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ScienceDirect

Fuzzy Sets and Systems 266 (2015) 144–156

FUZZY sets and systems

www.elsevier.com/locate/fss

A new index for bond management in an uncertain environment

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Received 14 May 2013; received in revised form 3 July 2014; accepted 6 August 2014

Available online 13 August 2014

Abstract

Within the framework of Assets Liability Management, we understand that immunization is the main method to assure a certain yield in a future date departing from an initial portfolio. Although the objective of passive strategies is to design a portfolio that will achieve the performance of a predetermined benchmark, active bond management strategies rely on expectations of interest rate movements or changes in yield–spread relationships. However, the variation of the duration increases the risk of a portfolio, that why the decision maker will have to choose the combination of expected return and risk which provides the higher utility. Finally, the construction of a fuzzy return risk map will allow the decision maker to know the over risk and the over return as regards immunization strategy for each duration and for each risk aversion of the decision maker. The construction of a risk return map presents the results in an appropriate way. It will help the decision maker to choose the best duration for the decision maker interest rate forecast. Finally, this methodology is applied to the Spanish debt so that the decision maker can select the duration that brings him a greater preference.

Keywords: Finance; Active strategies; Duration; Immunization; Portfolio; Passive strategies

1. Introduction

One of the most important research fields in modern finance theory [1] has been the mean-variance methodology for the portfolio selection problem proposed by Markowitz [2], in which the investor is averse to risk. This model considers, for a given return that is multivariate distributed, the decision maker (DM) has a quadratic utility function [3]. In contrast, Konno and Yamazaki [4] proposed the first linear model for portfolio selection, the L_1 risk model.

Asset-Liability Management can be traced back to Redington [5] who suggests a parallel treatment to the assets and liabilities in actuarial valuation. This topic has been broadly dealt with in financial literature. In Van Der Meer and Smink [6] an extensive revision of these techniques is considered. It divides them in static methods (cash flow

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http://dx.doi.org/10.1016/j.fss.2014.08.002
0165-0114/

payment, gap analysis, segmentation and cash floor matching) and dynamic ones (passive, such as immunization or active, such as contingent immunization).

However, some authors, like Gerber [7], point out that the use of stochastic models is not suitable for the prediction of long term interest rates. They consider it is much more realistic the use of discount rates based on fuzzy numbers (FN), since only data available is the one facilitated by experts. For applications of the theory of the fuzzy sets in finance, the following can be considered: Kaufmann [8], Li Calzi [9], Gil [10], Ostaszewski [11], Terceño et al. [12, 13], De Andrés and Terceño [14–16] and Terceño and Brotons [17].

However, the bibliography on immunization in a fuzzy environment is very scarce. Derrig and Ostaszewski [18] uses fuzzy mathematics to estimate the effective tax rate and the tax-free discount rate in an insurance company with a portfolio formed with assets and liabilities. In our opinion, the use of FN presents important advantages: as bond prices oscillate between a maximum and a minimum price, we consider much more appropriate to consider the price, and thus, the corresponding internal rate return (IRR), as fuzzy magnitudes. Considering this, we can use all the information that the market provides. When the problem to investigate is economic or social, the observations are consequence of the interaction among the expectations and the opinions of the economic agents, which are highly vague and imprecise.

Terceño et al. [19], Brotons and Terceño [20,21] and Brotons [22] apply immunization strategy in a fuzzy environment. However, our goal is to design active bond management in a fuzzy environment, in order to anticipate changes in interest rates. From a starting immunized portfolio, the DM will have to decide whether to increase the expectation return (modifying portfolio duration), increasing the risk as well, or not. The use of utility functions and the building of risk return maps will improve decision-making.

One reasonable function that is commonly employed by financial theorists [23] assigns a risky portfolio P with risky rate of return r_p, an expected rate of return $E(r_p)$ and a variance of the rate of return $\sigma^2(r_p)$ the utility score $U(P) = E(r_p) - 0.005 \cdot A \cdot \sigma^2(r_p)$ where A is an index of the DM's risk aversion. Carlsson et al. [24], in a fuzzy environment, uses $E(\cdot)$ as a measure of return and $\sigma^2(\cdot)$ as a measure of risk.

In practical applications, the use of utility theory has proved to be problematic, which should be more serious than having axiomatic problems: (i) utility measures cannot be validated inter-subjectively, (ii) the consistency of utility measures cannot be validated across events or contexts for the same subject, (iii) utility measures show discontinuities in empirical tests (as shown by [25]), which should not happen with rational DMs if the axiomatic foundation is correct, and (iv) utility measures are artificial and thus hard to use on an intuitive basis.

Because of the limitations of probability assessment and utility theory we deal with fuzzy interest and we propose the use of possibility theory. In literature, in the works of fuzzy utility [26–30] fuzzy parameters are assumed to be known membership functions. However, it is actually not always easy for a DM to specify the membership function or probability distribution in an inexact environment [24]. At least in some of the cases, the use of interval coefficients may serve the purpose better.

Consequently, the main goal of the present work is to design active management strategies in a fuzzy environment, using Sengupta's methodology to get the return and the risk of a portfolio. Vercher et al. [31] uses this methodology to optimize a fuzzy portfolio under downside risk measures. Interest rate will have to be forecast by the DM, and as a result, the portfolio duration will have to be modified in order to increase the portfolio return, in exchange for a higher risk (wrong estimation will decrease portfolio return).

