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GLOBAL ATTRACTORS FOR WEAK SOLUTIONS OF THE THREE-DIMENSIONAL NAVIER-STOKES EQUATIONS WITH DAMPING

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Dedicated to Professor Peter Kloeden on his 70th birthday

Abstract. In this paper we obtain the existence of global attractors for the dynamical systems generated by weak solution of the three-dimensional Navier-Stokes equations with damping. We consider two cases, depending on the values of the parameter β controlling the damping term. First, we prove that for $\beta \geq 4$ weak solutions are unique and establish the existence of the global attractor for the corresponding semigroup. Second, for $3 \leq \beta < 4$ we define a multivalued dynamical systems and prove the existence of the global attractor as well. Finally, some numerical simulations are performed.

1. **Introduction.**

The three-dimensional Navier-Stokes equations with damping have been studied intensively over the last years. They describe the situation where there exists resistance to the motion of a flow. One outstanding model in which a damping term appears comes from the flow of cerebrospinal fluid inside the porous brain tissues [12]. Such dissipative damping is also common in many different models. For example, compressible Euler equations with damping describe the flow of a compressible gas through a porous medium [8], whereas Saint-Venant equations are used in oceanography to describe the flow of viscous shallow water with friction [1].

In this paper we study the asymptotic behaviour of weak solutions of the following equation

$$u_t - \mu\Delta u + (u\cdot\nabla)\, u + \alpha\, |u|^{\beta-1}\, u + \nabla p = f,\ \ (x,t) \in \Omega \times (0,T)\,, \tag{1}$$

where $\Omega \subset \mathbb{R}^3$, $\beta \geq 1$, $\mu, \alpha > 0$, $\mu > 0$ is the kinematic viscosity and u is the velocity vector of an incompressible fluid satisfying Dirichlet boundary conditions.

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We would like to point out that the damping term is very helpful from the mathematical point of view, as it allows us to obtain solutions more regular than in the standard Navier-Stokes equations without damping (that is, when $\alpha = 0$). For this reason it is possible to prove the existence of global attractors for (1), at least in a given range of the parameter β, whereas to date this problem remains open for the standard three-dimensional Navier-Stokes equations.

Existence of weak solutions for problem (1) with initial condition in the space of free-divergence square integrable functions was established at first in [2, Theorem 1] for $\beta \geq 1$ and $\Omega = \mathbb{R}^3$ and in [15, Theorem 2.1] for bounded domains Ω. Also, in [2] and [15] existence of global strong solutions of (1) with more regular initial conditions for $\beta \geq 7/2$ and uniqueness for $7/2 \leq \beta \leq 5$ were proved. Existence and uniqueness of global strong solutions were extended for $3 < \beta < 7/2$ in [23] and [17], for $\beta = 3$, $\alpha = \mu = 1$, $\Omega = \mathbb{R}^3$ in [25] and for $\beta = 3$, $\alpha\mu > \frac{1}{4}$, Ω bounded in [9]. On top of that, in [25] uniqueness of strong solutions was proved to be true for all $\beta \geq 1$ when $\Omega = \mathbb{R}^3$. If either the initial condition u_0 is small enough or the viscosity μ is large, existence of global strong solutions was established in [24] for $1 \leq \beta < 3$, $\Omega = \mathbb{R}^3$. Also, existence of T-periodic solutions was studied in [9], and regularity of weak solutions in [25], [22].

Therefore, when $3 < \beta \leq 5$ strong solutions of (1) generate a dynamical system. When the domain Ω is bounded in the autonomous case the existence of a global attractor for $7/2 \leq \beta \leq 5$ was proved in [15, Theorem 2.1], and this result was extended to the nonautonomous case in [16], where the existence of a uniform attractor was established. In addition, in [17] existence of pullback attractors in the nonautonomous case was studied in the more general situation where $3 < \beta \leq 5$.

It is worth noting that so far there are no results in the literature either on uniqueness or existence of attractors for weak solutions. In Section 2 we prove first that weak solutions are unique if $\beta \geq 4$. In this way, in Section 3 we are able to define a semigroup of operators and prove that a global attractor exists. Moreover, if $\beta \geq 3$ we obtain that every weak solution is continuous, and using this property we are able to prove for $3 \leq \beta < 4$ that the multivalued semiflow generated by the weak solutions possesses a global attractor.

In the case of strong solutions it would be desirable to get rid of the restriction $\beta \leq 5$, which was assumed in all the previous papers. In Section 4 we obtain a conditional result stating that if every strong solution is continuous with values in the space $L^{\beta+1}$, then the global attractor exists for $\beta > 5$ as well.

In Section 5 we present some numerical simulations to illustrate some aspects about the asymptotic behaviour of model (1) for a particular flow domain and initial conditions. Specifically, we analyze how the input parameters α and β affect the behaviour of the fluid flow evolution over a sufficiently long term.

2. Uniqueness and regularity of weak solutions.

Consider a bounded open set $\Omega \subset \mathbb{R}^3$ with smooth boundary $\partial\Omega$. We study the three-dimensional Navier-Stokes equations with damping

$$\left\{ \begin{array}{c} u_t - \mu \Delta u + (u \cdot \nabla)\, u + \alpha \left| u \right|^{\beta - 1} u + \nabla p = f, \ \ (x,t) \in \Omega \times (0,T)\,, \\ \operatorname{div} u = 0, \ \ (x,t) \in \Omega \times (0,T)\,, \\ u\mid_{\partial\Omega} = 0, \ \ t \in (0,T)\,, \\ u\mid_{t=0} = u_0, \ x \in \Omega, \end{array} \right. \tag{2}$$

where $\mu > 0$ is the kinematic viscosity and f is an external force. Also, $\beta \geq 1$ and $\alpha > 0$ are given constants. The functions $u(x,t) = (u_1(x,t), u_2(x,t), u_3(x,t))$,

$p(x,t)$ stand for the velocity field and the pressure, respectively. Here and further, $|\cdot|$ denotes in general the norm in $\mathbb{R}^d$ for any $d \geq 1$.

We define the usual function spaces

$$\begin{aligned} \mathcal{V} &= \{u \in (C_0^\infty(\Omega))^3 : div\ u = 0\}, \\ H &= cl_{(L^2(\Omega))^3}\mathcal{V}, \\ V &= cl_{(H_0^1(\Omega))^3}\mathcal{V}, \end{aligned}$$

where cl_X denotes the closure in the space X. It is well known that H, V are separable Hilbert spaces and identifying H and its dual we have $V \subset H \subset V'$ with dense and continuous injections. We denote by $(\cdot,\cdot)$, $\|\cdot\|_H$ and $((\cdot,\cdot))$, $\|\cdot\|_V$ the inner product and norm in H and V, respectively, and by $\langle\cdot,\cdot\rangle$ duality between V' and V. Let H_w be the space H endowed with the weak topology. As usual, we define the continuous trilinear form $b : V \times V \times V \to \mathbb{R}$ by

$$b(u,v,w) = \sum_{i,j=1}^{3} \int_\Omega u_i \frac{\partial v_j}{\partial x_i} w_j dx.$$

It is well-known that $b(u,v,v) = 0$, if $u \in V, v \in \left(H_0^1(\Omega)\right)^3$. For $u, v \in V$ we denote by $B(u,v)$ the element of V' defined by $\langle B(u,v), w\rangle = b(u,v,w)$, for all $w \in V$.

The norm in the spaces $L^p(\Omega)$, $(L^p(\Omega))^3$, $p \geq 1$, will be denoted indistinctly by $\|\cdot\|_p$.

Let P be the orthogonal projection from $(L^2(\Omega))^3$ onto H and $Au = -P\Delta u$ be the Stokes operator, defined by $\langle Au, v\rangle = ((u,v))$ for $u, v \in V$. Since the boundary $\partial\Omega$ is smooth, $D(A) = \left(H^2(\Omega)\right)^3 \cap V$ and $\|Au\|_2$ defines a norm in $D(A)$ which is equivalent to the norm in $\left(H^2(\Omega)\right)^3$.

For $u_0 \in H,\ f \in H$ the function

$$u \in L^\infty(0,T;H) \cap L^2(0,T;V) \cap L^{\beta+1}\left(0,T;\left(L^{\beta+1}(\Omega)\right)^3\right)$$

is said to be a weak solution to problem (2) on $(0,T)$ if $u(0) = u_0$ and

$$\frac{d}{dt}(u,v) + \mu((u,v)) + b(u,u,v) + \alpha\left(|u|^{\beta-1}u, v\right) = (f,v), \tag{3}$$

for any $v \in V \cap \left(L^{\beta+1}(\Omega)\right)^3$, in the sense of scalar distributions.

We recall the following well-known result on existence of weak solutions.

