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Article

Extended Fuzzy Analytic Hierarchy Process (E-FAHP): A general approach

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Abstract: Fuzzy Analytic Hierarchy Process (FAHP) methodologies have witnessed a growing development from the late 1980s until now, and countless FAHP based applications have been published in many fields including economics, finance, environment or engineering. In this context, the FAHP methodologies have been generally restricted to fuzzy numbers with linear type of membership functions (triangular numbers-TN- and trapezoidal numbers –TrN-). This paper proposes an extended FAHP model (E-FAHP) where pairwise fuzzy comparison matrices are represented by a special type of fuzzy numbers referred to as (m.n)-trapezoidal numbers (TrN (m,n)) with nonlinear membership functions. It is then demonstrated that there are a significant number of FAHP approaches that can be reduced to the proposed E-FAHP structure. A comparative analysis of E-FAHP and Mikhailov's model is illustrated with a case study showing that E-FAHP includes linear and non-linear fuzzy numbers.

Keywords: AHP, Fuzzy AHP, Fuzzy numbers, (m.n)-trapezoidal numbers, MCDM

1. Introduction

One of the most frequently used MCDM tools which has been employed to solve intricate decision making problems over the past years has been the Analytic Hierarchy Process (AHP), proposed by Saaty [1]. The judgments made by the decision makers rely on pairwise comparisons given by the relative weights of the criteria that appear in the intermediate steps of AHP. These judgments are based on information and knowledge on the problem provided by decision makers (DMs). Therefore, the comparisons involve subjectivity in interpreting and assessing the problem, which means that the DMs standpoints have a profound effect on the final results [2].

There is widespread literature addressing the situation in which uncertainty stemming from imprecision and subjectivity in the evaluation process makes conventional AHP an inadequate tool. This is especially true in cases in which vagueness inherent in linguistic assessment[3]. This limitation, however, vanishes when fuzzy logic is included into the AHP methodology, which leads to Fuzzy Analytic Hierarchy Process (FAHP). In fact, a considerable number of research papers deal with the efficiency and applicability of FAHP, whether on its own or combined with different MCDM techniques. Such studies are closely related to fuzzy numbers having linear membership functions, that is to say, triangular number (TN) and in some cases, trapezoidal number (TrN). By using fuzzy numbers with linear membership functions, complex nonlinear computations are avoided[4,5]. It should be noted that one of the main drawbacks when using linear membership functions lies with the problems related to finding a solution to a problem. These authors emphasize the importance of using a membership function which can be easily adjusted.

The purpose of this paper is to go one step further by proposing an extended framework which can provide insight into the presentation of FAHP approaches. In other words, the paper aims to provide a unifying basis for FAHP, starting from Mikhailov's Fuzzy Preference Programming (FPP) method [6]. The basis of the FPP method is the fuzzy geometrical representations of the prioritization problem, which can be resolved as a standard linear program with no difficulties. Some interesting properties of the method are worth mentioning, such as natural consistency index as well as good rank preservation and precision. Besides, it is regarded as a suitable alternative to other well-known prioritization methods, primarily when the decision maker's preferences are highly inconsistent. The method we propose is called Extended FAHP (E-FAHP), which uses a special fuzzy number written as (m,n)-trapezoidal number ($TrN_{(m,n)}$) [7] , this number having a nonlinear membership function.

The proposed E-FAHP model can be extended to different types of nonlinear fuzzy numbers, which renders the model a practical tool to allow decision makers to express their judgments.

The paper is organized as follows. Section 2. reviews the recent uses of FAHP methodologies and presents the foundation of Mikhailov's Fuzzy Preference Programming (FPP) method and describes the E-FAHP methodology using (m.n)-trapezoidal numbers. In Section 3., we develop an illustrative example based on Mikahilov and Tsvetinov [8] case study. Finally, the conclusion of the paper appears in Section 4.

2. Materials and Methods

2.1. Background and literature review

Analytic Hierarchy Process (AHP) is a commonly used MCDM technique originally proposed by Saaty [1]. However, it has been subject to criticism since it employs an unbalanced scale of judgments and it is unable to handle imprecision and uncertainty in the pairwise comparison process [9]. In order to address these shortcomings, FAHP was developed to solve the hierarchical problems arising from the fact that decision makers usually find that giving interval judgments is more accurate than giving fixed value judgments. As a result, FAHP uses both, fuzzy set theory and fuzzy numbers in order to express the uncertain comparison of opinions and it enables the incorporation of the incomplete, unquantifiable and non-obtainable information into the decision making process.