This way of proceed does not allow to choose the risk in a suitable way, so one of the major points of the present paper is the construction of return risk maps in a fuzzy environment, using fuzzy utility functions that allows DMs to choose the most appropriate combination of risk and return, modifying portfolio duration. In order to allow that DM can choose between different portfolios, we propose the construction of an index that allows him to set the preference of one option over another when, because of uncertainty, both are represented through Triangular Fuzzy Numbers (TFN), and we will use their α-cuts. That is, we will propose an index that allows DM to order intervals (α-cuts) to a certain level of presumption.

The structure of the paper is as follows. Section 2 is devoted to describing the preliminary concepts. In Section 3, we introduce the concept of bond and portfolio's duration that we extend to the fuzzy environment. Then, in Section 4 we expose the passive management. Risk and return of a portfolio using active bond management are obtained in Section 5. The next section presents the construction of a new index of preference for portfolio selection investment. We illustrate our approach with an application to the Spanish Public Debt market using numerical examples in Section 7.

2. Preliminary concepts

Since we are interested in comparing the risk and the return of portfolios in a fuzzy environment, we will study the relationship between these interval-valued expectations by using properly ordered relations. The ordering of intervals is a complex issue extensively treated in the literature [32–38].

In the strict sense could establish the ordering of intervals as

Definition 1. Let A and B be two closed intervals with $A = [a_L, a_R]$ and $B = [b_L, b_R]$, then $A \le B$ iff $a_R \le b_L$.

Assuming that using this criterion most intervals could not be ordered, Ishibuchi and Tanaka [33] defined the following criterion when the previous is not applicable:

Definition 2. Let A and B be two closed intervals on the real line, then

$$A \le_{LR} B \quad \text{iff} \quad a_L \le b_L \quad \text{and} \quad a_R \le b_R$$

When this criterion does not allow ordering two intervals, the same authors propose:

Definition 3.

$$A \le_{mw} B \quad \text{iff} \quad m(A) \le m(B) \quad \text{and} \quad hw(A) \ge hw(B).$$

Where $m(I)$ is the midpoint of I, and $hw(I)$ the half-width of I.

This definition does not allow either ordering all the intervals, since it can happen that $m(A) \le m(B)$ and $hw(A) < hw(B)$. Therefore, Sengupta and Pal [36] defined the acceptability index as:

Definition 4. Let A and B be two closed intervals such that $m(A) \le m(B)$. Denote by $\prec$ an order relation, in such a way that we say A is inferior to B, $A \prec B$, in terms of the value of the acceptability index

$$A_{\prec}(A, B) = \frac{m(B) - m(A)}{hw(B) + hw(A)}, \quad \text{where } hw(B) + hw(A) \neq 0$$

$A_{\prec}(A, B)$ may be interpreted as the grade of acceptability of the first interval to be inferior to the second interval.

The grade of acceptability of $A_{\prec}(A, B)$ may be classified and interpreted further on the basis of comparative position of mean of interval B with respect to those of interval A as follows:

$$A_{\prec}(A, B) \begin{cases} 0 & \text{if } m(A) = m(B) \\ 0 < A_{\prec}(A, B) < 1 & \text{if } m(A) < m(B) \text{ and } a_R > b_L \\ \ge 1 & \text{if } m(A) < m(B) \text{ and } a_R \le b_L \end{cases}$$

If $A_{\prec}(A, B) = 0$, then the premise "A is inferior to B" is not accept. If $0 < A_{\prec}(A, B) < 1$, then the interpreter accepts the premise A is inferior to B with different degree of satisfaction ranging from zero to one (excluding zero and one). If $A_{\prec}(A, B) \ge 1$, the interpreter is absolutely satisfied with the premise A is inferior to B.

Except for Definition 1, the rest of criteria do not establish a strict order, in the sense that there may be values of an interval around a small value that would exceed values of an interval around a greater value.

3. Bond and portfolio's duration

Before approaching the goals that we have intended, it is our desire to expose some basic financial concepts for the development of the present work. The effective spot rate at time t corresponding to a term $[t, t + n]$ that we will denote as ${}_t i_n$ is defined as internal rate of return of a zero-coupon bond of maximum credit quality for the maturity $t + n$.

If the spot rate takes the constant value i for any term, the expression of the duration at time 0 of a bond that generates the stream of payments, $\{(C_1, t_1), (C_2, t_2), \dots, (C_n, t_n)\}$ is:

$$D = \frac{\sum_{s=1}^{n} t_s C_s (1+i)^{-t_s}}{\sum_{s=1}^{n} C_s (1+i)^{-t_s}}$$

being t_s the maturity of the cash flow C_s. The maturity, the cash flows and the yield of a bond are known beforehand, and premature paying-off does not exist.

For the case in that the interest rates are defined by a FN, $\tilde{i}(x)$, is transformed into,

$$\tilde{D}(x) = \frac{\sum_{s=1}^{n} t_s C_s (1+\tilde{i}(x))^{-t_s}}{\sum_{s=1}^{n} C_s (1+\tilde{i}(x))^{-t_s}}$$

and the total duration of a portfolio, $\tilde{D}^F(x)$, if we suppose that it is formed by N_k bonds of the type k, $k = 1, \dots, m$, being C_s^k the cash flow of the bond k in the period t_s:

$$\tilde{D}^F(x) = \frac{\sum_{k=1}^{m} \sum_{s=1}^{n} t_s^k N_k C_s^k (1+\tilde{i}(x))^{-t_s^k}}{\sum_{k=1}^{m} \sum_{s=1}^{n} N_k C_s^k (1+\tilde{i}(x))^{-t_s^k}}$$

