(\omega)$ is the radius of the absorbing ball $B_\delta(\omega)$, defined in (5.13). First, from (5.20), it is clear that

$$\lim_{t\to\infty} e^{-ct} 2|\phi|^2 \sup_{0<\delta\le\delta_0} |y_\delta(\theta_{-t}\omega)|^2 = 0.$$

Since

$$\begin{aligned} 2\|\phi y_\delta(\theta_{-t+\cdot}\omega)\|^2_{L^2_V} &= 2\,\|\phi\|^2 \int_{-\infty}^0 e^{\gamma s}\,|y_\delta(\theta_{-t+s}\omega)|^2\,ds \\ &\le 4\,\|\phi\|^2 \left(t^2 \int_{-\infty}^0 e^{\gamma s} ds + \int_{-\infty}^0 e^{\gamma s}\,|s|^2\,ds\right) = R_1(1+t^2), \end{aligned}$$

for any $t \ge t_0,\ 0 < \delta \le \delta_0$ and some positive constants t_0, δ_0 and R_1, we infer

$$\lim_{t\to\infty} e^{-ct} 2 \sup_{0<\delta\le\delta_0} \|\phi y_\delta(\theta_{-t+\cdot}\omega)\|^2_{L^2_V} = 0.$$

Next, we need to analyze the integral term in $\rho_\delta^2(\omega)$. By (5.25), we deduce there are positive constants R_2, R_3 and t_0, such that for $t \ge t_0$,

$$\begin{aligned} e^{-ct} \int_{-\infty}^0 e^{\gamma s}\Theta_{1,\delta}(\theta_{-t+s}\omega) ds &\le R_2 e^{-ct} \int_{-\infty}^0 e^{\gamma s}\left(1 + |-t+s|^{4p^2}\,ds\right) \\ &\le R_3 e^{-ct}\left(|t|^{2p} \int_{-\infty}^0 e^{\gamma s} ds + \int_{-\infty}^0 e^{\gamma s}\,|s|^{4p^2}\,ds\right) \to 0, \end{aligned}$$

as $t \to +\infty$. Thus, (5.35) holds, namely, $\overline{B}$ is tempered for some $\delta_0 > 0$. □

Corollary 5.13 *The set*

$$\overline{\mathcal{A}}(\omega)=\cup_{0<\delta\leq\delta_0}\mathcal{A}_\delta(\omega),$$

is tempered for some $\delta_0>0$.

6 Upper-Semicontinuity

In this section, we consider the limiting behavior of the random attractor $\mathcal{A}_\delta$ of the stochastic nonlocal PDEs with long time memory driven by colored noise (5.1) when $\delta\to 0$.

First, observe that the nonlinear term f satisfies

$$|f'(u)|\leq\widetilde{\beta}(1+|u|^{2p-2}), \tag{6.1}$$

for some constants $\widetilde{\beta}>0$. The results of this section are valid for a general function $f\in C^1(\mathbb{R})$ satisfying (3.8), (3.18) and (6.1).

Lemma 6.1 *Assume that* (h_1)*-*(h_2) *hold true, and let* $\phi\in V\cap H^2(\mathcal{O})\cap L^{2p}(\mathcal{O})$ *be such that* $\Delta\phi\in L^{2p}(\mathcal{O})$*. Then, for every* $\omega\in\Omega$*, we have*

$$\|\eta_0-\eta_{0,\delta}\|^2_{L^2_\mu(\mathbb{R}^+;V)}\leq 2K_\mu\left(\|\varphi-\varphi_\delta\|^2_{L^2_V}+\|(z_*(\theta.\omega)-y_\delta(\theta.\omega))\phi\|^2_{L^2_V}\right),$$

where φ *and* η_0 *appear in (2.6) and* φ_δ *and* $\eta_{0,\delta}$ *appear in (5.10), respectively.* K_μ *is the same constant as in Corollary 3.2.*

Proof The proof follows the same lines of Corollary 3.2. We omit the details here. □

Lemma 6.2 *Assume that (1.11), (2.8) and* (h_1) *-*(h_2) *hold. Let* $\phi\in V\cap H^2(\mathcal{O})\cap L^{2p}(\mathcal{O})$ *be such that* $\Delta\phi\in L^{2p}(\mathcal{O})$*, let a be a locally Lipschitz function. Suppose* u_δ *and u be solutions to problems (5.1) and (1.10) with initial data* $u_{0,\delta}$ *and* u_0 *in* H*, and the initial functions* φ_δ *and* φ *in* L^2_V*, respectively. Then, for every* $\omega\in\Omega$*,* $T>0$ *and* $\varepsilon\in(0,1)$*, there exists* $\delta_0=\delta_0(\omega,T,\varepsilon)$ *and* $c=c(\omega,T,\varepsilon,\sup_{t\in[0,T]}|z_*(\theta_t\omega)|,\phi)$ *such that, for all* $0<\delta<\delta_0$ *and* $t\in[0,T]$*,*

$$\begin{aligned}&\|(u_\delta(t;\omega,(u_{0,\delta},\varphi_\delta)),u_{\delta,t}(\cdot;\omega,(u_{0,\delta},\varphi_\delta)))-(u(t;\omega,(u_0,\varphi)),u_t(\cdot;\omega,(u_0,\varphi)))\|^2_X\\&\leq c\|(u_{0,\delta},\varphi_\delta)-(u_0,\varphi)\|^2_X+c\varepsilon\Bigg(1+|u_0|^2+|u_{0,\delta}|^2+\|\varphi\|^2_{L^2_V}+\|\varphi_\delta\|^2_{L^2_V}\\&\quad+\int_0^t\Theta_1(\theta_r\omega)dr+\int_0^t\Theta_{1,\delta}(\theta_r\omega)dr\Bigg),\end{aligned} \tag{6.2}$$

where $\Theta_1(\omega)$ *and* $\Theta_{1,\delta}(\omega)$ *are the same constants as in (3.13) and (5.16), respectively.*

Proof Let $\xi_\delta = v - v_\delta, \theta_\delta^t = \eta^t - \eta_\delta^t$ and $q_\delta = (\xi_\delta, \theta_\delta^t)$ with $q_{0,\delta} = (v_0 - v_{0,\delta}, \eta_0 - \eta_{0,\delta})$. By (2.6) and (5.10), we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|\xi_\delta|^2+\frac{1}{2}\frac{d}{dt}\|\theta_\delta^t\|_\mu^2+(-a(l(v)+l(\phi)z_*(\theta_t\omega))\Delta v+a(l(v_\delta)+l(\phi)y_\delta(\theta_t\omega))\Delta v_\delta,\xi_\delta)\\
&\quad+(-a(l(v)+l(\phi)z_*(\theta_t\omega))z_*(\theta_t\omega)\Delta\phi+a(l(v_\delta)+l(\phi)y_\delta(\theta_t\omega))y_\delta(\theta_t\omega)\Delta\phi,\xi_\delta)\\
&\quad+(f(v+\phi z_*(\theta_t\omega))-f(v_\delta+\phi y_\delta(\theta_t\omega)),\xi_\delta)\\
&=(z_*(\theta_t\omega)-y_\delta(\theta_t\omega))(\phi,\xi_\delta)+(z_k^\phi(t,\omega)-z_{k,\delta}^\phi(t,\omega),\xi_\delta)-(((\theta_\delta^t)',\theta_\delta^t))_\mu. \qquad (6.3)
\end{aligned}$$

For the third term of left hand side of the above equality, we have

$$\begin{aligned}
&(-a(l(v)+l(\phi)z_*(\theta_t\omega))\Delta v+a(l(v_\delta)+l(\phi)y_\delta(\theta_t\omega))\Delta v_\delta,\xi_\delta)\\
&\quad=(-a(l(v)+l(\phi)z_*(\theta_t\omega))\Delta v+a(l(v)+l(\phi)z_*(\theta_t\omega))\Delta v_\delta,\xi_\delta)\\
&\qquad+(-a(l(v)+l(\phi)z_*(\theta_t\omega))\Delta v_\delta+a(l(v_\delta)+l(\phi)y_\delta(\theta_t\omega))\Delta v_\delta,\xi_\delta)\\
&\quad\geq a(l(v)+l(\phi)z_*(\theta_t\omega))\|\xi_\delta\|^2\\
&\qquad-|(-a(l(v)+l(\phi)z_*(\theta_t\omega))+a(l(v_\delta)+l(\phi)y_\delta(\theta_t\omega)))|\,\|v_\delta\|\|\xi_\delta\|. \qquad (6.4)
\end{aligned}$$