Theorem 2.1. *([2, Theorem 1] and [15, Theorem 2.1]) For any $u_0 \in H,\ f \in H$, $\beta \geq 1$ there exists at least one weak solution u to problem (2).*

Let $Y = V' + \left(L^{\frac{\beta+1}{\beta}}(\Omega)\right)^3$, the dual space of $V \cap \left(L^{\beta+1}(\Omega)\right)^3$. We note that by standard estimates on B for any weak solution we have that

$$\begin{aligned} Au &\in L^2(0,T;V'), \\ B(u,u) &\in L^{\frac{4}{3}}(0,T;V'), \\ |u|^{\beta-1}u &\in L^{\frac{\beta+1}{\beta}}\left(0,T;\left(L^{\beta+1}(\Omega)\right)^3\right), \end{aligned}$$

which implies in particular that

$$-\mu Au - B(u,u) - \alpha \left|u\right|^{\beta-1} u + f \in L^{\frac{4}{3}}\left(0,T;V'\right) + L^{\frac{\beta+1}{\beta}}\left(0,T;\left(L^{\frac{\beta+1}{\beta}}\left(\Omega\right)\right)^{3}\right)$$
$$\subset L^{1}(0,T;Y).$$

It follows from equality (3) and a standard result [18, p.250, Lemma 1.1] that

$$\frac{du}{dt} = -\mu Au - B(u,u) - \alpha \left|u\right|^{\beta-1} u + f \tag{4}$$

in the sense of Y-valued distributions. Hence, the derivate $\frac{du}{dt}$ belongs to the space

$$Y = L^{\frac{4}{3}}\left(0,T;V'\right) + L^{\frac{\beta+1}{\beta}}\left(0,T;\left(L^{\frac{\beta+1}{\beta}}\left(\Omega\right)\right)^{3}\right)$$

and equality (4) is satisfied in the space Y for a.a. $t \in (0,T)$.

In order to obtain good estimates of weak solutions we need $\frac{du}{dt}$ to be more regular. We can obtain such a result for $\beta \geq 3$.

Lemma 2.2. *Let u be a weak solution to (2) such that $u \in L^{q}\left(0,T;\left(L^{q}\left(\Omega\right)\right)^{3}\right)$ with $q \geq 4$. Then*

$$\frac{du}{dt} \in L^{2}\left(0,T;V'\right) + L^{\frac{\beta+1}{\beta}}\left(0,T;\left(L^{\frac{\beta+1}{\beta}}\left(\Omega\right)\right)^{3}\right), \tag{5}$$

$$u \in C\left([0,T],H\right), \tag{6}$$

the map $t \mapsto \left\|u\left(t\right)\right\|_{H}^{2}$ is absolutely continuous and

$$\frac{d}{dt}\left\|u\left(t\right)\right\|_{H}^{2} = 2\left\langle u, \frac{du}{dt}\right\rangle \text{ for a.a. } t \in (0,T). \tag{7}$$

Proof. Using the well-known inequality (see [18, p.297])

$$\left|b\left(u,u,v\right)\right| \leq C\left\|u\right\|_{4}^{2}\left\|v\right\|_{V}, \ \forall u,v \in V,$$

and $u \in L^{q}\left(0,T;\left(L^{q}\left(\Omega\right)\right)^{3}\right) \subset L^{4}\left(0,T;\left(L^{4}\left(\Omega\right)\right)^{3}\right)$, we have

$$B\left(u,u\right) \in L^{2}\left(0,T;V'\right),$$

so (5) follows.

Properties (6)-(7) follow from [3, Chapter II, Theorem 1.8]. □

Corollary 1. *If $\beta \geq 3$, then any weak solution to (2) satisfies (5)-(7).*

Proof. Since $u \in L^{\beta+1}\left(0,T;\left(L^{\beta+1}\left(\Omega\right)\right)^{3}\right)$ and $\beta \geq 3$, we obtain that u belongs to $L^{q}\left(0,T;\left(L^{q}\left(\Omega\right)\right)^{3}\right)$ with $q \geq 4$. □

Lemma 2.3. *If $\beta \geq 3$, then any weak solution satisfies the estimates*

$$\|u(t)\|_{H}^{2} \leq e^{-\mu\lambda_{1}t}\left\|u_{0}\right\|_{H}^{2} + \frac{\|f\|_{H}^{2}}{\mu^{2}\lambda_{1}^{2}}, \tag{8}$$

$$\mu\int_{s}^{t}\|u\|_{V}^{2}\,d\tau + 2\alpha\int_{s}^{t}\|u\|_{\beta+1}^{\beta+1}\,d\tau \leq \left\|u_{0}\right\|_{H}^{2} + \frac{\|f\|_{H}^{2}}{\mu^{2}\lambda_{1}^{2}} + \frac{1}{\mu\lambda_{1}}\|f\|_{H}^{2}\left(t-s\right), \tag{9}$$

for any $t \geq s \geq 0$.

Proof. Multiplying equality (4) by u and using (7) and $b(u,u,u)=0$ we have

$$\frac{1}{2}\frac{d}{dt}\|u\|_H^2+\mu\|u\|_V^2+\alpha\|u\|_{\beta+1}^{\beta+1}=(f,u)\le\frac{\mu\lambda_1}{2}\|u\|_H^2+\frac{1}{2\mu\lambda_1}\|f\|_H^2. \tag{10}$$

As $\mu\|u\|_V^2\ge\mu\lambda_1\|u\|_H^2$, we deduce that

$$\frac{d}{dt}\|u\|_H^2+\mu\lambda_1\|u\|_H^2\le\frac{1}{\mu\lambda_1}\|f\|_H^2 \tag{11}$$

and Gronwall's lemma yields

$$\|u(t)\|_H^2\le e^{-\mu\lambda_1 t}\|u_0\|_H^2+\frac{\|f\|_H^2}{\mu^2\lambda_1^2}.$$

By $\mu\|u\|_V^2\ge\frac{\mu}{2}\|u\|_V^2+\frac{\mu\lambda_1}{2}\|u\|_H^2$, integrating over the interval (s,t) in (10) it follows that

$$\begin{aligned}\mu\int_s^t\|u\|_V^2\,d\tau+2\alpha\int_s^t\|u\|_{\beta+1}^{\beta+1}\,d\tau&\le\|u(s)\|_H^2+\frac{1}{\mu\lambda_1}\|f\|_H^2(t-s)\\&\le e^{-\mu\lambda_1 s}\|u_0\|_H^2+\frac{\|f\|_H^2}{\mu^2\lambda_1^2}+\frac{1}{\mu\lambda_1}\|f\|_H^2(t-s),\end{aligned}$$

so the lemma is proved. □

Further, when $\beta\ge4$ we can prove that the weak solution is unique.

Theorem 2.4. *Let $\beta\ge4$. Then for any $u_0\in H$ there exists a unique weak solution $u(\cdot)$ to problem (2), which is continuous with respect to the initial datum u_0.*

Proof. Let u_1,u_2 be two weak solutions such that $u_1(0)=u_2(0)=u_0$. Denote $w=u_1-u_2$.

We multiply equality (4) by w for each solution u_1,u_2. Hence,

$$\left\langle\frac{du_1}{dt},w\right\rangle+\mu((u_1,w))+b(u_1,u_1,w)+\alpha\left(|u_1|^{\beta-1}u_1,w\right)=(f,w),$$

$$\left\langle\frac{du_2}{dt},w\right\rangle+\mu((u_2,w))+b(u_2,u_2,w)+\alpha\left(|u_2|^{\beta-1}u_2,w\right)=(f,w).$$

Taking the difference of these expressions and using (7) we get

$$\begin{aligned}&\tfrac{1}{2}\tfrac{d}{dt}\|w\|_H^2+\mu\|w\|_V^2+b(u_1,u_1,w)-b(u_2,u_2,w)\\&+\alpha\left(|u_1|^{\beta-1}u_1,w\right)-\alpha\left(|u_2|^{\beta-1}u_2,w\right)=0.\end{aligned}$$

For the nonlinear terms we have

$$b(u_1,u_1,w)-b(u_2,u_2,w)=b(w,u_1,w)+b(u_2,w,w)=-b(w,w,u_1),$$

$$\begin{aligned}&\alpha\left(|u_1|^{\beta-1}u_1,w\right)-\alpha\left(|u_2|^{\beta-1}u_2,w\right)\\&=\alpha\int_\Omega\left(|u_1|^{\beta-1}u_1-|u_2|^{\beta-1}u_2\right)(u_1-u_2)\,dx\ge0,\end{aligned}$$

where we have used the properties of the operator b and the fact that the function $f(x)=|x|^{\beta-1}x$ is increasing.

Thus,

$$\frac{d}{dt}\|w\|_H^2+2\mu\|w\|_V^2\le2b(w,w,u_1).$$

By Hölder's inequality we have

$$\left|b\left(w,w,u_{1}\right)\right|\leq C_{1}\left\|w\right\|_{\rho}\left\|u_{1}\right\|_{r}\left\|w\right\|_{V},$$

where $\frac{1}{r}+\frac{1}{\rho}=\frac{1}{2}$, $r\geq 5$, so that $2\leq\rho\leq\frac{10}{3}$. Now the interpolation inequality and the embedding $H^{1}\left(\Omega\right)\subset L^{6}\left(\Omega\right)$ gives

$$\left\|w\right\|_{\rho}\leq\left\|w\right\|_{2}^{1-\frac{3}{r}}\left\|w\right\|_{6}^{\frac{3}{r}}\leq C_{2}\left\|w\right\|_{2}^{1-\frac{3}{r}}\left\|w\right\|_{V}^{\frac{3}{r}},$$

as $\frac{1}{\rho}=\frac{1-\frac{3}{r}}{2}+\frac{\frac{3}{r}}{6}$. Then, by Young's inequality we have

$$\begin{aligned}2\left|b\left(w,w,u_{1}\right)\right|&\leq C_{3}\left\|w\right\|_{H}^{1-\frac{3}{r}}\left\|w\right\|_{V}^{1+\frac{3}{r}}\left\|u_{1}\right\|_{r}\\&\leq\mu\left\|w\right\|_{V}^{2}+C_{4}\left\|w\right\|_{H}^{2}\left\|u_{1}\right\|_{r}^{\frac{2r}{r-3}}.\end{aligned}$$

Therefore,

$$\frac{d}{dt}\left\|w\right\|_{H}^{2}\leq C_{4}\left\|w\right\|_{H}^{2}\left\|u_{1}\right\|_{r}^{\frac{2r}{r-3}}.$$

Since $u_{1}\in L^{r}\left(0,T;L^{r}\left(\Omega\right)\right)$ and $r\geq 5$ imply that $M\left(t\right)=\left\|u_{1}\left(t\right)\right\|_{r}^{\frac{2r}{r-3}}\in L^{1}\left(0,T\right)$, by Gronwall's lemma we obtain that

$$\left\|w\left(t\right)\right\|_{H}^{2}\leq\left\|w\left(0\right)\right\|_{H}^{2}e^{C_{4}\int_{0}^{t}M(s)ds},\tag{12}$$

so $u_{1}\equiv u_{2}$.