Along our work we will use TFNs. This way allows us to capture in a best way the information provided by the market, on the negotiation of fixed income bonds along the trading time. We will take as the center of the FN the weighted average price (WAP) of the trades, as right radius the difference between the maximum price and the WAP, and as left radius, the difference between the WAP and the minimum. This way, the nucleus of the FN is formed by a single value. We have discarded FN with a greater nucleus although it could take as center, for example, the range of values among the first and third quartile, but they are more difficult to understand and to obtain from the experts when they are asked for their opinion. In this way, the interest rate is denoted by $\tilde{i} = (i_C, l_i, r_i)$ and its membership function:

$$\mu_i(x) \begin{cases} 0 & x \le i_C - l_i \\ 1 - \frac{i_C - x}{l_i} & i_C - l_i \le x \le i_C \\ 1 - \frac{x - i_C}{r_i} & i_C \le x \le i_C + r_i \\ 0 & i_C + r_i \le x \end{cases}$$

Terceño et al. [39–41] and Perrone and La Diega [42] show that expressions of financial valuation of a set of capitals (current value and final value) are well approximated by a TFN when interest rates are TFN and financial capitals are crisp numbers. For the duration, which decreases (increases) as bond yield increases (decreases) [43], we have calculated the present value, so we can approximate it through a TFN whose lower extreme corresponds to the duration with the greatest interest, the maximum presumption with the interest of maximum presumption and the upper extreme with the lowest interest. Therefore,

$$\tilde{D}(x) = (D_C, l_D, r_D)$$

where,

$$D_C = D(i_C)$$
$$l_D = D(i_C) - D(i_C + r_i)$$
$$r_D = D(i_C - l_i) - D(i_C)$$

And the membership function:

$$\mu_D(x) = \begin{cases} 0 & x \le i_C - l_i \\ 1 - \frac{D(x) - D(i_C)}{D(i_C - l_i) - D(i_C)} & i_C - l_i \le x \le i_C \\ 1 - \frac{D(i_C) - D(x)}{D(i_C) - D(i_C + r_i)} & i_C \le x \le i_C + r_i \\ 0 & x \ge i_C + r \end{cases}$$

In our case, if we assume m titles to form the portfolio, the duration of this, $\tilde{D}^F$, would be:

$$\tilde{D}^F = x_1\tilde{D}_1 + \ldots + x_m\tilde{D}_m$$

where:

x_k: proportion of investment of the bond k in portfolio
$\tilde{D}_k$: duration of the bond k

We work with the α-cuts of an approximate TFN, so we will be working with confidence intervals; the level to be used will depend on the risk aversion of the investor (the greater risk aversion, the lower level), so the decision should be made comparing intervals.

4. Portfolio immunization

An immunization strategy is aimed to determine the kind and the number of securities that a DM should acquire to assure a capital for a certain term, which we will denominate Investor Planning Horizon (IPH).

According to the condition of immunization, a portfolio is immunized when the IPH is equal to the duration of the portfolio. So, it is necessary to find the proportion of investment in each bond that provides a duration equal to the IPH.

If $\tilde{D}_k$ and $\tilde{D}^F$ can be approximated by TFN, we can express them as:

$$\tilde{D}_k = \left(D_C^k, l_D^k, r_D^k\right)$$

$$\tilde{D}^F = \left(\sum_{k=1}^{m} x_k D_C^k, \sum_{k=1}^{m} x_k l_D^k, \sum_{i=1}^{m} x_k r_D^k\right)$$

Being the α-cuts of the duration of a bond k:

$$\tilde{D}_k(\alpha) = \left[D_C^k - (1-\alpha)l_D^k, D_C^k + (1-\alpha)r_D^k\right]$$

and the α-cuts of the duration of the portfolio:

$$\tilde{D}^F(\alpha) = \left[\sum_{k=1}^{m} x_k D_C^k - (1-\alpha)\sum_{k=1}^{m} x_k l_D^k, \sum_{k=1}^{m} x_k D_C^k + (1-\alpha)\sum_{k=1}^{m} x_k r_D^k\right]$$

The condition of immunization also presents a risk, the risk of immunization. For this reason, we will consider the equality between IPH and duration as the highest level of presumption for which it is possible to immunize the portfolio, that is to say, the highest alpha-cut that contains inside the IPH, being the IPH a known value. Because of the condition of maximizing the alpha-cut, the IPH must be placed in one extreme of it. Depending on the situation of IPH respect to the center of the TFN, this optimum can be obtained in the left or right part of the membership function. By solving the two programs of optimization we will obtain the optimal in each side, and the maximum of both will be the solution.

To the right side,

$$\begin{aligned} \max_{x_1,\ldots,x_m} \quad & \alpha = \frac{\sum_{k=1}^{m} x_k D_C^k + \sum_{k=1}^{m} x_k r_D^k - \text{IPH}}{\sum_{k=1}^{m} x_k r_D^k} \\ \text{s.t.} \quad & x_1 + \ldots + x_m = 1 \\ & x_k \geq 0, k = 1, \ldots m \end{aligned}$$

and to the left side:

$$\begin{aligned} \max_{x_1,\ldots,x_m} \quad & \alpha = \frac{\text{IPH} - \sum_{k=1}^{m} x_k D_C^k + \sum_{k=1}^{m} x_k l_D^k}{\sum_{k=1}^{m} x_k l_D^k} \\ \text{s.t.} \quad & x_1 + \ldots + x_m = 1 \\ & x_k \geq 0, k = 1, \ldots m \end{aligned}$$