Several authors have proposed Fuzzy Analytic Hierarchy Process (FAHP) [6,10–14], since it represents a systematic approach to the selection of alternatives and the resolution of problems by applying fuzzy set theory, which helps to express the uncertain comparison of opinions through the use of fuzzy numbers and AHP. The methods employed by Van Laarhoven and Pedrcyz [10], Buckley [11], Enea and Piazza [13] and Krejčí et al.[14] derive fuzzy priorities represented as fuzzy numbers or fuzzy sets. On the other hand, Chang [12], Mikhailov [6] obtain crisp priorities from fuzzy comparisons.

FAHP is frequently applied along with other tools, namely, Goal Programming (GP), Fuzzy Programming Lineal (FPL), Fuzzy Dematel (FD), Moora and Fuzzy Moora (FMoora), Topsis and Fuzzy Topsis (FTopsis), Vikor, Strengths-weaknesses-Opportunities-Threats (SWOT) analysis, Grey Relational Analysis (GRA), Fuzzy Comprehensive Evaluation Method (FCEM, Particle Swarm Optimization (PSO) and DEA. In Table 1, we display some relevant FAHP applications in which Fuzzy numbers with linear membership functions, that is, triangular numbers (TN) are the main membership function used, followed by trapezoidal number (TrN).

Table 1. Fuzzy AHP application areas, methods and types of fuzzy numbers

PAPERS	AREA	METHOD	FUZZY NUMBER
[15]	Inventory classification system	FAHP	TN
[16]	Transportation		TN
[17]	Service quality in health		TN

[18]	Job security		TrN
[19]	Intellectual capital management		TN
[20]	Current bank account selection		TN
[21]	Mining Project		TN
[22]	Evaluation of the university business incubators		TN
[23]	Designing environment friendly products		TN
[24]	Process engineering		TN
[25]	Supplier choice in airline retail industry		TN
[26]	Evaluation on self-ignition risks of coal stockpiles		TN
[27]	Investment project selection		TN/TrN
[28]	Selection among energy alternative		TYPE-2
[29]	Risk evaluation		TN
[30]	Urban land-use planning		TN
[31]	Service quality in health		TN
[32]	Application to 3PSP selection		TN
[33]	Integrated manufacture planning	FAHP/GP	TN
[34]	Supply chain	FAHP/FLP	TN
[35]	Human resources management	FAHP/FD	TrN
[36]	Choice of ERP software system	FAHP/FMOORA	TN
[37]	Industrial engineering sector choosing		TN
[38]	Failure modes and effect analysis	FAHP/FTOPSIS	TN
[39]	Healthcare industry		TN
[40]	Construction project		TN
[41]	Knowledge management		TN
[42]	Cloud service selection		TN
[43]	Financial performance of industrial sector	FAHP/TOPSIS/VIKOR	TN
[44]	E-book business model	FAHP/TOPSIS/VIKOR/GRA	TN
[45]	Financial performance of Banks	FAHP/TOPSIS	TN
[46]	Supply chain management		TrN
[3]	Outsourcing reverse Logistic	FAHP/SWOT	TIN
[47]	Evaluating teaching performance	FAHP/FCEM	TN
[48]	Nonlinear optimization	FAHP/PSO	TN
[49]	Bank loan decision for enterprises	FAHP/DEA	TN

Source: Own Elaboration from ISI Web of Knowledge Database

As Table 1 shows, a great number of contributions only apply FAHP. In other cases, however, when we apply FAHP combined with other methodologies, a first step is to determine weights for each criterion using FAHP, while a second step entails establishing a ranking using some of the aforementioned methods. These techniques are primarily MCDM methodologies which complement FAHP and have been applied to many fields such as economics, finance, environment or engineering.

2.2. Mikhailov's model: Fuzzy Preference Programming (FPP)

FAHP models operate basically using triangular or trapezoidal fuzzy numbers with linear membership functions, which involves the subsequent limitation for the decision makers when their opinions have to be represented.

The main steps in FAHP are the following:

Just like in classical AHP, obtain a hierarchical structure from a decision making problem.

The next step is to develop pairwise fuzzy comparison matrices. Take a prioritization problem with *n* components, where fuzzy numbers denote pairwise fuzzy comparisons. As in classical AHP, every set of comparisons for each level need n (n-1) / 2 judgments, these being used to build a *positive fuzzy reciprocal comparison matrix* $\tilde{A}=\{\tilde{a}_{ij}\}$ that:

$$\begin{pmatrix} \tilde{a}_{11} & \cdots & \tilde{a}_{1t} \\ \vdots & \ddots & \vdots \\ \tilde{a}_{t1} & \cdots & \tilde{a}_{tt} \end{pmatrix} \quad (1)$$

The third step is control of coherence and resulting priorities, which evaluates consistency and also obtains priorities from the pairwise fuzzy matrices.