Hence, both uniqueness and continuity of the solution $u\left(\cdot\right)$ with respect to u_{0} follow. □

3. Global attractor for weak solutions.

Our aim now is to prove the existence of the global attractor for the weak solutions of problem (2).

We shall divide this section into two cases: a) $\beta\geq 4$; b) $3\leq\beta<4$. In the first one uniqueness of weak solutions implies that we can define a semigroup of operators, to which we can apply the classical theory of attractors for semigroups. In the second one more than one solution can possibly exist for a given initial datum, and then we need to make use of the theory of attractors for multivalued semiflows.

3.1. Case $\beta\geq 4$.

In view of Theorem 2.4 we can define the semigroup of operators $S:\mathbb{R}^{+}\times H\rightarrow H$ by

$$S(t,u_{0})=u\left(t\right),$$

where $u\left(\cdot\right)$ is the unique solution to problem (2) with initial condition u_{0}. It is straightforward to see that S satisfies the semigroup properties: $S\left(0,u_{0}\right)=u_{0}$, for any $u_{0}\in H$, and $S\left(t+s,u_{0}\right)=S\left(t,S\left(s,u_{0}\right)\right)$, for any $u_{0}\in H$, $t,s\geq 0$. Also, making use again of Theorem 2.4 we obtain that $S(t,u_{0})$ is continuous with respect to the initial condition u_{0} for fixed $t\geq 0$.

Lemma 3.1. *If $\beta\geq 4$, then any weak solution of (2) with initial data such that $\left\|u_{0}\right\|_{H}\leq R$ satisfies the estimate*

$$\left\|u\left(t+r\right)\right\|_{V}^{2}+\left\|u\left(t+r\right)\right\|_{\beta+1}^{\beta+1}\leq C\left(R,r\right),$$

for any $r>0$ and $t\geq 0$, where $C\left(R,r\right)$ is such that $C\left(R,r\right)\rightarrow\infty$ if $r\rightarrow 0^{+}$ or $R\rightarrow+\infty$.

Proof. The following calculations are formal, but they can be justified using Galerkin approximations.

Let $R>0$ and $\|u_0\| \leq R$. We multiply the equation by u_t and $-\Delta$ and integrate over Ω. Then

$$\begin{aligned}
&\frac{\mu}{2}\frac{d}{dt}\|u\|_V^2+\frac{\alpha}{\beta+1}\frac{d}{dt}\|u\|_{\beta+1}^{\beta+1}+\|u_t\|_H^2=-\int_\Omega(u\cdot\nabla)\,uu_tdx+(f,u_t)\,,\\
&\frac{1}{2}\frac{d}{dt}\|u\|_V^2+\mu\|\Delta u\|_2^2+\alpha\int_\Omega|u|^{\beta-1}|\nabla u|^2dx+\frac{\alpha(\beta-1)}{4}\int_\Omega|u|^{\beta-3}\left|\nabla|u|^2\right|^2dx\\
&\qquad\qquad=-\int_\Omega(u\cdot\nabla)\,u\Delta udx-(f,\Delta u)\,.
\end{aligned}$$

Summing up these expressions and using Hölder's and Young's inequalities we get

$$\begin{aligned}
&\frac{\mu+1}{2}\frac{d}{dt}\|u\|_V^2+\frac{\alpha}{\beta+1}\frac{d}{dt}\|u\|_{\beta+1}^{\beta+1}+\frac{1}{2}\|u_t\|_H^2+\frac{\mu}{2}\|\Delta u\|_2^2\\
&+\alpha\int_\Omega|u|^{\beta-1}|\nabla u|^2dx+\frac{\alpha(\beta-1)}{4}\int_\Omega|u|^{\beta-3}\left|\nabla|u|^2\right|^2dx\\
&\leq\left(1+\frac{1}{\mu}\right)|u\cdot\nabla u|_2^2+\left(1+\frac{1}{\mu}\right)|f|_H^2\,.
\end{aligned}\tag{13}$$

For $\beta\geq 4$ the term $J=\left(1+\frac{1}{\mu}\right)|u\cdot\nabla u|_2^2$ was estimated in [2, p.806] as follows:

$$J\leq\frac{\mu}{4}\|\Delta u\|_2^2+R_1\|u\|_{\beta+1}^{\frac{4(\beta+1)}{\beta-2}}\|u\|_H^2\,,$$

for some constant $R_1>0$. If $\beta\geq 6$, then $\frac{4(\beta+1)}{\beta-2}\leq\beta+1$ and Young's inequality implies

$$J\leq\frac{\mu}{4}\|\Delta u\|_2^2+R_2\|u\|_{\beta+1}^{\beta+1}\|u\|_H^2+R_3\|u\|_H^2\,.$$

If $4\leq\beta\leq 6$, then

$$J\leq\frac{\mu}{4}\|\Delta u\|_2^2+R_1\|u\|_{\beta+1}^{\beta+1}\|u\|_{\beta+1}^{\frac{5\beta-\beta^2+6}{\beta-2}}\|u\|_H^2$$

and since $\frac{5\beta-\beta^2+6}{\beta-2}\leq\beta+1$, applying again Young's inequality we have

$$J\leq\frac{\mu}{4}\|\Delta u\|_2^2+R_4\|u\|_{\beta+1}^{\beta+1}\|u\|_{\beta+1}^{\beta+1}\|u\|_H^2+R_5\|u\|_{\beta+1}^{\beta+1}\|u\|_H^2\,.$$

Joining these inequalities we get

$$J\leq\frac{\mu}{4}\|\Delta u\|_2^2+R_6\left(\|u\|_{\beta+1}^{\beta+1}\left(1+\|u\|_{\beta+1}^{\beta+1}\right)\|u\|_H^2+\|u\|_H^2\right).\tag{14}$$

Thus,

$$\begin{aligned}
&\frac{\mu+1}{2}\frac{d}{dt}\|u\|_V^2+\frac{\alpha}{\beta+1}\frac{d}{dt}\|u\|_{\beta+1}^{\beta+1}\\
&\leq R_6\left(\|u\|_{\beta+1}^{\beta+1}\left(1+\|u\|_{\beta+1}^{\beta+1}\right)\|u\|_H^2+\|u\|_H^2\right)+R_7,
\end{aligned}$$

where $R_7=\left(1+\frac{1}{\mu}\right)|f|_2^2$. Let us denote $y(t)=\frac{\mu+1}{2}\|u\|_V^2+\frac{\alpha}{\beta+1}\|u\|_{\beta+1}^{\beta+1}$. From Lemma 2.3 there exists $C_R>0$ such that $\|u(t)\|_H\leq C_R$ for all $t\geq 0$. Then

$$\frac{dy}{dt}\leq R_8\left(1+\|u\|_{\beta+1}^{\beta+1}\right)y+R_9.\tag{15}$$

In view of Lemma 2.3,

$$\int_t^{t+r} R_8 \left(1+\|u(s)\|_{\beta+1}^{\beta+1}\right) ds$$
$$\leq R_8 r + \frac{R_8}{2\alpha}\left(R^2 + \frac{\|f\|_H^2}{\mu^2\lambda_1^2} + \frac{1}{\mu\lambda_1}\|f\|_H^2 r\right) = a_1(r,R),$$

$$\int_t^{t+r} y(s)\,ds$$
$$\leq \left(\frac{\mu+1}{2} + \frac{\alpha}{\beta+1}\right)\left(R^2 + \frac{\|f\|_H^2}{\mu^2\lambda_1^2} + \frac{1}{\mu\lambda_1}\|f\|_H^2 r\right)\max\{\frac{1}{\mu},\frac{1}{2\alpha}\} = a_3(r,R).$$

for any $t \geq 0,\ r > 0$. By the uniform Gronwall lemma [19] we obtain

$$y(t+r) \leq \left(\frac{a_3(r,R)}{r} + R_9 r\right) e^{a_1(r,R)}, \text{ for all } t \geq 0,\ r > 0.$$

□

As a consequence of this lemma and the compact embedding $V \subset H$ we obtain the following result.