5. Risk and return in active management

For an IPH of T years, and assuming that the DM has two zero coupon bonds with maturities t_1 and t_2, the proportion of the investment in each kind of bond has to fulfil the set of equations:

$$x_1 + x_1 = 1$$
$$x_1 t_1 + x_2 t_2 = T$$

whose solution is $(x_1, x_2) = (\frac{t_2-T}{t_2-t_1}, \frac{T-t_1}{t_2-t_1})$. So, being the price of the portfolio in 0, P_0^F, the DM has to buy $n_1 = \frac{P_0^F x_1}{P_0^1}$ bonds of the first kind and $n_2 = \frac{P_0^F x_2}{P_0^2}$ of the second kind. The prices of the two bound are the crisp numbers P_0^1 and P_0^2.

$$P_0^F = n_1 P_0^1 + n_2 P_0^2$$

And the value at the IPH, $\tilde{P}_{\mathrm{IPH}}^F$, is:

$$\tilde{P}_{\mathrm{IPH}}^F(\tilde{i}) = \sum_{s=1}^{2} n_s \tilde{P}_{\mathrm{IPH}}^s = \sum_{s=1}^{2} n_s P_0^s (1+\tilde{i})^T$$

According to dynamic immunization theorem [44], if there is a variation on the interest rates, changing from i to $\tilde{i} = (i_C, l_i, r_i)$, and assuming, without loss of generality, that $i_C + r_i < i$, the value of the portfolio in the moment IPH will be at least $P_{\mathrm{IPH}}^F(i)$. The value of the portfolio is $\tilde{P}_{\mathrm{IPH}}^F(i) = (P_{\mathrm{IPH}_C}^F, l_{P_{\mathrm{IPH}}}^F, r_{P_{\mathrm{IPH}}}^F)$, where

$$P_{\mathrm{IPH}_C}^F = \sum_{s=1}^{2} n_s P_{t_s}^s (1+i_C)^{T-t_s}$$
$$l_{P_{\mathrm{IPH}}}^F = \sum_{s=1}^{2} n_s P_{t_s}^s \left[(1+i_C+r_i)^{T-t_s} - (1+i_C)^{T-t_s}\right]$$
$$r_{P_{\mathrm{IPH}}}^F = \sum_{s=1}^{2} n_s P_{t_s}^s \left[(1+i_C)^{T-t_s} - (1+i_C-l_i)^{T-t_s}\right]$$

Note if there is no change in interest rates the accumulate portfolio value remains constant. However, for any changes in it, the portfolio value increases, both for interest rates increases and decreases (due to bond convexity). Now, we define portfolio's return as,

$$\tilde{h}_{\mathrm{IPH}} = (h_{\mathrm{IPH}_C}, l_{h_{\mathrm{IPH}}}, r_{h_{\mathrm{IPH}}})$$

where,

$$h_{\mathrm{IPH}_C} = \left(P_{\mathrm{IPH}_C}^F / P_0^F\right)^{1/T} - 1,$$
$$l_{h_{\mathrm{IPH}}} = \left(\left(P_{\mathrm{IPH}_C}^F + r_{P_{\mathrm{IPH}}}^F\right)/P_0^F\right)^{1/T} - \left(P_{\mathrm{IPH}_C}^F / P_0^F\right)^{1/T}$$
$$r_{h_{\mathrm{IPH}}} = \left(P_{\mathrm{IPH}_C}^F / P_0^F\right)^{1/T} - \left(\left(P_{\mathrm{IPH}_C}^F + l_{P_{\mathrm{IPH}}}^F\right)/P_0^F\right)^{1/T}$$

Consequently, the midpoint of the portfolio return is $m(h_{\mathrm{IPH}}) = h_{\mathrm{IPH},C} + (l_{h_{\mathrm{IPH}}} + r_{h_{\mathrm{IPH}}})/2$ and the width $w(h_{\mathrm{IPH}}) = l_{h_{\mathrm{IPH}}} + r_{h_{\mathrm{IPH}}}$. However, the width for an immunized portfolio is nearly zero.

A DM who has the two kinds of bonds abovementioned, in order to immunize its portfolio, he will take the assets proportion (x_1, x_2). Therefore, the duration will be equal to IPH. For changes in the level of interest rates, active bond management can be used to take advantage from an expected change in interest rates. To enhance return, for lower expected interest rate, the DM will increase the duration of the portfolio; conversely, for an expected rise of them, the DM would shorten portfolio duration. In both situations, the risk portfolio will increase.

- If rates are expected to rise from i to $\tilde{i}'$, with $\tilde{i}' > i$, portfolio duration will be shorten from $D = \text{IPH}$ to D', with $D' < \text{IPH}$. For a right forecast of the interest rate, the present portfolio value will be reduced $\tilde{P}^F_{\text{IPH},\tilde{i}'}(D) < \tilde{P}^F_{\text{IPH},\tilde{i}'}(D')$ in a smaller amount than for a duration $D = \text{IPH}$, and as a result the portfolio value at IPH will be higher than the one for the immunized portfolio. Otherwise, for a wrong interest forecast with interest rate $\tilde{i}^*$, with $\tilde{i}^* < i$, the present portfolio value will increase $\tilde{P}^F_{\text{IPH},\tilde{i}^*}(D) < \tilde{P}^F_{\text{IPH},\tilde{i}^*}(D')$, but as the reinvest rate is lower, the portfolio value at time T will be lower than its value for $D = \text{IPH}$.
- Otherwise, if rates are expected to drop from i to $\tilde{i}^*$, with $\tilde{i}^* < i$, the DM will increase the portfolio duration from D to D^*, with $D^* > \text{IPH}$, so that the present portfolio value will increase, $\tilde{P}^F_{\text{IPH},\tilde{i}^*}(D) < \tilde{P}^F_{\text{IPH},\tilde{i}^*}(D^*)$, more than for $D = \text{IPH}$; As a result, the portfolio value at IPH will be higher than the one for an immunized portfolio. Otherwise, for a wrong interest forecast with interest rate $\tilde{i}'$, with $\tilde{i}' > i$ the strong drop of the present portfolio will not be compensate for the higher reinvestment rate $\tilde{P}^F_{\text{IPH},\tilde{i}'}(D) <^F_{\text{IPH},\tilde{i}'} (D^*)$, $D^* > \text{IPH}$.

