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ORIGINAL PAPER

Distinguished $C_p(X)$ spaces and the strongest locally convex topology

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Abstract
Since Tychonoff spaces X serve as continuous Hamel bases for the strong dual $L_\beta(X)$ of $C_p(X)$, an old splitting theorem proves: *$C_p(X)$ is distinguished $\Leftrightarrow$ $L_\beta(X)$ has the strongest locally convex topology* (slctop) [Ferrando/Kąkol]. Our new splitting theorem: *The span $L_X(Y)$ of $Y \subseteq X$ complements $L_X(X \backslash Y)$ in $L_\beta(X)$*. Thereby we prove *If $X = Y_1 \cup \cdots \cup Y_n$ and each $C_p(Y_j)$ is distinguished, then so is $C_p(X)$, provided either* (i) *all Y_j are G_δ sets, or* (ii) *all are F_σ sets. Hence, provided* (iii) *all Y_j are open, or* (iv) *all are closed*. Parts (ii)/(iv) extend to countable unions (known). Part (i) does not, via Michael's line. Countable case (iii) remains open. A dozen recent related results are proved/improved in our slctop analysis of $L_\beta(X)$.

Keywords Distinguished · Barrelled · φ-complemental · Stationary sets · Bidual · $\Sigma(X)$

Mathematics Subject Classification 46A03 · 46A08 · 46E10 · 54C35

1 Introduction

The strongest locally convex topology (slctop) for a real linear space E with algebraic dual E^* is $\beta(E, E^*)$. The strong dual φ of the Fréchet space $\mathbb{R}^{\mathbb{N}}$ is the simplest nontrivial example. Indeed, φ has its slctop and is the only $\aleph_0$-dimensional (Hausdorff) barrelled space, up to isomorphism. A subset of a topological space is assumed to carry the relative topology unless otherwise indicated. A *subspace* of a linear space is a subset closed under vector addition and scalar multiplication. If E has its slctop, then every $\aleph_0$-dimensional subspace is a copy (= isomorphic image) of φ that is complemented in E [37, II, Ex. 7(a)]. A space E

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is distinguished if and only if its strong dual $\left(E', \beta\left(E', E\right)\right)$ is barrelled [25]. We say E is *flat* if $E' = E^*$ [32, 36]. Every space with the slctop is flat.

Recent distinguished $C_p(X)$ study [8–10, 12, 14, 15, 18–20, 26, 27, 29] begins with the slctop [9, Corollary 3.4]. Yet we just learned that, a *half-century* ago, Prof. Denny Gulick [16] proved $C_p(X)$ is distinguished if and only if its bidual is $\mathbb{R}^X$, and gave nice examples of (non)distinguished $C_p(X)$ spaces. He used neither the word *distinguished* nor the notion of slctop. He tacitly reminds us that slctop was little regarded (see [31, pp. 88, 89]) before we proved barrelled spaces are φ-complemental, φ generates the second smallest variety, nuclear spaces often contain φ, strict (LF)-spaces always contain φ complemented, etc. [6, 21, 22, 31, 33, 34, 36].

A locally convex space E is *φ-complemental* [35] if, for each $\aleph_0$-codimensional closed subspace F in E, every algebraic complement of F is a topological complement isomorphic to φ. Generally, *φ-complemental* is far from *barrelled* [35]. In fact, the weak barrelledness table [35, Fig. 1.3] presents *barrelled* $\Rightarrow$ *$\aleph_0$-barrelled* $\Rightarrow$ *C- barrelled* $\Rightarrow$ *property (L)* $\Rightarrow$ *inductive* $\Rightarrow$ *φ-complemental* $\Rightarrow$ *primitive* as one of four seven-step paths from *barrelled* to *primitive* in which *φ-complemental* is the penultimate step. However, *barrelled* and *φ- complemental* are equivalent for $L_\beta(X)$ spaces, wherein complemented copies of φ abound (Theorems 5, 10). And $\Sigma(X) \subseteq M(X)$ (definitions below) if and only if *every* $\aleph_0$-dimensional subspace of $L_\beta(X)$ is a complemented copy of φ (Theorem 16), which seems very near to *φ- complemental*.

But $\Sigma(\omega_1) \subseteq M(\omega_1)$ (Corollary 17), and $L_\beta(\omega_1)$ is not barrelled (Theorem 18 has proof much simpler than Ką kol-Leiderman's original [18]). Hence $L_\beta(\omega_1)$ is not φ-complemental, but 'very nearly' so, which recalls the titular question of [12]. Theorem 5 characterizes sets $Y \subseteq X$ whose span $L_X(Y)$ in $L_\beta(X)$ is barrelled, generalizing the case $Y = X$ given independently by K-L and F-S [12]. In Sect. 5, our slctop theory unifies/extends most known results of a certain interesting type (see Abstact).

2 Structural notions for the strong dual $L_\beta(X)$

We write $E = F \oplus G$ to indicate that the locally convex space E is the (topological) direct sum of algebraically complementary subspaces F and G. A subspace F in E is *complemented* if there exists a subspace G with $E = F \oplus G$. A subset S in E is *bounded away from zero* if some 0-neighborhood in E misses S. And E is *feral* [24] if every linearly independent sequence is unbounded.

We list some possible properties for a locally convex space E.

