On the limit of solutions for a reaction-diffusion equation containing fractional Laplacians

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Abstract

A kind of nonlocal reaction-diffusion equations on an unbounded domain containing a fractional Laplacian operator is analyzed. To be precise, we prove the convergence of solutions of the equation governed by the fractional Laplacian to the solutions of the classical equation governed by the standard Laplacian, when the fractional parameter grows to 1. The existence of global attractors is investigated as well. The novelty of this paper is concerned with the convergence of solutions when the fractional parameter varies, which, as far as the authors are aware, seems to be the first result of this kind of problems in the literature.

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1 Introduction

For a given initial time $\tau \in \mathbb{R}$, our aim in this paper is to analyze several interesting aspects related to the following problem on $\mathbb{R}^m$ $(m \geq 1)$,

$$\begin{cases} \dfrac{\partial u}{\partial t} + (-\Delta)^\gamma u = f(t,x,u) + h(t,x), \quad (x,t) \in \mathbb{R}^m \times (\tau, \infty), \\ u(\tau, x) = u_\tau(x), \quad x \in \mathbb{R}^m, \end{cases} \tag{1.1}$$

where $(-\Delta)^\gamma$, $\gamma \in (0,1)$, stands for the fractional Laplacian operator, $h \in L^2_{loc}(\mathbb{R}; L^2(\mathbb{R}^m))$ and f is a continuous function satisfying some appropriate assumptions as specified later. Precisely, under the same assumptions imposed on f and h, we are interested in studying the convergence

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of solutions of (1.1), when γ grows to 1^-, to the corresponding ones with $\gamma = 1$, that is, the classical reaction-diffusion problem (see, e.g., [2], [27]),

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u = f(t,x,u) + h(t,x), \quad (x,t) \in \mathbb{R}^m \times (\tau, \infty), \\ u(\tau, x) = u_\tau(x), \quad x \in \mathbb{R}^m. \end{cases} \tag{1.2}$$

The well-posedness of problem (1.1) has already been stated (for instance, in [15] it is proved for a stochastic equation which contains our deterministic model as a particular case), as well as the existence of global attractors (see also [8], [29, Theorem 2.3] for the case of bounded domains). Several stochastic variants have also been studied (see [31, 32, 33] and the references therein), in which the existence and upper-semicontinuity of random attractors were presented when some parameters in the noise term vary. We are interested in the convergence of solutions when the fractional parameter γ goes to 1^-. This is an extremely challenging task which will be tackled in the current paper successfully. We observe that in the papers [3] and [4] this question was studied for an abstract parabolic equation and the Schrödinger equation, respectively. However, as far as we are aware, there are no studies yet for the particular problem (1.1).

The behavior of the operator $(-\Delta)^\gamma$ as $\gamma \to 1^-$ has been studied for example in [8, 16, 20]. In [20, Proposition 4.4] it is shown that $(-\Delta)^\gamma u(x)$ converges to $-\Delta u(x)$ as $\gamma \to 1$, for all $x \in \mathbb{R}^m$, when $u \in C_0^\infty(\mathbb{R}^m)$. In [8, Proposition 2.3] the authors proved that for every $u \in C_0^\infty(\mathbb{R}^m)$ and $v \in W_0^{1,2}(\mathcal{O})$,

$$\lim_{\gamma \to 1^-} \int_{\mathcal{O}} v \, (-\Delta)^\gamma \, u dx = - \int_{\mathcal{O}} v \Delta u dx.$$

Also, in [16, Proposition 3.1.1] the convergence was established within the framework of the equivalent definition of the fractional operators $(-\Delta)^\gamma$ given by Balakrishnan. We extend these results, and several novel properties related to fractional Laplacian operators are established in different phase spaces, which play a key role to demonstrate the main result in the present paper.

The reasons to study this kind of problems in our investigation are as follows. Fractional problem (1.1) and its stationary version have been considered to illustrate and model the motion of nonlinear deflects in crystals within the area of dislocation dynamics (see, for example, [5, 7, 12, 24, 25]). Also, in the phase-field and interfacial dynamics framework, this equation is usually known as the fractional Allen-Cahn equation (see, for example, [9, 19]). However, it is necessary to emphasize that, to analyze the problem of anomalous diffusion in physics, probability, finance and other fields of science, the linear fractional parabolic equation $\partial_t u + (-\Delta)^\gamma u = 0$ with $\gamma \in (0,1)$ is frequently used instead of the standard parabolic equation $\partial_t u - \Delta u = 0$ (see, for example, [1, 6, 10, 17, 22, 34, 35, 36]). There is an interesting work dealing with different aspects of the normal and anomalous diffusion, see [26] and the references therein. Therefore, we have a great interest in knowing if the convergence of the anomalous diffusion problem to the normal one takes place in a smooth and reasonable way. In our opinion, this would confirm that the mathematical modeling of the problems is appropriate to describe the real phenomena.

It is worth highlighting that $\frac{1}{C(m,\gamma)}$ (cf. (2.4)) is not uniformly bounded with respect to $\gamma \in (0,1)$, see [20, Corollary 4.2]. Consequently, we cannot use the standard arguments when proving existence of weak solutions to obtain that u_n converges to u strongly in $L^2(\tau, T; L^2(\mathbb{R}^m))$. The lack of this strong convergence prevents us from taking the convergence of f even if f is a continuous function. To overcome this difficulty, we have to pay a price, that is, suppose that f is a sublinear function so that the solution of the limit problem is regular enough. In this way, we fill the gap of showing the limit of f by applying a classical monotone method.

The outline of this paper is as follows. Section 2 is devoted to recall the concept of fractional Laplacian operator, set up the problem, introduce hypotheses and notation, as well as the definition of weak solutions. In Section 3, we perform a rigorous analysis of the fractional Laplacian operators $(-\Delta)^\gamma$ with $\gamma \in (0,1)$, which are the crucial tools used to prove the convergence of solutions. Section 4 is fully dedicated to address the main theorems of the paper, concerning the convergence of solutions to the equation governed by fractional Laplacian to the ones of the classical reaction-diffusion equation as $\gamma \to 1^-$, when the external term f satisfies a sublinear condition. Finally, Section 5 deals with the existence of global attractors to the fractional reaction-diffusion equations, when f contains more general cases and h is independent of t.

2 Preliminaries

In this section, we will recall the concept of fractional Laplacian operator on $\mathbb{R}^m$, enumerate the assumptions imposed on nonlinear terms f and h for the fractional PDEs under investigation and introduce the definitions of solutions to problems (1.1) and (1.2), respectively.

2.1 Fractional setting

Let $\mathcal{S}$ be the Schwartz space of rapidly decaying C^∞ functions on $\mathbb{R}^m$. For any fixed $0 < \gamma < 1$, the fractional Laplacian operator $(-\Delta)^\gamma$ of $u \in \mathcal{S}$ at the point x is defined by,

$$(-\Delta)^\gamma u(x) = -\frac{1}{2} C(m,\gamma) \int_{\mathbb{R}^m} \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^{m+2\gamma}} dy, \qquad x \in \mathbb{R}^m, \tag{2.1}$$

where $C(m,\gamma)$ is a positive constant given by

$$C(m,\gamma) = \frac{\gamma 4^\gamma \Gamma(\frac{m+2\gamma}{2})}{\pi^{\frac{m}{2}} \Gamma(1-\gamma)}. \tag{2.2}$$

For any $0 < \gamma < 1$, the fractional Sobolev space $W^{\gamma,2}(\mathbb{R}^m) := H^\gamma(\mathbb{R}^m)$ is defined by

$$H^\gamma(\mathbb{R}^m) = \left\{ u \in L^2(\mathbb{R}^m) : \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} \frac{|u(x) - u(y)|^2}{|x-y|^{m+2\gamma}} dx dy < \infty \right\},$$

endowed with the norm

$$\|u\|_{H^\gamma(\mathbb{R}^m)} = \left(\int_{\mathbb{R}^m} |u(x)|^2 dx + \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} \frac{|u(x) - u(y)|^2}{|x-y|^{m+2\gamma}} dx dy \right)^{\frac{1}{2}}.$$

From now on, we denote by $|\cdot|$ and $\|\cdot\|_p$ the norms in $\mathbb{R}^m$ and $L^p(\mathbb{R}^m)$ for $p \neq 2$, respectively, whereas we denote the norm and the inner product of $L^2(\mathbb{R}^m)$ by $\|\cdot\|$ and $(\cdot,\cdot)$, respectively. With some abuse of notation, $(\cdot,\cdot)$ will stand also for the pairing between $L^q(\mathbb{R}^m)$ and $L^p(\mathbb{R}^m)$, where q is the conjugate number of p. Moreover, the Gagliardo semi-norm of $H^\gamma(\mathbb{R}^m)$, denoted by $\|\cdot\|_{\dot{H}^\gamma(\mathbb{R}^m)}$, is written as

$$\|u\|^2_{\dot{H}^\gamma(\mathbb{R}^m)} = \int_{\mathbb{R}^m}\int_{\mathbb{R}^m} \frac{|u(x)-u(y)|^2}{|x-y|^{m+2\gamma}} dxdy, \quad u \in H^\gamma(\mathbb{R}^m). \tag{2.3}$$

Thus, $\|u\|^2_{H^\gamma(\mathbb{R}^m)} = \|u\|^2 + \|u\|^2_{\dot{H}^\gamma(\mathbb{R}^m)}$ for all $u \in H^\gamma(\mathbb{R}^m)$. Note that $H^\gamma(\mathbb{R}^m)$ is a Hilbert space with the inner product,

$$\begin{aligned}(u,v)_{H^\gamma(\mathbb{R}^m)} = &\int_{\mathbb{R}^m} u(x)v(x)dx \\ &+ \int_{\mathbb{R}^m}\int_{\mathbb{R}^m} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{m+2\gamma}} dxdy, \qquad \forall u,v \in H^\gamma(\mathbb{R}^m).\end{aligned}$$

By [20], we know that the norm $\|u\|_{H^\gamma(\mathbb{R}^m)}$ is equivalent to $\left(\|u\|^2 + \|(-\Delta)^{\frac{\gamma}{2}}u\|^2\right)^{\frac{1}{2}}$ for $u \in H^\gamma(\mathbb{R}^m)$. More precisely, we have

$$\|u\|^2_{H^\gamma(\mathbb{R}^m)} = \|u\|^2 + \frac{2}{C(m,\gamma)}\|(-\Delta)^{\frac{\gamma}{2}}u\|^2, \qquad \forall u \in H^\gamma(\mathbb{R}^m). \tag{2.4}$$

We will use the notation $V = H^1(\mathbb{R}^m)$, $H = L^2(\mathbb{R}^m)$, $V_\gamma = H^\gamma(\mathbb{R}^m)$, while their dual spaces are denoted by V^*, H^* and V_γ^*, respectively. Moreover, the norms in V and V_γ will be denoted by $\|\cdot\|_V$ and $\|\cdot\|_{V_\gamma}$. It follows from [20] that the embeddings $V \subset V_{\gamma_2} \subset V_{\gamma_1}$ are continuous for $0 < \gamma_1 \le \gamma_2 < 1$. Identifying H with its dual, we obtain the following chain of continuous embeddings

$$V \subset V_\gamma \subset H = H \subset V_\gamma^* \subset V^*, \quad \text{for } \gamma \in (0,1)\,.$$

Let $b_\gamma : V_\gamma \times V_\gamma \to \mathbb{R}$ be a bilinear form given by

$$b_\gamma(v_1,v_2) = \frac{1}{2}C(m,\gamma)\int_{\mathbb{R}^m}\int_{\mathbb{R}^m} \frac{(v_1(x)-v_1(y))(v_2(x)-v_2(y))}{|x-y|^{m+2\gamma}} dxdy, \quad \forall v_1,v_2 \in V_\gamma, \tag{2.5}$$

where $C(m,\gamma)$ is the constant in (2.2). We associate the operator $A^\gamma : V_\gamma \to V_\gamma^*$ with b_γ by

$$< A^\gamma(v_1), v_2 >_{(V_\gamma^*,V_\gamma)} = b_\gamma(v_1,v_2), \qquad \forall v_1,v_2 \in V_\gamma,$$

where $< \cdot,\cdot >_{(V_\gamma^*,V_\gamma)}$ is the duality pairing of V_γ^* and V_γ. Similarly, pairing between spaces V and V^* will be denoted by $< \cdot,\cdot >_{(V^*,V)}$.