Corollary 2. *For any $r > 0$ the map $u_0 \mapsto S(r,u_0)$ maps bounded subsets of H onto bounded subsets of $V \cap L^{\beta+1}(\Omega)$. Hence, $S(r)$ is a compact operator, i.e., it maps bounded subsets of H onto relatively compact ones.*

We recall that the set $\mathcal{A}$ is said to be a global attractor for S if it is invariant, i.e. $S(t,\mathcal{A}) = \mathcal{A}$, for all $t \geq 0$, and it attracts every bounded subset B of the phase space H, which means that

$$dist_H(S(t,B),\mathcal{A}) \to 0 \text{ as } t \to +\infty,$$

where $dist_X(C,A) = \sup_{x\in C}\inf_{y\in A}\|x-y\|_X$ is the Hausdorff semidistance between subsets of the Banach space X.

Usually in the literature a global attractor is supposed to be compact as well. However, we prefer to use this more general definition and add compactness as an additional property, as generally speaking a global attractor does not have to be bounded (see [20] for a non-trivial example of an unbounded non-locally compact attractor).

A set B_0 is called absorbing if for any bounded set B there exists a time $T(B)$ such that

$$S(t,B) \subset B_0 \text{ for any } t \geq T.$$

We put

$$B_0 = \left\{u \in H : \|u\|_H^2 \leq 1 + \frac{\|f\|_H^2}{\mu^2\lambda_1^2}\right\}.$$

Lemma 3.2. *If $\beta \geq 4$, the ball B_0 is absorbing.*

Proof. This result follows directly from (8). □

The semigroup S is said to be asymptotically compact if for any bounded subset B every sequence $y_k \in S(t_k,B)$, where $t_k \to +\infty$, is relatively compact in H.

Lemma 3.3. *If $\beta \geq 4$, there exists a compact absorbing set K.*

Proof. Let $K = \overline{S(1, B_0)}$, which is a compact set in H due to Lemma 3.1 and the compact embedding $V \subset H$. Then since B_0 is absorbing, for any bounded set B there exists $T(B) > 0$ such that

$$S(t, B) = S(1, S(t-1, B)) \subset S(1, B_0) \subset K \text{ for all } t \geq T.$$

□

Corollary 3. *If $\beta \geq 4$, the semigroup S is asymptotically compact.*

The following two conditions guarantee the existence of the global compact attractor [10]:

1. There exists a bounded absorbing set B_0.
2. S is asymptotically compact.

Theorem 3.4. *If $\beta \geq 4$, the semigroup S possesses the global compact connected attractor $\mathcal{A}$.*

Proof. The existence of the global compact attractor is a consequence of Lemma 3.2 and Corollary 3. Since the space H is connected, the connectedness of $\mathcal{A}$ follows from [5, p.4]. □

It is possible to prove that the global attractor is more regular if $4 \leq \beta < 5$. Indeed, let us check that $\mathcal{A}$ is in fact bounded in the space $\left(H^2(\Omega)\right)^3$, and then compact in V and $\left(L^{\beta+1}(\Omega)\right)^3$.

Lemma 3.5. *If $\beta \geq 4$, then any weak solution of (2) with initial data such that $\|u_0\|_H \leq R$ satisfies the estimate*

$$\left\| \frac{du}{dt}(r) \right\|_2 \leq D(R, r),$$

for any $r > 0$, where $D(R, r)$ is such that $D(R, r) \to \infty$ if $r \to 0^+$ or $R \to +\infty$.

Proof. As before, the calculations here are formal, but they can be justified via Galerkin approximations.

Integrating (13) over the interval $\left(t, t + \frac{r}{2}\right)$, with $t \geq \frac{r}{2}$, and using inequality (14) and Lemmas 2.3, 3.1 we have

$$\begin{aligned}
&\frac{1}{2} \int_t^{t+\frac{r}{2}} \left\| \frac{du}{ds} \right\|_H^2 ds + \frac{\mu}{4} \int_t^{t+\frac{r}{2}} \|\Delta u\|_2^2 \, ds \\
&\leq \frac{\mu + 1}{2} \|u(t)\|_V^2 + \frac{\alpha}{\beta + 1} \|u(t)\|_{\beta+1}^{\beta+1} + \frac{r}{2} \left(1 + \frac{1}{\mu}\right) |f|_H^2 \\
&+ R_6 \int_t^{t+\frac{r}{2}} \left(\|u\|_{\beta+1}^{\beta+1} \left(1 + \|u\|_{\beta+1}^{\beta+1}\right) \|u\|_H^2 + \|u\|_H^2 \right) ds \\
&\leq D_1(r, R),
\end{aligned} \tag{16}$$

where D_1 satisfies the above properties for D.

Further, differentiating (4) with respect to the time variable and multiplying by $\frac{du}{dt}$ we obtain

$$\frac{1}{2} \frac{d}{dt} \left\| \frac{du}{dt} \right\|_2^2 + \mu \left\| \frac{du}{dt} \right\|_V^2 = b(u_t, u, u_t) - \int_\Omega J_F(u) u_t \cdot u_t dx,$$

where J_F is the Jacobian matrix of $F(u) = \alpha \left|u\right|^{\beta-1} u$, which is definite positive [15, Lemma 2.4]. Hence, the last term is non-positive. By the standard estimate [18, p.297]

$$\left|b(u_t, u, u_t)\right| \leq K_1 \left\|u_t\right\|_2^{\frac{1}{2}} \left\|u_t\right\|_V^{\frac{3}{2}} \left\|u\right\|_V$$

we infer by Young's inequality that

$$\frac{1}{2}\frac{d}{dt}\left\|\frac{du}{dt}\right\|_2^2 + \frac{\mu}{2}\left\|\frac{du}{dt}\right\|_V^2 \leq K_2 \left\|\frac{du}{dt}\right\|_2^2 \left\|u\right\|_V^4 .$$

Thus, from Lemma 3.1 we get

$$\frac{d}{dt}\left\|\frac{du}{dt}\right\|_2^2 \leq D_2(r,R)\left\|\frac{du}{dt}\right\|_2^2,$$

where again D_2 satisfies the above properties for D.

Now, integrating the last expression over (s,r), with $\frac{r}{2} \leq s \leq r$, and using (16) we have

$$\left\|\frac{du}{dt}(r)\right\|_2^2 \leq \left\|\frac{du}{dt}(s)\right\|_2^2 + D_2(r,R)\int_s^r \left\|\frac{du}{dt}\right\|_2^2 dt \leq \left\|\frac{du}{dt}(s)\right\|_2^2 + 2D_1(r,R)D_2(r,R).$$

Integrating again but now with respect to s over $\left(\frac{r}{2}, r\right)$ and applying again (16) we get

$$\begin{aligned}\frac{r}{2}\left\|\frac{du}{dt}(r)\right\|_2^2 &\leq \int_{\frac{r}{2}}^r \left\|\frac{du}{dt}(s)\right\|_2^2 ds + rD_1(r,R)D_2(r,R)\\ &\leq 2D_1(r,R) + rD_1(r,R)D_2(r,R).\end{aligned}$$

Thus, the proof is finished. □

Lemma 3.6. *If $4 \leq \beta < 5$, then any weak solution of (2) with initial data such that $\left\|u_0\right\|_H \leq R$ satisfies the estimate*

$$\left\|Au(r)\right\|_2 \leq K(R,r),$$

for any $r > 0$, where $K(R,r)$ is such that $K(R,r) \to \infty$ if $r \to 0^+$ or $R \to +\infty$.

Proof. We follow the same steps as in [15, Proposition 5].

By Proposition 9.2 in [14] we have

$$\left\|B(u)\right\|_2 \leq d_1 \left\|u\right\|_V^{\frac{3}{2}} \left\|Au\right\|_2^{\frac{1}{2}} \leq \frac{\mu}{4}\left\|Au\right\|_2 + d_2 \left\|u\right\|_V^3,$$

whereas Gagliardo-Nirenberg inequality and $\beta < 5$ gives

$$\begin{aligned}\alpha\left\|\left|u\right|^{\beta-1}u\right\|_2 = \alpha\left\|u\right\|_{2\beta}^{\beta} &\leq d_3 \left\|Au\right\|_2^{\frac{3(\beta-1)}{\beta+7}} \left\|u\right\|_{\beta+1}^{\frac{\beta^2+4\beta+3}{\beta+7}}\\ &\leq \frac{\mu}{4}\left\|Au\right\|_2 + d_4 \left\|u\right\|_{\beta+1}^{\frac{\beta^2+4\beta+3}{10-2\beta}} .\end{aligned}$$

Hence, from (4) we get

$$\frac{\mu}{2}\left\|Au(r)\right\|_2 \leq \left\|\frac{du}{dt}(r)\right\|_2 + d_2\left\|u(r)\right\|_V^3 + d_4\left\|u(r)\right\|_{\beta+1}^{\frac{\beta^2+4\beta+3}{10-2\beta}} + \left\|f\right\|_H,$$

so the results follows by applying Lemmas 3.1, 3.5. □

Theorem 3.7. *If $4 \leq \beta < 5$, then the global attractor from Theorem 3.4 is bounded in $\left(H^2(\Omega)\right)^3$, and then compact in V and $\left(L^{\beta+1}(\Omega)\right)^3$. Moreover,*

$$dist_{V \cap (L^{\beta+1}(\Omega))^3}(S(t,B), \mathcal{A}) \to 0 \text{ as } t \to +\infty, \tag{17}$$

for any B bounded in H.