One last step is aggregation of priorities and classification of alternatives. By applying a simple weighted sum, we aggregate the local priorities computed in the distinct levels of the hierarchy of decisions. The global priorities thus obtained provide the final ranking and the selection of the best alternative.

The reason why Mikhailov's methodology[50] has been selected is because it helps us evaluate consistency of the decision makers' opinions by using the so-called λ or "index of consistency" [48]. According to this methodology, Fuzzy Preference Programming (FPP) is proposed to obtain priorities from the fuzzy comparison judgments, which removes some of the drawbacks of the fuzzy prioritization methods currently employed. This proposed approach does not involve the building up of complete fuzzy comparison matrices, and besides it allows us to derive priorities from an incomplete set of fuzzy judgments. Moreover, the approach remains invariant to the precise shape of the fuzzy sets that have been employed in the representation of judgments[48].

By employing α-cuts, initial fuzzy judgments are converted into a series of interval judgments. The method is used to transform the FPP priority allocation problem into a fuzzy program. This allows us to derive clear priorities from interval judgments, which correspond to each α-level cut. Therefore, the need for another fuzzy classification procedure disappears.

Take a fuzzy judgment matrix $\tilde{A}=\{\tilde{a}_{ij}\}$ that is constructed as in (1). The components of the pairwise fuzzy comparison matrix are expressed by triangular numbers $T\tilde{a}_{ij}=\left(a_{ij}^1,a_{ij}^2,a_{ij}^3\right)$, where $i,j=1,\ldots t$. Besides,

If $i\neq j, a_{ij}^1<a_{ij}^2<a_{ij}^3$

If $i=j,\ \tilde{a}_{ij}=\tilde{a}_{ji}=(1,1,1)$

where $w=\left(w_1,w_2,\ldots,w_t\right)^T$ is the vector of exact priorities.

The FPP priority allocation problem consists in solving the following program[6]:

$$\begin{aligned} & \textit{Maximize } \lambda \\ & \textit{Subject to} \\ & \left(a_{ij}^2-a_{ij}^1\right)\lambda w_j - w_i + a_{ij}^1 w_j \le 0 \\ & \left(a_{ij}^3-a_{ij}^2\right)\lambda w_j + w_i - a_{ij}^3 w_j \le 0 \\ & \sum_{k=1}^{t} w_k = 1; w_k > 0; k=1,2,\ldots,t \\ & i=1,2,\ldots,t-1; j=1,2,3,\ldots,t; j>i \end{aligned} \quad (2)$$

Mikhailov denotes λ^* as "consistency index", which is used to evaluate the satisfaction level of the optimal priority vector w^*. When λ^* is positive, all the solution coefficients entirely satisfy fuzzy opinions. This means that the initial set of fuzzy judgments is significantly consistent. Conversely, a negative value of λ^* shows that the fuzzy judgments are highly inconsistent, that is to say, we can employ the optimal value of λ^*

as a consistency measure of the initial set of fuzzy judgments.

2.3. Extended FAHP (E-FAHP) with (m,n)-trapezoidal numbers

An extension of FAHP Mikhailov's model for its application with (m,n)-trapezoidal numbers called Extended FHP (E-FAHP) is proposed. Before establishing the E-FAHP model, let us begin with a basic definition for (m,n)-trapezoidal number.

Definition 1. (m,n)-trapezoidal number . Let us now define a type of fuzzy number called (m.n)-trapezoidal number, $Tr\tilde{A}_{(m,n)} = \left(a^1,\ a^2,\ a^3,\ a^4\right)_{(m,n)}$ where $a^1 \le a^2 \le a^3 \le a^4 \epsilon\ X$. Its membership function is provided by Appadoo [7]:

$$Tr\tilde{A}_{(m,n)}(x) = \begin{cases} 0, & a^1 \le 0 \\ 1-\left(\dfrac{a^2-x}{a^2-a^1}\right)^m, & a^1 \le x \le a^2 \\ 1, & a^2 \le x \le a^3 \\ 1-\left(\dfrac{x-a^3}{a^4-a^3}\right)^n & a^3 \le x \le a^4 \\ 0, & a^4 \ge 0 \end{cases} \quad (3)$$

The representation of $Tr\tilde{A}_{(m.n)}$, from the α-cuts is:

$$Tr\tilde{A}_{(m.n)}(\alpha) = \left[a_L(\alpha), a_H(\alpha)\right] = \left[a^2-\left(a^2-a^1\right)(1-\alpha)^{1/m}, a^3+\left(a^4-a^3\right)(1-\alpha)^{1/n}\ \right]\ \forall \alpha \in [0,1]$$