Assuming, without loss of generality, the DM expects a reduction of the interest rate, for example, varying from i to $\tilde{i} = (i_C, l_i, r_i)$, and considering $i_C + r_i < i$, an increase of the duration will be required in order to get a higher return, but the risk will be enlarged. For a duration equal to D^*, with $D^* > D$, the proportion of the assets must be $(x_1^*, x_2^*) = (\frac{t_2 - D^*}{t_2 - t_1}, \frac{D^* - t_1}{t_2 - t_1})$, and the number of each kind of bonds that a DM needs to buy is $(n_1^*, n_2^*) = (\frac{P_0^F x_1^*}{P_0^1}, \frac{P_0^F x_2^*}{P_0^2})$. Hence, the value of the portfolio at the IPH is a function of D^* and $\tilde{i} = (i_C, l_i, r_i)$,

$$\tilde{P}^F_{\text{IPH}}(D^*) = \left(P^F_{\text{IPH}_C}(D^*), l^F_{P_{\text{IPH}}}(D^*), r^F_{P_{\text{IPH}}}(D^*)\right),$$

where

$$P^F_{\text{IPH}_C}(D^*) = n_1^* P^1_{t_1}(1 + i_C)^{T - t_1} + n_2^* P^2_{t_2}(1 + i_C)^{-(t_2 - T)}$$

$$l^F_{P_{\text{IPH}}}(D^*) = \begin{cases} \sum_{s=1}^2 (-1)^{s+1} n_s^* P^s_{t_s}[(1 + i_C)^{T - t_s} - (1 + i_C + r_i)^{T - t_s}] & D^* > T \\ \sum_{s=1}^2 (-1)^{s+1} n_s^* P^s_{t_s}[(1 + i_C)^{(-1)^{s+1}(T - t_s)} - (1 + i_C + r_i)^{(-1)^{s+1}(T - t_s)}] & D^* \le T \end{cases}$$

$$r^F_{P_{\text{IPH}}}(D^*) = \begin{cases} \sum_{s=1}^2 (-1)^{s+1} n_s^* P^s_{t_s}[(1 + i_C - l_i)^{(-1)^{s+1}(T - t_s)} - (1 + i_C)^{(-1)^{s+1}(T - t_s)}] & D^* > T \\ \sum_{s=1}^2 (-1)^{s+1} n_s^* P^s_{t_s}[(1 + i_C - l_i)^{(T - t_s)} - (1 + i_C)^{(T - t_s)}] & D^* \le T \end{cases}$$

The return for the new interest rate $\tilde{i} = (i_C, l_i, r_i)$ is:

$$\begin{aligned} \tilde{h}_{\text{IPH},D^*}(\tilde{i}) &= (h_{\text{IPH},D^*_C}, l_{h_{\text{IPH},D^*}}, r_{h_{\text{IPH},D^*}}) \\ &= \left(\left(\frac{P^F_{\text{IPH}_C}(D^*)}{P^F_0}\right)^{1/T} - 1, \left(\frac{P^F_{\text{IPH}_C}(D^*)}{P^F_0}\right)^{1/T} - \left(\frac{P^F_{\text{IPH}_C}(D^*) - l^F_{P_{\text{IPH}}}(D^*)}{P^F_0}\right)^{1/T}, \right. \\ &\quad \left. \left(\frac{P^F_{\text{IPH}_C}(D^*) + r^F_{P_{\text{IPH}}}(D^*)}{P^F_0}\right)^{1/T} - \left(\frac{P^F_{\text{IPH}_C}(D^*)}{P^F_0}\right)^{1/T}\right) \end{aligned}$$

The midpoint of the portfolio return is:

$$m\left(h_{\text{IPH}_C}(D^*)\right) = h_{\text{IPH}_C}(D^*) + \frac{r_{\text{IPH}}(D^*) - l_{\text{IPH}}(D^*)}{2}$$

And the width:

$$w\left(h_{\text{IPH}_C}(D^*)\right) = r_{\text{IPH}}(D^*) - l_{\text{IPH}}(D^*)$$

Finally, our goal is to compare these results with those of an immunized portfolio.

6. The choice between active and passive bond management

Once the set of eligible portfolio has been constructed, the DM will have to choose one alternative: immunization, or active bond management, and in the second case, he will have to select the duration that provides the best combination between risk and return. For this purpose, in the crisp environment, utility functions have been widely studied (see for example [45]). For an approximation to the possibilistic risk premium associated with a fuzzy number, see [24,46].

In Section 2, we have presented different criteria to order intervals (in case of working with FN, we have already established that we will use α-cuts, so the ordering will correspond to intervals for a certain level of presumption). Despite the exposed criteria, we can find situations in which values are not comparable, even more in our case when we compare two values, return and risk. In order to solve this situation, we propose a new index to evaluate intervals in general that allows us to choose between risk and return. Obviously, this index is just for situations in which there is an overlap between two intervals.