For the fourth term of left hand side of Eq. (6.3), by the Lipschitz condition of the function a and (1.11), we deduce

$$\begin{aligned}
&(-a(l(v)+l(\phi)z_*(\theta_t\omega))z_*(\theta_t\omega)\Delta\phi+a(l(v_\delta)+l(\phi)y_\delta(\theta_t\omega))y_\delta(\theta_t\omega)\Delta\phi,\xi_\delta)\\
&\quad=(-a(l(v)+l(\phi)z_*(\theta_t\omega))z_*(\theta_t\omega)\Delta\phi+a(l(v)+l(\phi)z_*(\theta_t\omega))y_\delta(\theta_t\omega)\Delta\phi,\xi_\delta)\\
&\qquad+(-a(l(v)+l(\phi)z_*(\theta_t\omega))y_\delta(\theta_t\omega)\Delta\phi+a(l(v_\delta)+l(\phi)z_*(\theta_t\omega))y_\delta(\theta_t\omega)\Delta\phi,\xi_\delta)\\
&\qquad+(-a(l(v_\delta)+l(\phi)z_*(\theta_t\omega))y_\delta(\theta_t\omega)\Delta\phi+a(l(v_\delta)+l(\phi)y_\delta(\theta_t\omega))y_\delta(\theta_t\omega)\Delta\phi,\xi_\delta)\\
&\quad=a(l(v)+l(\phi)z_*(\theta_t\omega))(z_*(\theta_t\omega)-y_\delta(\theta_t\omega))(\nabla\phi,\nabla\xi_\delta)\\
&\qquad+(-a(l(v)+l(\phi)z_*(\theta_t\omega))y_\delta(\theta_t\omega)\Delta\phi+a(l(v_\delta)+l(\phi)z_*(\theta_t\omega))y_\delta(\theta_t\omega)\Delta\phi,\xi_\delta)\\
&\qquad+(-a(l(v_\delta)+l(\phi)z_*(\theta_t\omega))y_\delta(\theta_t\omega)\Delta\phi+a(l(v_\delta)+l(\phi)y_\delta(\theta_t\omega))y_\delta(\theta_t\omega)\Delta\phi,\xi_\delta)\\
&\quad\leq M|z_*(\theta_t\omega)-y_\delta(\theta_t\omega)|\|\phi\|\|\xi_\delta\|+L_a(R)|l||y_\delta(\theta_t\omega)||\xi_\delta|\|\phi\|\|\xi_\delta\|\\
&\qquad+L_a(R)|l||\phi||z_*(\theta_t\omega)-y_\delta(\theta_t\omega)||y_\delta(\theta_t\omega)|\|\phi\|\|\xi_\delta\|, \qquad (6.5)
\end{aligned}$$

for some $R > 0$, which is chosen in a similar way as in the proof of Lemma 4.3 as v_δ are bounded in $C([0,T],H)$ (see Lemma 5.7). For the last term of left hand side of Eq. (6.3), we have

$$\begin{aligned}
&\int_{\mathcal{O}}(f(v+\phi z_*(\theta_t\omega))-f(v_\delta+\phi y_\delta(\theta_t\omega)))\,\xi_\delta dx\\
&\quad=\int_{\mathcal{O}}(f(v+\phi z_*(\theta_t\omega))-f(v+\phi y_\delta(\theta_t\omega)))\,\xi_\delta dx\\
&\qquad+\int_{\mathcal{O}}(f(v+\phi y_\delta(\theta_t\omega))-f(v_\delta+\phi y_\delta(\theta_t\omega)))\,\xi_\delta dx
\end{aligned}$$

$$= \int_{\mathcal{O}} f'(\vartheta_1(x))\xi_\delta\phi(z_*(\theta_t\omega) - y_\delta(\theta_t\omega))dx + \int_{\mathcal{O}} f'(\vartheta_2(x))|\xi_\delta|^2 dx, \quad (6.6)$$

where $\vartheta_1(x) = v(x) + \theta_1(x)\phi(x)z_*(\theta_t\omega) + (1-\theta_1(x))\phi(x)y_\delta(\theta_t\omega)$, $\vartheta_2(x) = \phi(x)y_\delta(\theta_t\omega) + \theta_2(x)v_\delta(x) + (1-\theta_2(x))v(x)$, $\theta_i(x) \in [0,1]$. Then, recalling that $f'(u) \geq -\frac{\sigma}{2}$ for some $\sigma > 0$ (cf. (3.18)), by means of (6.1) and the Young inequality, we obtain that there exist constants $\widetilde{c}, c > 0$, such that

$$\begin{aligned}
&\int_{\mathcal{O}} (f(v + \phi z_*(\theta_t\omega)) - f(v_\delta + \phi y_\delta(\theta_t\omega)))\,\xi_\delta dx \\
&\quad \geq -\widetilde{c}\int_{\mathcal{O}} \left(1 + |v|^{2p-2} + |\phi z_*(\theta_t\omega)|^{2p-2} + |\phi y_\delta(\theta_t\omega)|^{2p-2}\right)|\xi_\delta|\,|\phi|\,|z_*(\theta_t\omega) \\
&\qquad - y_\delta(\theta_t\omega)|dx - \frac{\sigma}{2}\,|\xi_\delta|^2 \\
&\quad \geq -c|\xi_\delta||\phi||z_*(\theta_t\omega) - y_\delta(\theta_t\omega)| - c\left(|z_*(\theta_t\omega)|^{2p-2} + |y_\delta(\theta_t\omega)|^{2p-2} + 1\right) \\
&\qquad \times \left(\|\phi\|_{2p}^{2p} + \|\xi_\delta\|_{2p}^{2p} + \|v\|_{2p}^{2p}\right)|z_*(\theta_t\omega) - y_\delta(\theta_t\omega)| - \frac{\sigma}{2}\,|\xi_\delta|^2\,. \qquad (6.7)
\end{aligned}$$

For the first term of right hand side of (6.3), by the Young inequality, we infer

$$(z_*(\theta_t\omega) - y_\delta(\theta_t\omega))(\phi, \xi_\delta) \leq c|z_*(\theta_t\omega) - y_\delta(\theta_t\omega)|^2|\phi|^2 + |\xi_\delta|^2. \quad (6.8)$$

For the second term of right hand side of (6.3), we deduce that

$$(z_k^\phi(t,\omega) - z_{k,\delta}^\phi(t,\omega), \xi_\delta) = \int_{-\infty}^t k(t-s)(z_*(\theta_s\omega) - y_\delta(\theta_s\omega))ds(\Delta\phi, \xi_\delta). \quad (6.9)$$

It follows from (2.2) that for any $\varepsilon > 0$, there exists $T_1 < 0$ such that for all $t \leq T_1$,

$$|z_*(\theta_t\omega)| \leq \varepsilon|t|.$$

Similarly, (5.6) implies that for any $\varepsilon > 0$, there exists $T_2 < 0$ such that for all $t \leq T_2$, $0 < \delta \leq \widetilde{\sigma}$,

$$|y_\delta(\theta_t\omega)| \leq \varepsilon|t|.$$

Notice that

$$\begin{aligned}
\int_{-\infty}^t k(t-s)|z_*(\theta_s\omega) - y_\delta(\theta_s\omega)|ds &= \int_{-\infty}^T k(t-s)|z_*(\theta_s\omega) - y_\delta(\theta_s\omega)|ds \\
&\quad + \int_T^t k(t-s)|z_*(\theta_s\omega) - y_\delta(\theta_s\omega)|ds.
\end{aligned} \qquad (6.10)$$

On the one hand, let $T=\min\{T_1,T_2\}$. We can assume that $t-T\geq 1$. By Remark 2.1, we arrive at

$$\begin{aligned}
&\int_{-\infty}^{T} k(t-s)|z_*(\theta_s\omega)-y_\delta(\theta_s\omega)|ds\\
&\quad\leq \int_{-\infty}^{T} k(t-s)|z_*(\theta_s\omega)|ds+\int_{-\infty}^{T} k(t-s)|y_\delta(\theta_s\omega)|ds\\
&\quad\leq 2\varepsilon\int_{-\infty}^{T} k(t-s)|s|ds=2\varepsilon\int_{t-T}^{\infty} k(s)|t-s|ds\\
&\quad\leq 2\varepsilon\int_{1}^{\infty}\frac{\mu(1)e^{-\varpi(s-1)}}{\varpi}|t-s|ds\leq c\varepsilon. \qquad (6.11)
\end{aligned}$$

On the other hand, by the continuity of $z_*(\theta_t\omega)$ and $y_\delta(\theta_t\omega)$ with respect to t, together with Remark 2.1, we obtain

$$\int_{T}^{t} k(t-s)|z_*(\theta_s\omega)-y_\delta(\theta_s\omega)|ds<\infty. \qquad (6.12)$$

Collecting (6.10)–(6.12), it is obvious that (6.9) can be bounded by

$$\begin{aligned}
&(z_k^{\phi}(t,\omega)-z_{k,\delta}^{\phi}(t,\omega),\xi_\delta)\\
&\quad\leq c\varepsilon\|\phi\|\|\xi_\delta\|+\int_{T}^{t} k(t-s)|z_*(\theta_s\omega)-y_\delta(\theta_s\omega)|ds\|\phi\|\|\xi_\delta\|. \qquad (6.13)
\end{aligned}$$

Finally, for the last term of (6.3), similar to (4.4), we find

$$-(((\theta_\delta^t)',\theta_\delta^t))_\mu=\int_0^\infty \mu'(s)|\nabla\theta_\delta^t(s)|^2ds\leq-\frac{\varpi}{2}\|\theta_\delta^t\|_\mu^2. \qquad (6.14)$$

Substituting (6.4)–(6.8) and (6.13)–(6.14) into (6.3), by (1.11), we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|q_\delta\|_{\mathcal{H}}^2+m\|\xi_\delta\|^2+\frac{\varpi}{2}\|\theta_\delta^t\|_\mu^2\\
&\quad\leq|-a(l(v)+l(\phi)z_*(\theta_t\omega))+a(l(v_\delta)+l(\phi)y_\delta(\theta_t\omega))|\|v_\delta\|\|\xi_\delta\|+M|z_*(\theta_t\omega)\\
&\qquad-y_\delta(\theta_t\omega)|\|\phi\|\|\xi_\delta\|+\frac{\sigma}{2}|\xi_\delta|^2+L_a(R)|l||y_\delta(\theta_t\omega)||\xi_\delta|\|\phi\|\|\xi_\delta\|\\
&\qquad+L_a(R)|l||\phi||z_*(\theta_t\omega)-y_\delta(\theta_t\omega)||y_\delta(\theta_t\omega)|\|\phi\|\|\xi_\delta\|\\
&\qquad+c|\xi_\delta||\phi||z_*(\theta_t\omega)-y_\delta(\theta_t\omega)|+|\xi_\delta|^2+c\varepsilon\|\phi\|\|\xi_\delta\|\\
&\qquad+c\left(|z_*(\theta_t\omega)|^{2p-2}+|y_\delta(\theta_t\omega)|^{2p-2}+1\right)\left(\|\phi\|_{2p}^{2p}+\|\xi_\delta\|_{2p}^{2p}+\|v\|_{2p}^{2p}\right)|z_*(\theta_t\omega)\\
&\qquad-y_\delta(\theta_t\omega)|+c|z_*(\theta_t\omega)-y_\delta(\theta_t\omega)|^2|\phi|^2\\
&\qquad+\int_{T}^{t} k(t-s)|z_*(\theta_s\omega)-y_\delta(\theta_s\omega)|ds\|\phi\|\|\xi_\delta\|.
\end{aligned}$$