Proof. Since the global attractor is invariant, $\mathcal{A} = S(r, \mathcal{A})$ for $r > 0$, so $\mathcal{A}$ is bounded in $\left(H^2(\Omega)\right)^3$ by Lemma 3.6. The compact embeddings $H^2(\Omega) \subset H^1(\Omega)$, $H^2(\Omega) \subset L^{\beta+1}(\Omega)$ imply the compactness of the attractor in V and $\left(L^{\beta+1}(\Omega)\right)^3$.

Let B_0 be the absorbing ball given in Lemma 3.2. By Lemma 3.6 the set $B_1 = S(r, B_0)$ is bounded in $\left(H^2(\Omega)\right)^3$ and

$$S(t,B) = S(r, S(t-r, B)) \subset B_1 \text{ for } t \geq t_0(B).$$

From here it is easy to deduce (17). □

3.2. **Case** $3 \leq \beta < 4$**.** Let us define the set

$$D_T(u_0) = \{u(\cdot) \text{ is a weak solution of (2) in the interval } (0,T)\}.$$

We know by Theorem 2.1 that for any $u_0 \in H$ and $T > 0$ the set $D_T(u_0)$ is non-empty.

We observe that as $3 \leq \beta < 4$, we have that

$$V \subset \left(L^{\beta+1}(\Omega)\right)^3 \subset \left(L^{\frac{\beta+1}{\beta}}(\Omega)\right)^3 \subset V',$$

so $V \cap \left(L^{\beta+1}(\Omega)\right)^3 = V$. Hence, equality (3) has to be satisfied just for $v \in V$. On top of that, as $\frac{\beta+1}{\beta} \leq \frac{4}{3}$, we have that

$$L^{\frac{\beta+1}{\beta}}\left(0,T; \left(L^{\frac{\beta+1}{\beta}}(\Omega)\right)^3\right) \subset L^{\frac{4}{3}}(0,T;V'),$$

so the derivative $\dfrac{du}{dt}$ belongs to $L^{\frac{4}{3}}(0,T;V')$.

Lemma 3.8. *Let $3 \leq \beta < 4$. If $u(\cdot) \in D_T(u_0)$, then for any $s \in (0,T)$, the function $w(\cdot) = u(\cdot + s)$ belongs to $D_{T-s}(u(s))$.*

If $u(\cdot) \in D_s(u_0)$ and $w(\cdot) \in D_{T-s}(u(s))$, then the function

$$z(t) = \begin{cases} u(t) & \text{if } t \in [0,s], \\ w(t-s) & \text{if } t \in [s,T], \end{cases}$$

belongs to $D_T(u_0)$.

Proof. Let $u(\cdot) \in D_T(u_0)$. Then it is obvious that

$$w(\cdot) = u(\cdot + s) \in L^\infty(0, T-s; H) \cap L^2(0, T-s; V) \cap L^{\beta+1}\left(0, T-s; L^{\beta+1}(\Omega)\right). \tag{18}$$

Also, (3) implies that for any $v\in V$, $\phi\in C_0^\infty(0,T-s)$ one has

$$\begin{aligned}
&-\int_0^{T-s}\left(w(\tau),v\right)\phi'(\tau)d\tau\\
&+\int_0^{T-s}\mu((w(\tau),v))+b\left(w(\tau),w(\tau),v\right)+\alpha\left(\left|w(\tau)\right|^{\beta-1}w(\tau),v\right)\phi(\tau)d\tau\\
&=-\int_s^T\left(u(r),v\right)\phi'(r-s)dr\\
&+\int_s^T\mu((u(r),v))+b\left(u(r),u(r),v\right)+\alpha\left(\left|u(r)\right|^{\beta-1}u(r),v\right)\phi(r-s)dr\\
&=\int_s^T\left(f,v\right)\phi(r-s)dr=\int_0^{T-s}\left(f,v\right)\phi(\tau)d\tau,
\end{aligned}$$

so w satisfies (3) in the interval $(0,T-s)$. We infer that $w\in D_{T-s}\left(u\left(s\right)\right)$.

Let now $u\left(\cdot\right)\in D_s\left(u_0\right)$ and $w\left(\cdot\right)\in D_{T-s}\left(u\left(s\right)\right)$. Arguing as in the previous case we obtain that $w(t-s)$ satisfies equality (3) in the interval (s,T). As the time derivative of a weak solution belongs to $L^{\frac{4}{3}}\left(0,T;V'\right)$, equality (3) is equivalent to saying that

$$\int_0^T\left(\left\langle\frac{du}{dt},\xi\right\rangle+\mu\left((u,\xi)\right)+\left\langle B(u,u),\xi\right\rangle\right)dt+\alpha\int_0^T\int_\Omega\left|u\right|^{\beta-1}u\xi dxdt$$
$$=\int_0^T\left(f,\xi\right)dt,$$

for any $\xi\in L^4\left(0,T;V\right)$. The function z satisfies (18) in the interval $(0,T)$ and this equality as well. Indeed, denoting $h(t)=w(t-s)$ we have

$$\begin{aligned}
&\int_0^T\left(\left\langle\frac{dz}{dt},\xi\right\rangle+\mu\left((z,\xi)\right)+\left\langle B(z,z),\xi\right\rangle\right)dt+\alpha\int_0^T\int_\Omega\left|z\right|^{\beta-1}z\xi dxdt\\
&=\int_0^s\left(\left\langle\frac{du}{dt},\xi\right\rangle+\mu\left((u,\xi)\right)+\left\langle B(u,u),\xi\right\rangle\right)dt+\alpha\int_0^s\int_\Omega\left|u\right|^{\beta-1}u\xi dxdt\\
&+\int_s^T\left(\left\langle\frac{dh}{dt},\xi\right\rangle+\mu\left((h,\xi)\right)+\left\langle B(h,h),\xi\right\rangle\right)dt+\alpha\int_s^T\int_\Omega\left|h\right|^{\beta-1}h\xi dxdt\\
&=\int_0^s\left(f,\xi\right)dt+\int_s^T\left(f,\xi\right)dt=\int_0^T\left(f,\xi\right)dt,
\end{aligned}$$

proving that z is really a weak solution. □

In view of this lemma every solution can be extended to a globally defined one, that is, a solution which exists for $t\in[0,+\infty)$. In this situation we denote by $D\left(u_0\right)$ the set of all globally defined solutions with initial condition u_0 and observe that for any $t\geq0$ the following equality holds:

$$\{u(t):u\in D\left(u_0\right)\}=\{u(t):u\in\cup_{T>0}D_T\left(u_0\right)\}.$$

Denote by $P\left(H\right)$ the set of all non-empty subsets of H. Let us define the following (possibly multivalued) family of operators $G:\mathbb{R}^+\times H\rightarrow P\left(H\right)$:

$$G\left(t,u_0\right)=\{y\in H:y=u\left(t\right),\,u\left(\cdot\right)\in D\left(u_0\right)\}.$$

Using Lemma 3.8 we can easily prove that G is a strict multivalued semiflow, that is, the following two properties hold:

- $G(0,u_0) = u_0$ for all $u_0 \in H$;
- $G(t+s,u_0) = G(t, G(s,u_0))$, for all $u_0 \in H$, $t,s \geq 0$.

The set $\mathcal{A}$ is a global attractor for G if:

- $\mathcal{A}$ is negatively invariant, i.e., $\mathcal{A} \subset G(t,\mathcal{A})$ for all $t \geq 0$;
- $\mathcal{A}$ attracts every bounded set of H, that is,

$$dist\left(G(t,B), \mathcal{A}\right) \to 0 \text{ as } t \to +\infty.$$

It is invariant if, moreover, $\mathcal{A} = G(t,\mathcal{A})$ for all $t \geq 0$.

The next lemma is crucial for proving the existence of a global attractor.

Lemma 3.9. *Assume that $3 \leq \beta < 4$. Let $u_0^n \to u_0$ weakly in H and let $u_n(\cdot) \in D(u_0^n)$. Then there exists a weak solution $u(\cdot)$ to (2) with $u(0) = u_0$ and a subsequence $u_{n_k}(\cdot)$ such that $u_{n_k} \to u$ in $C([\varepsilon,T],H)$ for all $0 < \varepsilon < T$.*

If, moreover, $u_0^n \to u_0$ strongly in H, then $u_{n_k} \to u$ in $C([0,T],H)$ for all $T > 0$.

Proof. We fix $T > 0$. We deduce from Lemma 2.3 that the sequence u_n is bounded in

$$L^\infty(0,T;H) \cap L^2(0,T;V) \cap L^{\beta+1}\left(0,T;\left(L^{\beta+1}(\Omega)\right)^3\right).$$

Also, using (4) and standard estimates (see [18, p.297]) we obtain that $\dfrac{du_n}{dt}$ is bounded in $L^{\frac{4}{3}}(0,T;V')$.