Let A and B be two bonds with returns to a certain level of presumption, $A_\alpha = [a_L, a_R]$ and $B_\alpha = [b_L, b_R]$, the range of the interval ($w(A)$ and $w(B)$) or the half-width ($hw(A)$ and $hw(B)$) correspond to a certain "measure" of risk of the bonds.

We want to measure the preference of investing in bond A concerning to the bond B. Therefore, we define four "submeasures" that will allow us, combining them, this measure of preference:

i) Comparison between returns,
ii) Comparison between risks, that is, the difference in widths,
iii) The part of the interval A beats B and
iv) The part of the interval B that beats A.

All these considerations must be weighted according to the DM risk aversion.

i) Comparison between returns (md):

$$md(A,B) = \begin{cases} 1 & a_L \geq b_R \\ -1 & a_R \leq b_L \\ \frac{m(A)-m(B)}{hw(A)+hw(B)} & \text{otherwise} \end{cases} \tag{1}$$

ii) Comparison between risks, that is, the difference in widths:

$$wd(A,B) = \frac{w(B)-w(A)}{w(A)+w(B)}, \quad -1 \leq wd(A,B) \leq 1 \tag{2}$$

iii) A beats B

$$F(A,B) = \begin{cases} 0 & a_R \leq b_R \\ \frac{a_R-b_R}{w(A)} & a_L < b_R < a_R \\ 1 & a_L \geq b_R \end{cases} \tag{3}$$

iv) B beats A

$$G(A,B) = \begin{cases} 0 & b_R \leq a_R \\ \frac{a_R-b_R}{w(B)} & b_L < a_R < b_R \\ -1 & b_L \geq a_R \end{cases} \tag{4}$$

Finally, the measure of preference $I(A,B)$ is:

$$I(A,B) = \omega_1 \cdot md(A,B) + \omega_2 \cdot wd(A,B) + \omega_3 \cdot \big(F(A,B) + G(A,B)\big), \quad \sum_{s=1}^{3} \omega_s = 1 \tag{5}$$

where

$$I(A,B) > 0 \to A \succ B$$
$$I(A,B) < 0 \to A \prec B$$
$$I(A,B) = 0 \to A = B$$

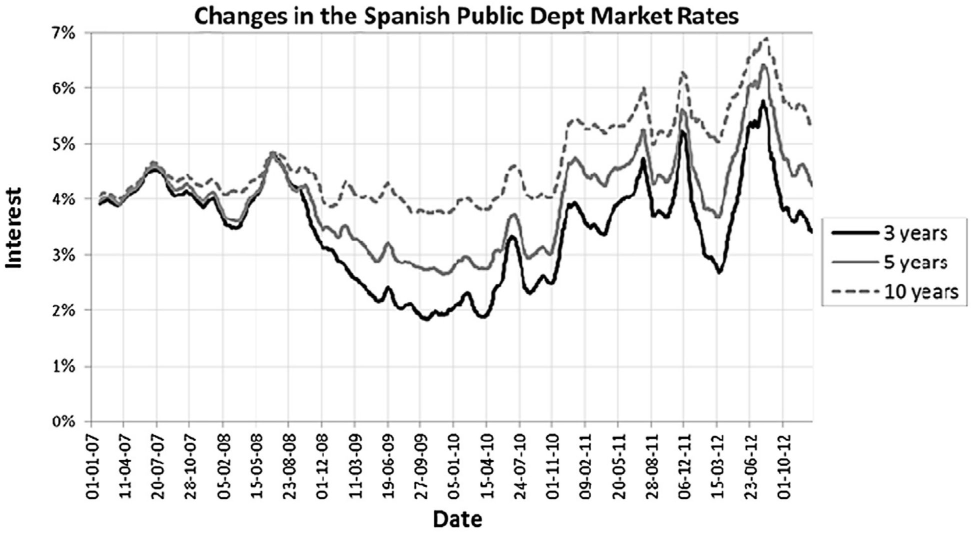

Fig. 1. Changes in the Spanish Public Debt Market rate for bond with maturities 3, 5 and 10 years.

As $F(A, B)$ and $G(A, B)$ cannot be positives at the same time, we weight them with the same coefficient, but it can be different.

As can be seen, this index is a generalization of the acceptability index of Definition 4 proposed by Sengupta and Pal [36], which coincides with this for the case where and are zero.

The DM can consider only the relation between risk and return, taking in consideration only the two first components, or he can consider the fact that a part of the interval be higher than the other. In that way, a very aggressive DM will prefer the alternative that provides the option that has the possibility to beat the other, although it was very little. Others DM do not consider this possibility and they prefer to value only the return difference, and if they are very similar, they take the option with less width.

Example 1. Being the returns $(a_L, a_R) = (3, 9)$, $(b_L, b_R) = (5, 7.5)$. In this case, we have $m_A = 6.0$, $m_B = 6.25$, $w_A = 6.0$ and $w_B = 2.5$. Consider that the DM has very little aversion to risk, and he considers the most important thing to get the alternative that provides the maximum return, although its possibility is very little. So he assigns the followings weights: $\omega_1 = 0.25$, $\omega_2 = 0.25$ and $\omega_3 = 0.5$.

According to Sengupta and Pal, $(b_L, b_R) = (5, 7.5)$ is preferred because its mean is higher and its width is smaller, but considering the measure of preference, we get $md(A, B) = -0.02$, $wd(A, B) = -0.41$, $F(A, B) = 0.25$ and $G(A, B) = 0$, so $I(A, B) = 0.02$. As a result, the DM will have to choose the first alternative $(a_L, a_R) = (3, 9)$.