The membership function of $Tr\tilde{A}_{(m.n)}$ is displayed in Fig. 1:

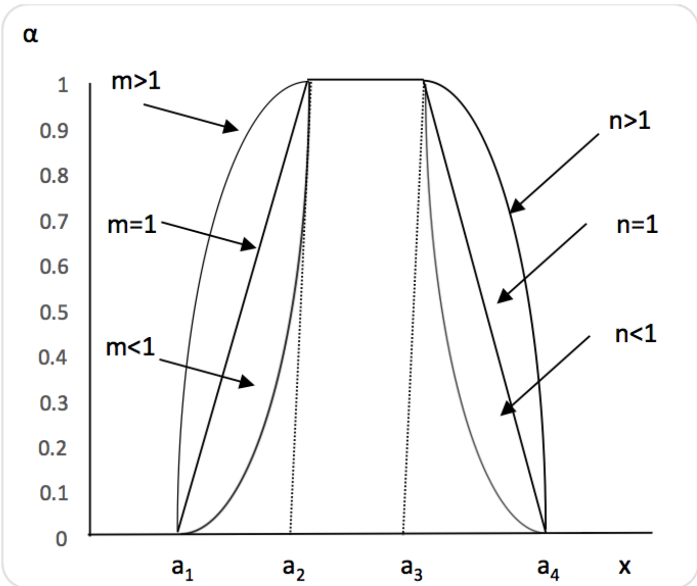

Figure 1. Membership function of $Tr\tilde{A}_{(m.n)}$

From $Tr\tilde{A}_{(m.n)}$, we can obtain a trapezoidal number $\left(Tr\tilde{A}\right)$, when *m*=*n*=*1*, that is:

$$Tr\tilde{A}(\alpha) = \left[a_L(\alpha), a_H(\alpha)\right] = \left[a^2-\left(a^2-a^1\right)(1-\alpha), a^3+\left(a^4-a^3\right)(1-\alpha)\ \right]\ \forall \alpha \in [0,1]$$

In a similar way we could obtain a triangular number $\left(T\tilde{A}\right)$, from $Tr\tilde{A}_{(m.n)}$, if *m*=*n*=*1*, and from $a^2 = a^3$, and we rewrite a^3 for a^4, that is:

$$T\tilde{A}(\alpha) = \left[a_L(\alpha), a_H(\alpha)\right] = \left[a^2 - \left(a^2 - a^1\right)(1-\alpha), a^2 + \left(a^3 - a^2\right)(1-\alpha)\ \right]\ \forall\alpha \in [0,1]$$

Next, we state the main operations, with $\tilde{A}$ and $\tilde{B}$ being two (m,n)-trapezoidal numbers,

$$Tr\,\tilde{A}_{(ma.na)}(\alpha) = \left[a^2 - \left(a^2 - a^1\right)(1-\alpha)^{1/ma}, a^3 + \left(a^4 - a^3\right)(1-\alpha)^{1/na}\ \right]\ \forall\alpha \in [0,1]$$

$$Tr\tilde{B}_{(mb.nb)}(\alpha) = \left[b^2 - \left(b^2 - b^1\right)(1-\alpha)^{1/mb}, b^3 + \left(b^4 - b^3\right)(1-\alpha)^{1/nb}\ \right]\ \forall\alpha \in [0,1]$$

The aggregation of $Tr\,\tilde{A}_{(ma.na)}$ and $Tr\tilde{B}_{(mb.nb)}$, will be given by:

$$\begin{gathered} Tr\tilde{A}_{(ma.na)}(\alpha) + Tr\tilde{B}_{(mb.nb)}(\alpha) = \\ = \left[\left(a^2 - \left(a^2 - a^1\right)(1-\alpha)^{1/ma}\right) + \left(b^2 - \left(b^2 - b^1\right)(1-\alpha)^{1/mb}\right), \left(a^3 + \left(a^4 - a^3\right)(1-\alpha)^{1/na}\right) + (b^3 + \left(b^4 - b^3\right)(1-\alpha)^{1/nb})\right] \\ \forall\alpha \in [0,1] \end{gathered}$$