2.2 Setting of the problem

Throughout this paper, we assume that $f : \mathbb{R}\times\mathbb{R}^m\times\mathbb{R} \to \mathbb{R}$ is continuous, and such that for all t, $u \in \mathbb{R}$ and $x \in \mathbb{R}^m$, $f(t,x,\cdot) \in C^1(\mathbb{R})$ and

$$\frac{\partial f}{\partial u}(t,x,u) \le \sigma, \tag{2.6}$$

$$f(t,x,u)u \leq -\mu|u|^2 + \psi_1(t,x), \tag{2.7}$$

$$|f(t,x,u)| \leq \psi_2(t,x)|u| + \psi_3(t,x), \tag{2.8}$$

where $\mu > 0,\ \sigma \geq 0,\ \psi_1 \in L^1_{loc}(\mathbb{R}; L^1(\mathbb{R}^m)),\ \psi_2 \in L^\infty_{loc}(\mathbb{R}; L^\infty(\mathbb{R}^m)),\ \psi_3 \in L^2_{loc}(\mathbb{R}; L^2(\mathbb{R}^m))$ and ψ_i are nonnegative for $i = 1, 2, 3$.

Let us establish the definitions of weak solutions to problems (1.1)-(1.2)

Definition 2.1 *Let $\tau \in \mathbb{R}$, $u_\tau \in H$, $h \in L^2_{loc}(\mathbb{R}; L^2(\mathbb{R}^m))$ and $\gamma \in (0,1)$. The function $u \in C([\tau,\infty), H)$ is said to be a weak solution to problem (1.1) if $u(\tau) = u_\tau$, $u \in L^2_{loc}(\tau,\infty; V_\gamma)$, $\dfrac{du}{dt} \in L^2_{loc}(\tau,\infty; V^*_\gamma)$ and u satisfies, for every $\xi \in V_\gamma$,*

$$\frac{d}{dt}(u,\xi) + \frac{1}{2}C(m,\gamma)\int_{\mathbb{R}^m}\int_{\mathbb{R}^m}\frac{(u(t,x)-u(t,y))(\xi(x)-\xi(y))}{|x-y|^{m+2\gamma}}dxdy$$
$$= \int_{\mathbb{R}^m}(f(t,x,u(t,x)) + h(t,x))\,\xi(x)\,dx, \tag{2.9}$$

in the sense of scalar distributions on (τ,∞).

It is known (see [15, Theorem 3.2]) that problem (1.1) possesses a unique weak solution for each $u_\tau \in H$ and $\gamma \in (0,1)$, which is continuous with respect to the initial datum u_τ. Moreover, it satisfies the energy equation,

$$\frac{d}{dt}\|u\|^2 + C(m,\gamma)\|u\|^2_{\dot{H}^\gamma(\mathbb{R}^m)} = 2\int_{\mathbb{R}^m}(f(t,x,u(t,x)) + h(t,x))\,u(t,x)\,dx, \quad \text{for a.a. } t \geq \tau.$$

Definition 2.2 *Let $\tau \in \mathbb{R}$, $u_\tau \in H$, $h \in L^2_{loc}(\mathbb{R}; L^2(\mathbb{R}^m))$ and $\gamma = 1$. The function $u \in C([\tau,\infty), H)$ is said to be a weak solution to problem (1.2) if $u(\tau) = u_\tau$, $u \in L^2_{loc}(\tau,\infty; V)$, $\dfrac{du}{dt} \in L^2_{loc}(\tau,\infty; V^*)$ and u satisfies, for every $\xi \in V$,*

$$\frac{d}{dt}(u,\xi) + \int_{\mathbb{R}^m}\nabla u\cdot\nabla\xi dx = \int_{\mathbb{R}^m}(f(t,x,u(t,x)) + h(t,x))\,\xi(x)\,dx, \tag{2.10}$$

in the sense of scalar distributions on (τ,∞).

As before, it is also well known (see e.g. [28]) that problem (1.2) possesses a unique weak solution for every $u_\tau \in H$, which is continuous with respect to the initial datum u_τ. Also, it satisfies the energy equation,

$$\frac{d}{dt}\|u\|^2 + 2\|u\|^2_V = 2\int_{\mathbb{R}^m}(f(t,x,u(t,x)) + h(t,x))\,u(t,x)\,dx, \quad \text{for a.a. } t \geq \tau.$$

Let us define the function $\widetilde{f} : \mathbb{R}\times\mathbb{R}^m\times\mathbb{R} \to \mathbb{R}$ by

$$\widetilde{f}(t,x,u) = f(t,x,u) - \sigma u.$$

It is clear that

$$\frac{\partial \widetilde{f}}{\partial u}(t,x,u) \leq 0, \tag{2.11}$$

$$\widetilde{f}(t,x,u)u \leq -(\mu+\sigma)|u|^2 + \psi_1(t,x), \tag{2.12}$$

$$|\widetilde{f}(t,x,u)| \leq (\psi_2(t,x)+\sigma)|u| + \psi_3(t,x). \tag{2.13}$$

In this way, problems (1.1) and (1.2) can be rewritten as

$$\begin{cases} \dfrac{\partial u}{\partial t} + (-\Delta)^\gamma u - \sigma u = \widetilde{f}(t,x,u) + h(t,x), \quad (x,t) \in \mathbb{R}^m \times (\tau,\infty)\,, \\ u(\tau,x) = u_\tau(x), \quad x \in \mathbb{R}^m, \end{cases} \tag{2.14}$$

and

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u - \sigma u = \widetilde{f}(t,x,u) + h(t,x), \quad (x,t) \in \mathbb{R}^m \times (\tau,\infty)\,, \\ u(\tau,x) = u_\tau(x), \quad x \in \mathbb{R}^m, \end{cases} \tag{2.15}$$

respectively. Further, we define the $\overline{f} : \mathbb{R} \times \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}$ by

$$\overline{f}(t,x,v) = \widetilde{f}(t,x,ve^{\sigma t})e^{-\sigma t}.$$

Consequently,

$$\frac{\partial \overline{f}}{\partial v}(t,x,v) \leq 0, \tag{2.16}$$

$$\begin{aligned} \overline{f}(t,x,v)v &= \widetilde{f}(t,x,ve^{\sigma t})ve^{-\sigma t} \\ &\leq -(\mu+\sigma)|v|^2 + \psi_1(t,x)e^{-2\sigma t}, \end{aligned} \tag{2.17}$$

$$|\overline{f}(t,x,v)| \leq (\psi_2(t,x)+\sigma)|v| + \psi_3(t,x)e^{-\sigma t}. \tag{2.18}$$

By performing the change of variable $v = ue^{-\sigma t}$, problems (1.1) and (1.2) formally become

$$\begin{cases} \dfrac{\partial v}{\partial t} + (-\Delta)^\gamma v = \overline{f}(t,x,v) + e^{-\sigma t}h(t,x), \quad (x,t) \in \mathbb{R}^m \times (\tau,\infty)\,, \\ v(\tau,x) = e^{-\sigma\tau}u_\tau(x), \quad x \in \mathbb{R}^m, \end{cases} \tag{2.19}$$

and

$$\begin{cases} \dfrac{\partial v}{\partial t} - \Delta v = \overline{f}(t,x,v) + e^{-\sigma t}h(t,x), \quad (x,t) \in \mathbb{R}^m \times (\tau,\infty)\,, \\ v(\tau,x) = e^{-\sigma\tau}u_\tau(x), \quad x \in \mathbb{R}^m. \end{cases} \tag{2.20}$$

Lemma 2.3 *The function $u\,(t)$ is a weak solution to problem (1.1) if and only if $v\,(t) = u(t)e^{-\sigma t}$ is a weak solution to problem (2.19).*

Proof. Let $u(t)$ be a weak solution to problem (1.1). Then it is clear that $v \in C([\tau,\infty),H)$, $v(\tau)=e^{-\sigma\tau}u_\tau$, $v \in L^2_{loc}(\tau,\infty;V_\gamma)$, $\dfrac{dv}{dt} \in L^2_{loc}(\tau,\infty;V_\gamma^*)$ and

$$\frac{dv}{dt}=\frac{du}{dt}e^{-\sigma t}-\sigma e^{-\sigma t}u.$$

Hence,

$$\begin{aligned}
&\frac{d}{dt}(v,\xi)+\frac{1}{2}C(m,\gamma)\int_{\mathbb{R}^m}\int_{\mathbb{R}^m}\frac{(v(t,x)-v(t,y))(\xi(x)-\xi(y))}{|x-y|^{m+2\gamma}}dxdy\\
&-\int_{\mathbb{R}^m}\left(\overline{f}(t,x,v(t,x))+e^{-\sigma t}h(t,x)\right)\xi(x)\,dx\\
&=e^{-\sigma t}\frac{d}{dt}(u,\xi)+e^{-\sigma t}\frac{1}{2}C(m,\gamma)\int_{\mathbb{R}^m}\int_{\mathbb{R}^m}\frac{(u(t,x)-u(t,y))(\xi(x)-\xi(y))}{|x-y|^{m+2\gamma}}dxdy\\
&\quad -e^{-\sigma t}\int_{\mathbb{R}^m}\Bigg(\underbrace{\sigma u(t,x)+\widetilde{f}(t,x,e^{\sigma t}v(t,x))}_{=f(t,x,u(t,x))}+h(t,x)\Bigg)\xi(x)\,dx\\
&=0.
\end{aligned}$$

The converse is proved in a similar way. ■

Likewise we prove the analogous statement for problem (1.2).

Lemma 2.4 *The function $u(t)$ is a weak solution to problem (1.2) if and only if $v(t)=u(t)e^{-\sigma t}$ is a weak solution to problem (2.20).*

3 Some properties of the fractional Laplacian operator $(-\Delta)^\gamma$

We establish in this section several properties for the fractional Laplacian operator $(-\Delta)^\gamma$, which are the essential tools to prove the key result of this manuscript. Let $C_0^\infty(X)$ be the space of infinitely differentiable functions $u: X\to\mathbb{R}$ with compact support. Denote by $\mathcal{D}'$ the space of distributions of $\mathcal{D}=C_0^\infty((\tau,\tau+T)\times\mathbb{R}^m)$ and by $<\cdot,\cdot>_{(\mathcal{D}',\mathcal{D})}$ the duality pairing of $\mathcal{D}'$ and $\mathcal{D}$. Besides, let us denote by B_R the ball in $\mathbb{R}^m$ centered at 0 with radius R.

Initially, recall the well-known limit result from [20, Proposition 4.4] that if $u\in C_0^\infty(\mathbb{R}^m)$, then

$$\lim_{\gamma\to 1^-}(-\Delta)^\gamma u(x)=-\Delta u(x),\qquad \text{for all } x\in\mathbb{R}^m. \tag{3.1}$$

Now, we will extend this convergence in several phase spaces.

Lemma 3.1 *For any $\tau\in\mathbb{R}$, $T>0$ and $u\in C_0^\infty((\tau,\tau+T)\times\mathbb{R}^m)$, the following statement holds:*

$$\lim_{\gamma\to 1^-}(-\Delta)^\gamma u=-\Delta u,\ \textit{strongly in } L^p((\tau,\tau+T)\times\mathbb{R}^m),\quad \forall p\geq 1.$$

In particular, $\lim_{\gamma\to 1^-}(-\Delta)^\gamma u=-\Delta u$ in the sense of distributions in $\mathcal{D}'$.

Proof. Due to the singularity of the kernel (cf. (2.1)) of the fractional Laplacian operator $(-\Delta)^\gamma$, we will split the estimates into two parts.