7. Spanish Public Debt Market

We shall illustrate the proposed methodology with an application to the Spanish Public Debt Market. For this purpose we choose the interest rates on the secondary market for government securities, in particular we take government bonds with maturities 3 and 10 years, from 1/1/2007 to 31/12/2012. This information has been taken from the web page of the Spanish Central Bank (www.bde.es).

Changes in the interest rate for the last seven years can be followed in Fig. 1. It is possible to check that in the summer of 2008 starts a decline in the return of the public debt till the end of 2010. This behavior is similar to other markets like the Euribor or the Spanish AIAF. At the beginning of 2010, the stress in the European Public Debt markets started, and as a result, this index rose shapely, like others similar.

We are going to consider the period from January 2007 to December 2012, an IPH of 5 years, and two zero coupon bonds with maturities 3 and 10 years.

There is a vast literature on measures of possibility in uncertain scenarios, for instance, Li et al. [47], Cummins and Derrig [48], Sadefo et al. [49] or Zhang et al. [50], but only two situations will be assumed, the first one that we

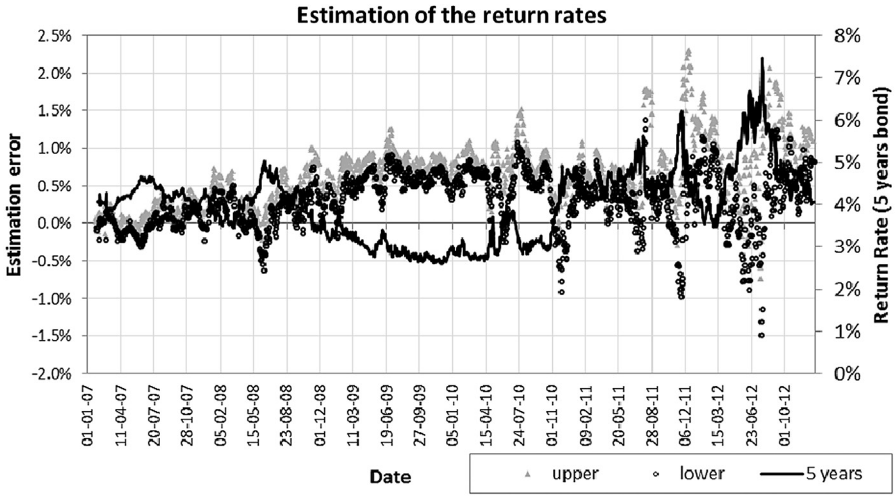

Fig. 2. Rate returns forecast and Public Debt return (5 years maturity).

will consider is the pessimistic one with a probability 0.20 with an interest rate of percentiles (0.4, 0.5, 0.6) of the last 20 traded sessions of the return of the bond with 3 years maturity. On the other hand, for the optimistic situation (obviously, probability 0.8) we are going to consider the same percentiles (0.4, 0.5, 0.6) of the last 20 trades sessions of the bonds with maturity equal to 10 years. The short maturity is used for the pessimistic situation due to the fact that the interest rate for bonds with 3 year maturity used to be smaller than the interest rate for bond with 10 years maturity.

Fig. 2 shows the difference between predicted interest rate for the following period and the real interest, being the grey triangles, the upper limit of the error, and the black segments, the lower limit of it, both of them in the left axis. We overlap the rates of interest of the Spanish Public Debt for maturities for 5 years (right axis). As our prediction for the following year is just a function of the current year, the error increases during 2011 and especially in 2012.

Each day, we have obtained the upper interest rate multiplying the upper value of the pessimistic case by 0.2 and the upper value of the optimistic case by 0.8. The same has been done for the lower case.

Considering durations from 3 to 10 years, we have calculated the expected return for each duration, using first the upper interest rate and second the lower interest rate. As in the theoretical explanation, we have obtained the midpoint of these two estimations, the excess of the midpoint from the immunization case ($D = 5$) for each duration and the difference between the return for the upper case and the lower case for each duration. So, for each duration, we have two points, the excess of return from the immunization case and the width of the interval.

We are going to obtain the measure of preference for durations equal to 4 and 6 years. The coefficients for this measure are for the difference in mean $\omega_1 = 0,6$, for the difference in width $\omega_2 = 0,3$, for the case that $D = 4$ beats $D = 6\omega_3 = 0,05$, and for the last case in which $D = 6$ beats $D = 4$, $\omega_4 = 0,05$.

The measure of preference can take values from -1 to 1. Positives values show that is better to use $D = 4$, and for negatives values is better to use $D = 6$. Fig. 3 shows the way in which $D = 4$ is preferred to $D = 6$ in the period from 01/01/2007 to 31/12/2012. The measure of preference has been worked out using expressions (1) to (5). The measure of preference is presented as a point and we have added a moving average of 50 trading days in order to show in a better way the changes in the measure of preference.

For this particular case, the measure of preference present negative values, so de DM will prefer $D = 6$, but there is some short periods in which the measure of preference takes values near 0 or even positive ($D = 4$ is preferred). This happens when the spread between the rate of interest of 3 and 10 years bond maturity is similar, or when there is an important change in the tendency.

The final step is to check if we get higher returns for those durations. For this purpose, we have assumed that the short time bone has a maturity of 3 years, and we reinvest it at the interest rate of the Public Debt existing till year 5. On the other hand, the long-term bond (5 years maturity) will be sold according to this interest rate. According to these premises, Fig. 4 shows the difference between the real return for each duration and the return of the immunized portfolio ($D = 5$). The better results are obtained for the higher durations.