The difference between $Tr\,\tilde{A}_{(ma.na)}$ and $Tr\tilde{B}_{(mb.nb)}$, will be given by:

$$\begin{gathered} Tr\tilde{A}_{(ma.na)}(\alpha) - Tr\tilde{B}_{(mb.nb)}(\alpha) = \\ = \left[\left(a^2 - \left(a^2 - a^1\right)(1-\alpha)^{1/ma}\right) - \left(b^3 + \left(b^4 - b^3\right)(1-\alpha)^{1/mb}\right), \left(a^3 + \left(a^4 - a^3\right)(1-\alpha)^{1/na}\right) - (b^2 - \left(b^2 - b^1\right)(1-\alpha)^{1/nb})\right] \\ \forall\alpha \in [0,1] \end{gathered}$$

The multiplication of $Tr\,\tilde{A}_{(ma.na)}$ and $Tr\tilde{B}_{(mb.nb)}$, will be given by:

$$\begin{gathered} Tr\tilde{A}_{(ma.na)}(\alpha)\, x\, Tr\tilde{B}_{(mb.nb)}(\alpha) = \\ = \left[\left(a^2 - \left(a^2 - a^1\right)(1-\alpha)^{1/ma}\right) x \left(b^2 - \left(b^2 - b^1\right)(1-\alpha)^{1/mb}\right), \left(a^3 + \left(a^4 - a^3\right)(1-\alpha)^{1/na}\right) x (b^3 + \left(b^4 - b^3\right)(1-\alpha)^{1/nb})\right] \\ \forall\alpha \in [0,1] \end{gathered}$$

The division between $Tr\,\tilde{A}_{(ma.na)}$. and $Tr\tilde{B}_{(mb.nb)}$, will be given by:

$$\frac{Tr\tilde{A}_{(ma.na)}(\alpha)}{Tr\tilde{B}_{(mb.nb)}(\alpha)} = \left[\frac{\left(a^2 - \left(a^2 - a^1\right)(1-\alpha)^{1/ma}\right)}{\left(b^3 + \left(b^4 - b^3\right)(1-\alpha)^{1/mb}\right)}, \frac{\left(a^3 + \left(a^4 - a^3\right)(1-\alpha)^{1/na}\right)}{\left(b^2 - \left(b^2 - b^1\right)(1-\alpha)^{1/nb}\right)}\right]\ \forall\alpha \in [0,1]$$

In our case, and unlike Mikhailov's model, let us suppose a fuzzy judgment matrix $\tilde{A} = \left\{\tilde{a}_{ij}\right\}$, built as in (1). We represent the components of the pairwise fuzzy comparison matrix by $Tr\tilde{a}_{ij(m,n)} = \left(a_{ij}^1, a_{ij}^2, a_{ij}^3, a_{ij}^4\right)_{(m.n)}$, where $i, j = 1, \ldots t$. Also,

If $i \neq j,\ a_{ij}^1 < a_{ij}^2 < a_{ij}^3 < a_{ij}^4$

If $i = j,\ \tilde{a}_{ij} = \tilde{a}_{ji} = (1, 1, 1, 1)_{(m,n)}$

As a result, an exact priority vector $w = \left(w_1, w_2, \ldots, w_t\right)^T$ which derives from $\tilde{A}$ should satisfy fuzzy inequations:

$$a_{ij}^{1} \tilde{\leq} \frac{w_i}{w_j} \tilde{\leq} a_{ij}^{4} \tag{4}$$

where $w_i > 0$, $w_j > 0, i \neq j$ and symbol $\tilde{\leq}$ represent "fuzzy less than or equal to".

In order to measure the satisfaction degree of different crisp relationships w_i / w_J as regards double side inequality in equation (4), we can define a new membership function from (3):

$$\mu_{ij}\left(\frac{w_i}{w_j}\right) = \begin{cases} 0, & a_{ij}^{1} \leq 0 \\ 1 - \left(\dfrac{a_{ij}^{2} - \left(\dfrac{w_i}{w_j}\right)}{a_{ij}^{2} - a_{ij}^{1}}\right)^{m} & a_{ij}^{1} \leq \left(\dfrac{w_i}{w_j}\right) \leq a_{ij}^{2} \\ 1, & a_{ij}^{2} \leq \left(\dfrac{w_i}{w_j}\right) \leq a_{ij}^{3} \\ 1 - \left(\dfrac{\left(\dfrac{w_i}{w_j}\right) - a_{ij}^{3}}{a_{ij}^{4} - a_{ij}^{3}}\right)^{n} & a_{ij}^{3} \leq \left(\dfrac{w_i}{w_j}\right) \leq a_{ij}^{4} \\ 0, & a_{ij}^{4} \geq 0 \end{cases} \tag{5}$$

The solution to the prioritization problem through FPP relies on two main assumptions [50].