(a) E has the slctop.
(b) $E = F \oplus G$ and F and G have the slctop whenever F and G are algebraic complements in E.
(c) $E = F \oplus G$ and G is a copy of φ whenever F and G are algebraic complements in E with $\dim G = \aleph_0$.
(d) G is a complemented copy of φ whenever G is an $\aleph_0$-dimensional subspace of E.
(e) G is a copy of φ whenever G is an $\aleph_0$ -dimensional subspace of E.
(f) Every linearly independent sequence in E is bounded away from zero.
(g) E is feral.
(*) E is φ-complemental.
(†) E is flat.

One easily observes that (a) ⇔ (b) ⇒ (c) ⇒ (d) ⇒ (e) ⇒ (f) ⇒ (g) (see [37, Ch. II, Ex. 7]). Also, (c) ⇒ (*) and (a) ⇒ (†) ⇒ (f). In fact, any linearly independent set S in E is part of a Hamel basis B for E; if E is flat and f is the unique linear form in $E^* = E'$ whose value on B is constantly 2, then the polar f° is a 0 -neighborhood which bounds S away from zero.

In the special case $E = L_\beta(X)$, condition (g) always holds, and we will see that:
I. For $E = L_\beta(X)$, conditions (a), (b), (c), (*), (†) are equivalent, and
II. For $E = L_\beta(X)$, conditions (d), (e), (f) are equivalent.

Moreover, the conditions in (I) are all equivalent to the condition that $C_p(X)$ be distinguished, and we prove the conditions in (II) are equivalent to the condition that the dual $M(X)$ of $L_\beta(X)$, when canonically identified as a dense subspace of $\mathbb{R}^X$, must contain the subspace $\Sigma(X)$ of $\mathbb{R}^X$ consisting of all countably supported functions on X. In the terminology of [1], $\Sigma(X)$ is the $\aleph_0$ -dual of $L(X)$ with respect to the Hamel basis X.

3 The ubiquitous component φ in $L_\beta(X)$

The (LF)-spaces are trichotomized by φ [28, 34]: $(LF)_1$-spaces contain (copies of) φ complemented; $(LF)_2$-spaces contain φ, but not φ complemented; $(LF)_3$-spaces contain no copies of φ. By contrast, we find that all $L_\beta(X)$ spaces contain copious complemented copies of φ (Theorem 10). The slctop exotica of $L_\beta(X)$ synergizes the study of local convexity and distinguished $C_p(X)$.

First, we observe

Lemma 1 *Let F be a closed subspace of a locally convex space E. The following two assertions are equivalent.*

1. *The quotient E/F has its slctop.*
2. *⟨Every⟩ [Some] algebraic complement G of F in E is a topological complement having its slctop.*

Proof If E/F has its slctop, the restriction $q|_G$ of the quotient map q is an isomorphism, so G has the slctop and the projection $(q|_G)^{-1} \circ q$ is continuous. The converse is obvious. □

Thus E is φ-complemental if and only if the quotient E/F is a copy of φ for each closed $\aleph_0$-codimensional subspace F in E, which illumines the path *barrelled* ⇒ φ *-complemental* [33] ⇒ *primitive* [35]. We [35, 36] defined E as *primitive* if $f \in E'$ whenever $f \in E^*$ and each restriction $f|_{E_n}$ is continuous for some increasing sequence $\{E_n : n \in \mathbb{N}\}$ of subspaces covering E.

Throughout, X denotes an infinite Tychonoff space, $C(X)$ the continuous real-valued functions on X, and $C_p(X)$ denotes the linear space $C(X)$ endowed with the topology of pointwise convergence. By means of the evaluation map, we may think of each x in X as a continuous linear form on $C_p(X)$, so that X becomes a Hamel basis for the dual $L(X)$ of $C_p(X)$. We write $L_\beta(X)$ to denote $L(X)$ with the strong topology $\beta(L(X), C(X))$: the strong dual of $C_p(X)$. The bidual $M(X)$ of $C_p(X)$ is the dual of $L_\beta(X)$. As linear forms are determined by their values on a fixed Hamel basis, we identify the algebraic dual $L_\beta(X)^*$ with $\mathbb{R}^X$, so that $M(X)$ becomes the subspace of $\mathbb{R}^X$ consisting of points in the closure in $\mathbb{R}^X$ of bounded sets in $C_p(X)$. (See the first paragraph of [11, Section 2].)

Trivially, every flat space is primitive. The converse is far from true in general. Yet,

Theorem 2 *Every primitive subspace G of $L_\beta(X)$ is flat.*

Proof We must show $G' \supseteq G^*$. As $G^* = L_\beta(X)^*|_G = \mathbb{R}^X|_G$, we let $h \in \mathbb{R}^X$ and show that $h|_G \in G'$. The sets $Y_n := \{x \in X : |h(x)| \leq n\}$ span increasingly large subspaces $L_X(Y_n)$ which cover $L(X)$, so the subspaces $G_n := G \bigcap L_X(Y_n)$ likewise cover G. Since G is primitive, we need only prove each $h|_{G_n} \in G_n'$. Choose $h_n \in \mathbb{R}^X$ with $h_n|_{Y_n} = h|_{Y_n}$ and $|h_n| \leq \boldsymbol{n}$ (the identically n function). As h_n is continuous, so is $h_n|_{G_n} = h|_{G_n}$. Indeed, $h_n \in \ell_\infty(X) \subseteq M(X)$ [12, Theorem 2]. □

For each $g \in \mathbb{R}^X$ define $P_g = \left\{f \in \mathbb{R}^X : |f| \leq |g|\right\}$, a bounded set in $\mathbb{R}^X$. For each bounded set A in $\mathbb{R}^X$ define $\phi_A \in \mathbb{R}^X$ by writing $\phi_A(x) = \sup_{f\in A}|f(x)|$ for each $x \in X$. For each bounded set B in $C_p(X)$, define another bounded set B^+ in $C_p(X)$ consisting of those h in $C(X)$ for which there exists a finite set $\mathcal{F} \subseteq B$ such that $|h(x)| \leq \max_{f\in\mathcal{F}}|f(x)|$ for all $x \in X$.