(i) On the one hand, by [20, (4.14)], we have

$$
\begin{aligned}
&C(m,\gamma)\left|\int_{B_1}\frac{u(t,x+y)+u(t,x-y)-2u(t,x)}{|y|^{m+2\gamma}}dy\right| \\
&\le C(m,\gamma)\frac{\omega_{m-1}\,\|u\|_{C^2((\tau,\tau+T)\times\mathbb{R}^m)}}{2(1-\gamma)}\le R_1,\quad \forall t\in(\tau,\tau+T),\ x\in\mathbb{R}^m,\ \gamma\ge\gamma_0,
\end{aligned}\tag{3.2}
$$

for some constants $R_1>0$ and $\gamma_0\in(0,1)$, where ω_{m-1} is the $(m-1)$-dimensional measure of the unit sphere S^{m-1}. In order to obtain the last inequality, we have used the following result from [20, Corollary 4.2]:

$$\lim_{\gamma\to 1^-}\frac{C(m,\gamma)}{1-\gamma}=\frac{4m}{\omega_{m-1}}.$$

On the other hand, since $u\in C_0^\infty((\tau,\tau+T)\times\mathbb{R}^m)$, there exists $R>0$ such that,

$$\int_{B_1}\frac{u(t,x+y)+u(t,x-y)-2u(t,x)}{|y|^{m+2\gamma}}dy=0,\qquad \text{for }\ x\notin B_R. \tag{3.3}$$

Combining (3.2) and (3.3) to define

$$v_1(t,x)=\begin{cases}R_1, & \text{if }\ |x|\le R,\\ 0, & \text{if }\ |x|>R,\end{cases}$$

it is immediate to know that

$$C(m,\gamma)\left|\int_{B_1}\frac{u(t,x+y)+u(t,x-y)-2u(t,x)}{|y|^{m+2\gamma}}dy\right|\le v_1(t,x),\qquad \text{for a.a. }\ x\in\mathbb{R}^m,$$

and $v_1\in L^p((\tau,\tau+T)\times\mathbb{R}^m)$.

(ii) Secondly, it follows from the Hölder inequality that there exists a positive constant R_2 such that for all $\gamma\ge\gamma_0$ with $\gamma_0\in(0,1)$, we have

$$
\begin{aligned}
&C(m,\gamma)\left|\int_{\mathbb{R}^m\setminus B_1}\frac{u(t,x+y)+u(t,x-y)-2u(t,x)}{|y|^{m+2\gamma}}dy\right| \\
&\le C(m,\gamma)\left(\int_{\mathbb{R}^m\setminus B_1}\frac{1}{|y|^{m+2\gamma}}dy\right)^{\frac{p-1}{p}}\left(\int_{\mathbb{R}^m\setminus B_1}\frac{(u(t,x+y)+u(t,x-y)-2u(t,x))^p}{|y|^{m+2\gamma}}dy\right)^{\frac{1}{p}} \\
&\le R_2\left(\frac{\omega_{m-1}}{2\gamma_0}\right)^{\frac{p-1}{p}}\left(\int_{\mathbb{R}^m\setminus B_1}\frac{(u(t,x+y)+u(t,x-y)-2u(t,x))^p}{|y|^{m+2\gamma_0}}dy\right)^{\frac{1}{p}}:=v_2(t,x),
\end{aligned}\tag{3.4}
$$

for all $t\in(\tau,\tau+T)$ and $x\in\mathbb{R}^m$. Now, we will check $v_2\in L^p((\tau,\tau+T)\times\mathbb{R}^m)$. In light of the

Hölder inequality, we deduce that there exist positive constants R_3 and R_4 such that,

$$\begin{aligned}
&\int_{\tau}^{\tau+T}\int_{\mathbb{R}^m}|v_2(t,x)|^p dxdt\\
\le\ & R_3\int_{\mathbb{R}^m\setminus B_1}\frac{1}{|y|^{m+2\gamma_0}}\int_{\tau}^{\tau+T}\int_{\mathbb{R}^m}(u(t,x+y)+u(t,x-y)-2u(t,x))^p\, dxdtdy\\
\le\ & R_4\|u\|^p_{L^p((\tau,\tau+T)\times\mathbb{R}^m)}\frac{\omega_{m-1}}{2\gamma_0}.
\end{aligned}$$

Hence, let $v(t,x)=\max\{v_1(t,x),v_2(t,x)\}$ for a.a. $(t,x)\in(\tau,\tau+T)\times\mathbb{R}^m$, we deduce from **(i)** and **(ii)** that $v\in L^p((\tau,\tau+T)\times\mathbb{R}^m)$, and

$$|(-\Delta)^\gamma u(t,x)|\le v(t,x),\qquad \forall\gamma\ge\gamma_0.$$

The first statement follows from (3.1) and Lebesgue's theorem immediately.

Furthermore, for arbitrary $\varphi\in\mathcal{D}$, we obtain

$$\langle(-\Delta)^\gamma u+\Delta u,\varphi\rangle_{(\mathcal{D}',\mathcal{D})}=\int_{\tau}^{\tau+T}\int_{\mathbb{R}^m}\left((-\Delta)^\gamma u(t,x)+\Delta u(t,x)\right)\varphi(t,x)dxdt\overset{\gamma\to1^-}{\longrightarrow}0.$$

The proof of this lemma is complete. ■

Now, in a similar way we prove the following result.

Lemma 3.2 *For any $u\in C_0^\infty(\mathbb{R}^m)$, $\lim_{\gamma\to1^-}(-\Delta)^\gamma u=-\Delta u$ strongly in $L^p(\mathbb{R}^m)$ for any $p\ge1$.*

We next establish the continuity of the operator $(-\Delta)^\gamma$ with respect to the parameter γ.

Lemma 3.3 *For any $\tau\in\mathbb{R}$, $T>0$, $u\in C_0^\infty((\tau,\tau+T)\times\mathbb{R}^m)$ and $\gamma_0\in(0,1)$, the following statement holds:*

$$\lim_{\gamma\to\gamma_0}(-\Delta)^\gamma u=(-\Delta)^{\gamma_0}u,\ \textit{strongly in}\ L^p((\tau,\tau+T)\times\mathbb{R}^m),\quad\forall p\ge1.$$

Proof. Let $\gamma\to\gamma_0\in(0,1)$. We first show $\lim_{\gamma\to\gamma_0}(-\Delta)^\gamma u(t,x)=(-\Delta)^{\gamma_0}u(t,x)$ for any $(t,x)\in(\tau,\tau+T)\times\mathbb{R}^m$. It is obvious that,

$$\frac{u(t,x+y)+u(t,x-y)-2u(t,x)}{|y|^{m+2\gamma}}\overset{\gamma\to\gamma_0}{\longrightarrow}\frac{u(t,x+y)+u(t,x-y)-2u(t,x)}{|y|^{m+2\gamma_0}},\quad\text{for a.a. } y\in\mathbb{R}^m.$$

For $\gamma\le\gamma_0+\alpha$, we have

$$\left|\frac{u(t,x+y)+u(t,x-y)-2u(t,x)}{|y|^{m+2\gamma}}\right|\le\frac{\|D^2u\|_{L^\infty((\tau,\tau+T)\times\mathbb{R}^m)}}{|y|^{m+2(\gamma_0-1+\alpha)}},\quad\text{if } |y|<1,$$

and for $\gamma\ge\gamma_0-\alpha$,

$$\left|\frac{u(t,x+y)+u(t,x-y)-2u(t,x)}{|y|^{m+2\gamma}}\right|\le\frac{3\|u\|_{L^\infty((\tau,\tau+T)\times\mathbb{R}^m)}}{|y|^{m+2(\gamma_0-\alpha)}},\quad\text{if } |y|\ge1,$$

where D^2u stands for the matrix of second derivatives of u. Hence, (2.1), the continuity of Gamma function for positive part and Lebesgue's theorem yield

$$\begin{aligned}(-\Delta)^{\gamma} u(t,x) &= -\frac{1}{2}C(m,\gamma)\int_{\mathbb{R}^m}\frac{u(t,x+y)+u(t,x-y)-2u(t,x)}{|y|^{m+2\gamma}}dy \\ &\overset{\gamma\to\gamma_0}{\longrightarrow} -\frac{1}{2}C(m,\gamma_0)\int_{\mathbb{R}^m}\frac{u(t,x+y)+u(t,x-y)-2u(t,x)}{|y|^{m+2\gamma_0}}dy \\ &= (-\Delta)^{\gamma_0} u(t,x).\end{aligned}$$

In what follows, we check the above convergence is true in $L^p((\tau,\tau+T)\times\mathbb{R}^m)$. Take a small $\alpha>0$ such that $|\gamma-\gamma_0|<\alpha$, namely, $\gamma_0-\alpha<\gamma<\gamma_0+\alpha$. Let us proceed likewise as in the proof of Lemma 3.1. Notice that

$$\begin{aligned}&C(m,\gamma)\left|\int_{|y|<1}\frac{u(t,x+y)+u(t,x-y)-2u(t,x)}{|y|^{m+2\gamma}}dy\right| \\ &= C(m,\gamma)\left|\int_{|y|<1}\frac{1}{|y|^{m+2\gamma}}\int_0^1\int_{-r}^{r}D^2u(t,x+sy)y\cdot ydsdrdy\right| \\ &\le K_1\int_{|y|<1}\frac{1}{|y|^{m+2(\gamma_0-1+\alpha)}}\int_{-1}^{1}\left|D^2u(t,x+sy)\right|dsdy := v_3(t,x),\end{aligned} \tag{3.5}$$

where K_1 is a positive constant. Thanks to $u\in C_0^\infty((\tau,\tau+T)\times\mathbb{R}^m)$, there exists a sufficiently large R such that, for any $y\in B_1$, $s\in[-1,1]$ and $t\in[\tau,\tau+T]$,

$$D^2u(t,x+sy)=0,\quad \forall x\notin B_R.$$

Also, we find a positive constant K_2 satisfying $\left|D^2u(t,x+sy)\right|\le K_2$. Thus, $v_3\in L^p((\tau,\tau+T)\times\mathbb{R}^m)$.

Similarly, for $y\in\mathbb{R}^m\backslash B_1$, there exists a positive constant K_3 such that

$$\begin{aligned}&C(m,\gamma)\left|\int_{|y|\ge 1}\frac{u(t,x+y)+u(t,x-y)-2u(t,x)}{|y|^{m+2\gamma}}dy\right| \\ &\le K_3\left|\int_{|y|\ge 1}\frac{u(t,x+y)+u(t,x-y)-2u(t,x)}{|y|^{m+2\gamma_0-2\alpha}}dy\right| := v_4(t,x),\end{aligned} \tag{3.6}$$

where the last inequality holds since

$$\frac{1}{|y|^{m+2\gamma}}\le\frac{1}{|y|^{m+2(\gamma_0-\alpha)}},\quad \text{if } |y|\ge 1.$$

The function v_4 belongs to $L^p((\tau,\tau+T)\times\mathbb{R}^m)$. Indeed, by the Hölder inequality, we derive

$$\begin{aligned}
&\int_\tau^{\tau+T}\int_{\mathbb{R}^m} v_4^p(t,x)dxdt\\
&\le \int_\tau^{\tau+T}\int_{\mathbb{R}^m} K_3^p\left(\int_{|y|\ge 1}\frac{u(t,x+y)+u(t,x-y)-2u(t,x)}{|y|^{m+2\gamma_0-2\alpha}}dy\right)^p dxdt\\
&\le K_3^p\left(\int_{|y|\ge 1}\frac{1}{|y|^{m+2\gamma_0-2\alpha}}dy\right)^{p-1}\\
&\quad\times\int_{|y|\ge 1}\frac{1}{|y|^{m+2\gamma_0-2\alpha}}\int_\tau^{\tau+T}\int_{\mathbb{R}^m}\left(u(t,x+y)+u(t,x-y)-2u(t,x)\right)^p dxdtdy\\
&\le K_4\left\|u\right\|^p_{L^p((\tau,\tau+T)\times\mathbb{R}^m)}.
\end{aligned} \tag{3.7}$$

Thus, Lebesgue's theorem, together with (3.5)-(3.7), concludes the proof of this lemma. ∎

In the same way, we can prove the following result.

Lemma 3.4 *For any $u\in C_0^\infty(\mathbb{R}^m)$, $\lim_{\gamma\to\gamma_0^-}(-\Delta)^\gamma u=(-\Delta)^{\gamma_0}u$ strongly in $L^p(\mathbb{R}^m)$ for any $p\ge 1$ and $\gamma_0\in(0,1)$.*

Our objective now is to obtain some properties of $(-\Delta)^{\frac{\gamma}{2}}u$, $\gamma\in(0,1)$, regarding the fractional Laplacian operators. Let

$$D_\gamma=D((-\Delta)^\gamma)=\{u\in L^2(\mathbb{R}^m):(-\Delta)^\gamma u\in L^2(\mathbb{R}^m)\}.$$

Lemma 3.5 *For any $u\in H^2(\mathbb{R}^m)$ and $\gamma\in(0,1)$, there exists a constant $C_\gamma>0$ such that*

$$\left\|(-\Delta)^\gamma u\right\|\le C_\gamma\left\|u\right\|_{H^2(\mathbb{R}^m)}.$$

In particular, the embedding $H^2(\mathbb{R}^m)\subset D_\gamma$ is continuous.