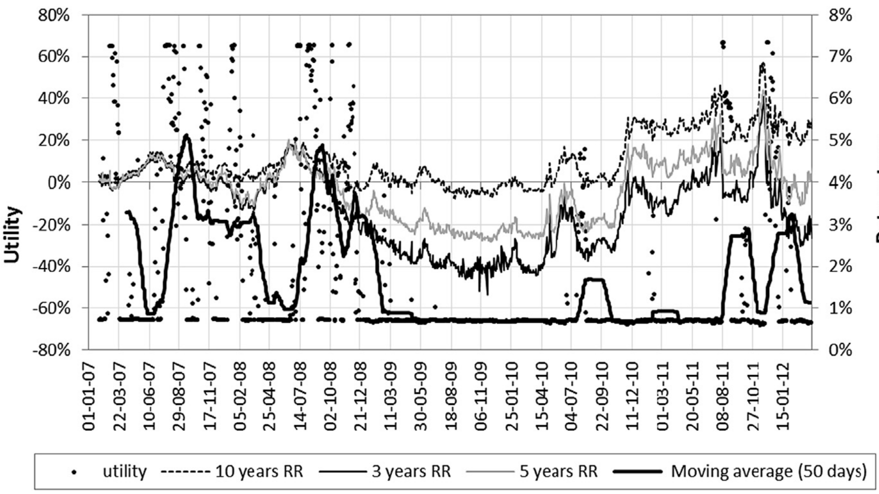

Fig. 3. Measure of preference and the moving average (50 days) in left axis. Rate return (*RR*) for bonds with 3, 5 and 10 years maturities.

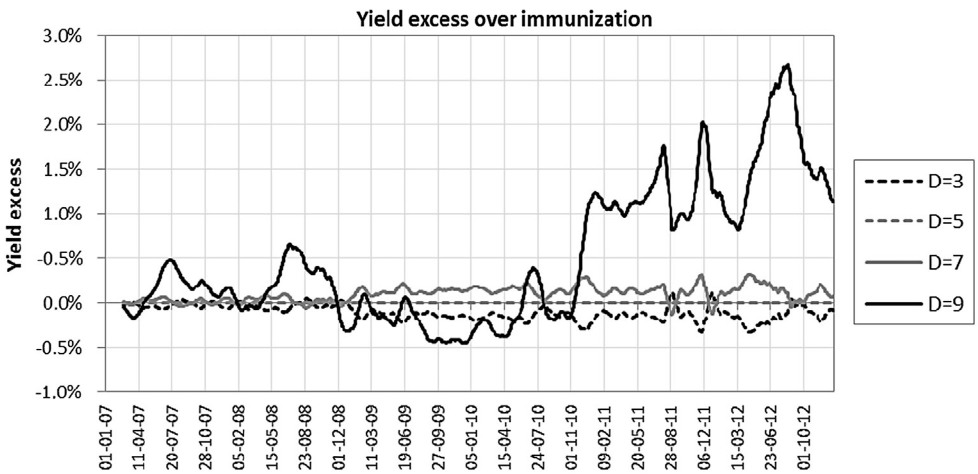

Fig. 4. Excess of the real return for each duration from the expected return (midpoint).

As a result, with this kind of measure of preference, the excess of mid return and the excess of width are not the only aspect to consider in the DM. He will considers others aspects like the possibility that the alternative chosen will have the possibility that some value of the interval will beat the others alternatives.

8. Conclusions

Traditional theory of passive bond management immunizes a portfolio in order to assure its value at the IPH. Otherwise, active bond management deals with portfolio duration in order to increase its value. The evolution of future interest rates is unknown a priori, so it cannot be said which kind of management will be preferable. Therefore, this has been approximated by a fuzzy number.

Active and passive portfolio management are based on duration. Assuming that in order to calculate the duration we should use future interest rates, which are uncertain, we start by estimating them through TFN. This duration will also be a FN, but not triangular, although it can be approximated by TFN since, according to Terceño et al. [39–41] and Perrone and La Diega [42], expressions of financial valuation are well approximated by a TFN.

Immunization of a portfolio is obtained by equating duration and IPH. Since the duration of the portfolio is approximated by TFN, our aim is to obtain the highest level of presumption to which the portfolio is immunized ($D = \mathrm{IPH}$). So, we have established an optimization program which allows the combination of bonds that maximize the level of presumption which immunizes the portfolio. Such combination will be calculated for the lower and the upper extreme of the α-cut, being one of them the solution according to the situation of the IPH respect to the center of the TFN, which represents the duration of the portfolio.

This study will be made on the basis of the α-cuts and the choice will be based on the classical duality of risk/return, where the expected return is measured through the central value of the interval (α-cut) and to risk use the half-width of this interval.

Because of the difficulty for ordering intervals, we propose a measure that allows the decision-maker to establish a degree of acceptability in the preference of an interval (α-cut) over another. In addition, the proposed measure allows the decision-maker to evaluate the importance of several aspects in strategies of active and passive portfolio management, such as the difference between the obtained yields, their different risk, or the possibility of reaching different unattainable values by the other alternative. The different weighting of all these possibilities allows to achieve a measure of preference for one option over the other so that the decision-maker can choose the most appropriate to its risk aversion. This will allow each agent to make different decisions based on your preferences.

By using the proposed methodology, we present an application to the Spanish debt market between years 1997 and 2012, for an IPH of 5 years and for a particular kind of decision-maker.

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