In an early version of [15], Saxon proved the following lemma. We add here equicontinuity of P_g, with easy proof: If $g \in M(X)$, there exists a bounded set B in $C_p(X)$ with $g \in \overline{B}$, closure in $\mathbb{R}^X$. Since $\phi_{\{g\}} = |g|$ and $P_{|g|} = P_g$, the lemma's first statement (proved in [12, 15]) assures $P_g \subseteq \overline{B^+}$. As B^+ is bounded in $C_p(X)$, the set P_g is equicontinuous.

Lemma 3 *If a subset A of $\mathbb{R}^X$ lies in the closure $\overline{B}$ in $\mathbb{R}^X$ of a bounded set B in $C_p(X)$, then $P_{\phi_A} \subseteq \overline{B^+}$. In particular, (i) $P_{\phi_B} \subseteq \overline{B^+}$ always holds, and (ii) if $g \in M(X)$, then $P_g \subseteq M(X)$; indeed, P_g is equicontinuous on $L_\beta(X)$.*

If $\mathbb{R}^{(X)}$ denotes the finitely supported functions in $\mathbb{R}^X$, $(ii) \Rightarrow \mathbb{R}^{(X)} \subseteq \ell_\infty(X) \subseteq M(X)$ [9, 12], since $\boldsymbol{n} \in C_p(X) \subseteq M(X)$ and $\ell_\infty(X) = \bigcup_{n\in\mathbb{N}} P_{\boldsymbol{n}}$. Now $\mathbb{R}^{(X)} \subseteq M(X)$ simply says the basis X has continuous coefficient functionals, so the splitting theorem [30, 36] proves $L_\beta(X)$ is barrelled ($C_p(X)$ is distinguished) if and only if $L_\beta(X)$ has the slctop [9, 12, 15]. Given $Y \subseteq X$, the subspace $L_X(Y)$ spanned by Y is closed in $L_\beta(X)$ since $\mathbb{R}^{(X)} \subseteq M(X)$, so $L_X(X\backslash Y)$ is a closed algebraic complement. Much more, in fact:

Theorem 4 (SPLITTING THEOREM) *If $Y \subseteq X$, then $L_\beta(X) = L_X(Y) \oplus L_X(X\backslash Y)$.*

Proof Let V be a closed absolutely convex neighborhood of the origin in $L_X(Y)$. Its polar V° in $L_X(Y)'$ is equicontinuous on $L_X(Y)$, with $V^{\circ\circ} = V$. The Hahn-Banach theorem provides a set D in $M(X)$ equicontinuous on $L_\beta(X)$ with set of restrictions $D|_{L_X(Y)} = V^\circ$. Equicontinuity of D means there is a bounded set B in $C_p(X)$ such that $D \subseteq \overline{B}$, closure in $\mathbb{R}^X$. Therefore $P_{\phi_D} \subseteq \overline{B^+}$ by Lemma 3. In particular,

$$A := \left\{f \in \mathbb{R}^X : f|_{L_X(Y)} \in V^\circ \text{ and } f|_{X\backslash Y} = 0\right\} \subseteq \overline{B^+}.$$

Hence A is equicontinuous on $L_\beta(X)$. Clearly, then, the polar $A^\bullet$ is a 0-neighborhood in $L_\beta(X)$, and $A^\bullet = V^{\circ\circ} + L_X(X\backslash Y) = V + L_X(X\backslash Y)$. Thus the projection of $L_\beta(X)$ onto $L_X(Y)$ along $L_X(X\backslash Y)$ is continuous. □

We generalize results from our previous paper [12].

Theorem 5 *If $Y \subseteq X$, the following six assertions are equivalent.*

1. *The subspace $L_X(Y)$ of $L_\beta(X)$ has the slctop.*
2. *The subspace $L_X(Y)$ is barrelled.*
3. *The subspace $L_X(Y)$ is primitive.*

4. *The subspace $L_X(Y)$ is flat.*
5. *The dual space $M(X)$ contains all functions in $\mathbb{R}^X$ which vanish off Y.*
6. *For each sequence $Y_1, Y_2, \ldots$ of pairwise disjoint subsets of Y, empty sets allowed, there exists a sequence $Q_1, Q_2, \ldots$ of open sets in X such that each $Q_n \supseteq Y_n$ and, for each $x \in X$, the set $\{n \in \mathbb{N} : x \in Q_n\}$ is finite.*

Proof Clearly, (1) ⇒ (2) ⇒ (3), and (3) ⇒ (4) by Theorem 2.

[(4) ⇒ (5)]. Suppose $f \in \mathbb{R}^X$ vanishes off Y. Then f vanishes on $L_X(X\backslash Y)$ and is continuous there. By (4), the restriction $f|_Y$ is continuous on $L_X(Y)$. Therefore f is continuous on $L_X(Y) \oplus L_X(X\backslash Y) = L_\beta(X)$ (Theorem 4); *i.e.*, $f \in M(X)$.