Proof. For $u\in H^2(\mathbb{R}^m)$, let $u_n\in C_0^\infty(\mathbb{R}^m)$ be a sequence converging to u in $H^2(\mathbb{R}^m)$. Observe the Hölder inequality implies that

$$\begin{aligned}
&\frac{1}{4}C^2(m,\gamma)\int_{\mathbb{R}^m}\left|\int_{|y|<1}\frac{u(x+y)+u(x-y)-2u(x)}{|y|^{m+2\gamma}}dy\right|^2 dx\\
\le\ &\frac{1}{4}C^2(m,\gamma)\int_{|y|<1}\frac{1}{|y|^{m+2(\gamma-1)}}dy\int_{\mathbb{R}^m}\int_{|y|<1}\frac{\left(u(x+y)+u(x-y)-2u(x)\right)^2}{|y|^{m+2\gamma+2}}dydx.
\end{aligned} \tag{3.8}$$

If the last integral is well defined and can be bounded by a suitable estimate in terms of the norm $\|u\|_{H^2(\mathbb{R}^m)}$, the result of this lemma holds.

First, it is clear for a.a. $x\in\mathbb{R}^m$ and $y\in B_1$ that

$$\frac{\left(u_n(x+y)+u_n(x-y)-2u_n(x)\right)^2}{|y|^{m+2\gamma+2}}\overset{n\to\infty}{\longrightarrow}\frac{\left(u(x+y)+u(x-y)-2u(x)\right)^2}{|y|^{m+2\gamma+2}}. \tag{3.9}$$

On the one hand, making use of a similar idea as in Lemma 3.3, we infer that there exist some positive constants C_1 and $C_{2,\gamma}$, such that

$$
\begin{aligned}
&\int_{\mathbb{R}^m}\int_{|y|<1}\frac{(u_n(x+y)+u_n(x-y)-2u_n(x))^2}{|y|^{m+2\gamma+2}}dydx\\
\le&\int_{\mathbb{R}^m}\int_{|y|<1}\frac{1}{|y|^{m+2\gamma+2}}\left(\int_0^1\int_{-r}^{r}D^2u_n(x+sy)y\cdot ydsdr\right)^2dydx\\
\le&\int_{\mathbb{R}^m}\int_{|y|<1}\frac{1}{|y|^{m+2(\gamma-1)}}\left(\int_{-1}^{1}\left|D^2u_n(x+sy)\right|ds\right)^2dydx\\
\le&\, C_1\int_{|y|<1}\frac{1}{|y|^{m+2(\gamma-1)}}\int_{-1}^{1}\int_{\mathbb{R}^m}\left|D^2u_n(x+sy)\right|^2dxdsdy\\
\le&\, C_1\int_{|y|<1}\frac{1}{|y|^{m+2(\gamma-1)}}\int_{-1}^{1}\int_{\mathbb{R}^m}\left|D^2u_n(x)\right|^2dxdsdy\\
\le&\, C_{2,\gamma}\int_{\mathbb{R}^m}|D^2u_n(x)|^2dx.
\end{aligned}
$$

Then, (3.9) and Fatou's lemma imply that there exist positive constants $C_{3,\gamma}$ such that

$$
\begin{aligned}
&\int_{\mathbb{R}^m}\int_{|y|<1}\frac{(u(x+y)+u(x-y)-2u(x))^2}{|y|^{m+2\gamma+2}}dydx\\
\le&\liminf_{n\to\infty}\int_{\mathbb{R}^m}\int_{|y|<1}\frac{(u_n(x+y)+u_n(x-y)-2u_n(x))^2}{|y|^{m+2\gamma+2}}dydx\\
\le&\, C_{2,\gamma}\lim_{n\to\infty}\int_{\mathbb{R}^m}\left|D^2u_n(x)\right|^2dx\\
=&\, C_{2,\gamma}\int_{\mathbb{R}^m}\left|D^2u(x)\right|^2dx\le C_{3,\gamma}\|u\|^2_{H^2(\mathbb{R}^m)}.
\end{aligned}
\tag{3.10}
$$

On the other hand, by the Hölder inequality, we know there exist some positive constants $C_{4,\gamma}$ and $C_{5,\gamma}$, such that

$$
\begin{aligned}
&\frac{1}{4}C^2(m,\gamma)\int_{\mathbb{R}^m}\left|\int_{|y|\ge1}\frac{u(x+y)+u(x-y)-2u(x)}{|y|^{m+2\gamma}}dy\right|^2dx\\
\le&\frac{1}{4}C^2(m,\gamma)\int_{\mathbb{R}^m}\int_{|y|\ge1}\frac{1}{|y|^{m+2\gamma}}dy\int_{|y|\ge1}\frac{(u(x+y)+u(x-y)-2u(x))^2}{|y|^{m+2\gamma}}dydx\\
\le&\, C_{4,\gamma}\int_{|y|\ge1}\frac{1}{|y|^{m+2\gamma}}\int_{\mathbb{R}^m}(u(x+y)+u(x-y)-2u(x))^2dxdy\\
\le&\, C_{5,\gamma}\|u\|^2.
\end{aligned}
\tag{3.11}
$$

Collecting together (3.8)-(3.11), we obtain that $(-\Delta)^\gamma u\in L^2(\mathbb{R}^m)$ and

$$
\|(-\Delta)^\gamma u\|\le C_\gamma\|u\|_{H^2(\mathbb{R}^m)},
$$

where $C_\gamma:=\max\{C_{3,\gamma},C_{5,\gamma}\}$. The proof of this lemma is complete. ■

Lemma 3.6 *If $u \in C_0^\infty(\mathbb{R}^m)$, then*

$$\left((-\Delta)^{\frac{\gamma}{2}} u, (-\Delta)^{\frac{\gamma}{2}} u\right) = \left((-\Delta)^{\gamma} u, u\right), \qquad \textit{for any } \gamma \in (0,1).$$

Proof. By Proposition 3.6 in [20], for any $u \in V_\gamma$, we have

$$\left((-\Delta)^{\frac{\gamma}{2}} u, (-\Delta)^{\frac{\gamma}{2}} u\right) = \left\|(-\Delta)^{\frac{\gamma}{2}} u\right\|^2 = \frac{1}{2}C(m,s)\,\|u\|^2_{\dot{H}^\gamma(\mathbb{R}^m)}.$$

Meanwhile, it follows from (2.3) that

$$\begin{aligned}\frac{1}{2}C(m,\gamma)\|u\|^2_{\dot{H}^\gamma(\mathbb{R}^m)} &= \frac{1}{2}C(m,\gamma)\int_{\mathbb{R}^m}\int_{\mathbb{R}^m}\frac{|u(x)-u(y)|^2}{|x-y|^{m+2\gamma}}dxdy \\ &= \langle A^\gamma(u), u\rangle_{(V_\gamma^*, V_\gamma)}.\end{aligned}$$

Hence, for $u \in C_0^\infty(\mathbb{R}^m)$, we have

$$\begin{aligned}&\left((-\Delta)^{\frac{\gamma}{2}} u, (-\Delta)^{\frac{\gamma}{2}} u\right) \\ &= \frac{1}{2}C(m,\gamma)\int_{\mathbb{R}^m}\int_{\mathbb{R}^m}\frac{|u(x)-u(y)|^2}{|x-y|^{m+2\gamma}}dxdy \\ &= C(m,\gamma)\int_{\mathbb{R}^m}u(x)P.V.\int_{\mathbb{R}^m}\frac{u(x)-u(y)}{|x-y|^{m+2\gamma}}dydx \\ &= \left((-\Delta)^\gamma u, u\right).\end{aligned}$$

Here $P.V.$ is a commonly used abbreviation for "in the principal value sense", for more details, see [20]. ■

Lemma 3.7 *If $u \in C_0^\infty(\mathbb{R}^m)$, then* $\left\|(-\Delta)^{\frac{1}{2}} u\right\| = \|\nabla u\|.$

Proof. Taking a sequence $\alpha_n \to 1^-$ as $n \to \infty$, making use of Lemma 3.2 and Lemma 3.6, we obtain

$$\left\|(-\Delta)^{\frac{\alpha_n}{2}} u\right\|^2 = \left((-\Delta)^{\frac{\alpha_n}{2}} u, (-\Delta)^{\frac{\alpha_n}{2}} u\right) = \left((-\Delta)^{\alpha_n} u, u\right) \xrightarrow[n\to\infty]{\alpha_n\to 1^-} (-\Delta u, u) = \|\nabla u\|^2.$$

Moreover, it follows from Lemma 3.4 that,

$$\left\|(-\Delta)^{\frac{\alpha_n}{2}} u\right\| \xrightarrow[n\to\infty]{\alpha_n\to 1^-} \left\|(-\Delta)^{\frac{1}{2}} u\right\|.$$

We conclude the proof by the uniqueness of the limit. ■

Lemma 3.8 *If $u \in H^2(\mathbb{R}^m)$, then* $\left\|(-\Delta)^{\frac{1}{2}} u\right\| = \|\nabla u\|.$

Proof. We take a sequence $\{u_n\} \subset C_0^\infty(\mathbb{R}^m)$ converging to u in $H^2(\mathbb{R}^m)$. First, it is easy to see from Lemma 3.7 that

$$\left\|(-\Delta)^{\frac{1}{2}} u_n\right\| = \|\nabla u_n\|.$$

Next, as $u_n \to u$ in V, we immediately derive that

$$\|\nabla u_n\| \overset{n\to\infty}{\longrightarrow} \|\nabla u\| .$$

At last, we deduce from Lemma 3.5 that

$$\left\|(-\Delta)^{\frac{1}{2}} u_n\right\| \overset{n\to\infty}{\longrightarrow} \left\|(-\Delta)^{\frac{1}{2}} u\right\| .$$

Therefore, $\left\|(-\Delta)^{\frac{1}{2}} u\right\| = \|\nabla u\|$. ■

4 Convergence of solutions

This section is devoted to proving the solutions of problem (1.1) converge as $\gamma \to 1^-$ to the solutions of the limit problem with $\gamma = 1$, that is, to the solution of the standard reaction-diffusion equation (1.2). To that end, we begin by proving the hemicontinuity of function f.

For each $t \in \mathbb{R}$, let $F(t, \cdot) : H \to H$ be the Nemytskii operator defined by $y = F(t, u)$, if $y(x) = F(t, u)(x) := f(t, x, u(x))$ for a.a. $x \in \mathbb{R}^m$. In view of (2.8), we know that $y \in H$ if $u \in H$. Hence, this operator is well defined.

Lemma 4.1 *For each $t \in \mathbb{R}$, the function $F(t, \cdot)$ is hemicontinuous, i.e., the real map $\lambda \longmapsto (F(t, u + \lambda v), w)$ is continuous for any $u, v, w \in H$.*

Proof. It is obvious that for a.a. $x \in \mathbb{R}^m$,

$$f(t, x, u(x) + \lambda v(x))w(x) \to f(t, x, u(x) + \lambda_0 v(x))w(x), \quad \text{as } \lambda \to \lambda_0 \in \mathbb{R}.$$

By (2.8), we know that

$$\begin{aligned} |f(t, x, u(x) + \lambda v(x))w(x)| &\le (\psi_3(t, x) + \psi_2(t, x) (|u(x)| + \lambda |v(x)|)) |w(x)| \\ &= r(t, x), \quad \text{for a.a. } x \in \mathbb{R}^m, \end{aligned}$$

where $r(t,\cdot) \in L^1(\mathbb{R}^m)$. Thus, the Lebesgue theorem implies that

$$\begin{aligned} (F(t, u + \lambda v), w) &= \int_{\mathbb{R}^m} f(t, x, u(x) + \lambda v(x))w(x)dx \\ &\overset{\lambda\to\lambda_0}{\longrightarrow} \int_{\mathbb{R}^m} f(t, x, u(x) + \lambda_0 v(x))w(x)dx = (F(t, u + \lambda_0 v), w) . \end{aligned}$$