By the Young inequality and the fact that a is locally Lipschitz, we derive

$$
\begin{aligned}
&\frac{d}{dt}\|q_\delta\|^2_{\mathcal{H}} + 2m\|\xi_\delta\|^2 + \varpi\|\theta^t_\delta\|^2_\mu \\
&\quad\le c|\xi_\delta|^2\|v_\delta\|^2 + \frac{m}{4}\|\xi_\delta\|^2 \\
&\qquad + c|\phi|^2|z_*(\theta_t\omega) - y_\delta(\theta_t\omega)|^2\|v_\delta\|^2 + \frac{m}{4}\|\xi_\delta\|^2 + c\|\phi\|^2|z_*(\theta_t\omega) \\
&\qquad - y_\delta(\theta_t\omega)|^2 + \frac{m}{4}\|\xi_\delta\|^2 + c|\xi_\delta|^2|y_\delta(\theta_t\omega)|^2\|\phi\|^2 + \frac{m}{4}\|\xi_\delta\|^2 \\
&\qquad + c|\phi|^2|y_\delta(\theta_t\omega)|^2\|\phi\|^2|z_*(\theta_t\omega) - y_\delta(\theta_t\omega)|^2 + \frac{m}{4}\|\xi_\delta\|^2 \\
&\qquad + \sigma|\xi_\delta|^2 + c|\phi|^2|z_*(\theta_t\omega) - y_\delta(\theta_t\omega)|^2 + 3|\xi_\delta|^2 \\
&\qquad + c\big(|z_*(\theta_t\omega)|^{2p-2} + |y_\delta(\theta_t\omega)|^{2p-2} + 1\big)\left(\|\phi\|^{2p}_{2p} + \|\xi_\delta\|^{2p}_{2p} + \|v\|^{2p}_{2p}\right)|z_*(\theta_t\omega) \\
&\qquad - y_\delta(\theta_t\omega)| + \frac{m}{4}\|\xi_\delta\|^2 + c\left(\int_T^t k(t-s)|z_*(\theta_s\omega) - y_\delta(\theta_s\omega)|ds\right)^2\|\phi\|^2 \\
&\qquad + \frac{m}{4}\|\xi_\delta\|^2 + c\varepsilon^2\,\|\phi\|^2 .
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\frac{d}{dt}\|q_\delta\|^2_{\mathcal{H}} + \frac{m}{4}\,\|\xi_\delta\|^2 + \varpi\|\theta^t_\delta\|^2_\mu \\
&\quad\le \left(c\|v_\delta\|^2 + \sigma + 3 + c|y_\delta(\theta_t\omega)|^2\|\phi\|^2\right)|\xi_\delta|^2 \\
&\qquad + c\left(|\phi|^2\|v_\delta\|^2 + \|\phi\|^2 + |\phi|^2|y_\delta(\theta_t\omega)|^2\|\phi\|^2 + |\phi|^2\right)|z_*(\theta_t\omega) - y_\delta(\theta_t\omega)|^2 \\
&\qquad + c\left(|z_*(\theta_t\omega)|^{2p-2} + |y_\delta(\theta_t\omega)|^{2p-2} + 1\right)\left(\|\phi\|^{2p}_{2p} + \|\xi_\delta\|^{2p}_{2p} + \|v\|^{2p}_{2p}\right) \\
&\qquad |z_*(\theta_t\omega) - y_\delta(\theta_t\omega)| \\
&\qquad + c\varepsilon^2\|\phi\|^2 + c\left(\int_T^t k(t-s)|z_*(\theta_s\omega) - y_\delta(\theta_s\omega)|ds\right)^2\|\phi\|^2. \qquad (6.15)
\end{aligned}
$$

With the help of (5.5), for every $\varepsilon > 0$, there exists $\delta_0 = \delta_0(\omega, T, \varepsilon) > 0$ such that for all $0 < \delta < \delta_0$ and $t \in [0, T]$,

$$
|z_*(\theta_t\omega) - y_\delta(\theta_t\omega)| \le \varepsilon. \qquad (6.16)
$$

Notice that, thanks to this fact, there exists a constant $c := c(\omega, T, \varepsilon)$ such that,

$$
\sup_{t\in[0,T]} |y_\delta(\theta_t\omega)| \le c \sup_{t\in[0,T]} |z_*(\theta_t\omega)|, \quad \text{for all } 0 < \delta < \delta_0. \qquad (6.17)
$$

Hence, it follows from Remark 2.1 that

$$\begin{aligned}
&\frac{d}{dt}\|q_\delta\|_{\mathcal{H}}^2 + \frac{m}{4}\|\xi_\delta\|^2 \\
&\le c\left(\|v_\delta\|^2 + 1 + |y_\delta(\theta_t\omega)|^2\|\phi\|^2\right)\|q_\delta\|_{\mathcal{H}}^2 \\
&\quad + c\varepsilon^2\left(|\phi|^2\|v_\delta\|^2 + \|\phi\|^2 + |\phi|^2|y_\delta(\theta_t\omega)|^2\|\phi\|^2 + |\phi|^2\right) \\
&\quad + c\varepsilon\left(|z_*(\theta_t\omega)|^{2p-2} + |y_\delta(\theta_t\omega)|^{2p-2} + 1\right)\left(\|\phi\|_{2p}^{2p} + \|\xi_\delta\|_{2p}^{2p} + \|v\|_{2p}^{2p}\right). \qquad (6.18)
\end{aligned}$$

Multiplying by $e^{-c\int_0^t(\|v_\delta\|^2+1+|y_\delta(\theta_\tau\omega)|^2\|\phi\|^2)d\tau}$ and integrating in (6.18), we deduce that, for all $0 < \delta < \delta_0$ and $t \in [0, T]$,

$$\begin{aligned}
&\|q_\delta\|_{\mathcal{H}}^2 + \frac{m}{4}\int_0^t e^{c\int_s^t(\|v_\delta\|^2+1+|y_\delta(\theta_\tau\omega)|^2\|\phi\|^2)d\tau}\|\xi_\delta\|^2 ds \\
&\le e^{c\int_0^t(\|v_\delta\|^2+1+|y_\delta(\theta_s\omega)|^2\|\phi\|^2)ds}\Bigg(\|q_{0,\delta}\|_{\mathcal{H}}^2 + \int_0^t e^{-c\int_0^s(\|v_\delta\|^2+1+|y_\delta(\theta_\tau\omega)|^2\|\phi\|^2)d\tau} \qquad (6.19) \\
&\times\Bigg(c\varepsilon^2\left(|\phi|^2\|v_\delta\|^2 + \|\phi\|^2 + |\phi|^2|y_\delta(\theta_s\omega)|^2\|\phi\|^2 + |\phi|^2\right) \\
&+ c\varepsilon\left(|z_*(\theta_s\omega)|^{2p-2} + |y_\delta(\theta_s\omega)|^{2p-2} + 1\right)\left(\|\phi\|_{2p}^{2p} + \|\xi_\delta\|_{2p}^{2p} + \|v\|_{2p}^{2p}\right)\Bigg)ds\Bigg).
\end{aligned}$$

In view of (5.17), $\phi \in V$ and (6.17), there is $c = c(T, \omega, \varepsilon, \|\phi\|)$ such that,

$$\int_0^T\left(\|v_\delta\|^2 + 1 + |y_\delta(\theta_s\omega)|^2\|\phi\|^2\right)ds \le c, \quad \text{if } 0 < \delta < \delta_0.$$

By (6.19), (4.5) and (5.17), there exist $\delta_1 \in (0, \delta_0)$ and $c := c(\omega, T, \varepsilon, \phi, \sup_{t\in[0,T]}|z_*(\theta_t\omega)|)$ such that, for all $0 < \delta < \delta_1$ and $t \in [0, T]$,

$$\begin{aligned}
\|q_\delta\|_{\mathcal{H}}^2 + \frac{m}{4}\int_0^t\|\xi_\delta\|^2 ds \le{}& c\left(|v_0 - v_{0,\delta}|^2 + \|\eta_0 - \eta_{0,\delta}\|_\mu^2\right) \\
&+ c\varepsilon\Bigg(1 + |v_0|^2 + |v_{0,\delta}|^2 + \|\eta_0\|_\mu^2 + \|\eta_{0,\delta}\|_\mu^2 \\
&+ \int_0^t\Theta_1(\theta_r\omega)dr + \int_0^t\Theta_{1,\delta}(\theta_r\omega)dr\Bigg).
\end{aligned}$$

Notice that

$$u_\delta(t;\omega,u_{0,\delta}) - u(t;\omega,u_0) = v_\delta(t;\omega,v_{0,\delta}) - v(t;\omega,v_0) + \phi y_\delta(\theta_t\omega) - \phi z_*(\theta_t\omega), \qquad (6.20)$$

where $u_{0,\delta} = v_{0,\delta} + \phi y_\delta(\omega)$ and $u_0 = v_0 + \phi z_*(\omega)$. It follows from the above equations, corollaries 3.2 and 5.4, Lemma 6.1 and (6.16), that there exist $\delta_2 \in (0, \delta_1)$

and $c := c(\omega, T, \varepsilon, \phi, \sup_{t\in[0,T]} |z_*(\theta_t\omega)|)$, such that for all $0 < \delta < \delta_2$ and $t \in [0, T]$,

$$\begin{aligned}
&|u_\delta - u|^2 + \int_0^t \|\xi_\delta\|^2\, ds \\
&\quad \le c\left(|u_0 - u_{0,\delta}|^2 + \|\varphi - \varphi_\delta\|^2_{L^2_V}\right) \\
&\qquad + c\varepsilon\left(1 + |u_0|^2 + |u_{0,\delta}|^2 + \|\varphi\|^2_{L^2_V} + \|\varphi_\delta\|^2_{L^2_V} + \int_0^t \Theta_1(\theta_r\omega)dr\right. \\
&\qquad \left. + \int_0^t \Theta_{1,\delta}(\theta_r\omega)dr\right).
\end{aligned} \tag{6.21}$$