[(5) ⇒ (6)]. Let disjoint subsets $Y_1, Y_2, \ldots$ of Y be given. Let h be the step function which vanishes off $\bigcup_{n\in\mathbb{N}} Y_n$ and has constant value n on Y_n for each $n \in \mathbb{N}$. Since $h \in M(X)$ by (5), there is a bounded set B in $C_p(X)$ with $h \in \overline{B}$, closure in $\mathbb{R}^X$. Each $Q_n := \bigcup \left\{ f^{-1}(n-1, n+1) : f \in B \right\}$ is an open set, possibly empty, containing Y_n by definition of h and $\overline{B}$. Moreover, when a point x is in Q_n, there is some $f_n \in B$ such that $f_n(x) > n-1$. Since B is bounded at x, the point x can be in only finitely many Q_n. Thus (6) holds.

[(6) ⇒ (5)]. Let u be a function in $\mathbb{R}^X$ which vanishes off Y. The disjoint sets $Y_n := \{x \in Y : n-1 \leq |u(x)| < n\}$ $(n \in \mathbb{N})$ cover Y. Let $Q_1, Q_2, \ldots$ be corresponding open sets in X satisfying (6). If

$$B_n := \{g \in C(X) : |g| < \mathbf{n} \text{ and } g \text{ vanishes off } Q_n\}$$

then $B := \bigcup_{n=1}^{\infty} B_n$ is bounded in $C_p(X)$. In fact, for each $x \in X$ we readily obtain

$$\sup_{g\in B} |g(x)| \leq \max\left(\{0\} \bigcup \{n \in \mathbb{N} : x \in Q_n\}\right) := m_x.$$

For a given finite set $\sigma \subseteq X$, if $x \in \sigma \cap Y$ then $x \in Y_n \subseteq Q_n$ for some $n \in \mathbb{N}$, so $|u(x)| < n \leq m_x \in \mathbb{N}$, and, routinely, there exists $g_x \in B_{m_x}$ with $g_x(x) = u(x)$. If, on the other hand, $x \in \sigma\backslash Y$, then $u(x) = 0$, and we let $g_x = \mathbf{0} \in B_1 \subseteq B$. Thus for all $x \in \sigma$ we have $g_x \in B$ and $|u(x)| \leq |g_x(x)|$. The finite $\mathcal{F} := \{g_x : x \in \sigma\} \subseteq B$ yields $h \in B^+$ agreeing with u on σ. Hence $u \in \overline{B^+}^{\mathbb{R}^X}$. Now $\overline{B^+}^{\mathbb{R}^X} \subseteq M(X)$ since B^+ is bounded in $C_p(X)$. Then $u \in M(X)$, proving (5) holds.

[(5) ⇒ (1)]. Let A be a bounded set in the algebraic dual $\mathbb{R}^Y$ of $L(Y)$. The polar A° in $L(Y)$ is a basic slctop 0-neighborhood. If B is the subset of $\mathbb{R}^X$ whose restriction sets $B|_Y$ and $B|_{X\backslash Y}$ are A and $\{\mathbf{0}\}$, respectively, then ϕ_B vanishes off Y, so $\phi_B \in M(X)$ by (5). By Lemma 3(*ii*), P_{ϕ_B} is equicontinuous on $L_\beta(X)$, as is its subset B. Therefore the polar $B^\bullet$ is a 0-neighborhood in $L_\beta(X)$, making $A^\circ = B^\bullet \cap L_X(Y)$ a 0-neighborhood in the subspace $L_X(Y)$. Consequently, $L_X(Y)$ has the slctop. □

When $Y = X$, then $L_X(Y) = L_\beta(X)$, and we have: *The condition* (2) *that $L_\beta(X)$ be barrelled, i.e., that $C_p(X)$ be distinguished, equals the condition that $L_\beta(X)$ bear the slctop; be primitive; be flat* [$M(X) = L_\beta(X)' = L_\beta(X)^* = \mathbb{R}^X$], which are fundamental results given in [9, 12, 15, 16]. Another such example: *If $|X| = \aleph_0$, then $C_p(X)$ is distinguished*, because, for $f \in \mathbb{R}^X$ with $|X| = \aleph_0$, one easily finds a bounded sequence $\{f_n : n \in \mathbb{N}\} \subseteq C_p(X)$ converging in $\mathbb{R}^X$ to f, so that $f \in M(X) = \mathbb{R}^X$.

Also immediate, when $Y = X$, is a Kąkol–Leiderman solution [18] to Problem 1 of [15], independently given in variant form by Ferrando–Saxon [12]:

Corollary 6 *$C_p(X)$ is distinguished if and only if, for each sequence $X_1, X_2, \ldots$ of pairwise disjoint subsets of X, there exists a sequence $Q_1, Q_2, \ldots$ of open sets in X such that Q_n contains X_n and, for each $x \in X$, the set $\{n \in \mathbb{N} : x \in Q_n\}$ is finite.*

If $Y \subseteq X$ satisfies the equivalent conditions of Theorem 5, then Corollary 6 readily shows that $C_p(Y)$ is distinguished. Indeed, we readily obtain two facts, one well-known:

(A) *If* $Y \subseteq X$ *and* $C_p(X)$ *is distinguished, then* $C_p(Y)$ *is distinguished* [9, 12, 15], and
(B) *If* $L_X(Y)$ *has the slctop, then* $C_p(Y)$ *is distinguished.*

Could the converse of (B) hold, adding a seventh equivalent condition to (1) - (6)? The answer is *No*. The space $C_p(Y)$ is distinguished for $|Y| = \aleph_0$. Yet $L_X(Y)$ fails to have the slctop for some such $Y \subseteq X$ precisely when $\Sigma(X) \nsubseteq M(X)$ (Theorem 16). Examples include $X = \mathbb{M}$ (Corollary 29), $X = \mathbb{S}$ (the Sorgenfrey line [12, 15]) and many others (Theorem 20).