Therefore, $F(t,\cdot)$ is hemicontinuous. ■

Theorem 4.2 *Let $u_\tau^n \to u_\tau$ in H as $n \to \infty$, and let $u_n(\cdot)$ be the solution of problem (1.1) for $\gamma = \gamma_n \in (0, 1)$ with initial value u_τ^n. If $\gamma_n \to 1^-$ as $n \to \infty$, then, $u_n \to u$ weak-star in $L^\infty(\tau, \tau + T; H)$ and weakly in $L^2(\tau, \tau + T; H)$ as $n \to \infty$ for any $T > 0$, where $u(\cdot)$ is the unique solution of problem (1.1) with $\gamma = 1$. Moreover, for any sequence $t_n \in [\tau, \tau + T]$ converging to $t_0 \in [\tau, \tau + T]$, we have*

$$u_n(t_n) \to u(t_0) \text{ weakly in } H.$$

Proof. Assume first that $\sigma = 0$ in (2.6). Let $\gamma_n \in (0,1)$ and $\gamma_n \to 1^-$ as $n \to \infty$. Denote by $u_n(\cdot)$ the unique solution to problem (1.1) with initial condition u_τ^n. Multiplying equation (1.1) by u_n, making use of (2.7) and the Young inequality, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u_n\|^2 + \mu\|u_n\|^2 + \frac{1}{2}C(m,\gamma_n)\|u_n\|^2_{\dot{H}^{\gamma_n}(\mathbb{R}^m)} &\le \int_{\mathbb{R}^m}(\psi_1(t,x)+h(t,x))\,u_n(t,x)dx \\ &\le \int_{\mathbb{R}^m}\psi_1(t,x)dx + \frac{1}{2\mu}\|h(t)\|^2 + \frac{\mu}{2}\|u_n\|^2.\end{aligned} \tag{4.1}$$

Consequently,

$$\frac{d}{dt}\|u_n\|^2 + \mu\|u_n\|^2 + C(m,\gamma_n)\|u_n\|^2_{\dot{H}^{\gamma_n}(\mathbb{R}^m)} \le 2\int_{\mathbb{R}^m}\psi_1(t,x)dx + \frac{1}{\mu}\|h(t)\|^2.$$

Multiplying the above inequality by $e^{\mu s}$ and integrating it over (τ,t) with $t>\tau$, we obtain

$$\|u_n(t)\|^2 \le \|u_\tau^n\|^2 e^{-\mu(t-\tau)} + 2\int_\tau^t e^{-\mu(t-s)}\int_{\mathbb{R}^m}\psi_1(s,x)dxds + \frac{1}{\mu}\int_\tau^t e^{-\mu(t-s)}\|h(s)\|^2 ds. \tag{4.2}$$

Subsequently, for $T>0$, there exist two positive constants M_T (which depends on T) and $\overline{M}_{T,\gamma_n}$ (which depends on T and γ_n), such that

$$\sup_{t\in[\tau,\tau+T]}\|u_n(t)\| \le M_T, \tag{4.3}$$

and

$$C(m,\gamma_n)\int_\tau^{\tau+T}\|u_n(s)\|^2_{\dot{H}^{\gamma_n}(\mathbb{R}^m)}ds \le \overline{M}_{T,\gamma_n}. \tag{4.4}$$

Thus, $\{u_n\}$ is bounded in $L^\infty(\tau,\tau+T;H)$. It also follows from (4.4) that $\{(-\Delta)^{\frac{\gamma_n}{2}}u_n\}$ is bounded in $L^2(\tau,\tau+T;H)$, therefore $\{A^{\gamma_n}(u_n)\}$ is bounded in $L^2(\tau,\tau+T;V^*)$. Moreover, by (4.3) and (2.8), we infer that $\{F(\cdot,u_n)\}$ is bounded in $L^2(\tau,\tau+T;H)$. Hence, $\{\frac{du_n}{dt}\}$ is bounded in $L^2(\tau,\tau+T;V^*)$. Then there exist functions u,χ and ζ such that, up to a subsequence (relabeled the same), the following convergences take place,

$$u_n \to u \text{ weak-star in } L^\infty(\tau,\tau+T;H), \tag{4.5}$$

$$u_n \to u \text{ weakly in } L^2(\tau,\tau+T;H), \tag{4.6}$$

$$F(\cdot,u_n) \to \chi \text{ weakly in } L^2(\tau,\tau+T;H), \tag{4.7}$$

$$A^{\gamma_n}(u_n) \to \zeta \text{ weakly in } L^2(\tau,\tau+T;V^*), \tag{4.8}$$

$$\frac{du_n}{dt} \to \frac{du}{dt} \text{ weakly in } L^2(\tau,\tau+T;V^*). \tag{4.9}$$

Let us first check $\zeta = -\Delta u$. To this end, for any $\varphi \in C_0^\infty((\tau,\tau+T)\times\mathbb{R}^m)$, by Lemma 3.1, we have

$$\begin{aligned}&< A^{\gamma_n}(u_n),\varphi >_{(\mathcal{D}',\mathcal{D})} = \int_\tau^{\tau+T} < A^{\gamma_n}(u_n),\varphi >_{(V^*,V)} dt \\ &= \int_\tau^{\tau+T} < u_n, A^{\gamma_n}(\varphi) >_{(V,V^*)} dt = \int_\tau^{\tau+T}\int_{\mathbb{R}^m} u_n(-\Delta)^{\gamma_n}\varphi dxdt \\ &\xrightarrow{\gamma_n\to 1^-} \int_\tau^{\tau+T}\int_{\mathbb{R}^m} u(-\Delta)\varphi dxdt = \int_\tau^{\tau+T} < u,(-\Delta)\varphi >_{(V,V^*)} dt =< (-\Delta)u,\varphi >_{(\mathcal{D}',\mathcal{D})},\end{aligned}$$

where the above convergence follows from the facts that $u_n \to u$ weakly in $L^2(\tau, \tau+T; H)$ and $(-\Delta)^{\gamma_n}\varphi \to (-\Delta)\varphi$ in $L^2(\tau, \tau+T; H)$. Therefore, $A^{\gamma_n}(u_n) \to -\Delta u$ as $\gamma^n \to 1^-$ in the sense of distributions, which implies that $\zeta = -\Delta u$. Therefore, for any $\eta \in L^2(\tau, \tau+T; V)$, the above convergences (4.5)-(4.9) imply that

$$\int_\tau^{\tau+T} < \frac{du_n}{dt}, \eta >_{(V^*,V)} dt + \int_\tau^{\tau+T} < A^{\gamma_n}(u_n), \eta >_{(V^*,V)} dt - \int_\tau^{\tau+T} (F(t,u_n) + h(t), \eta) dt$$

$$\xrightarrow[n\to\infty]{\gamma_n\to 1^-} \int_\tau^{\tau+T} < \frac{du}{dt}, \eta >_{(V^*,V)} dt + \int_\tau^{\tau+T} < -\Delta u, \eta >_{(V^*,V)} dt - \int_\tau^{\tau+T} (\chi + h, \eta) dt = 0.$$

This equality is equivalent to (2.9) (for more details on this fact, see, e.g., [11, p.43]).

In addition, we know that

$$u \in L^\infty(\tau, \tau+T; H),$$
$$\frac{du}{dt} \in L^2(\tau, \tau+T; V^*),$$
$$-\Delta u \in L^2(\tau, \tau+T; V^*).$$

Notice that, since $-\Delta u(t) \in V^*$ for a.a. $t \in (\tau, \tau+T)$, we obtain that $u(t) \in V$. Therefore,

$$\|\nabla u(t)\|^2 =< -\Delta u(t), u(t) >_{(V^*,V)} \leq \|\Delta u(t)\|_{V^*} \|\nabla u(t)\|,$$

which implies,

$$u \in L^2(\tau, \tau+T; V).$$

Thus, u is the unique weak solution of the problem

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u = g(t), & (x,t) \in \mathbb{R}^m \times (\tau, \infty), \\ u(x,\tau) = u_\tau(x), & x \in \mathbb{R}^m, \end{cases} \tag{4.10}$$

where $g(t) = \chi(t) + h(t) \in L^2(\tau, \tau+T; H)$.

Further, we need to prove that for any sequence $t_n \in [\tau, \tau+T]$ converging to $t_0 \in [\tau, \tau+T]$, we have

$$u_n(t_n) \to u(t_0) \text{ weakly in } H.$$

It is clear that there is a subsequence $\{u_{n_k}(t_{n_k})\}$ and $y \in H$ such that $u_{n_k}(t_{n_k}) \to y$ weakly in H. If we prove that $y = u(t_0)$, then the result follows by a standard contradiction argument.

For any $R > 0$, let us consider the spaces $H_0^1(B_R)$, $L^2(B_R)$, $(H_0^1(B_R))^*$. Then we have the following chains of continuous embeddings,

$$H_0^1(B_R) \subset V \subset H \subset V^* \subset (H_0^1(B_R))^*,$$

$$H_0^1(B_R) \subset L^2(B_R) \subset (H_0^1(B_R))^*,$$

where we consider that $z \in H_0^1(B_R)$ is an element of V by setting $z(x) = 0$ for $x \notin B_R$. Let $L_R z$ be the projection of $z \in H$ onto $L^2(B_R)$. Since the embedding $L^2(B_R) \subset (H_0^1(B_R))^*$ is compact, in view of estimate (4.2), we know that the sequence $\{L_R u_n(t)\}$ is relatively compact in $(H_0^1(B_R))^*$ for any $t \in (\tau, \tau+T)$. Furthermore, by means of the fact that $\{\frac{du_n(t)}{dt}\}$ is bounded in $L^2(\tau, \tau+T; V^*) \subset L^2(\tau, \tau+T; (H_0^1(B_R))^*)$, we infer there exists a constant C, such that

$$\|u_n(t) - u_n(s)\|_{\left(H_0^1(B_R)\right)^*} \leq \int_s^t \left\| \frac{du_n}{dt}(r) \right\|_{\left(H_0^1(B_R)\right)^*} dr \leq C\,(t-s)^{\frac{1}{2}}.$$

Hence, the sequence $\{L_R u_n(\cdot)\}$ is equicontinuous in $(H_0^1(B_R))^*$. The Ascoli-Arzelà theorem shows that, up to a subsequence (relabeled the same), $L_R u_n \to L_R u$ in $C([\tau, \tau+T], (H_0^1(B_R))^*)$. By the compact embedding $L^2(B_R) \subset (H_0^1(B_R))^*$ and a diagonal argument, we deduce that $L_R u_{n_k}(t_{n_k}) \to L_R u(t_0)$, $L_R u_{n_k}(t_{n_k}) \to L_R y$ in $(H_0^1(B_R))^*$ for any $R > 0$. Therefore, $L_R u(t_0) = L_R y$ in $(H_0^1(B_R))^*$ for any $R > 0$. We obtain then immediately that $u(t_0)$ and y are the same distribution, hence, $y = u(t_0)$.

At last, let us prove that $\chi = F(\cdot, u(\cdot))$. If $u_\tau \in V$, it is known (see Lemma 4.4 below) that the unique solution $u(\cdot)$ to problem (4.10) belongs to $L^2(\tau, \tau+T; H^2(\mathbb{R}^m)) \cap L^\infty(\tau, \tau+T; V)$. As the unique weak solution $u(\cdot)$ of (4.10) satisfies that $u(t) \in V$ for a.a. $t \in (\tau, \tau+T)$, it is easy to see that

$$u \in L^2(\tau+\delta, \tau+T; H^2(\mathbb{R}^m)), \quad \forall \delta \in (0, T). \tag{4.11}$$

Now multiplying (1.1) by u_n, we obtain

$$\frac{1}{2}\frac{d}{dt}\|u_n\|^2 + \frac{1}{2}C(m,\gamma_n)\|u_n\|^2_{\dot{H}^{\gamma_n}(\mathbb{R}^m)} = (F(t, u_n(t)) + h(t,x), u_n(t)).$$

Then, integrating the above equality over $(\tau, \tau+T)$, we find

$$\begin{aligned}\int_\tau^{\tau+T} (F(t, u_n(t)), u_n(t))\,dt &= \frac{1}{2}\|u_n(\tau+T)\|^2 - \frac{1}{2}\|u_n(\tau)\|^2 \\ &\quad + \frac{1}{2}C(m,\gamma_n)\int_\tau^{\tau+T} \|u_n(t)\|^2_{\dot{H}^{\gamma_n}(\mathbb{R}^m)} dt \\ &\quad - \int_\tau^{\tau+T} (h(t,x), u_n(t))\,dt.\end{aligned} \tag{4.12}$$