By (6.16) and (6.20)–(6.21), we obtain

$$\begin{aligned}
&\int_0^t \|u_\delta(s) - u(s)\|^2 ds \\
&\quad \le 2\int_0^t \|\xi_\delta(s)\|^2 ds + 2\int_0^t \|\phi\|^2 |y_\delta(\theta_s\omega) - z_*(\theta_s\omega)|^2 ds \\
&\quad \le c\left(|u_0 - u_{0,\delta}|^2 + \|\varphi - \varphi_\delta\|^2_{L^2_V}\right) \\
&\qquad + c\varepsilon\left(1 + |u_0|^2 + |u_{0,\delta}|^2 + \|\varphi\|^2_{L^2_V} + \|\varphi_\delta\|^2_{L^2_V} + \int_0^t \Theta_1(\theta_r\omega)dr\right. \\
&\qquad \left. + \int_0^t \Theta_{1,\delta}(\theta_r\omega)dr\right).
\end{aligned} \tag{6.22}$$

Hence, for every $\omega \in \Omega$ and $t \in [0, T]$, we have

$$\begin{aligned}
&\|u_{\delta,t} - u_t\|^2_{L^2_V} = \int_{-\infty}^0 e^{\gamma s}\|u_\delta(t+s) - u(t+s)\|^2 ds \\
&= \int_{-\infty}^0 e^{-\gamma(t-s)}\|\varphi_\delta(s) - \varphi(s)\|^2 ds + \int_0^t e^{-\gamma(t-s)}\|u_\delta(s) - u(s)\|^2 ds \\
&\le \|\varphi_\delta - \varphi\|^2_{L^2_V} + \int_0^t \|u_\delta(s) - u(s)\|^2 ds \\
&\le c\left(|u_0 - u_{0,\delta}|^2 + \|\varphi - \varphi_\delta\|^2_{L^2_V}\right) \\
&\quad + c\varepsilon\left(1 + |u_0|^2 + |u_{0,\delta}|^2 + \|\varphi\|^2_{L^2_V} + \|\varphi_\delta\|^2_{L^2_V} + \int_0^t \Theta_1(\theta_r\omega)dr\right. \\
&\quad \left. + \int_0^t \Theta_{1,\delta}(\theta_r\omega)dr\right),
\end{aligned} \tag{6.23}$$

which, together with (6.21), finishes the proof of this lemma. □

Remark 6.3 The constant c depends continuously on ε in Lemma 6.2.

As a consequence of Lemma 6.2, we obtain the following covergence of solutions to (5.1) when δ approaches to zero.

Corollary 6.4 *Assume that (1.11), (2.8),* (h_1)-(h_2) *hold and* $\delta_n \to 0$ *as* $n \to \infty$. *Let* $\phi \in V \cap H^2(\mathcal{O}) \cap L^{2p}(\mathcal{O})$ *be such that* $\Delta\phi \in L^{2p}(\mathcal{O})$ *and a be a locally Lipschitz function. Suppose that* u_{δ_n} *and* u *are the solutions of (5.1) and (2.6) with initial data* u_{0,δ_n} *and* u_0 *in* H, *and the initial functions* φ_{δ_n} *and* φ *in* L^2_V, *respectively. If* $u_{0,\delta_n} \to u_0$ *in* H *and* $\varphi_{\delta_n} \to \varphi$ *in* L^2_V *as* $n \to \infty$, *then for every* $\omega \in \Omega$ *and* $t > 0$,

$$u_{\delta_n}(t;\omega,(u_{0,\delta_n},\varphi_{\delta_n})) \to u(t;\omega,(u_0,\varphi)) \quad in \quad H, \quad as \quad n\to\infty,$$

and

$$u_{\delta_n,t}(\cdot;\omega,(u_{0,\delta_n},\varphi_{\delta_n})) \to u_t(\cdot;\omega,(u_0,\varphi)) \quad in \quad L^2_V, \quad as \quad n\to\infty.$$

The above convergence is uniform with respect to $t \in [0, T]$.

We also need the following weak convergence of solutions to prove the upper-semicontinuity of random attractors in this section.

Lemma 6.5 *Under assumptions of Corollary 6.4, suppose that* $\{\delta_n\}_{n=1}^\infty$ *is a sequence such that* $\delta_n \to 0$ *as* $n \to \infty$. *Let* v_{δ_n} *and* v *be the solutions of (5.8) and (2.4) with initial data* v_{0,δ_n} *and* v_0 *in* H, *and the initial functions* $\varphi_{v,\delta_n} = \varphi_{\delta_n} - \phi y_{\delta_n}(\theta.\omega)$ *and* $\varphi_v = \varphi - \phi z_*(\theta.\omega)$, *respectively. If* $v_{0,\delta_n} \to v_0$ *weakly in* H *and* $\varphi_{v,\delta_n} \to \varphi_v$ *weakly in* L^2_V *as* $n \to \infty$. *Then for every* $\omega \in \Omega$,

$$\begin{aligned}&\Big(v_{\delta_n}(r;0,\omega,(v_{0,\delta_n},\mathcal{J}^{\delta_n}_{\omega,0}\varphi_{\delta_n})), v_{\delta_n,r}(\cdot;0,\omega,(v_{0,\delta_n},\mathcal{J}^{\delta_n}_{\omega,0}\varphi_{\delta_n}))\Big)\\ &\quad\to \big(v(r;0,\omega,(v_0,\mathcal{J}_{\omega,0}\varphi)), v_r(\cdot;0,\omega,(v_0,\mathcal{J}_{\omega,0}\varphi))\big) \;\; weakly\ in \;\; X, \quad \forall r\ge 0;\end{aligned} \tag{6.24}$$

$$\begin{aligned}&v_{\delta_n}(\cdot;0,\omega,(v_{0,\delta_n},\mathcal{J}^{\delta_n}_{\omega,0}\varphi_{\delta_n}))\\ &\to v(\cdot;0,\omega,(v_0,\mathcal{J}_{\omega,0}\varphi)) \;\; weakly\ in \;\; L^{2p}(0,T;L^{2p}(\mathcal{O})), \quad \forall T>0;\end{aligned} \tag{6.25}$$

$$\begin{aligned}&v_{\delta_n}(\cdot;0,\omega,(v_{0,\delta_n},\mathcal{J}^{\delta_n}_{\omega,0}\varphi_{\delta_n}))\\ &\to v(\cdot;0,\omega,(v_0,\mathcal{J}_{\omega,0}\varphi)) \;\; weakly\ in \;\; L^2(0,T;V), \quad \forall T>0;\end{aligned} \tag{6.26}$$

$$\eta_{0,\delta_n} := \mathcal{J}^{\delta_n}_{\omega,0}\varphi_{\delta_n} \to \eta_0 := \mathcal{J}_{\omega,0}\varphi \;\; weakly\ in \;\; L^2_\mu(\mathbb{R}^+;V), \tag{6.27}$$

as $n \to \infty$.

Proof The proof of this lemma follows the same arguments as [19, Lemma 3.5], so we omit the details here. □

Recall that for each $\delta > 0$, $\mathcal{A}_\delta$ is the unique $\mathcal{D}$-random attractor of Ξ_δ in X. To establish the upper semicontinuity of these attractors as $\delta \to 0$, we need the following compactness result.

Lemma 6.6 *Suppose that conditions of Corollary 6.4 hold. Let* $\omega \in \Omega$ *be fixed. If* $\delta_n \to 0$ *as* $n \to \infty$ *and* $(u^n, \varphi^n) \in \mathcal{A}_{\delta_n}(\omega)$, *then the sequence* $\{(u^n, \varphi^n)\}_{n=1}^\infty$ *has a convergent subsequence in* X.

Proof Since $\delta_n \to 0$, by (5.14) we find that for every $\omega \in \Omega$, there exists $N_1 = N_1(\omega)$ such that for all $n \geq N_1$,

$$\rho^2_{\delta_n}(\omega) \leq 2\rho^2(\omega). \tag{6.28}$$

Due to $(u^n, \varphi^n) \in \mathcal{A}_{\delta_n}(\omega)$ and $\mathcal{A}_{\delta_n}(\omega) \subset B_{\delta_n}(0, \rho^2_{\delta_n}(\omega))$, by (6.28) we obtain that, for all $n \geq N_1$,

$$\|(u^n, \varphi^n)\|^2_X \leq 2\rho^2(\omega). \tag{6.29}$$

It follows from (6.29) that the sequence $\{(u^n, \varphi^n)\}_{n=1}^{\infty}$ is bounded in X, hence, there exists $(u_0, \varphi_0) \in X$ such that, up to a subsequence,

$$(u^n, \varphi^n) \to (u_0, \varphi_0), \quad \text{weakly in } X \text{ as } \ n \to \infty. \tag{6.30}$$

In what follows, we will prove that the weak convergence in (6.30) is actually a strong one in X. Since $(u^n, \varphi^n) \in \mathcal{A}_{\delta_n}(\omega)$, by the invariance of $\mathcal{A}_{\delta_n}$, for every $k \geq 1$, there exists $(u^{n,k}, \varphi^{n,k}) \in \mathcal{A}_{\delta_n}(\theta_{-k}\omega)$ such that,

$$\begin{aligned}(u^n, \varphi^n) &= \Xi_{\delta_n}(k, \theta_{-k}\omega, (u^{n,k}, \varphi^{n,k}))\\ &= (u_{\delta_n}(0; -k, \omega, (u^{n,k}, \varphi^{n,k})), u_{\delta_n,0}(\cdot; -k, \omega, (u^{n,k}, \varphi^{n,k}))).\end{aligned} \tag{6.31}$$