In Sect. 5 we find a useful quasi-converse of (B). Here we extend Theorem 5:

Theorem 7 *If* G *is a complemented subspace of* $L_\beta(X)$*, then statements* (1)–(4) *of Theorem 5 remain equivalent when* G *replaces* $L_X(Y)$.

Proof For $n = 1, 2, 3, 4$ let (n') denote statement (n) with G replacing $L_X(Y)$. Clearly, as before, $(1') \Rightarrow (2') \Rightarrow (3') \Rightarrow (4')$. By hypothesis, $L_\beta(X) = F \oplus G$ for some subspace F. It is (easily) known in general that *if* $G \approx E/F$ *is flat, then every algebraic complement* H *of* F *is flat* (see [25, 10.8.2]). Indeed, let $h \in E^*$ be arbitrary on H and vanish on F. Then there exists $h' \in (E/F)^* = (E/F)'$ with $h = h' \circ q \in E'$, where q is the quotient map. Thus the arbitrary point $h|_H$ in H^* is also in H'; i.e., $H^* = H'$, as desired. In our particular case, $[G \approx L_\beta(X)/F$ *is flat*$] \Rightarrow [L_X(Y)$ *is flat*$]$, where we choose Y to be a maximally large subset of X such that $L_X(Y)$ is transverse to F. Thus $(4') \Rightarrow (4)$ for this Y. By Theorem 5 we have $(4) \Rightarrow (1)$. Finally, $(1) \Rightarrow (1')$ via Lemma 1. □

Theorem 5 has a convenient countable version:

Theorem 8 *Let* Y *consist of distinct points* $x_1, x_2, \ldots$ *in* X*. The following six assertions are equivalent.*

1. *The subspace* $L_X(Y)$ *is flat.*
2. *The subspace* $L_X(Y)$ *is a copy of* φ.
3. *The subspace* $L_X(Y)$ *is a complemented copy of* φ.
4. *The quotient space* $L_\beta(X)/L_X(X \backslash Y)$ *is a copy of* φ.
5. *The dual space* $M(X)$ *contains all functions in* $\mathbb{R}^X$ *which vanish off* Y.
6. *There exists a sequence* $Q_1, Q_2, \ldots$ *of open sets in* X *such that* $x_n \in Q_n$ *for each* $n \in \mathbb{N}$ *and, for every* $x \in X$*, the set* $\{n \in \mathbb{N} : x \in Q_n\}$ *is finite.*

Proof The equivalence of (1)–(5) follows from Theorems 4 and 5.

$[(5) \Rightarrow (6)]$. Apply $[(5) \Rightarrow (6)]$ of Theorem 5 with each $Y_n = \{x_n\}$.

$[(6) \Rightarrow (5)]$. Let h be a function in $\mathbb{R}^X$ which vanishes off Y, and let $Q_1, Q_2, \ldots$ be as in (6). For each $n \in \mathbb{N}$, choose $g_n \in C(X)$ vanishing off Q_n with $g_n(x_n) = h(x_n)$. For each $x \in X$, we have $g_n(x) = 0$ for almost all $n \in \mathbb{N}$. Therefore $B := \{g_n : n \in \mathbb{N}\}$ is bounded in $C_p(X)$. Comparing values both on and off Y, we see that $|h| \leq \phi_B$; i.e., $h \in P_{\phi_B}$. Lemma 3(i) says $P_{\phi_B} \subseteq \overline{B^+}^{\mathbb{R}^X}$. And $\overline{B^+}^{\mathbb{R}^X} \subseteq M(X)$ since B^+ is bounded in $C_p(X)$. Thus $h \in M(X)$, which proves (5) holds. □

Corollary 9 *If* $Y = \{x_n : n \in \mathbb{N}\}$ *is a relatively discrete set of distinct points in* X*, its span* $L_X(Y)$ *is a copy of* φ *complemented in* $L_\beta(X)$.

Proof There are open sets U_n in X such that each $U_n \cap Y = \{x_n\}$. Being regular, X admits open sets V_n with $x_n \in V_n$ and $\overline{V}_n \subseteq U_n$. If $Q_1 = V_1$ and $Q_{k+1} = V_{k+1} \backslash \bigcup_{j \leq k} \overline{V}_j$ for $k \geq 1$, the open disjoint sets Q_n verify part (6) of Theorem 8, and thus part (3). □

Theorem 10 *Each infinite linearly independent set S in $L_\beta(X)$ has a subset S_0 whose span* $\mathrm{sp}(S_0)$ *is a complemented copy of φ.*

Proof Zorn's lemma produces a unique subset Z of the Hamel basis X whose span $L_X(Z)$ algebraically complements $\mathrm{sp}(S)$ in $L(X)$. Since $|X\backslash Z| = |S|$ is infinite and X is regular, routine induction yields a denumerable relatively discrete set $Y \subseteq X\backslash Z$. By splitting Theorem 4 and Corollary 9,

$$L_\beta(X) = L_X(Y) \oplus L_X(X\backslash Y) \quad \text{and} \quad L_X(Y) \text{ is a copy of } \varphi.$$

Moreover, $[Y \subseteq X\backslash Z]$ ensures $[Z \subseteq X\backslash Y]$. Therefore $L(X) = L_X(Z) + \mathrm{sp}(S) = L_X(X\backslash Y) + \mathrm{sp}(S)$. Again, elementary algebra produces a unique subset S_0 of the linearly independent set S whose span $\mathrm{sp}(S_0)$ is an algebraic complement to the closed $L_X(X\backslash Y)$. So $|S_0| = |Y| = \aleph_0$, and by Lemma 1, $\mathrm{sp}(S_0)$ is a topological complement to $L_X(X\backslash Y)$ and has its slctop; i.e., $\mathrm{sp}(S_0)$ is a complemented copy of φ. □

Does *every* linearly independent sequence in $L_\beta(X)$ span a (complemented) copy of φ? No, not always (*e.g.*, not when $X = \mathbb{M}$, Michael's line), but yes, precisely when $\Sigma(X) \subseteq M(X)$. Such questions anticipate Theorem 16, an expansion of Theorem 8.