For every $v \in L^2(\tau, \tau+T; H)$, define the sequence

$$X_n := \int_\tau^{\tau+T} (F(t, u_n(t)) - F(t, v(t)), u_n(t) - v(t))\,dt.$$

Therefore, it follows from (4.12) and the above equality that

$$\begin{aligned}X_n = &\frac{1}{2}\|u_n(\tau+T)\|^2 - \frac{1}{2}\|u_\tau^n\|^2 + \frac{1}{2}C(m,\gamma_n)\int_\tau^{\tau+T} \|u_n(t)\|^2_{\dot{H}^{\gamma_n}(\mathbb{R}^m)} dt \\ &- \int_\tau^{\tau+T} (h(t), u_n(t))\,dt - \int_\tau^{\tau+T} (F(t, u_n(t)), v(t))\,dt - \int_\tau^{\tau+T} (F(t, v(t)), u_n(t) - v(t))\,dt.\end{aligned}$$

On the one hand, by (2.6), (4.6), (4.7), $u_\tau^n \to u_\tau$ in H and $u_n(\tau+T) \to u(\tau+T)$ weakly in H as $n \to \infty$, we obtain

$$\begin{aligned} 0 \geq \liminf_{n\to\infty} \ X_n \geq \frac{1}{2}\left\|u(\tau+T)\right\|^2 - \frac{1}{2}\left\|u_\tau\right\|^2 + \frac{1}{2}\liminf_{n\to\infty} C(m,\gamma_n)\int_\tau^{\tau+T} \left\|u_n(t)\right\|^2_{\dot{H}^{\gamma_n}(\mathbb{R}^m)} dt \\ - \int_\tau^{\tau+T} (h(t),u(t))\,dt - \int_\tau^{\tau+T} (\chi(t),v(t))\,dt - \int_\tau^{\tau+T} (F(t,v(t)),u(t)-v(t))\,dt. \end{aligned} \tag{4.13}$$

On the other hand, multiplying (4.10) by u and integrating it over $(\tau, \tau+T)$, we have

$$\frac{1}{2}\left\|u(\tau+T)\right\|^2 - \frac{1}{2}\left\|u_\tau\right\|^2 - \int_\tau^{\tau+T} (h(t),u(t))\,dt = \int_\tau^{\tau+T} (\chi(t),u(t))\,dt - \frac{1}{2}\int_\tau^{\tau+T} \left\|\nabla u\right\|^2 dt. \tag{4.14}$$

Thus, combining (4.13) with (4.14), we deduce

$$\begin{aligned} 0 \geq &\int_\tau^{\tau+T} (\chi(t)-F(t,v(t)),u(t)-v(t))\,dt \\ &+ \frac{1}{2}\liminf_{n\to\infty} C(m,\gamma_n)\int_\tau^{\tau+T} \left\|u_n(t)\right\|^2_{\dot{H}^{\gamma_n}(\mathbb{R}^m)} dt - \frac{1}{2}\int_\tau^{\tau+T} \left\|\nabla u\right\|^2 dt. \end{aligned} \tag{4.15}$$

Observe that, for any $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that

$$\frac{1}{2}\int_\tau^{\tau+\delta} \left\|\nabla u\right\|^2 dt \leq \varepsilon.$$

Thus, (4.15) and the above inequality imply that

$$\begin{aligned} \varepsilon \geq &\int_\tau^{\tau+T} (\chi(t)-F(t,v(t)),u(t)-v(t))\,dt \\ &+ \frac{1}{2}\liminf_{n\to\infty} C(m,\gamma_n)\int_{\tau+\delta}^{\tau+T} \left\|u_n(t)\right\|^2_{\dot{H}^{\gamma_n}(\mathbb{R}^m)} dt - \frac{1}{2}\int_{\tau+\delta}^{\tau+T} \left\|\nabla u\right\|^2 dt. \end{aligned} \tag{4.16}$$

Next, we want to show that

$$\int_{\tau+\delta}^{\tau+T} \left\|\nabla u\right\|^2 dt \leq \liminf_{n\to\infty} C(m,\gamma_n)\int_{\tau+\delta}^{\tau+T} \left\|u_n(t)\right\|^2_{\dot{H}^{\gamma_n}(\mathbb{R}^m)} dt. \tag{4.17}$$

It follows from (4.4) and (2.4) that, up to a subsequence, as $\gamma_n \to 1^-$ and $n \to \infty$,

$$(-\Delta)^{\frac{\gamma_n}{2}} u_n \to y, \quad \text{weakly in } L^2(\tau,\tau+T;H). \tag{4.18}$$

If we are able to prove that $\|y\,(t)\| = \|\nabla u\,(t)\|$ for a.a. $t \in (\tau+\delta, \tau+T)$, then (4.16) follows. To this end, for any $\vartheta \in C_0^\infty((\tau,\tau+T)\times\mathbb{R}^m)$, we have

$$\int_{\tau+\delta}^{\tau+T} \Big((-\Delta)^{\frac{\gamma_n}{2}} u_n(t), \vartheta(t)\Big)\,dt = \int_{\tau+\delta}^{\tau+T} \Big(u_n(t), (-\Delta)^{\frac{\gamma_n}{2}} \vartheta(t)\Big)\,dt. \tag{4.19}$$

On the one hand, by (4.18), we derive

$$\int_{\tau+\delta}^{\tau+T} \left((-\Delta)^{\frac{\gamma_n}{2}} u_n(t), \vartheta(t)\right) dt \to \int_{\tau+\delta}^{\tau+T} (y(t), \vartheta(t))dt, \quad \text{as } \gamma_n \to 1^-(n \to \infty). \tag{4.20}$$

On the other hand, we deduce from Lemma 3.3 that

$$\int_{\tau+\delta}^{\tau+T} \left(u_n(t), (-\Delta)^{\frac{\gamma_n}{2}} \vartheta(t)\right) dt \to \int_{\tau+\delta}^{\tau+T} \left(u(t), (-\Delta)^{\frac{1}{2}} \vartheta(t)\right) dt, \quad \text{as } \gamma_n \to 1^-(n \to \infty). \tag{4.21}$$

Moreover, by (4.11), we know that $u(t) \in H^2(\mathbb{R}^m)$ for a.a. $t \in (\tau+\delta, \tau+T)$, which, together with Lemma 3.5, imply that $(-\Delta)^{\frac{1}{2}} u(t) \in H$. Therefore,

$$\int_{\tau+\delta}^{\tau+T} \left(u(t), (-\Delta)^{\frac{1}{2}} \vartheta(t)\right) dt = \int_{\tau+\delta}^{\tau+T} \left((-\Delta)^{\frac{1}{2}} u(t), \vartheta(t)\right) dt. \tag{4.22}$$

Thus, (4.19)-(4.22) yield $y = (-\Delta)^{\frac{1}{2}} u$ in $L^2(\tau+\delta, \tau+T; H)$. Then Lemma 3.8 shows

$$\|\nabla u(t)\| = \left\|(-\Delta)^{\frac{1}{2}} u(t)\right\| = \|y(t)\|, \quad \text{for a.a. } t \in (\tau+\delta, \tau+T),$$

as desired.

Thus, from (4.16) and (4.17), we infer that

$$\int_{\tau}^{\tau+T} (\chi(t) - F(t, v(t)), u(t) - v(t))\, dt \le \varepsilon, \qquad \text{for any } \varepsilon > 0,$$

which implies,

$$\int_{\tau}^{\tau+T} (\chi(t) - F(t, v(t)), u(t) - v(t))\, dt \le 0.$$

Letting $v(t) = u(t) - \theta z(t)$, where $\theta > 0$ and $z \in L^2(\tau, \tau+T; H)$, we have

$$\int_{\tau}^{\tau+T} (\chi(t) - F(t, u(t) - \theta z(t)), z(t))\, dt \le 0.$$

Since $F(t,\cdot)$ is hemicontinuous (see Lemma 4.1), passing to the limit as $\theta \to 0$ and using the Lebesgue theorem, we derive

$$\int_{\tau}^{\tau+T} (\chi(t) - F(t, u(t)), z(t))\, dt \le 0.$$

As z is arbitrary, we deduce that $\chi = F(\cdot, u(\cdot))$.

Therefore, $u(\cdot)$ is the weak solution to problem (1.1) with $\gamma = 1$. Since every converging subsequence has the same limit, the whole sequence satisfies the above convergences.

If $\sigma > 0$ in (2.6), then by Lemma 2.3 we consider the sequence of solutions $v_n(t) = u_n(t)e^{-\sigma t}$ of problem (2.19) which, by the above, converges to the unique solution $v(\cdot)$ to problem (2.20) with $v(\tau) = u_\tau e^{-\sigma\tau}$ in the sense given in the statement. Hence, by Lemma 2.4 it readily follows that the convergences are true for the sequence $\{u_n(\cdot)\}$ as well. ■

By a standard contradiction argument, we deduce the following result.

Corollary 4.3 *Let $u_\tau^\gamma \to u_\tau$ as $\gamma \to 1^-$. Let $u_\gamma(\cdot)$ be the solution to problem (1.1) with initial value u_τ^γ. Then $u_\gamma \to u$ as $\gamma \to 1^-$ in the sense given in Theorem 4.2, where $u(\cdot)$ is the unique solution to problem (1.1) with initial value u_τ and $\gamma = 1$.*

Lemma 4.4 *If $u_\tau \in V$, then the unique solution $u(\cdot)$ to problem (4.10) belongs to $L^2(\tau, \tau + T; H^2(\mathbb{R}^m)) \cap L^\infty(\tau, \tau + T; V)$.*

Proof. Making use of the standard procedures (see, e.g., the proof of [18, Theorem 5]), we can approximate the solution $u(\cdot)$ by solutions of the problem in a bounded domain with Dirichlet boundary conditions. Precisely, let us consider the following parabolic equation, for any $R > 0$,

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u = g(t), & (x,t) \in B_R \times (\tau, \infty), \\ u = 0, & (x,t) \in \partial B_R \times (\tau, \infty), \\ u(x,\tau) = u_\tau^R(x), & x \in B_R, \end{cases} \tag{4.23}$$

where $u_\tau^R(x) = \psi_R(|x|)u_\tau(x)$, with ψ_R being a smooth function such that

$$\psi_R(s) = \begin{cases} 1, & \text{if } 0 \le s \le R-1, \\ 0 \le \xi(s) \le 1, & \text{if } R-1 < s < R, \\ 0, & \text{if } s \ge R, \end{cases}$$

and $|\psi_R'(s)|$, $|\psi_R''(s)| \le L$, for all $s \ge 0$, $R > 0$ and a positive constant L. If $u_\tau \in H$, then problem (4.23) has a unique weak solution $u_R(\cdot)$. We extend the function u_R to the whole space by defining

$$\overline{u}_R(x) = \begin{cases} \psi_R(|x|)u_R(x), & \text{if } x \in B_R, \\ 0, & \text{otherwise.} \end{cases}$$

Then, using the same arguments as in [18], we obtain

$$\begin{aligned} &\overline{u}_R \to u \ \text{ weak-star in } \ L^\infty(\tau, \tau+T; H), \\ &\overline{u}_R \to u \ \text{ weakly in } \ L^\infty(\tau, \tau+T; V), \end{aligned}$$

as $R \to +\infty$.

If $u_\tau \in V$, then $u_R(\cdot) \in L^\infty(\tau, \tau+T; H_0^1(B_R)) \cap L^2(\tau, \tau+T; H^2(B_R) \cap H_0^1(B_R))$ (see, for example, [23, p.70]). Multiplying the equation in (4.23) by $-\Delta u_R$, we obtain

$$\frac{d}{dt} \|u_R\|^2_{H_0^1(B_R)} + \|\Delta u_R\|^2_{L^2(B_R)} \le \|g(t)\|^2_{L^2(B_R)}.$$

Therefore, there exists a constant $K > 0$ such that,

$$\|u_R(t)\|^2_{H_0^1(B_R)} + \int_\tau^{\tau+T} \|\Delta u_R\|^2_{L^2(B_R)}\, ds \le \left\|u_\tau^R\right\|^2_{H_0^1(B_R)} + \int_\tau^{\tau+T} \|g(s)\|^2_{L^2(B_R)}\, ds \le K.$$

It follows that $\overline{u}_R$ is bounded in $L^\infty(\tau, \tau+T; V) \cap L^2(\tau, \tau+T; H^2(\mathbb{R}^m))$. Hence,

$$\overline{u}_R \to u \ \ \text{weak-star in} \ \ L^\infty(\tau, \tau+T; V),$$
$$\overline{u}_R \to u \ \ \text{weakly in} \ \ L^2(\tau, \tau+T; H^2(\mathbb{R}^m)),$$

as $R \to +\infty$, and the result follows. ■

5 Existence of global attractors

We will focus on the existence of global attractors to problem (1.1) in this section for a more general nonlinear term f. To show the main ideas, assume that the functions $h \in H$ and $f : \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}$ do not depend on time t. Namely, we will study the autonomous case of (1.1) and take $\tau = 0$. In addition, suppose that f is a continuous function such that, for all $x \in \mathbb{R}^m$ and $u \in \mathbb{R}$, $f(x, \cdot) \in C^1(\mathbb{R})$ and

$$\frac{\partial f}{\partial u}(x,u) \le \sigma, \tag{5.1}$$

$$f(x,u)u \le -\beta |u|^p + \psi_1(x), \tag{5.2}$$

$$|f(x,u)| \le \psi_2(x)|u|^{p-1} + \psi_3(x), \tag{5.3}$$

where $\beta > 0$, $\sigma \ge 0$, $p \ge 2$ are constants, $\psi_1 \in L^1(\mathbb{R}^m)$, $\psi_2 \in L^\infty(\mathbb{R}^m)$, $\psi_3 \in L^q(\mathbb{R}^m)$ and q is the conjugate number of p, that is, $1/p + 1/q = 1$.