On the one hand, since $(u^{n,k}, \varphi^{n,k}) \in \mathcal{A}_{\delta_n}(\theta_{-k}\omega)$ and $\mathcal{A}_{\delta_n}(\theta_{-k}\omega) \subset B_{\delta_n}(0, \rho_{\delta_n}(\theta_{-k}\omega))$, by (6.28), we infer that for each $k \geq 1$ and $n \geq N_1(\theta_{-k}\omega)$,

$$\|(u^{n,k}, \varphi^{n,k})\|^2_X \leq 2\rho^2(\theta_{-k}\omega). \tag{6.32}$$

On the other hand, by (5.2), denoting $\varphi_v^{n,k} := \varphi^k_{v,\delta_n}$, we have

$$\begin{aligned}&\Big(v_{\delta_n}(0; -k, \omega, (v^{n,k}, \varphi_v^{n,k})), v_{\delta_n,0}(\cdot; -k, \omega, (v^{n,k}, \varphi_v^{n,k}))\Big)\\ &\quad= \Big(u_{\delta_n}(0; -k, \omega, (u^{n,k}, \varphi^{n,k})), u_{\delta_n,0}(\cdot; -k, \omega, (u^{n,k}, \varphi^{n,k}))\Big)\\ &\qquad-(\phi y_{\delta_n}(\omega), \phi y_{\delta_n}(\theta_{\cdot}\omega)),\end{aligned} \tag{6.33}$$

where

$$(v^{n,k}, \varphi_v^{n,k}) = (u^{n,k}, \varphi^{n,k}) - (\phi y_{\delta n}(\theta_{-k}\omega), \phi y_{\delta n}(\theta_{-k+\cdot}\omega)). \tag{6.34}$$

By (6.31) and (6.33), we obtain

$$\begin{aligned}(u^n, \varphi^n) \ &= \Big(v_{\delta_n}(0; -k, \omega, (v^{n,k}, \varphi_v^{n,k})), v_{\delta_n,0}(\cdot; -k, \omega, (v^{n,k}, \varphi_v^{n,k}))\Big)\\ &\quad+(\phi y_{\delta_n}(\omega), \phi y_{\delta_n}(\theta_{\cdot}\omega)).\end{aligned} \tag{6.35}$$

By (6.32) and (6.34), we have, for $n \geq N_1(\theta_{-k}\omega)$,

$$\|(v^{n,k}, \varphi_v^{n,k})\|_X^2 \leq 4\left(\rho^2(\theta_{-k}\omega) + |\phi|^2 y_{\delta_n}^2(\theta_{-k}\omega) + \|\phi y_{\delta_n}(\theta.\omega)\|_{L_V^2}^2\right). \tag{6.36}$$

Now, by (5.5), Remark 5.2 and (6.36), we find that there exists $N_2 = N_2(\omega, k) \geq N_1$, such that for every $k \geq 1$ and $n \geq N_2$,

$$\|(v^{n,k}, \varphi_v^{n,k})\|_X^2 \leq 4\rho^2(\theta_{-k}\omega) + 8|\phi|^2 z_*^2(\theta_{-k}\omega) + 8\|\phi z_*(\theta.\omega)\|_{L_V^2}^2. \tag{6.37}$$

Note that (6.30), (6.35), (5.5) and Remark 5.2 imply that, as $n \to \infty$,

$$(v_{\delta_n}(0; -k, \omega, (v^{n,k}, \varphi_v^{n,k})), v_{\delta_n,0}(\cdot; -k, \omega, (v^{n,k}, \varphi_v^{n,k}))) \to (v_0, \varphi_{v,0}) \text{ weakly in } X, \tag{6.38}$$

with

$$(v_0, \varphi_{v,0}) = (u_0, \varphi_0) - (\phi z_*(\omega), \phi z_*(\theta.\omega)). \tag{6.39}$$

By (6.37), we find that for each fixed $k \geq 1$, the sequence $\{(v^{n,k}, \varphi_v^{n,k})\}$ is bounded in X, and hence, there is a subsequence (not relabeled) such that for every $k \geq 1$, there exists $(\tilde{v}^k, \tilde{\varphi}_v^k) \in X$ such that

$$(v^{n,k}, \varphi_v^{n,k}) \to (\tilde{v}^k, \tilde{\varphi}_v^k) \quad \text{weakly in } X \text{ as } n \to \infty. \tag{6.40}$$

By (6.40) and Lemma 6.5, we find

$$v_{\delta_n}(0; -k, \omega, (v^{n,k}, \varphi_v^{n,k})) \to v(0; -k, \omega, (\tilde{v}^k, \tilde{\varphi}_v^k)) \quad \text{weakly in } H \text{ as } n \to \infty, \tag{6.41}$$

and

$$v_{\delta_n,0}(\cdot; -k, \omega, (v^{n,k}, \varphi_v^{n,k})) \to v_0(\cdot; -k, \omega, (\tilde{v}^k, \tilde{\varphi}_v^k)) \quad \text{weakly in } L_V^2 \text{ as } n \to \infty. \tag{6.42}$$

Now, by (6.38) and (6.41)–(6.42), we have

$$(v_0, \varphi_{v,0}) = (v(0; -k, \omega, (\tilde{v}^k, \tilde{\varphi}_v^k)), v_0(\cdot; -k, \omega, (\tilde{v}^k, \tilde{\varphi}_v^k))). \tag{6.43}$$

We need to prove that, up to a subsequence, the convergence (6.30) is also true with respect to the strong topology. We will do it in several steps.

Statement 1. We have

$$v_{\delta_n}(0; -k, \omega, (v^{n,k}, \varphi_v^{n,k})) \to v(0; -k, \omega, (\tilde{v}^k, \tilde{\varphi}_v^k)) \quad \text{strongly in } H \text{ as } n \to \infty. \tag{6.44}$$

In a similar way as in Lemma 4.3, we obtain that

$$v_{\delta_n}(\cdot; -k, \omega, (v^{n,k}, \varphi_v^{n,k})) \to v(\cdot; -k, \omega, (\tilde{v}^k, \tilde{\varphi}_v^k)) \quad \text{in } L^2(-k, 0; H). \tag{6.45}$$

Thus,

$$v_{\delta_n}(t; -k, \omega, (v^{n,k}, \varphi_v^{n,k}))\,(x) \to v(t; -k, \omega, (\tilde{v}^k, \tilde{\varphi}_v^k))\,(x) \quad \text{for a.a. } (t, x) \in (-k, 0) \times \mathcal{O}.$$

Let us denote by

$$z_{\delta_n}(t) = \left(v_{\delta_n}(t), \eta_{\delta_n}^t(s)\right),$$

the solution to problem (5.10) with initial condition $(v^{n,k}, \mathcal{J}_{\omega,0}^{\delta_n}\varphi^{n,k})$ on $t = -k$. Integrating in (5.17) over $(-k, t)$ for $t \in [-k, 0)$, we have

$$\|z_{\delta_n}(t)\|_{\mathcal{H}}^2 + \frac{m}{2}\int_{-k}^{t} \left\|v_{\delta_n}(s)\right\|^2 ds \le \|z_{\delta n}(-k)\|_{\mathcal{H}}^2 + \int_{-k}^{t} \Theta_{1,\delta_n}(\theta_s\omega)ds \le R_1(\omega, k), \tag{6.46}$$

where $\Theta_{1,\delta_n}(\omega)$ is defined in (5.16). Observe that the existence of the bound $R_1(\omega, k)$ follows from (6.40), (5.5), (5.21) and (5.23).

In view of (6.46), $v^n(t) = v_{\delta_n}(t; -k, \omega, (v^{n,k}, \varphi_v^{n,k}))$ is bounded in $C([-k, 0], H)$, so the same arguments as in Lemma 4.3 show that

$$v^n \to v \text{ in } C([-k, 0], V^* + L^q(\mathcal{O})),$$

where $v(t) = v(t; -k, \omega, (\tilde{v}^k, \tilde{\varphi}_v^k))$. Then if $t_n \to t_0$, $t_n \in [-k, 0]$, $t_0 \in (-k, 0]$, we obtain

$$v^n(t_n) \to v(t_0) \text{ weakly in } H,$$

and

$$|v(t_0)| \le \liminf_{n\to\infty} |v_n(t_n)|.$$

Let us prove that $v^n(t_n) \to v(t_0)$ strongly in H. Using Lemma 3.6, we deduce that v^n are weak solutions to problem (5.8). Multiplying the equation by v^n, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|v^n(t)|^2 + m\|v^n\|^2 + (f(v^n + \phi y_\delta(\theta_t\omega)), v^n) \\
&\quad \le M|y_\delta(\theta_t\omega)|\|\phi\|\|v^n\| + \left(\int_{-\infty}^{t} k(t-s)\Delta v^n(s)ds, v^n(t)\right) + (h, v^n(t)) \\
&\qquad + |y_\delta(\theta_t\omega)||\phi||v^n| + (z_{k,\delta}^{\phi}(\theta_t\omega), v^n).
\end{aligned}$$

By similar arguments as in Lemma 5.7 and the Young inequality, we obtain

$$\begin{aligned}
&\frac{d}{dt}|v^n|^2 + m\|v^n\|^2 + \frac{f_0}{2D}\|v^n\|_{2p}^{2p} \le 2\alpha|\mathcal{O}| + C_{1,\delta}(\theta_t\omega)(1+\|\phi\|_{2p}^{2p})\\
&\quad + \frac{4M^2}{m}|y_\delta(\theta_t\omega)|^2\|\phi\|^2 + \frac{4}{m\lambda_1}|h|^2 + \frac{4}{m\lambda_1}|y_\delta(\theta_t\omega)|^2|\phi|^2\\
&\quad + \frac{C_{2,\delta}(\theta_t\omega)}{m}\|\phi\|^2 + 2\int_{-\infty}^t k(t-s)\|v^n(s)\|ds\|v^n(t)\|.
\end{aligned}$$