4 Ferality, $\Sigma(X) \subseteq M(X)$, bounded away from zero

A result/comment from [13, 24] solves [23, Problem 1]. Our brief organic proof translates weak (quasi)barrelled spaces into feral duals. Defining *feral* is key.

Theorem 11 *A locally convex space E has its weak topology $\sigma(E, E')$ and is* [*quasibarrelled*] ⟨*barrelled*⟩ *if and only if the* [*strong dual* $(E', \beta(E', E))$] ⟨*weak* dual* $(E', \sigma(E', E))$⟩ *is feral.*

Proof (new) E has its weak topology and is [quasibarrelled] ⟨ barrelled⟩ ⇔ [each bornivorous barrel] ⟨each barrel⟩ contains a finite-codimensional subspace (see [39, Lemma 1.2]) ⇔ [each $\beta(E', E)$-bounded set] ⟨ each $\sigma(E', E)$ -bounded set⟩ is finite-dimensional ⇔ [the strong dual] ⟨the weak* dual⟩ is feral. □

Important known facts easily follow: Since φ is feral [37, II, Ex. 7(b)], so is $L_\beta(X)$ by Theorem 10, which means $C_p(X)$ is quasibarrelled by Theorem 11.

Corollary 12 $L_\beta(X)$ *is feral* [16, Corollary 4] *and* $C_p(X)$ *is quasibarrelled* [4, 17].

Credit for Theorem 11. Our Theorem 11 has already appeared, in [24], in [23], and in a circulated preprint. On page 498 of [24], Saxon (a) defined feral spaces, which oppose docile spaces and determine (quasi)barrelled enlargements, and (b) gave Theorem 11, with proof (see formal **Proof** on p. 498 and the sentence thereafter). Now, nineteen years later, Kąkol/Śliwa [23] have reproduced the quasibarrelled part, adding two more (known) equivalent conditions and a much longer proof. More than two years earlier, on Jan. 2, 2021, Saxon emailed Ferrando, Kąkol, Leiderman and others a preprint that included the quasibarrelled version of Theorem 11. The half-century-old claims of Gulick and Buchwalter/Schmets to Corollary 12 are iron-clad. But Theorem 11, we believe, belongs to Saxon.

Clearly, $C_p(X)$ is *non*feral. Hence $C_p(X)$ admits quasibarrelled countable enlargements [39, Corollary 3.4]. The (quasi)barrelled enlargement issues are moot for $L_\beta(X)$: either the dual $L_\beta(X)' = \mathbb{R}^X = L_\beta(X)^*$ cannot be enlarged at all, or $L_\beta(X)$ is not even quasibarrelled [9, Corollaries 3.2, 3.4].

We know $L_\beta(X)$ is distinguished with Baire-like strong dual $M(X)$ [9, 12]. Moreover, and more generally,

Theorem 13 *Each feral space E is semi-reflexive with Baire-like strong dual E'.*

Proof Ferality means $\left(E', \sigma\left(E', E\right)\right) = \left(E', \beta\left(E', E\right)\right)$. The dual of the latter is the bidual E'' of E, and the dual of the former is E. Therefore $E'' = E$, proving E is semi-reflexive. Thus E' is barrelled [25, 23.3.4] under its only admissible topology $\sigma\left(E', E\right)$. Now the 0 -neighborhoods of E', unlike those of φ, always contain a finite-codimensional subspace. Hence the barrelled space E' cannot contain φ, and must be Baire-like [31]. □

Corollary 14 *Always, $L_\beta(X)$ is semi-reflexive with Baire-like strong dual $M(X)$.*

If $Y \subseteq X$, Arkhangel'skiĭ [2] denotes by $C_p(Y|X)$ the subspace of $C_p(Y)$ comprised of functions on Y having extensions belonging to $C(X)$. In a companion paper [11] we prove $C_p(Y|X)$ is a large subspace of $C_p(Y)$, so both spaces have the same (feral) strong dual $L_\beta(Y)$, and thus $C_p(Y|X)$ is always quasibarrelled. The weak* dual of $C_p(Y|X)$ is dominated by the weak* dual $L_p(Y)$ of $C_p(Y)$. If the coarser topology is feral, so is the finer. Hence $C_p(Y)$ *is barrelled if* $C_p(Y|X)$ *is*, by Theorem 11, or by the simple fact that $C_p(Y|X)$ is dense in $C_p(Y)$. Similarly, since $L_p(Y)$ dominates a subspace of $L_p(X)$, the former is feral if the latter is, and thus $C_p(Y)$ *is barrelled if* $C_p(X)$ *is* (well-known, or use Theorem 11). The question arises: *If $C_p(Y)$ is barrelled, must $C_p(Y|X)$ be, also?*