Thus, making use of the compactness theorem [11] we obtain a function $u(\cdot)$ and a subsequence (denoted again by u_n) such that

$$\begin{aligned} u_n &\to u \text{ weakly star in } L^\infty(0,T;H), \\ u_n &\to u \text{ weakly in } L^2(0,T;V), \\ u_n &\to u \text{ weakly in } L^{\beta+1}\left(0,T;\left(L^{\beta+1}(\Omega)\right)^3\right), \\ \frac{du_n}{dt} &\to \frac{du}{dt} \text{ weakly in } L^{\frac{4}{3}}(0,T;V'), \\ u_n &\to u \text{ strongly in } L^2(0,T;H), \\ u_n(t,x) &\to u(t,x) \text{ for a.a. } (t,x). \end{aligned} \tag{19}$$

Let us prove that

$$u_n(t_n) \to u(t_0) \text{ weakly in } H \tag{20}$$

for any sequence $\{t_n\}$ such that $t_n \to t_0$, where $t_n, t_0 \in [0,T]$. Denote $q = \frac{\beta+1}{\beta}$. The time derivatives are bounded in the space $L^{\frac{4}{3}}(0,T;V')$, which implies readily that the sequence $u_n(\cdot)$ is equicontinuous in the space V'. Moreover, $u_n(t_n)$ is bounded in H, and then the compact embedding $H \subset V'$ yields that it is relatively compact in V'. Hence, by Ascoli-Arzelà's theorem we have $u_n \to u$ in $C([0,T],V')$. Thus, by a contradiction argument we obtain that $u_n(t_n) \to u(t_0)$ weakly in H. In particular, we have that $u(0) = u_0$.

Further, we need to check that $u(\cdot)$ is a weak solution to problem (2).

The sequence $h(u_n(\cdot)) = |u_n(\cdot)|^{\beta-1} u_n(\cdot)$ is bounded in $L^q\left(0,T;(L^q(\Omega))^3\right)$ and $h(u_n(t,x)) \to h(u(t,x))$ for a.a. (t,x). Hence, $h(u_n(\cdot)) \to h(u(\cdot))$ weakly in $L^q\left(0,T;(L^q(\Omega))^3\right)$ [14, Lemma 8.3].

In order to show that u is a weak solution it remains to pass to the limit in the term B. Since $u_n \to u$ in $L^2(0,T;H)$ implies that $u_{ni}u_{nj} \to u_i u_j$ in $L^1(0,T;L^1(\Omega))$, for any $\zeta \in \mathcal{V}$, $\phi \in C_0^\infty(0,T)$ we have

$$\begin{aligned}\int_0^T (b(u_n,u_n,\zeta)-b(u,u,\zeta))\,\phi dt &= -\int_0^T (b(u_n,\zeta,u_n)-b(u,\zeta,u))\,\phi dt\\ &= -\sum_{i,j=1}^3 \int_0^T \int_\Omega (u_{ni}u_{nj}-u_iu_j)\frac{\partial \zeta_j}{\partial x_i}\phi dx dt \to 0,\end{aligned}$$

as $n \to \infty$.

We conclude that equality (3) is satisfied for the function u for all $\zeta \in \mathcal{V}$, and by density of $\mathcal{V}$ in V we obtain that (3) holds true. Thus, u is a weak solution.

Finally, we will prove that $u_n \to u$ in $C([\varepsilon,T],H)$ for all $0<\varepsilon<T$. From (11) we get

$$\|u_n(t)\|_H^2 \le \|u_n(s)\|_H^2 + \frac{1}{\mu\lambda_1}\|f\|_H^2(t-s)\,,\text{ for any } s\le t,$$

and the same inequality is true for u. Hence, the functions $J_n(t) = \|u_n(t)\|_H^2 - \frac{1}{\mu\lambda_1}\|f\|_H^2 t$, $J(t) = \|u(t)\|_H^2 - \frac{1}{\mu\lambda_1}\|f\|_H^2 t$ are non-increasing and continuous. Take a sequence $t_n \to t_0$ with $t_n, t_0 \in [\varepsilon, T]$. We know that $u_n(t_n) \to u(t_0)$ weakly in H, so

$$\|u(t_0)\|_H \le \liminf \ \|u(t_n)\|_H\,. \tag{21}$$

It is a consequence of (19) that $J_n(t) \to J(t)$ for a.a. t. Then we can choose $t_k < t_0$ as close to t_0 as we wish such that $J_n(t_k) \to J(t_k)$, and we can assume without loss of generality that $t_k < t_n$. Therefore,

$$\begin{aligned}J_n(t_n) - J(t_0) &= J_n(t_n) - J_n(t_k) + J_n(t_k) - J(t_k) + J(t_k) - J(t_0)\\ &\le |J_n(t_k) - J(t_k)| + |J(t_k) - J(t_0)|\,.\end{aligned}$$

Since $u(\cdot)$ is continuous, for any $\delta > 0$ there exists t_k and $N(t_k)$ such that $|J(t_k) - J(t_0)| \le \delta/2$ and $|J_n(t_k) - J(t_k)| \le \delta/2$ for all $n \ge N$. This implies that

$$\limsup \ \|u(t_n)\|_H \le \|u(t_0)\|_H\,. \tag{22}$$

Joining (21) and (22) we deduce that $\|u(t_n)\|_H \to \|u(t_0)\|_H$ and then $u(t_n) \to u(t_0)$ in H.

Since $T > 0$ is arbitrary, by a diagonal arguments we obtain a common subsequence on an arbitrary interval $[\varepsilon, T]$.

The first part of the lemma is proved.

For the second part, we need to prove only that $u(t_n) \to u(0)$ if $t_n \to 0$, $t_n \ge 0$. For this aim we repeat the above argument with $t_k = 0 = t_0$. Hence,

$$J_n(t_n) - J(t_0) = J_n(t_n) - J_n(0) + J_n(0) - J(0) \le |J_n(0) - J(0)| \to 0 \text{ as } n \to \infty,$$

because $u_n(0) \to u(0)$ in H. Then we obtain the result arguing in the same way as above. □

Corollary 4. *Assume that* $3 \le \beta < 4$. *For any* $t \ge 0$ *the map* $u_0 \mapsto G(t,u_0)$ *has compact values and closed graph. In addition, for any* $t_0 > 0$ *the map* $G(t_0,\cdot)$ *is compact.*

The map $u_0 \mapsto G(t,u_0)$ is said to be upper semicontinuous if for all $u_0 \in H$ and any neighborhood O of u_0 in H there exists $\delta > 0$ such that $G(t,u) \subset O$ for all u satisfying $\|u - u_0\| < \delta$.

Lemma 3.10. *Assume that* $3 \le \beta < 4$. *For any* $t \ge 0$ *the map* $u_0 \mapsto G(t,u_0)$ *is upper semicontinuous.*

Proof. If not, there exist $u_0 \in H$, $t > 0$, sequences $u_n^0 \to u_0$, $y_n \in G(t, u_0^n)$ and a neighborhood O of $G(t, u_0)$ such that $y_n \notin O$. Let $y_n = u_n(t)$, where $u_n(\cdot) \in D(u_0^n)$. Then by Lemma 3.9 there is a subsequence y_{n_k} satisfying $y_{n_k} \to y \in G(t, u_0)$, which is a contradiction. $\square$

For a multivalued semiflow G the concepts of absorbing set and asymptotically compactness are given in exactly the same way as for semigroups.

The following conditions are sufficient in order to obtain a global compact invariant minimal attractor $\mathcal{A}$ for a strict multivalued semiflow G [13, Theorem 3 and Remark 8]:

1. G possesses a bounded absorbing set B_0;
2. G is asymptotically compact;
3. G has closed values;
4. the map $u_0 \mapsto G(t, u_0)$ is upper semicontinuous.

Theorem 3.11. *Assume that $3 \leq \beta < 4$. Then G has a global invariant compact attractor $\mathcal{A}$, which is minimal among all closed attracting sets.*

Proof. We need to check the four aforementioned conditions.

It follows from (8) that the ball $B_0 = \{u \in H : \|u\|_H^2 \leq 1 + \frac{\|f\|_H^2}{\mu^2 \lambda_1^2}\}$ is absorbing.

In view of Corollary 4 and Lemma 3.10 G has compact (and then closed) values and the map $u_0 \mapsto G(t, u_0)$ is upper semicontinuous.

Finally, again by Corollary 4 the operator $G(1, \cdot)$ is compact. Hence, for any bounded set B an arbitrary sequence $y_n \in G(t_n, B)$, which belongs to

$$G(1, G(t_n - 1, B)) \subset G(1, B_0), \text{ for all } n \geq N,$$

is relatively compact in H, so G is asymptotically compact. $\square$

We can give also some information about the structure of the global attractor in terms of bounded complete trajectories, which are continuous functions $\gamma : \mathbb{R} \to H$ such that $u(\cdot) = \gamma(\cdot + s)$ belongs to $D(u_0)$ for all $s \in \mathbb{R}$ and satisfying that $\cup_{t \in \mathbb{R}} \gamma(t)$ is a bounded set. Indeed, by Lemmas 3.8, 3.9 we can apply Theorems 9, 10 from [6] and obtain that

$$\mathcal{A} = \{\gamma(0) : \gamma \in \mathbb{K}\}, \tag{23}$$

where is $\mathbb{K}$ the set of all bounded complete trajectories.

Finally, in a similar way as in [7] let us prove that the global attractor is stable, which means that for any $\varepsilon > 0$ there is $\delta > 0$ such that

$$G(t, O_\delta(\mathcal{A})) \subset O_\varepsilon(\mathcal{A}) \text{ for all } t \geq 0, \tag{24}$$

where $O_\eta(\mathcal{A}) = \{z \in H : dist(z, \mathcal{A}) < \eta\}$.