By the same arguments in Lemma 4.3, we have

$$\int_{-\infty}^t k(t-s)\|v^n(s)\|ds \le \frac{M_1^{\frac{1}{2}}\mu^{\frac{1}{2}}(t)\|\varphi_v^{n,k}\|_{L_V^2}}{\varpi^{\frac{1}{2}}(\varpi-\gamma)^{\frac{1}{2}}} + M_1M''\sqrt{t}.$$

Therefore, using (4.24)–(4.25), we deduce

$$\begin{aligned}
&\frac{d}{dt}|v^n(t)|^2 + \frac{m}{2}\|v^n(t)\|^2 + \frac{f_0}{2D}\|v^n(t)\|_{2p}^{2p} \le 2\alpha|\mathcal{O}| + C_{1,\delta}(\theta_t\omega)(1+\|\phi\|_{2p}^{2p})\\
&\quad + \frac{4M^2}{m}|y_\delta(\theta_t\omega)|^2\|\phi\|^2 + \frac{4}{m\lambda_1}|h|^2 + \frac{4}{m\lambda_1}|y_\delta(\theta_t\omega)|^2|\phi|^2\\
&\quad + \frac{C_{2,\delta}(\theta_t\omega)}{m}\|\phi\|^2 + \frac{4M_1\mu(t)\|\varphi_v^{n,k}\|_{L_V^2}^2}{\varpi m(\varpi-\gamma)} + \frac{4(M_1)^2(M'')^2T}{m}.
\end{aligned} \tag{6.47}$$

The function v satisfies the same inequality but replacing y_δ by z_*, $C_{i,\delta}$ by C_i, $i=1,2$ and $\varphi_v^{n,k}$ by $\widetilde{\varphi}_v^k$. We define the functions

$$\begin{aligned}
J_n(t) &= |v^n(t)|^2 - 2\alpha|\mathcal{O}|t - \frac{4(M_1)^2(M'')^2T}{m}t - \int_0^t C_{1,\delta}(\theta_r\omega)(1+\|\phi\|_{2p}^{2p})dr\\
&\quad - \int_0^t\left(\frac{4M^2}{m}|y_\delta(\theta_r\omega)|^2\|\phi\|^2 + \frac{4}{m\lambda_1}|z_*(\theta_r\omega)|^2|\phi|^2\right)dr\\
&\quad - \frac{4\,|h|^2}{m\lambda_1}t - \frac{4M_1\|\varphi_v^{n,k}\|_{L_V^2}^2}{\varpi m(\varpi-\gamma)}\int_0^t\mu(r)dr - \int_0^t\frac{C_2(\theta_r\omega)}{m}\|\phi\|^2\,dr,\\
J_t(t) &= |v(t)|^2 - 2\alpha|\mathcal{O}|t - \frac{4(M_1)^2(M'')^2T}{m}t - \int_0^t C_1(\theta_r\omega)(1+\|\phi\|_{2p}^{2p})dr\\
&\quad - \int_0^t\left(\frac{4M^2}{m}|z_*(\theta_r\omega)|^2\|\phi\|^2 + \frac{4}{m\lambda_1}|z_*(\theta_r\omega)|^2|\phi|^2\right)dr\\
&\quad - \frac{4\,|h|^2}{m\lambda_1}t - \frac{4M_1\|\widetilde{\varphi}_v^k\|_{L_V^2}^2}{\varpi m(\varpi-\gamma)}\int_0^t\mu(r)dr - \int_0^t\frac{C_2(\theta_r\omega)}{m}\|\phi\|^2\,dr.
\end{aligned}$$

From the regularity of v and all v^n, together with (5.21), (5.23), (6.45) and (6.47), it holds that these functions J and J_n are continuous and non-increasing on $[-k, 0]$, and

$$J_n(s) \to J(s) \ \text{ a.e. } \ s \in [-k, 0] \text{ as } n \to \infty.$$

Then the same argument as in Lemma 4.3 implies that $v^n(t_n) \to v(t_0)$ strongly in H, and thus (6.44) follows.

Statement 2. The following inequality holds true:

$$\limsup_{n\to\infty} \left\| v_{\delta_n,0} - v_0 \right\|^2_{L^2_V} \le M e^{-\gamma k} \limsup_{n\to\infty} \left(\left| u^{n,k} - \tilde{u}^k \right|^2 + \left\| \varphi^{n,k} - \tilde{\varphi}^k \right\|^2_{L^2_V} \right), \tag{6.48}$$

for any $k > 0$, where M is a positive constant and $\tilde{u}^k = \tilde{v}^k + \phi z_*(\theta_{-k}\omega)$.

Define the functions $\overline{z}_{\delta_n} = \left(\overline{v}_{\delta_n}, \overline{\eta}_{\delta_n}\right) = z_{\delta_n} - z$, where $z(t) = (v(t), \eta^t(s))$ is the solution to problem (2.6) with initial condition $(v^k, \mathcal{J}_{\omega,0}\varphi^k)$ on $t = -k$. Arguing as in Lemma 4.3, we have

$$\begin{aligned}
\frac{d}{dt}\|\overline{z}_{\delta_n}\|^2_{\mathcal{H}} + 2(((\bar{\eta}^t_{\delta_n})', \eta^t_{\delta_n}))_\mu \le{} & -2\int_{\mathcal{O}} (f(v_{\delta_n} + y_{\delta_n}(\theta_t\omega)) \\
& -f(v + \phi z_*(\theta_t\omega)))(v_{\delta_n} - v)dx \\
& -2\int_{\mathcal{O}} \left(a(l(v_{\delta_n}) + l(\phi)y_{\delta_n}(\theta_t\omega))\nabla v_{\delta_n} - a(l(v) + l(\phi)z_*(\theta_t\omega))\nabla v\right) \cdot \nabla(v_{\delta_n} - v)dx \\
& +2\int_{\mathcal{O}} (a(l(v_{\delta_n}) + l(\phi)y_{\delta_n}(\theta_t\omega))y_{\delta_n}(\theta_t\omega) - a(l(v) \\
& +l(\phi)z_*(\theta_t\omega))z_*(\theta_t\omega))\Delta\phi(v_{\delta_n} - v)dx \\
& +\int_{\mathcal{O}} \left(y_{\delta_n}(\theta_t\omega) - z_*(\theta_t\omega)\right)\phi\left(v_{\delta_n} - v\right)dx \\
& +2\int_{\mathcal{O}} \left(z^{\phi}_{k,\delta_n}(t,\omega) - z^{\phi}_k(t,\omega)\right)\left(v_{\delta_n} - v\right)dx.
\end{aligned}$$

Since a is a locally Lipschitz function, $\phi \in V$, $z_*(\theta_t\omega)$ is uniformly bounded for any $\omega \in \Omega$ on $[-k, 0]$, making use of (5.5), we deduce that

$$\begin{aligned}
\{l(v_{\delta_n}(t)) + l(\phi)y_n(\theta_t\omega)\}_{t\in[-k,0],0<\delta\le\overline{\sigma}} &\subset [-R, R], \\
\{l(v(t)) + l(\phi)z_*(\theta_t\omega)\}_{t\in[-k,0],0<\delta\le\delta_0} &\subset [-R, R],
\end{aligned}$$

for some $R, \delta_0 > 0$, and

$$\begin{aligned}
\left|a(l(v_{\delta_n}) + l(\phi)y_{\delta_n}(\theta_t\omega)) - a(l(v) + l(\phi)z_*(\theta_t\omega))\right| \le L_a(R)\,|l|\,(\left|v_{\delta_n} - v\right| \\
+ \left|y_{\delta_n}(\theta_t\omega) - z_*(\theta_t\omega)\right| |\phi|).
\end{aligned}$$

Hence, by (1.11) and the Young inequality, we infer that

$$\begin{aligned}
&-2\int_{\mathcal{O}}\big(a(l(v_{\delta_n})+l(\phi)y_{\delta_n}(\theta_t\omega))\nabla v_{\delta_n}-a(l(v)+l(\phi)z_*(\theta_t\omega))\nabla v\big)\cdot\nabla(v_{\delta_n}-v)dx\\
&\quad\le -2m\|v_{\delta_n}-v\|^2+2L_a(R)|l|\left(|v^n-v|\right.\\
&\qquad\left.+\left|y_{\delta_n}(\theta_t\omega)-z_*(\theta_t\omega)\right||\phi|\right)\|v\|\|v_{\delta_n}-v\|\\
&\quad\le(\alpha-2m)\|v_{\delta_n}-v\|^2+\frac{2L_a^2(R)|l|^2}{\alpha}\Big(|v_{\delta_n}-v|^2\\
&\qquad+\left|y_{\delta_n}(\theta_t\omega)-z_*(\theta_t\omega)\right|^2|\phi|^2\Big)\|v\|^2,
\end{aligned}$$

where $\alpha\le(m\lambda_1-\gamma)/\lambda_1$. By the above estimates, we deduce that