Define the Nemytskii operator $F : L^p(\mathbb{R}^m) \to L^q(\mathbb{R}^m)$ by $y = F(u)$, if $y\,(x) = f(x, u(x))$ for a.a. $x \in \mathbb{R}^m$. Now, we consider the problem

$$\begin{cases} \dfrac{\partial u}{\partial t} + (-\Delta)^\gamma u + \mu u = f(x,u) + h(x), \quad (x,t) \in \mathbb{R}^m \times (0,\infty), \\ u(x,0) = u_0(x), \quad x \in \mathbb{R}^m, \end{cases} \tag{5.4}$$

where $\mu > 0$.

Definition 5.1 *Let $u_0 \in H$ and $\gamma \in (0,1)$. A continuous function $u : [0,\infty) \to H$ is said to be a weak solution of problem (5.4) if $u(0) = u_0$, $u \in L^2_{loc}(0,\infty; V_\gamma) \cap L^p_{loc}(0,\infty; L^p(\mathbb{R}^m))$, $\dfrac{du}{dt} \in L^2_{loc}(0,\infty; V_\gamma^*) + L^q_{loc}(0,\infty; L^q(\mathbb{R}^m))$, and u satisfies, for every $\xi \in V_\gamma \cap L^p(\mathbb{R}^m)$,*

$$\begin{aligned} \frac{d}{dt}(u,\xi) + \frac{1}{2}C(m,\gamma)\int_{\mathbb{R}^m}\int_{\mathbb{R}^m} \frac{(u(x)-u(y))(\xi(x)-\xi(y))}{|x-y|^{m+2\gamma}}dxdy + \mu(u,\xi) \\ = \int_{\mathbb{R}^m} \left(f(x,u(t,x)) + h(x)\right)\xi\,(x)\,dx, \end{aligned} \tag{5.5}$$

in the sense of scalar distributions in $(0,\infty)$.

It is known (see, for example, [15, Theorem 3.2]) that problem (5.4) possesses a unique weak solution for any $u_0 \in H$ and $\gamma \in (0,1)$, which is continuous with respect to the initial datum u_0. For any $\gamma \in (0,1)$, let us define the family of operators $T_\gamma : \mathbb{R}^+ \times H \to H$, given by

$$T_\gamma(t, u_0) = u_\gamma(t),$$

where $u_\gamma(\cdot)$ is the unique weak solution to problem (5.4) with initial datum $u(0) = u_0$. It is not difficult to see that the map $(t, u_0) \mapsto T_\gamma(t, u_0)$ is continuous.

Before showing the main results, let us first recall some definitions which are essential for our proofs.

Definition 5.2 *The compact set $\mathcal{A}_\gamma$ is said to be a global attractor for T_γ, if the following conditions are fulfilled:*

1. *$T_\gamma(t, \mathcal{A}_\gamma) = \mathcal{A}_\gamma$, for all $t \geq 0$ (invariance);*
2. *$\mathcal{A}_\gamma$ attracts any bounded set B of H, that is:*

$$dist(T_\gamma(t, B), \mathcal{A}_\gamma) \to 0 \text{ as } t \to +\infty,$$

where $dist(C, D) = \sup_{x \in C} \inf_{y \in D} \; \|x - y\|$ is the Hausdorff semidistance in H.

Definition 5.3 *The map $\phi : \mathbb{R} \to H$ is called a complete trajectory of T_γ, if $\phi(t) = T_\gamma(t-s, \phi(s))$ for any $t \geq s$.*

Lemma 5.4 *Any solution $u(\cdot)$ of (5.4) satisfies*

$$\|u(t)\|^2 + C(m,\gamma) \int_0^t e^{-\mu(t-s)} \|u\|^2_{\dot{H}^\gamma(\mathbb{R}^m)} ds \leq \|u_0\|^2 e^{-\mu t} + \frac{2}{\mu} \int_{\mathbb{R}^m} \psi_1(x) dx + \frac{\|h\|^2}{\mu^2}. \tag{5.6}$$

Proof. Multiplying equation (5.4) by u, making using of (5.2) and the Young inequality, we have

$$\frac{d}{dt} \|u\|^2 + \mu \|u\|^2 + C(m,\gamma) \|u\|^2_{\dot{H}^\gamma(\mathbb{R}^m)} + 2\beta \|u\|_p^p \leq 2 \int_{\mathbb{R}^m} \psi_1(x) dx + \frac{1}{\mu} \|h\|^2. \tag{5.7}$$

Multiplying the above inequality by $e^{\mu s}$ and integrating over $(0, t)$ with $t > 0$, we obtain (5.6). ■

Lemma 5.5 *For any $R_1 > 0$, there exists $T(R_1) > 0$ such that,*

$$\|T_\gamma(t, u_0)\| \leq R_0 := \sqrt{1 + \frac{2}{\mu} \int_{\mathbb{R}^m} \psi_1(x) dx + \frac{\|h\|^2}{\mu^2}}, \qquad \forall t \geq T,$$

for any u_0 satisfying $\|u_0\| \leq R_1$.

Proof. This lemma follows easily from (5.6), so we omit the details of the proof here. ■

Remark 5.6 *The constants $T(R_1)$ and R_0 do not depend on γ.*

It follows from Lemma 5.5 that the ball $B_0 = \{x : \|x\| \le R_0\}$ is absorbing for T_γ. Next, we will obtain an estimate of the tails of solutions.

Lemma 5.7 *For any $R_1 > 0$, $\varepsilon > 0$, there are $T(\varepsilon, R_1)$ and $K(\varepsilon, R_1)$ such that, any solution $u(\cdot)$ of (5.4) starting at B_{R_1} satisfies*

$$\int_{|x|\ge k} |u(t,x)|^2\,dx \le \varepsilon, \qquad \text{for all } t \ge T,\ k \ge K.$$

Proof. Let $\theta : \mathbb{R}^+ \to \mathbb{R}^+$ be a smooth function such that

$$\theta(s) = \begin{cases} 0, & \text{if } 0 \le s \le \dfrac{1}{2}, \\ 1, & \text{if } s \ge 1. \end{cases}$$

Multiplying the equation in (5.4) by $\theta\left(\frac{|x|}{k}\right)u$, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^m} \theta\left(\frac{|x|}{k}\right)|u(t,x)|^2\,dx + \mu\int_{\mathbb{R}^m}\theta\left(\frac{|x|}{k}\right)|u(t,x)|^2\,dx \\ &= -\frac{1}{2}C(m,\gamma)\int_{\mathbb{R}^m}\int_{\mathbb{R}^m}\frac{\left(\theta\left(\frac{|x|}{k}\right)u(t,x) - \theta\left(\frac{|y|}{k}\right)u(t,y)\right)(u(t,x)-u(t,y))}{|x-y|^{m+2\gamma}}dydx \\ &+ \int_{\mathbb{R}^m}\theta\left(\frac{|x|}{k}\right)(f(x,u(t,x)) + h(x))u(t,x)dx. \end{aligned} \tag{5.8}$$

For the first term of right hand side of (5.8), by the Hölder inequality, we have

$$\begin{aligned} &-\int_{\mathbb{R}^m}\int_{\mathbb{R}^m}\frac{\left(\theta\left(\frac{|x|}{k}\right)u(t,x) - \theta\left(\frac{|y|}{k}\right)u(t,y)\right)(u(t,x)-u(t,y))}{|x-y|^{m+2\gamma}}dydx \\ &\le -\int_{\mathbb{R}^m}\int_{\mathbb{R}^m}\frac{\left(\theta\left(\frac{|x|}{k}\right) - \theta\left(\frac{|y|}{k}\right)\right)(u(t,x)-u(t,y))u(t,y)}{|x-y|^{m+2\gamma}}dxdy \\ &\le \|u(t)\|\left(\int_{\mathbb{R}^m}\left(\int_{\mathbb{R}^m}\frac{\left|\theta\left(\frac{|x|}{k}\right) - \theta\left(\frac{|y|}{k}\right)\right||u(t,x)-u(t,y)|}{|x-y|^{m+2\gamma}}dx\right)^2 dy\right)^{\frac{1}{2}} \\ &\le \|u(t)\|\left(\int_{\mathbb{R}^m}\int_{\mathbb{R}^m}\frac{\left|\theta\left(\frac{|x|}{k}\right) - \theta\left(\frac{|y|}{k}\right)\right|^2}{|x-y|^{m+2\gamma}}dx\int_{\mathbb{R}^m}\frac{|u(t,x)-u(t,y)|^2}{|x-y|^{m+2\gamma}}dxdy\right)^{\frac{1}{2}}. \end{aligned}$$

Since $|\theta'(s)| \le L_1$ for some $L_1 > 0$, by means of the change of variable $z = \frac{x}{k}$, we know that there exists a constant $L_2 > 0$ such that,

$$\int_{\mathbb{R}^m}\frac{\left|\theta\left(\frac{|x|}{k}\right) - \theta\left(\frac{|y|}{k}\right)\right|^2}{|x-y|^{m+2\gamma}}dx$$

$$
\begin{aligned}
&= \frac{1}{k^{2\gamma}} \int_{\mathbb{R}^m} \frac{\left|\theta\left(|z|\right) - \theta\left(\frac{|y|}{k}\right)\right|^2}{\left|z - \frac{y}{k}\right|^{m+2\gamma}} dz = \frac{1}{k^{2\gamma}} \int_{\mathbb{R}^m} \frac{\left|\theta\left(\left|\xi + \frac{y}{k}\right|\right) - \theta\left(\frac{|y|}{k}\right)\right|^2}{|\xi|^{m+2\gamma}} d\xi \\
&\le \frac{L_1^2}{k^{2\gamma}} \int_{|\xi|\le 1} \frac{1}{|\xi|^{m+2\gamma-2}} d\xi + \frac{1}{k^{2\gamma}} \int_{|\xi|\ge 1} \frac{1}{|\xi|^{m+2\gamma}} d\xi \le \frac{L_2}{k^{2\gamma}}.
\end{aligned}
$$

Thus, the above estimate and Lemma 5.4 imply that there exists a constant $L_3 > 0$, such that

$$
\begin{aligned}
&-\frac{1}{2} C(m,\gamma) \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} \frac{\left(\theta\left(\frac{|x|}{k}\right) u(t,x) - \theta\left(\frac{|y|}{k}\right) u(t,y)\right)(u(t,x) - u(t,y))}{|x-y|^{m+2\gamma}} dydx \\
&\le \frac{\sqrt{L_2}}{2k^\gamma} C(m,\gamma) \|u(t)\| \|u(t)\|_{\dot{H}^\gamma(\mathbb{R}^m)} \\
&\le \frac{L_3 C(m,\gamma)}{k^\gamma} \left(1 + \|u(t)\|^2_{\dot{H}^\gamma(\mathbb{R}^m)}\right). \qquad (5.9)
\end{aligned}
$$

For the second term of right hand side of (5.8), using (5.2) and the Young inequality, we find

$$
\begin{aligned}
&\int_{\mathbb{R}^m} \theta\left(\frac{|x|}{k}\right) (f(x,u(t,x)) + h(x)) u(t,x) dx \qquad (5.10) \\
&\le \frac{\mu}{2} \int_{\mathbb{R}^m} \theta\left(\frac{|x|}{k}\right) |u(t,x)|^2 dx + \int_{\mathbb{R}^m} \theta\left(\frac{|x|}{k}\right) \psi_1(x) dx + \frac{1}{2\mu} \int_{\mathbb{R}^m} \theta\left(\frac{|x|}{k}\right) |h(x)|^2 dx.
\end{aligned}
$$