Assumption 1. This assumption requires the existence of non-empty fuzzy feasible area $\tilde{P}$ on the $(n-1)$ –dimensional simplex Q^{n-1}

$$Q^{n-1} = \left\{ (w_1, w_2, \ldots, w_t) \middle| w_i \rangle 0, \sum_{1}^{t} w_i = 1 \right\} \tag{6}$$

Being defined as an intersection of the membership functions, similar to (5) and the simplex hyperplane (6), the membership function of the fuzzy feasible area $\tilde{P}$ is given by:

$$\mu_P(w) = \min_{ij} \left\{ \mu_{ij}(w) \middle| i = 1, \ldots, t-1; j = 2, \ldots, t; j \rangle i \right\} \tag{7}$$

Once membership functions (5) are defined as L-fuzzy sets, we can relax the assumption of non-emptiness of $\tilde{P}$ on the simplex. If fuzzy judgments are significantly inconsistent, then $\mu_P(w)$ could take negative values for all normalized priority vectors $w \in Q^{n-1}$.

Assumption 2. The second assumption incorporates a selection rule determining a priority vector which has the maximum degree of membership in aggregate membership function (7). It can be easily proven that $\mu_P(w)$ is a convex set and therefore priority vector $w^* \in Q^{n-1}$ always has the highest degree of membership.

$$\mu_P(w^*) = max \min_{ij} \left\{ \mu_{ij}(w) \mid w \in Q^{n-1} \right\} \tag{8}$$

Let us represent the maximin of prioritization problem (8) as follows:

$$
\begin{aligned}
&\textit{Maximize } \lambda \\
&\textit{Subject to} \\
&\lambda \le \mu_{ij}\left(w\right), i = 1,2,\dots,t-1; j = 1,2,3,\dots,t; j > i \\
&\sum_{l=1}^{t} w_l = 1; w_l > 0; l = 1,2,\dots,t
\end{aligned}
\tag{9}
$$

Taking into account the particular form of membership function (5), problem (9) can be converted into the E-FAHP preference programming:

$$
\begin{aligned}
&\textit{Maximize } \lambda \\
&\textit{Subject to} \\
&\left[w_j\left(a_{ij}^2 - a_{ij}^1\right)\right]^m \left(\lambda - 1\right) + \left(a_{ij}^2 w_j - w_i\right)^m \le 0 \\
&\left[w_j\left(a_{ij}^4 - a_{ij}^3\right)\right]^n \left(\lambda - 1\right) + \left(w_i - a_{ij}^3 w_j\right)^n \le 0 \\
&\sum_{k=1}^{t} w_k = 1; i = 1,2,\dots,t-1; j = 1,2,3,\dots,t; j > i\, w_k > 0; k = 1,2,\dots,t
\end{aligned}
\tag{10}
$$

If the elements of the pairwise fuzzy comparison matrix were represented by trapezoidal numbers $Tr\tilde{a}_{ij} = \left(a_{ij}^1, a_{ij}^2, a_{ij}^3, a_{ij}^4\right)$, where $i, j = 1,\dots t$, that is *m*=*n*=*1*, then the problem would become:

$$
\begin{aligned}
&\textit{Maximize } \lambda \\
&\textit{Subject to} \\
&\left(a_{ij}^2 - a_{ij}^1\right)\lambda w_j - w_i + a_{ij}^1 w_j \le 0 \\
&\left(a_{ij}^4 - a_{ij}^3\right)\lambda w_j + w_i - a_{ij}^4 w_j \le 0 \\
&\sum_{k=1}^{t} w_k = 1; w_k > 0; k = 1,2,\dots,t \\
&i = 1,2,\dots,t-1; j = 1,2,3,\dots,t; j > i
\end{aligned}
\tag{11}
$$

3. Results

In this section, we will illustrate our approach by solving a practical case of fuzzy AHP problem given in Mikhailov and Tsvetinov [8] with the help of E-FAHP. The problem is to assess three potential service providers considering three main criteria, namely, pricing, service quality and delivery time. Additionally, each main criterion is divided into two subcriteria, which are Cost-based and Demand-based Pricing, Reliable and Responsive Service Quality and Immediate and Negotiable Delivery, as Fig. 2 shows:

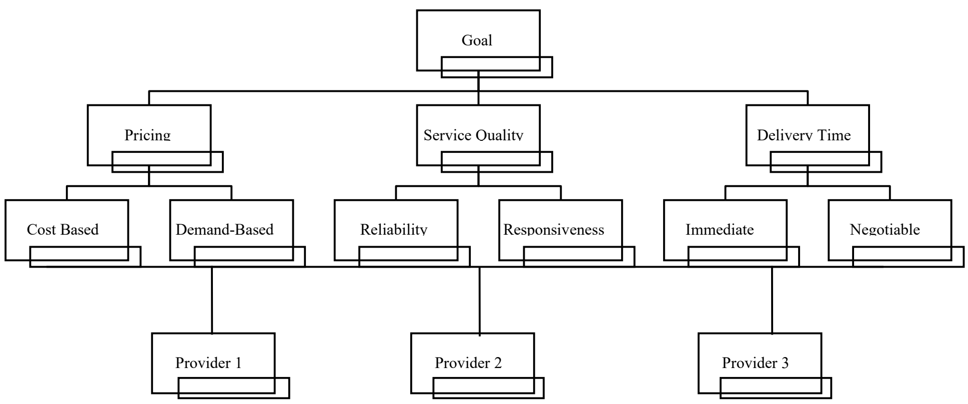

Figure 2. Decision Hierarchy [8]

The aim is to choose a service provider which satisfies all criteria in an optimal way. Table 2 displays the fuzzy pairwise comparison judgments of the main criteria.

Table 2. Mikhailov and Tsvetinov[8] pairwise comparison matrix

Goal	Pricing	Service Quality	Delivery Time
Pricing	1	(2, 3, 4)	(1, 2, 3)
Service Quality	$\frac{1}{(2,3,4)}$	1	$\left(\frac{1}{3},\frac{1}{2},1\right)$
Delivery Time	$\frac{1}{(1,2,3)}$	$\frac{1}{\left(\frac{1}{3},\frac{1}{2},1\right)}$	1

By applying Mikhailov's model (2), the corresponding criteria weights yield:

$w_1(pricing) = 0.538$

$w_2(Service\,Quality) = 0.170$

$w_3(Delivery\,Time) = 0.292$

$\lambda = 0.838$

To apply E-FAHP model (10), first we express the triangular numbers (TN) as (m.n)-trapezoidal numbers.

$Tr\tilde{a}_{ij(m,n)} = \left(a_{ij}^1, a_{ij}^2, a_{ij}^3, a_{ij}^4\right)_{(m,n)}$.

That is: $a_{ij}^2 = a_{ij}^3$ and m = n = 1.

In Table 3, the corresponding trapezoidal numbers when *m*=*n*=*1* and $a_{ij}^2 = a_{ij}^3$ are specified.

Table 3. Fuzzy pairwise comparison matrix using $Tr\tilde{a}_{ij(m,n)}$ when *m* = *n* = *1* and $a_{ij}^2 = a_{ij}^3$

Goal	Pricing	Service Quality	Delivery Time
Pricing	1	$(2, 3, 3, 4)_{(1,1)}$	$(1, 2, 2, 3)_{(1,1)}$
Service Quality	$\frac{1}{(2,3,3,4)_{(1,1)}}$	1	$\left(\frac{1}{3},\frac{1}{2},\frac{1}{2},1\right)_{(1,1)}$
Delivery Time	$\frac{1}{(1,2,2,3)_{(1,1)}}$	$\frac{1}{\left(\frac{1}{3},\frac{1}{2},\frac{1}{2},1\right)_{(1,1)}}$	1

To obtain the weights for each criterion, we apply E-FAHP model (10). In this case, we can check that weights of the main criteria correspond with the results obtained by Mikhailov and Tsevetinov [8].

$w_1(pricing) = 0.538$

$w_2(Service\,Quality) = 0.170$

$w_3(Delivery\,Time) = 0.292$

$\lambda = 0.838$

Next, we propose the same case study with different $a_{ij}^2 < a_{ij}^3$ and *m*= *n* =*1*. That is a trapezoidal number:

$Tr\tilde{a}_{ij} = Tr\tilde{a}_{ij(1,1)} = \left(a_{ij}^1, a_{ij}^2, a_{ij}^3, a_{ij}^4\right)_{(1,1)}$

With the fuzzy pairwise comparison matrix shown in Table 4.

Table 4. Fuzzy pairwise comparison matrix using $Tr\tilde{a}_{ij(m,n)}$ when $a_{ij}^2 < a_{ij}^3$ and *m= n=1*

Goal	Pricing	Service Quality	Delivery Time
Pricing	1	$(1, 2, 3, 4)_{(1,1)}$	$\left(1, \frac{3}{2}, 2, 3\right)_{(1,1)}$
Service Quality	$\frac{1}{(1, 2, 3, 4)_{(1,1)}}$	1	$\left(\frac{1}{5}, \frac{1}{3}, \frac{1}{2}, 1\right)_{(1,1)}$
Delivery Time	$\frac{1}{\left(1, \frac{3}{2}, 2, 3\right)_{(1,1)}}$	$\frac{1}{\left(\frac{1}{5}, \frac{1}{3}, \frac{1}{2}, 1\right)_{(1,1)}}$	1