$$\begin{aligned}
&\frac{d}{dt}\|\bar{z}_{\delta_n}\|_{\mathcal{H}}^2+\gamma\|\bar{z}_{\delta_n}\|_{\mathcal{H}}^2+m\|v_{\delta_n}-v\|^2\\
&\quad\le\frac{d}{dt}\|\bar{z}_{\delta_n}\|_{\mathcal{H}}^2+(2m-\alpha)\|v_{\delta_n}-v\|^2+\varpi\int_0^\infty\mu(s)|\nabla\bar{\eta}_{\delta_n}^t(s)|^2ds\\
&\quad\le\frac{2L_a^2(R)|l|^2}{\alpha}\Big(|v_{\delta_n}-v|^2+\left|y_{\delta_n}(\theta_t\omega)-z_*(\theta_t\omega)\right|^2|\phi|^2\Big)\|v\|^2\\
&\qquad-2\int_{\mathcal{O}}(f(v_{\delta_n}+\phi y_{\delta_n}(\theta_t\omega))-f(v+\phi z_*(\theta_t\omega)))(v_{\delta_n}-v)dx\\
&\qquad+2\int_{\mathcal{O}}(a(l(v_{\delta_n})+l(\phi)y_{\delta_n}(\theta_t\omega))y_{\delta_n}(\theta_t\omega)\\
&\qquad-a(l(v)+l(\phi)z_*(\theta_t\omega))z_*(\theta_t\omega))\Delta\phi(v_{\delta_n}-v)dx\\
&\qquad+2\int_{\mathcal{O}}\left(y_{\delta_n}(\theta_t\omega)-z_*(\theta_t\omega)\right)\phi\left(v_{\delta_n}-v\right)dx+2\int_{\mathcal{O}}\left(z_{k,\delta_n}^{\phi}-z_k^{\phi}\right)\left(v_{\delta_n}-v\right)dx,
\end{aligned}$$

where we have used that $0<\gamma\le\min\{(m-\alpha)\lambda_1,\delta\}$ by the choice of α. Multiplying by $e^{\gamma t}$ on both sides of the above inequality and integrating over $(-k,0)$, we obtain

$$\begin{aligned}
&\|\bar{z}_{\delta_n}(t)\|_{\mathcal{H}}^2+m\int_{-k}^0e^{\gamma s}\|v_{\delta_n}(s)-v(s)\|^2ds\\
&\quad\le e^{-\gamma k}\|\bar{z}_{\delta_n}(-k)\|_{\mathcal{H}}^2+\frac{2L_a^2(R)|l|^2}{\alpha}\int_{-k}^0e^{\gamma s}(|v_{\delta_n}-v|^2+\left|y_{\delta_n}(\theta_s\omega)-z_*(\theta_s\omega)\right|)\|v\|^2ds\\
&\qquad-2\int_{-k}^0e^{\gamma s}\int_{\mathcal{O}}(f(v_{\delta_n}+\phi y_{\delta_n}(\theta_s\omega))-f(v+\phi z_*(\theta_s\omega)))(v_{\delta_n}-v)dxds\\
&\qquad+2\int_{-k}^0e^{\gamma s}\int_{\mathcal{O}}(a(l(v_{\delta_n})+l(\phi)y_{\delta_n}(\theta_s\omega))y_{\delta_n}(\theta_s\omega)-a(l(v)\\
&\qquad+l(\phi)z_*(\theta_s\omega))z_*(\theta_s\omega))\Delta\phi(v_{\delta_n}-v)dxds\\
&\qquad+2\int_{-k}^0e^{\gamma s}\int_{\mathcal{O}}\left(y_{\delta_n}(\theta_s\omega)-z_*(\theta_s\omega)\right)\phi\left(v_{\delta_n}-v\right)dxds\\
&\qquad+2\int_{-k}^0e^{\gamma s}\int_{\mathcal{O}}\left(z_{k,\delta_n}^{\phi}(s,\omega)-z_k^{\phi}(s,\omega)\right)\left(v_{\delta_n}-v\right)dxds.
\end{aligned}$$

By a similar argument as in Lemma 4.3, we obtain that the second and fourth terms of the right-hand side of the above inequality converge to zero. Also, by (5.5) and (6.45), it is easy to see that the fifth term goes to zero as well. On the other hand, by (5.5)–(5.6), we deduce easily that $z^{\phi}_{k,\delta_n}(s,\omega) \to z^{\phi}_{k}(s,\omega)$ uniformly on $[-k,0]$. Hence, (6.45) implies that the last term of the above inequality also converges to zero.

It remains to analyze the third term. On the one hand, as in the proof of Lemma 4.3, we obtain that $f(v^n+\phi y_{\delta_n}(\theta_t\omega)) \to f(v+\phi z_*(\theta_t\omega))$ weakly in $L^q(-k,0;L^q(\mathcal{O}))$. Thus,

$$\int_{-k}^{0} e^{\gamma s}\int_{\mathcal{O}} (f(v_{\delta_n}+\phi y_{\delta_n}(\theta_s\omega)) - f(v+\phi z_*(\theta_s\omega)))v dxds \to 0.$$

On the other hand, by the same calculations as in Lemma 4.3, we infer that there are positive constants $\kappa_i\ (\omega,\phi,k)$, $i=1,2$, such that

$$f(v^n(s,x,\omega)+\phi y_n(\theta_s\omega))v^n(s,x,\omega) \geq -\kappa_1+\kappa_2\left|v^n(s,,x,\omega)+\phi y_n(\theta_s\omega)\right|^{2p}.$$

Then the Fatou-Lebesgue lemma implies that

$$\begin{aligned}&\limsup_{n\to\infty}\left(-2\int_{-k}^{0} e^{\gamma s}\int_{\mathcal{O}} f(v_{\delta_n}+\phi y_{\delta_n}(\theta_s\omega))v_{\delta_n} dxds\right)\\ &\quad\leq -2\int_0^t e^{\gamma s}\int_{\mathcal{O}} \liminf_{n\to\infty} f(v_{\delta_n}+\phi y_{\delta_n}(\theta_s\omega))v_{\delta_n} dxds\\ &\quad= -2\int_0^t e^{\gamma s}\int_{\mathcal{O}} f(v+\phi z_*(\theta_s\omega))v dxds.\end{aligned}$$

Since $f(v+\phi z_*(\theta.\omega)) \in L^q(-k,0;L^q(\mathcal{O}))$ and $v_{\delta_n} \to v$ weakly in $L^{2p}(-k,0;L^{2p}(\mathcal{O}))$, (4.28) holds true. Hence,

$$\begin{aligned}&\limsup_{n\to\infty}\left(-2\int_{-k}^{0} e^{\gamma s}\int_{\mathcal{O}} (f(v_{\delta_n}+\phi y_{\delta_n}(\theta_s\omega)) - f(v+\phi z_*(\theta_s\omega)))(v_{\delta_n}-v) dxds\right)\\ &\quad\leq 0.\end{aligned}$$

By Lemma 3.1, the proof of Corollary 3.2, (5.5), Remark 5.2 and the above convergences, we arrive at

$$\begin{aligned}&\limsup_{n\to\infty}\int_{-k}^{0} e^{\gamma s}\|v_{\delta_n}(s)-v(s)\|^2 ds\\ &\quad\leq \frac{1}{m}e^{-\gamma k}\limsup_{n\to\infty}\|\overline{z}_{\delta_n}(-k)\|^2_{\mathcal{H}}\\ &\quad\leq \frac{1}{m}e^{-\gamma k}\limsup_{n\to\infty}\left(\left|v^{n,k}-\tilde{v}^k\right|^2+\int_0^\infty \mu(s)\left\|\mathcal{J}^{\delta_n}_{\omega,0}\varphi^{n,k}-\mathcal{J}_{\omega,0}\widetilde{\varphi}^k\right\|^2 ds\right)\end{aligned}$$

$$
\begin{aligned}
&\le \frac{1}{m}e^{-\gamma k}\limsup_{n\to\infty}\left(\left|v^{n,k}-\tilde{v}^k\right|^2+2\int_0^\infty \mu(s)\left\|\mathcal{J}\left(\varphi^{n,k}-\widetilde{\varphi}^k\right)\right\|^2 ds\right.\\
&\quad\left.+2\int_0^\infty \mu(s)\left(\int_{-s}^0 \left|y_{\delta_n}(\theta_r\omega)-z_*(\theta_r\omega)\right|\|\phi\|\,ds\right)^2\right)\\
&\le \frac{1}{m}e^{-\gamma k}\limsup_{n\to\infty}\left(2\left|u^{n,k}-\tilde{u}^k\right|^2+2K_\mu\int_{-\infty}^0 e^{\gamma s}\left\|\varphi^{n,k}(s)-\widetilde{\varphi}^k(s)\right\|^2 ds\right.\\
&\quad\left.+2\left|y_{\delta_n}(\theta_{-k}\omega)-z_*(\theta_{-k}\omega)\right|^2|\phi|^2+2K_\mu\int_{-\infty}^0 e^{\gamma s}\|\phi\|^2\left\|y_{\delta_n}(\theta_s\omega)-z_*(\theta_s\omega)\right\|^2 ds\right)\\
&= \frac{1}{m}e^{-\gamma k}\limsup_{n\to\infty}\left(2\left|u^{n,k}-\tilde{u}^k\right|^2+2K_\mu\int_{-\infty}^0 e^{\gamma s}\left\|\varphi^{n,k}(s)-\widetilde{\varphi}^k(s)\right\|^2 ds\right).
\end{aligned}
$$

Then, the same arguments as in Lemma 4.3 imply (6.48) immediately.