Since $\psi_1 \in L^1(\mathbb{R}^m)$ and $h \in H$, we deduce that for $\varepsilon' > 0$, there exists $K(\varepsilon') > 0$ such that for all $k \ge K$, we have

$$
\begin{aligned}
\int_{\mathbb{R}^m} \theta\left(\frac{|x|}{k}\right) \psi_1(x) dx &\le \varepsilon', \\
\frac{1}{2\mu} \int_{\mathbb{R}^m} \theta\left(\frac{|x|}{k}\right) |h(x)|^2 dx &\le \varepsilon', \\
\max\left\{\frac{L_3}{k^\gamma}, \frac{L_3 C(m,\gamma)}{k^\gamma}\right\} &\le \varepsilon'. \qquad (5.11)
\end{aligned}
$$

Collecting (5.8)-(5.11), it yields

$$
\frac{d}{dt} \int_{\mathbb{R}^m} \theta\left(\frac{|x|}{k}\right) |u(t,x)|^2 dx + \mu \int_{\mathbb{R}^m} \theta\left(\frac{|x|}{k}\right) |u(t,x)|^2 dx \le 6\varepsilon' + 2\varepsilon' C(m,\gamma) \|u(t)\|^2_{\dot{H}^\gamma(\mathbb{R}^m)}.
$$

Multiplying the above inequality by $e^{\mu s}$ and integrating it over $(0,t)$, together with estimate (5.6), we infer there exists a positive constant L_4 such that,

$$
\begin{aligned}
\int_{\mathbb{R}^m} \theta\left(\frac{|x|}{k}\right) |u(t,x)|^2 dx &\le e^{-\mu t} \int_{\mathbb{R}^m} \theta\left(\frac{|x|}{k}\right) |u_0(x)|^2 dx \\
&\quad + \frac{6\varepsilon'}{\mu} + 2\varepsilon' C(m,\gamma) \int_0^t e^{-\mu(t-s)} \|u(s)\|^2_{\dot{H}^\gamma(\mathbb{R}^m)} ds \\
&\le e^{-\mu t} \int_{\mathbb{R}^m} \theta\left(\frac{|x|}{k}\right) |u_0(x)|^2 dx + L_4 \varepsilon'.
\end{aligned}
$$

We can choose $T(\varepsilon') > 0$ such that $e^{-\mu t} \int_{\mathbb{R}^m} \theta\left(\frac{|x|}{k}\right) |u_0(x)|^2\, dx \le \varepsilon'$ for all $t \ge T$, and ε' satisfies $(1 + L_4)\,\varepsilon' < \varepsilon$. Therefore,

$$\int_{|x|\ge k} |u(t,x)|^2\, dx \le \int_{\mathbb{R}^m} \theta\left(\frac{|x|}{k}\right) |u(t,x)|^2\, dx \le \varepsilon, \quad \forall t \ge T, \forall k \ge K.$$

The proof is complete. ■

Lemma 5.8 *The operator T_γ is asymptotically compact, that is, for any bounded set $B \in H$, every sequence $y_\gamma^n = T_\gamma(t_n, x_n)$ with $x_n \in B$ is relatively compact in H when $t_n \to +\infty$.*

Proof. In view of (5.6), the sequence $\{y_\gamma^n\}$ is bounded in H. Hence, up to a subsequence (relabeled the same), $y_\gamma^n \to y$ weakly in H. We need to prove that the convergence is in fact strong. As $y_\gamma^n = T_\gamma(1, T_\gamma(t_n - 1, x_n))$, there exist solutions $u_\gamma^n(\cdot)$ such that $u_\gamma^n(0) = z_\gamma^n = T_\gamma(t_n - 1, x_n)$ and $u_\gamma^n(1) = y_\gamma^n$. The sequence z_γ^n converges weakly in H to some z as well.

It follows from (5.6)-(5.7) that there is $M > 0$ such that,

$$\sup_{t\in[0,1]} \left\|u_\gamma^n(t)\right\| \le M,$$

$$\int_0^1 \left(C(m,\gamma)\|u_\gamma^n(s)\|^2_{\dot{H}^\gamma(\mathbb{R}^m)} + 2\beta \left\|u_\gamma^n(s)\right\|_p^p \right) ds \le M.$$

Thus, $\{u_\gamma^n\}$ is bounded in $L^\infty(0,1;H) \cap L^p(0,1;L^p(\mathbb{R}^m)) \cap L^2(0,1;V_\gamma)$. It also follows that $\{A^\gamma(u_\gamma^n)\}$ is bounded in $L^2(0,1;V_\gamma^*)$. Moreover, by (5.3), we deduce that $\{F(u_\gamma^n)\}$ is bounded in $L^q(0,1;L^q(\mathbb{R}^m))$. Hence, $\left\{\dfrac{du_\gamma^n}{dt}\right\}$ is bounded in $L^q(0,1;L^q(\mathbb{R}^m)) + L^2(0,1;V_\gamma^*)$. Then, using the Aubin-Lions lemma, there exists a function $u_\gamma \in L^\infty(0,1;H) \cap L^p(0,1;L^p(\mathbb{R}^m)) \cap L^2(0,1;V^*)$ with $\dfrac{du_\gamma}{dt} \in L^q(0,1;L^q(\mathbb{R}^m)) + L^2(0,1;V_\gamma^*)$ and $\chi \in L^q(0,1;L^q(\mathbb{R}^m))$, such that, up to a subsequence (relabeled the same), the following convergences hold:

$$u_\gamma^n \to u_\gamma \text{ weak-star in } L^\infty(0,1;H), \tag{5.12}$$

$$u_\gamma^n \to u_\gamma \text{ weakly in } L^2(0,1;V_\gamma), \tag{5.13}$$

$$u_\gamma^n \to u_\gamma \text{ weakly in } L^p(0,1;L^p(\mathbb{R}^m)), \tag{5.14}$$

$$A^\gamma(u_\gamma^n) \to A^\gamma(u_\gamma) \text{ weakly in } L^2(0,1;V_\gamma^*), \tag{5.15}$$

$$\frac{du_\gamma^n}{dt} \to \frac{du_\gamma}{dt} \text{ weakly in } L^q(0,1;L^q(\mathbb{R}^m)) + L^2(0,1;V_\gamma^*), \tag{5.16}$$

$$F(u_\gamma^n) \to \chi \text{ weakly in } L^q(0,1;L^q(\mathbb{R}^m)). \tag{5.17}$$

We need to prove that $u_\gamma(\cdot)$ is a weak solution of problem (5.4) satisfying $u_\gamma(0) = z$ and $u_\gamma(1) = y$. To this end, for $R > 0$, we define the projections $u_{\gamma,R}^n = L_R u_\gamma^n$, where $L_R : H \to L^2(B_R)$ is defined by $L_R v(x) = v(x)$ for a.a. $x \in B_R$. It is easy to see that the above convergences

(5.12)-(5.17) still hold, if we replace u_γ^n by $u_{\gamma,R}^n$, u_γ by $L_R u_\gamma$, $\mathbb{R}^m$ by B_R, H by $L^2(B_R)$, V_γ by $H^\gamma(B_R)$, V_γ^* by $(H^\gamma(B_R))^*$ and χ by $L_R\chi$, respectively.

Since the embedding $H_\gamma(B_R) \subset L^2(B_R)$ is compact (cf. [20]) and the embedding $L^2(B_R) \subset (H^\gamma(B_R) \cap L^p(B_R))^*$ is continuous, a standard Compactness Theorem [21, Theorem 8.1] implies that

$$u_{\gamma,R}^n \to L_R u_\gamma \text{ strongly in } L^2(0,1;L^2(B_R)), \tag{5.18}$$

$$u_{\gamma,R}^n(t,x) \to L_R u_\gamma(t,x) \text{ for a.a. } t \in (0,1), x \in B_R. \tag{5.19}$$

Hence, $f(x, u_{\gamma,R}^n(t,x)) \to f(x, L_R u_\gamma(t,x))$ for a.a. $t \in (0,1)$ and $x \in B_R$. It follows from [14, Lemma 1.3] that $L_R\chi = F_R(L_R u_\gamma)$, where $F_R : L^p(B_R) \to L^q(B_R)$ is the corresponding Nemytskii operator associated to f. In a standard way (see, e.g., [18, Lemma 8]), one can show that u_γ satisfies the equality in (5.4) in the sense of distributions. Then equality (5.5) is true as well, so u_γ is a solution of (5.4).

Further, we need to prove that $u_\gamma(0) = z$ and $u_\gamma(1) = y$. As the embedding $L^2(B_R) \subset (H^\gamma(B_R) \cap L^p(B_R))^*$ is compact and $\frac{du_{\gamma,R}^n}{dt}$ is bounded in $L^q(0,1;(H^\gamma(B_R) \cap L^p(B_R))^*)$, the Ascoli-Arzelà Theorem implies that $u_{\gamma,R}^n \to L_R u_\gamma$ strongly in $C([0,1],(H^\gamma(B_R) \cap L^p(B_R))^*)$. From this and (5.6), we deduce immediately that $u_{\gamma,R}^n(0) \to L_R u_\gamma(0)$, $u_{\gamma,R}^n(1) \to L_R u_\gamma(1)$ weakly in $L^2(B_R)$. Hence, $L_R z = L_R u_\gamma(0)$ and $L_R y = L_R u_\gamma(1)$ for any $R > 0$. Thus, the result follows.

On the one hand, integrating in (5.7) over $(0,t)$ with $t > 0$, we find that there exists a constant $C > 0$ such that the functions $J(t) = \|u_\gamma(t)\|^2 - Ct$, $J_n(t) = \left\|u_\gamma^n(t)\right\|^2 - Ct$ are non-increasing. On the other hand, in terms of (5.18), for any $R > 0$, we have that $L_R u_\gamma^n(t) \to L_R u_\gamma(t)$ in $L^2(B_R)$ for a.a. $t \in (0,1)$. For any $\varepsilon > 0$, Lemma 5.7 implies the existence of $R_2(\varepsilon) > 0$ and $N_1(\varepsilon) > 0$ such that

$$\int_{|x|>R_2} \left|u_\gamma^n(s,x)\right|^2 dx \le \varepsilon, \quad \text{for all } s \in [0,1] \text{ and } n \ge N_1.$$

Take a sequence $t_m \to 1^-$ such that $L_R u_\gamma^n(t_m) \to L_R u_\gamma(t_m)$ in $L^2(B_R)$ for any $t_m \in (0,1)$. Then by the monotonicity and the continuity of J_n and J, we infer that there is $N_2(\varepsilon) \ge N_1(\varepsilon)$ such that,

$$\begin{aligned} J_n(1) - J(1) &= J_n(1) - J_n(t_m) + J_n(t_m) - J(t_m) + J(t_m) - J(1) \\ &\le 2\varepsilon + \left| \int_{|x|\le R_2} \left|u_\gamma^n(t_m,x)\right|^2 dx - \int_{|x|\le R_2} |u_\gamma(t_m,x)|^2 dx \right| \\ &+ \int_{|x|>R_2} \left|u_\gamma^n(t_m,x)\right|^2 dx + \int_{|x|>R_2} |u_\gamma(t_m,x)|^2 dx \\ &\le 5\varepsilon. \end{aligned}$$

Hence, $\limsup_{n\to\infty} J_n(1) \le J(1)$, which implies that $\limsup_{n\to\infty} \left\|u_\gamma^n(1)\right\| \le \|u_\gamma(1)\|$. Since $\|y\| \le \liminf_{n\to\infty} \|y_n\|$, we obtain that $y_n \to y$ strongly in H. ∎

Theorem 5.9 *T_γ has a connected global attractor $\mathcal{A}_\gamma$, which is characterized by*

$$\mathcal{A}_\gamma = \{\phi(0) : \phi \textit{ is a bounded complete trajectory of } T_\gamma\}.$$

Moreover, it is the minimal closed set attracting every bounded set.

Proof. In view of Lemmas 5.5 and 5.8, it follows from standard results (see e.g. [13, Theorem 3.1]). ■

Lemma 5.10 *The set $\cup_{\gamma\in(0,1)}\mathcal{A}_\gamma$ is bounded in H.*

Proof. It follows from the inclusion $\mathcal{A}_\gamma \subset B_0$ for any $\gamma \in (0,1)$. ■

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Data availability

No data has been used in the development of the research in this paper.

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