Lemma 3.12. *Assume that $3 \leq \beta < 4$. The global attractor $\mathcal{A}$ given in Theorem 3.11 is stable.*

Proof. By contradiction if (24) does not hold, then there exist $\varepsilon > 0$ and sequences $\delta_k \to 0$, $x_k \in O_{\delta_k}(\mathcal{A})$, $t_k \geq 0$, $y_k \in G(t_k, x_k)$ such that

$$dist(y_k, \mathcal{A}) \geq \varepsilon. \tag{25}$$

We consider two cases: 1) $t_k \to +\infty$ for some subsequence; 2) $t_k \leq C$.

In the first situation, as the sequence $\{x_k\}$ belongs to a bounded set, by the definition of global attractor we get that $dist(y_k, \mathcal{A}) \to 0$, which contradicts (25).

In the second one, up to a subsequence $t_k \to t_0,\ x_k \to x_0 \in \mathcal{A}$, so by Lemma 3.9 and the invariance of $\mathcal{A}$ we obtain that

$$y_k \to y \in G(t_0, x_0) \subset G(t_0, \mathcal{A}) \subset \mathcal{A},$$

which is again a contradiction. □

Remark 1. Formula (23) and stability are also true for the global attractor $\mathcal{A}$ given in Theorem 3.4 for $\beta \geq 4$.

4. Global attractor for strong solutions.

In this section we are going to define a semigroup generated by strong solutions of (2).

We recall that a function $u : [0, T] \to V \cap \left(L^{\beta+1}(\Omega)\right)^3$ is called a strong solution of (2) if u is a weak solution and

$$u \in L^\infty(0, T; V) \cap L^2(0, T; \left(H^2(\Omega)\right)^3) \cap L^\infty(0, T; (L^{\beta+1}(\Omega))^3).$$

Theorem 4.1. *Let $u_0 \in V \cap \left(L^{\beta+1}(\Omega)\right)^3$, $f \in H$ and $\beta > 3$. Then there exists a unique strong solution of (2). Moreover, it satisfies*

$$\frac{du}{dt} \in L^2(0, T; \left(L^2(\Omega)\right)^3).$$

Existence of strong solutions for $\beta \geq 7/2$ and uniqueness for $7/2 \leq \beta \leq 5$ were proved at first in [2] and [15]. In [23] and [17] existence and uniqueness were extended for $3 < \beta < 7/2$. Finally, in [25] uniqueness of strong solutions was established for all $\beta \geq 1$. We observe also that for $\beta \geq 4$ uniqueness of strong solutions follows from Theorem 2.4.

From a standard result (see e.g. [14, Corollary 7.3]) and $u \in L^2(0, T; \left(H^2(\Omega)\right)^3)$, $\dfrac{du}{dt} \in L^2(0, T; L^2(\Omega))$ we obtain that $u \in C([0, T], V)$. If $3 < \beta \leq 5$, then $V \cap \left(L^{\beta+1}(\Omega)\right)^3 = V$, so the operator $S_V : \mathbb{R}^+ \times V \to V$ given by $S_V(t, u_0) = u\,(t)$, where $u\,(\cdot)$ is the unique strong solution to problem (2) with initial condition u_0, defines a semigroup having a global compact attractor in V. This result was proved in [15] for $7/2 \leq \beta \leq 5$ and in [17] for $3 < \beta \leq 5$. Although in the last paper the nonautonomous case is studied, for a time-independent external force f the autonomous attractor is obtained as a particular case.

In this section we will give an alternative proof of the existence of the strong attractor for $\beta = 5$. The reason is that in the above cited papers there is an argument which is correct for $\beta < 5$ but it is unclear when $\beta = 5$. Namely, if we take for example the paper [15], then in Proposition 5 inequality (28) in page 247 is correct for $\beta < 5$, but not for $\beta = 5$.

Also, we prove a conditional theorem on the existence of the global attractor for $\beta > 5$ in the phase space $Z = V \cap \left(L^{\beta+1}(\Omega)\right)^3$. We observe that properties $u \in L^\infty(0, T; (L^{\beta+1}(\Omega))^3)$, $u \in C([0, T], H)$ imply that $t \mapsto u(t)$ is continuous with respect to the weak topology of $(L^{\beta+1}(\Omega))^3$ [3, p.33, Theorem 1.7]. However, we need continuity with respect to the strong topology, that is, $u \in C([0, T], (L^{\beta+1}(\Omega))^3)$. Assuming that any strong solution satisfies this property, we are able to obtain the result.

When $\beta \geq 5$ we define the operator $S_X : \mathbb{R}^+ \times X \to X$, where $X = V \cap \left(L^{\beta+1}(\Omega)\right)^3$, by

$$S(t, u_0) = u(t),$$

where $u(\cdot)$ is the unique strong solution to problem (2) with initial condition u_0. By the same argument given in Lemma 3.8 one can check that S is a semigroup.

Lemma 4.2. *Let $\beta \geq 5$. Assume additionally when $\beta > 5$ that every strong solution $u(\cdot)$ with initial condition in X satisfies that $u \in C([0,T],(L^{\beta+1}(\Omega))^3)$ for any $T > 0$. If $u_0^n \to u_0$ weakly in X, then $S_X(t,u_0^n) \to S_X(t,u_0)$ strongly in X for any $t > 0$. This implies, in particular, that the operator $S_X(t,\cdot)$ is compact for $t > 0$ and continuous with respect to the initial condition for all $t \geq 0$.*

Proof. Let $T > t$. In view of (12) $u^n(\cdot) = S_X(\cdot,u_0^n)$ converges to $u(\cdot) = S_X(\cdot,u_0)$ in $C([0,T],H)$. Thus, $u^n(t) \to u(t)$ in H. Further, Lemmas 3.1, 3.5 and inequality (16) imply that

$$\begin{aligned}
u^n(t) &\to u(t) \text{ weakly in } V \cap (L^{\beta+1}(\Omega))^3,\\
u^n &\to u \text{ weakly star in } L^\infty(\varepsilon,T;V\cap(L^{\beta+1}(\Omega))^3),\\
u^n &\to u \text{ weakly } L^2(\varepsilon,T;(H^2(\Omega))^3),\\
\frac{du^n}{dt} &\to \frac{du}{dt} \text{ weakly in } L^2(\varepsilon,T;(L^2(\Omega))^3),
\end{aligned}$$

for any $0 < \varepsilon < T$. Hence, the compactness theorem [11] implies

$$\begin{aligned}
u^n &\to u \text{ strongly in } L^2(\varepsilon,T;V\cap(L^{\beta+1}(\Omega))^3),\\
u^n(t) &\to u(t) \text{ in } V\cap(L^{\beta+1}(\Omega))^3 \text{ for a.a. } t\in(0,T).
\end{aligned}$$

Now, using (15) and Lemma 3.1 we have

$$y(r) \leq y(s) + K(\varepsilon)(r-s),$$

for all $0 < \varepsilon \leq s < r \leq T$. Thus, the functions $J^n(r) = \frac{\mu+1}{2}\|u^n(r)\|_V^2 + \frac{\alpha}{\beta+1}\|u^n(r)\|_{\beta+1}^{\beta+1} + K(\varepsilon)r$, $J(r) = \frac{\mu+1}{2}\|u(r)\|_V^2 + \frac{\alpha}{\beta+1}\|u(r)\|_{\beta+1}^{\beta+1} + K(\varepsilon)r$ are non-increasing in $[\varepsilon,T]$. Since $u \in C([0,T];V\cap(L^{\beta+1}(\Omega))^3)$, these functions are continuous in $[\varepsilon,T]$. Also, $J^n(r) \to J(r)$ for a.a. $r \in (\varepsilon,T)$. Arguing as in the proof of (22) we obtain that

$$\begin{aligned}
&\limsup\left(\frac{\mu+1}{2}\|u^n(t)\|_V^2 + \frac{\alpha}{\beta+1}\|u^n(t)\|_{\beta+1}^{\beta+1}\right)\\
&\leq \frac{\mu+1}{2}\|u(t)\|_V^2 + \frac{\alpha}{\beta+1}\|u(t)\|_{\beta+1}^{\beta+1}.
\end{aligned}$$

But $u^n(t) \to u(t)$ weakly in $V\cap(L^{\beta+1}(\Omega))^3$ implies that

$$\begin{aligned}
&\frac{\mu+1}{2}\|u(t)\|_V^2 + \frac{\alpha}{\beta+1}\|u(t)\|_{\beta+1}^{\beta+1}\\
&\leq \liminf\left(\frac{\mu+1}{2}\|u^n(t)\|_V^2 + \frac{\alpha}{\beta+1}\|u^n(t)\|_{\beta+1}^{\beta+1}\right).
\end{aligned}$$

Making use of these two inequalities we infer that

$$\frac{\mu+1}{2}\|u^n(t)\|_V^2 + \frac{\alpha}{\beta+1}\|u^n(t)\|_{\beta+1}^{\beta+1} \to \frac{\mu+1}{2}\|u(t)\|_V^2 + \frac{\alpha}{\beta+1}\|u(t)\|_{\beta+1}^{\beta+1},$$

so

$$\begin{aligned}
\|u^n(t)\|_V &\to \|u(t)\|_V,\\
\|u^n(t)\|_{\beta+1} &\to \|u(t)\|_{\beta+1}.
\end{aligned}$$

Thus, taking into account that the space X is uniformly convex, we get that

$$S_X(t,u_0^n) \to S_X(t,u_0) \text{ strongly in } X.$$

$\square$

Theorem 4.3. *Assume the conditions of Lemma 4.2. Then the semigroup S_X possesses the global compact connected attractor $\mathcal{A}$.*

Proof. Let B_0 be the absorbing ball given in Lemma 3.2. By Lemma 3.1 the set $B_1 = S(r,B_0)$ is bounded in X and

$$S_X(t,B) = S_X(r,S_X(t-r,B)) \subset B_1 \text{ for } t \geq t_0(B),$$

where B is a bounded set in X. Hence, B_1 is a bounded absorbing set in X.