Statement 3. There is a subsequence such that

$$
(u^n,\varphi^n)\to(u_0,\varphi_0),\quad \text{strongly in } X \text{ as } n\to\infty. \tag{6.49}
$$

By Lemma 5.12, we know that the family $\overline{B}(\omega)$ is tempered in X. By (6.48), $(u^{n,k},\varphi^{n,k})$, $(\tilde{u}^k,\tilde{\varphi}^k)\in\overline{B}(\theta_{-k}\omega)$ and choosing some $0<c<\gamma$, there is a constant $R>0$ such that

$$
\begin{aligned}
\limsup_{n\to\infty}\left\|v_{\delta_n,0}-v_0\right\|^2_{L^2_V} &\le Me^{-(\gamma-c)k}e^{-ck}\limsup_{n\to\infty}\left(\left|u^{n,k}-\tilde{u}^k\right|^2+\left\|\varphi^{n,k}-\tilde{\varphi}^k\right\|^2_{L^2_V}\right)\\
&\le Re^{-(\gamma-c)k},
\end{aligned}
$$

for all $k\ge 1$. Further, for every $d\in\mathbb{N}$, there is $k_0(d)$ such that,

$$
\limsup_{n\to\infty}\left\|v_{\delta_n,0}-v_0\right\|^2_{L^2_V}\le\frac{1}{d},\qquad \forall k\ge k_0.
$$

Taking $d\to\infty$ and using a diagonal argument, we deduce the existence of a subsequence $\{v_{\delta_{n_d},0}\}$ such that $v_{\delta_{n_d},0}\to v_0$ in L^2_V. Together with (6.44) and $\phi y_{\delta_n}(\theta.\omega)\to\phi z_*(\theta.\omega)$ in X, it shows that (6.49) is true. The proof of this lemma is complete. □

Theorem 6.7 *Assume that (1.11), (2.8) and* (h_1)*-*(h_2) *hold. Let* $\phi\in V\cap H^2(\mathcal{O})\cap L^{2p}(\mathcal{O})$ *be such that* $\Delta\phi\in L^{2p}(\mathcal{O})$*,* $h\in H$ *and a be a locally Lipschitz function. Then, for all* $\omega\in\Omega$*,*

$$
\lim_{\delta\to 0} dist_X(\mathcal{A}_\delta(\omega),\mathcal{A}(\omega))=0.
$$

Proof By (5.14), we have

$$
\lim_{\delta\to 0}\|B_\delta(\omega)\|_X=\|B(\omega)\|_X,\qquad \text{for all }\ \omega\in\Omega,
$$

where for a set $S\subset X$, we denote $\|S\|_X=\sup_{y\in S}\|y\|_X$. This, together with Corollary 6.4 and Lemma 6.6, finishes the proof of this theorem by [30, Theorem 3.1]. □

7 Appendix

Let us consider Eq. (1.10) in the deterministic case, that is,

$$\begin{cases} \dfrac{\partial u}{\partial t} - a(l(u))\Delta u - \displaystyle\int_{-\infty}^{t} k(t-s)\Delta u(x,s)ds + f(u) = h, & \text{in } \mathcal{O}\times(\tau,\infty), \\ u(x,t) = 0, & \text{on } \partial\mathcal{O}\times(\tau,\infty), \\ u(x,0) = u_0(x), & \text{in } \mathcal{O} \\ u(x,t) = \phi(x,t), & \text{in } \mathcal{O}\times(-\infty,0], \end{cases} \tag{7.1}$$

where $\mathcal{O} \subset \mathbb{R}^N$ is a bounded domain with regular boundary. We assume that (1.11), (2.8) and (h_1)-(h_2) hold. Also, let $h \in H$ and a be a locally Lipschitz function.

As before, we consider Dadermos' tranform

$$\eta^t(s,x) = \int_{t-s}^{t} u(r,x)dr, \quad \text{for } s \geq 0,$$

which gives rise to the system,

$$\begin{cases} \dfrac{\partial u}{\partial t} - a(l(u))\Delta u - \displaystyle\int_{0}^{\infty} \mu(s)\Delta\eta^t(s)ds + f(u) = h, & \text{in } \mathcal{O}\times(\tau,\infty), \\ \dfrac{\partial}{\partial t}\eta^t(s) = u - \dfrac{\partial}{\partial s}\eta^t(s), & \text{in } \mathcal{O}\times(\tau,\infty)\times\mathbb{R}^+, \\ u(x,t) = \eta^t(x,s) = 0, & \text{on } \partial\mathcal{O}\times\mathbb{R}, s>0, \\ u(x,0) = u_0(x), & \text{in } \mathcal{O}, \\ \eta^0(x,s) = \eta_0(x,s), & \text{in } \mathcal{O}\times\mathbb{R}^+, \end{cases} \tag{7.2}$$

where

$$\eta^0(x,s) = \int_{-s}^{0} u(x,r)dr = \int_{-s}^{0} \phi(x,r)dr := \eta_0(x,s).$$

For any $(u_0, \phi) \in X = H \times L^2_V$, there exists a unique weak solution $z(\cdot) = (u(\cdot), \eta^{\cdot})$ to problem (7.2) [37, Theorem 3.4]. This is also a particular case of the result in Theorem 3.4 with $\phi = 0$.

Lemma 7.1 *Let $\{(u_0^n, \phi^n)\}$ be a sequence such that $(u_0^n, \phi^n) \to (u_0, \phi)$ weakly in X. Then*

$$u^n \to u \text{ in } C([r,T],H), \quad \text{for all } 0 < r < T, \tag{7.3}$$

where $z^n(\cdot) = (u^n(\cdot), \eta_n^{\cdot})$, $z(\cdot) = (u(\cdot), \eta^{\cdot})$ are the solutions to problem (7.2) corresponding to (u_0^n, ϕ^n) and (u_0, ϕ), respectively.

Proof It is known (see the proof of Lemma 3.9 in [37]) that,

$$\begin{aligned}
u^n &\to u \quad \text{weak-}*\ \text{in} \quad L^\infty(0,T;H);\\
u^n &\to u \quad \text{weakly in} \quad L^2(0,T;V);\\
u^n &\to u \quad \text{strongly in} \quad L^2(0,T;H);\\
\frac{du^n}{dt} &\to \frac{du}{dt} \quad \text{weakly in} \quad L^2(0,T;V^*)+L^q\left(0,T;L^q(\Omega)\right);\\
u^n(t) &\to u(t) \text{ in } H \quad \text{for a.a. } t\in(0,T)\,.
\end{aligned}$$

By the same arguments in the proof of Lemma 4.3, we obtain

$$u_n \to u \quad \text{in } C([-k,0],V^*+L^q(\mathcal{O})).$$

Then, if $t_n \to t_0, t_n \in [0,T],\ t_0\in(0,T]$, we infer

$$v_n(t_n) \to v(t_0) \text{ weakly in } H,$$

and

$$|v(t_0)| \le \liminf_{n\to\infty} \ |v_n(t_n))|\,.$$

We need to prove that $v^n(t_n) \to v(t_0)$ strongly in H. By Corollary 3.7, we know u^n are weak solutions of the equation

$$u^n_t - a(l(u))\Delta u^n - \int_{-\infty}^t k(t-s)\Delta u^n ds + f(u^n) = h.$$

Then,

$$\frac{1}{2}\frac{d}{dt}|u^n(t)|^2 + m\|u^n\|^2 + (f(u^n),u^n) \le \left(\int_{-\infty}^t k(t-s)\Delta u^n(s)ds, u^n(t)\right) + (h,u^n(t)).$$

By (3.8) and the Young inequality, we obtain

$$\frac{d}{dt}|u^n|^2 + m\|u^n\|^2 + f_0\|u^n\|_{2p}^{2p} \le 2\alpha|\mathcal{O}| + \frac{1}{m\lambda_1}|h|^2 + 2\int_{-\infty}^t k(t-s)\|u^n(s)\|ds\|u^n(t)\|.$$

By the arguments in Lemma 4.3, we have

$$\int_{-\infty}^t k(t-s)\|u^n(s)\|ds \le \frac{M_1^{\frac{1}{2}}\mu^{\frac{1}{2}}(t)\|\phi^n\|_{L^2_V}}{\varpi^{\frac{1}{2}}(\varpi-\gamma)^{\frac{1}{2}}} + M_1M''\sqrt{t}.$$

Thus,

$$\frac{d}{dt}|u^n(t)|^2 + m\|u^n(t)\|^2 + f_0\|u^n(t)\|_{2p}^{2p}$$
$$\leq 2\alpha|\mathcal{O}| + \frac{1}{m\lambda_1}|h|^2 + 2\left(\frac{M_1^{\frac{1}{2}}\mu^{\frac{1}{2}}(t)\|\phi^n\|_{L_V^2}}{\varpi^{\frac{1}{2}}(\varpi-\gamma)^{\frac{1}{2}}} + M_1 M'{\prime}\sqrt{T}\right)\|u^n(t)\|.$$

Therefore,

$$\frac{d}{dt}|u^n(t)|^2 + \frac{m}{2}\|u^n(t)\|^2 + f_0\|u^n(t)\|_{2p}^{2p}$$
$$\leq 2\alpha|\mathcal{O}| + \frac{1}{m\lambda_1}|h|^2 + \frac{4M_1\mu(t)\|\phi^n\|_{L_V^2}^2}{\varpi m(\varpi-\gamma)} + \frac{4(M_1)^2(M'')^2T}{m}.$$

The function u satisfies the same inequality but replacing ϕ^n by ϕ. We define the functions

$$J_n(t)7|u^n(t)|^2 - 2\alpha|\mathcal{O}|t - \frac{4(M_1)^2(M'')^2T}{m}t - \frac{|h|^2}{m\lambda_1}t - \frac{4M_1\|\phi^n\|_{L_V^2}^2}{\varpi m(\varpi-\gamma)}\int_0^t \mu(r)dr,$$
$$J(t) = |u(t)|^2 - 2\alpha|\mathcal{O}|t - \frac{4(M_1)^2(M'')^2T}{m}t - \frac{|h|^2}{m\lambda_1}t - \frac{4M_1\|\phi\|_{L_V^2}^2}{\varpi m(\varpi-\gamma)}\int_0^t \mu(r)dr.$$

These functions are continuous and non-increasing on $[0, T]$, and

$$J_n(s) \to J(s) \quad \text{for a.a. } s \in [0, T] \text{ as } n \to \infty.$$

Then the same argument as in Lemma 4.3 ensures that $v^n(t_n) \to v(t_0)$ strongly in H, and therefore (7.3) follows. □

Remark 7.2 The convergence (7.3) was stated in Lemma 3.9 from [37]. However, the proof of this result is incorrect there and we provided here a correct one.

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Declarations

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