The answer is *No*, even when Y is assumed to be dense in X. For a counterexample, let X be the Michael line $\mathbb{M}$ and Y its dense subset $\mathbb{P}$ of irrationals, and argue as follows:

Example 15 Recall the Michael line $\mathbb{M}$. It is the Tychonoff space $(\mathbb{R}, \tau)$, where τ is the coarsest topology on the real line $\mathbb{R}$ that (a) is finer than the usual topology on $\mathbb{R}$ and (b) induces the discrete topology on the irrationals $\mathbb{P}$. Now $\mathbb{M}$ induces the usual topology on the rationals $\mathbb{Q}$. Furthermore, $C_p(\mathbb{Q})$ and $C_p(\mathbb{P})$ *are distinguished*, because $\mathbb{Q}$ is countable and $\mathbb{P}$ is discrete [9, 15]. Yet $C_p(\mathbb{M})$ *is not distinguished*. In fact, $\Sigma(\mathbb{M}) \not\subseteq M(\mathbb{M})$ [12, 15].

Every τ-neighborhood of a rational q contains an open interval about q, thus meets $\mathbb{P}$, proving $\mathbb{P}$ *is dense in* $\mathbb{M}$. And since $\mathbb{P}$ is discrete, we have $C_p(\mathbb{P}) = \mathbb{R}^{\mathbb{P}}$ *is barrelled.* But *not the large subspace* $C_p(\mathbb{P}|\mathbb{M})$. Indeed, let S be a sequence in $\mathbb{P}$ convergent (in both $\mathbb{R}$ and $\mathbb{M}$) to some $q \in \mathbb{Q}$; e.g. , take $S = \left\{\sqrt{2}/n : n \in \mathbb{N}\right\}$, $q = 0$. Clearly, S serves as an infinite, linearly independent, bounded set in the weak* dual of $C_p(\mathbb{P}|\mathbb{M})$, so *said dual is not feral.* Since $C_p(\mathbb{P}|\mathbb{M})$ has its weak topology, $C_p(\mathbb{P}|\mathbb{M})$ *is not barrelled,* by Theorem 11. ◊

If $\{x_n\}_n$ is a bounded sequence in E, the null sequence $\{x_n/n\}_n$ cannot be bounded away from zero. Hence Sect. 2 observes (f) ⇒ (g). Prof. Jerzy Kąkol asked if the converse holds for $E = L_\beta(X)$; *i.e.*, must linearly independent sequences in the ever-feral $L_\beta(X)$ be bounded away from zero? Our next theorem answers *No*, since it is known that $\Sigma(\mathbb{M}) \not\subseteq M(\mathbb{M})$ [12, 15].

Theorem 16 *The following six assertions are equivalent.*

1. $\Sigma(X) \subseteq M(X)$.
2. *Every linearly independent sequence in $L_\beta(X)$ is bounded away from zero.*
3. *Every linearly independent sequence in $L_\beta(X)$ spans a copy of φ.*
4. *Every linearly independent sequence in $L_\beta(X)$ spans a complemented copy of φ.*

5. *If* $|X\backslash Z| = \aleph_0$*, then the quotient* $L_\beta(X)/L_X(Z)$ *is a copy of* φ*.*
6. *For each sequence of distinct points* $x_1, x_2, \ldots$ *in* X *there are open neighborhoods* Q_n *of* x_n *such that, for each* x *in* X*, the set* $\{n \in \mathbb{N} : x \in Q_n\}$ *is finite.*

Proof [(1) $\Rightarrow$ (4)]. If $\{v_n : n \in \mathbb{N}\}$ is linearly independent in $L(X)$ with span G_1, there is a sequence of distinct points $x_1, x_2, \ldots$ in X whose span G_2 contains G_1. By (1), the space $M(X)$ contains all functions in $\mathbb{R}^X$ which vanish on $Z := X\backslash\{x_n : n \in \mathbb{N}\}$. Therefore Theorem 8 implies G_2 is a copy of φ complemented in $L_\beta(X)$. Thus G_1 is a copy of φ complemented in G_2. Hence G_1 is also complemented in $L_\beta(X)$; i.e., (4) holds.

[(4) $\Rightarrow$ (3)]. Trivially.

[(3) $\Rightarrow$ (2)]. Every linearly independent sequence in φ is bounded away from zero.

[(2) $\Rightarrow$ (1)]. Let $f \in \Sigma(X)$. There are distinct points $x_1, x_2, \ldots$ in X, off of which f vanishes. If $y_n = (1 + |f(x_n)|)^{-1} \cdot x_n$, hypothesis (2) provides a bounded set B in $C_p(X)$ whose polar B° in $L(X)$ misses $\{y_n : n \in \mathbb{N}\}$. Thus for each n, there is some $g_n \in B$ with $|g_n(y_n)| > 1$, which means $|g_n(x_n)| > 1 + |f(x_n)|$. Clearly, then, $\phi_B \geq |f|$. Hence $f \in P_{\phi_B} \subseteq \overline{B^+} \subseteq M(X)$ by Lemma 3(*i*), so that (1) holds.

[(1) $\Leftrightarrow$ (6)]. Immediate from [Theorem 8, (6) $\Leftrightarrow$ (5)].

We now have the equivalence of (1), (2), (3), (4), and (6).

[(3) $\Rightarrow$ (5)]. If $|X\backslash Z| = \aleph_0$, set $Y = X\backslash Z$. By Theorem 4 and hypothesis (3), $L_\beta(X) = L_X(Y) \oplus L_X(Z) \approx \varphi \oplus L_X(Z)$. Therefore $L_\beta(X)/L_X(Z) \approx \varphi$; i.e., (5) holds.