To obtain the weights for each criterion, we apply E-FAHP model (10) or (11) developed in this paper:

$w_1(pricing) = 0.500$

$w_2(Service\,Quality) = 0.167$

$w_3(Delivery\,Time) = 0.333$

$\lambda = 1$

To conclude, the following case study is presented:

$Tr\tilde{a}_{ij(m,n)} = \left(a_{ij}^1, a_{ij}^2, a_{ij}^3, a_{ij}^4\right)_{(m,n)}$. where, $a_{ij}^2 < a_{ij}^3$ and m ≠ n. See data in Table 5.

Table 5 Fuzzy pairwise comparison matrix using $Tr\tilde{a}_{ij(m,n)}$ when m ≠ n and $a_{ij}^2 < a_{ij}^3$

Goal	Pricing	Service Quality	Delivery Time
Pricing	1	$(1, 2, 3, 4)_{(6,5)}$	$\left(1, \frac{3}{2}, 2, 3\right)_{(3,2)}$
Service Quality	$\frac{1}{(1, 2, 3, 4)_{(6,5)}}$	1	$\left(\frac{1}{5}, \frac{1}{3}, \frac{1}{2}, 1\right)_{(4,7)}$
Delivery Time	$\frac{1}{\left(1, \frac{3}{2}, 2, 3\right)_{(3,2)}}$	$\frac{1}{\left(\frac{1}{5}, \frac{1}{3}, \frac{1}{2}, 1\right)_{(4,7)}}$	1

To obtain the weights for each criterion, we apply the proposed E-FAHP model (10):

$w_1(pricing) = 0.471$

$w_2(Service\,Quality) = 0.163$

$w_3(Delivery\,Time) = 0.366$

$\lambda = 0.493$

Table 6. Results from Mikhailov and Tsevetinov, and E-FAHP model

	FAHP	E-FAHP		
	Mikhailov and Tsevetinov ($T\tilde{a}_{ij}$)	*m = n = 1* and $a_{ij}^2 = a_{ij}^3$ ($T\tilde{a}_{ij}$)	*m = n = 1* and $a_{ij}^2 < a_{ij}^3$ ($Tr\tilde{a}_{ij}$)	*m ≠ n* and $a_{ij}^2 < a_{ij}^3$ ($Tr\tilde{a}_{ij(m,n)}$)
w_1	0.538	0.538	0.500	0.471
w_2	0.170	0.170	0.167	0.163
w_3	0.292	0.292	0.333	0.366
λ	0.838	0.838	1	0.493

Table 6, bottom row, displays the value of the consistency index λ for each optimal solution. From this row we can see that the fuzzy judgments when m = n = 1 and $a_{ij}^2 < a_{ij}^3 (Tr\tilde{a}_{ij})$ are the most consistent $\lambda = 1$. Then, the solution ratio w_i / w_j for all scores coincides with the highest level of the membership functions of the fuzzy comparison judgments as shown in Table 6, that is, (w_1 / w_2) = (0.500/0.167) = 3, (w_1 / w_3) = (0.500/0333) = 1.5 and (w_2 / w_3) = (0.167/0.333) = 0.5.

4. conclusions

The general approach E-FAHP proposed in this paper is regarded as tentative for the following reasons. Firstly, the fuzzy prioritization method herein proposed enables us to obtain clear priorities based on a nonlinear optimization model for consistent and inconsistent pairwise judgments. In this way priority fuzzy computations and fuzzy classification techniques can be avoided. And secondly, in the proposed nonlinear optimization method, pairwise opinions are expressed as (m,n)-trapezoidal numbers. This is an appropriate formulation for priority allocation problems in which opinions are expressed as fuzzy numbers, regardless of the form adopted by fuzzy judgments (linear or nonlinear). Additionally, this formulation allows one to perform prioritization problem resolution in which judgments are represented by different types of fuzzy numbers (linear and nonlinear) or crisp numbers.

Despite the fact that FAHP technique is a well-known MCDM methodology, its integration into a unifying approach for both linear and non linear fuzzy numbers helps clarify the close relationship between them.

In the illustrative example, it is then demonstrated that different FAHP approaches can be reduced to the E-FAHP structure when the pairwise judgments are represented by (m,n)-trapezoidal numbers.

Practitioners should be aware that, whatever the FAHP model they are building, they are actually formulating a particular case of E-FAHP. Therefore, E-FAHP can be seen as a general framework that can lead to a better understanding and presentation of the different FAHP approaches.

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