Let B be bounded in X and $t_n \to \infty$, $y_n \in S_X(t_n,B)$. By Lemma 4.2 the operator $S_X(1,\cdot)$ is compact, which implies using

$$y_n \in S_X(1,S_X(t_n-1,B)) \subset S_X(1,B_1),\ \forall n \geq N,$$

that y_n is relatively compact in X.

The existence of the global compact attractor follows from [10] and its connectivity from [5, p.4]. $\square$

5. Numerical simulations.

We shall now focus on solving numerically the equation (2) employing computational fluid dynamics (CFD) to visualize scenarios in which the evolution of the fluid flow converges to a steady state. It is important to stress that the examples reported below are only intended for showing the asymptotic behaviour of the fluid flow numerically when taking different values of the parameters α and β in the momentum equation of (2), but no conclusive results should be deduced from the numerical simulations.

The geometry of the flow domain used in all our numerical experiments is a sphere Ω of radius 6 centered at the origin. We also take the source term f in (2) as

$$f(x) = \begin{cases} (0,2,0) & \text{if } x \in C \\ (0,0,0) & \text{if } x \in \Omega \setminus C \end{cases}$$

where C is a cylinder, with both radius and height of 4, within the flow domain symmetrically located at the center of the sphere Ω in such a way that the base of the cylinder is parallel to the xz-plane as in Figure 1. Observe that $f(x,t)$ can be seen as a constant source force within the cylinder C propelling the fluid flow upwards.

Numerical simulations were all performed using the CFD package OpenFOAM®, which is the acronym of *Open Source Field Operation and Manipulation*. It is an open-source CFD software based on C++ that contains a toolbox for tailored numerical solvers for a wide variety of problems relevant to the industry and scientific community. The solvers implemented in OpenFOAM® uses the Finite Volume Method (FVM) to discretize the governing equations on unstructured meshes (see [4, 21]). The solver used to integrate our model numerically was *pimpleFoam*, which combines the two most common algorithms for solving the Navier-Stokes equations, namely, SIMPLE and PISO algorithms. The pimpleFoam code is inherently transient, requiring an initial condition and boundary conditions. OpenFOAM® includes pre-processing and post-processing capabilities such as snappyHexMesh and ParaFoam for meshing and visualization, respectively.

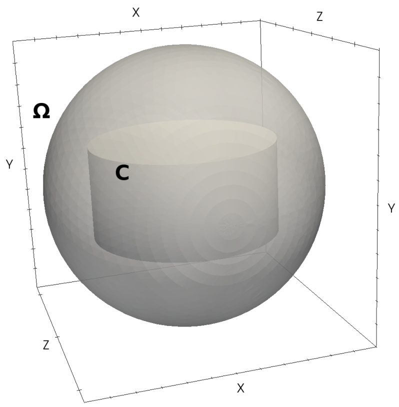

Figure 1. Flow domain

Figures 2, 3 and 4 show the steady state for the numerical solution of the equations (2) for experiments with different values of the parameters α and β. In those experiments we have set the initial condition $u_0(x) = (0,0,0)$, and the images represent the velocity vector field in the xy-section at $z = 0$. The darker areas in the images are those where the magnitude for the velocity vector u is smaller, while the lighter ones represent the areas where the velocity is higher. According to the results of the experiments, when the magnitude of the velocity vector is greater than 1 and the parameter β becomes bigger, the medium provides increased resistance to movement, so the fluid flow slows down more quickly. On the contrary, when the magnitude of the velocity vector is less than 1 and the parameter β becomes smaller, then medium provides decreased resistance to movement, so the fluid flow spreads further through the medium. The effect of the parameter α does not depend of the u magnitude, acting proportionally, i.e., the larger the value of α, the higher resistance to motion of the fluid flow. It has also been observed that convergence speed to the steady state is higher as α and β increase. Therefore, when α and β are small, a higher period of time to get convergence to the steady state is required. In fact, we have also performed simulations (not shown here) for values of α close to 0 (also for $\alpha = 0$) and a low value of β (for instance, $\beta = 1$), but we did not achieve convergence to a steady state for an approachable (from a computational point of view) time value. It is likely that for such values of the parameters the global attractor (if it exists!) is more complex than a fixed point.

Lastly, an experiment with a non-vanishing initial condition was carried out. The performance was made by taking $u_0(x) = (1,0,0)$, $\alpha = 0.2$ and $\beta = 1$, and the results of the model is shown in Figure 5. As one might expect, the steady state does not depend of the initial condition, hence, it is the same as taking u_0 equal to zero (compare the right panel of Figure 5 to the left panel of Figure 2).

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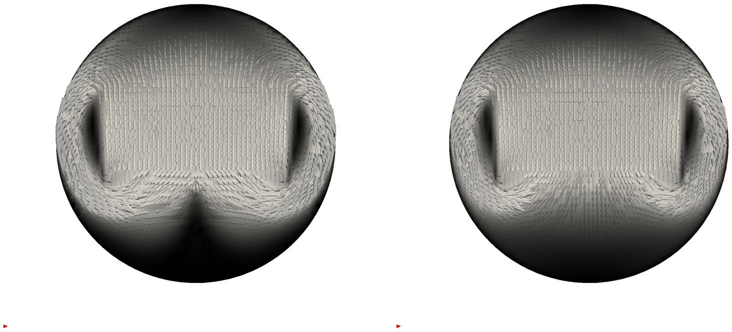

FIGURE 2. Flow velocity u for the steady state in the xy-section at $z = 0$. The darker areas mean lower fluid flow speed. The initial velocity u_0 is identically zero. Left panel parameters: $\alpha = 0.2$; $\beta = 1$. Right panel parameters: $\alpha = 0.5$; $\beta = 1$.

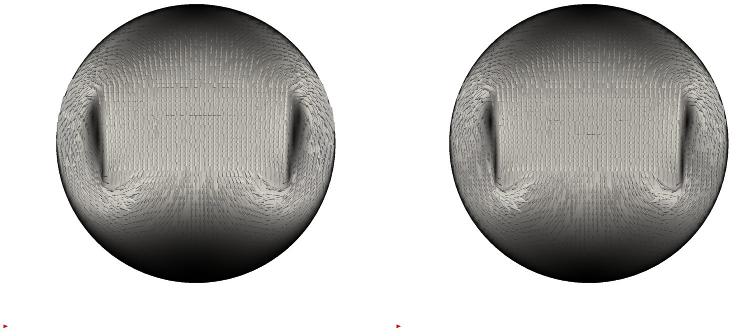

FIGURE 3. Flow velocity u for the steady state in the xy-section at $z = 0$. The darker areas mean lower fluid flow speed. The initial velocity u_0 is identically zero. Left panel parameters: $\alpha = 0.2$; $\beta = 2$. Right panel parameters: $\alpha = 0.5$; $\beta = 2$.

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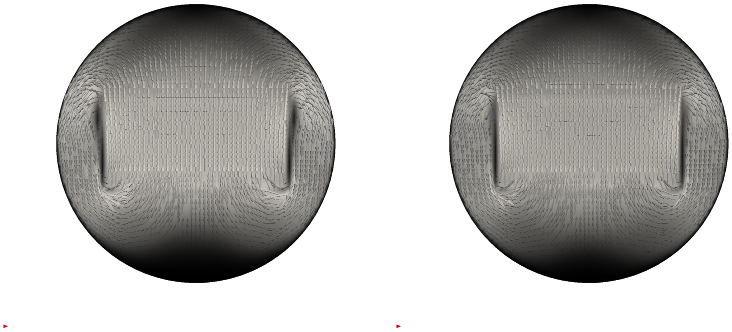

FIGURE 4. Flow velocity u for the steady state in the xy-section at $z = 0$. The darker areas mean lower fluid flow speed. The initial velocity u_0 is identically zero. Left panel parameters: $\alpha = 0.2$; $\beta = 4$. Right panel parameters: $\alpha = 0.5$; $\beta = 4$.

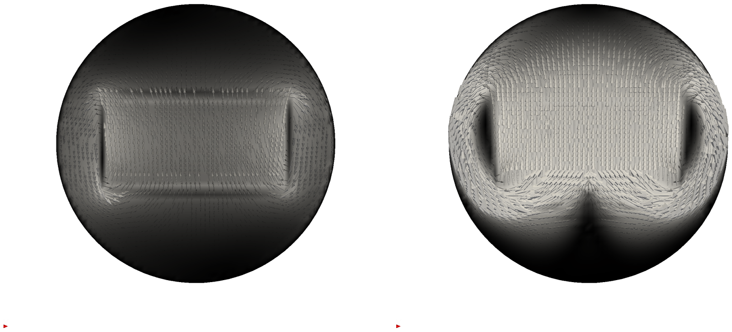

FIGURE 5. Flow velocity u in the xy-section at $z = 0$ for $\alpha = 0.2$ and $\beta = 1$. The darker areas mean lower fluid flow speed. The initial velocity $u_0 = (1, 0, 0)$. Left panel: state when $t = 0.1$. Right panel: steady state (t large enough).

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