[(5) $\Rightarrow$ (3)]. Let F be a subspace of $L_\beta(X)$. Choose $Z \subseteq X$ such that $L_X(Z)$ algebraically complements F. If $\dim(F) = \aleph_0$, then $|X\backslash Z| = \aleph_0$ and $L_\beta(X)/L_X(Z) \approx \varphi$ by (5). Hence $F \approx \varphi$ by Lemma 1; i.e., (3) holds. □

Theorem 8 or 16 and Corollary 9 improve [12, Theorem 22 and Corollary 23]:

(C) $\Sigma(X) \subseteq M(X)$ *if countable sets in* X *are relatively discrete; thus, if* X *is a* P*-space.*

Also, from [(1) $\Leftrightarrow$ (6)] of Theorem 16, we observe

(D) $\Sigma(Y) \subseteq M(Y)$ *if* $Y \subseteq X$ *and* $\Sigma(X) \subseteq M(X)$*.*

Corollary 17 *If* ω_1 *is the space of countable ordinals, then* $\Sigma(\omega_1) \subseteq M(\omega_1)$ [12].

Proof If $0 \leq \alpha_1 < \alpha_2 < \alpha_3 < \cdots$ is a sequence in ω_1, let $\beta = \sup\{\alpha_n : n \in \mathbb{N}\}$. With $Q_1 := [0, \beta)$ and $Q_n := (\alpha_{n-1}, \beta)$ for $n = 2, 3, \ldots$, apply [(6) $\Rightarrow$ (1)] of Theorem 16. □

A subset of ω_1 is a *stationary* set if it meets every **cl**osed **unb**ounded set (every *club set*) in ω_1. Infinite families of pairwise disjoint stationary sets exist [5, 4.3.2]. We greatly simplify the original Kąkol-Leiderman proof of

Theorem 18 *The space* $C_p(\omega_1)$ *is not distinguished* [18].

Proof Let $S_1, S_2, \ldots$ be a sequence of pairwise disjoint stationary sets in ω_1. Let Q_n be an open set containing S_n. The closed set $\omega_1\backslash Q_n$ must be bounded by some $\delta_n \in \omega_1$, since otherwise it is a club set which misses the stationary set S_n $(\subseteq Q_n)$, an impossibility. Therefore $(\delta_n, \omega_1) \subseteq Q_n$. Hence the successor δ of $\sup\{\delta_n : n \in \mathbb{N}\}$ is a point of ω_1 in Q_n for infinitely many (indeed, for all) $n \in \mathbb{N}$. Clearly, then, $X = \omega_1$ does not satisfy the condition of Corollary 6. Equivalently, $C_p(\omega_1)$ is not distinguished.

Corollary 19 *An ordinal* α *is countable if and only if* $C_p(\alpha)$ *is distinguished.*

For $L_\beta(X)$ spaces, *barrelled* $\Leftrightarrow$ *φ-complemental* $\Leftrightarrow$ *primitive* ([12], or invoke Theorem 5 with $Y = X$). In (5) of Theorem 16, divisors are characteristically (i) *closed*, (ii) $\aleph_0$-*codimensional*, and (iii) *of the form* $L_X(Z)$. We could omit (i), since (iii) $\Rightarrow$ (i). We cannot omit (iii), delimiting divisors by (i) and (ii) only. Because thus altered, (5) would simply mean, via Lemma 1, that $L_\beta(X)$ *is φ-complemental*, and $X = \omega_1$ would contradict [(1) $\Rightarrow$ (5)]. So, strangely enough, $L_\beta(\omega_1)$ is not φ-complemental, yet every $\aleph_0$-dimensional subspace of $L_\beta(\omega_1)$ is a complemented copy of φ (Corollary 17, Theorems 16, 18).

Theorem 18 is the Kąkol-Leiderman solution of [15, Problem 64]. Profs. Mikolaj Krupski and Roman Pol solve [12, Problem 24], giving a P-space X with $C_p(X)$ nondistinguished [29]. For all such X, the feral space $L_\beta(X)$ is not flat [12, 15], yet bounds every linearly independent sequence away from zero (Theorem 16 and (C)).

More abstractly, one could start with a linear space E having uncountable Hamel basis B, let E' denote the $\aleph_0$-dual with respect to B [1], and impose any locally convex topology on E compatible with the dual pair $\langle E, E'\rangle$. The resulting feral, non-flat space E clearly bounds linearly independent sequences away from zero. Being primitive, E is not a subspace of any $L_\beta(X)$ space (Theorem 2).

If, on the other hand, we desire natural examples of feral spaces that admit linearly independent sequences *not* bounded away from zero, we may choose spaces $L_\beta(X)$ precisely when $\Sigma(X) \nsubseteq M(X)$, again by Theorem 16. There are many such choices besides $L_\beta(\mathbb{M})$ and $L_\beta(\mathbb{S})$:

Theorem 20 *Suppose X is either (a) locally compact or (b) a complete metric space. If X contains a separable set S having no relative isolated points, then $\Sigma(X) \nsubseteq M(X)$.*

Proof I. Assume that $S = X$ is separable with no isolated points.

Fix a set Y of distinct points $x_1, x_2, \dots$ dense in X. By [(1) $\Rightarrow$ (3)] of Theorem 16 it suffices to show that $L_X(Y)$ is not a copy of φ; i.e., does not have the slctop. Suppose, by way of contradiction, that $L_X(Y)$ is a copy of φ. Theorem 8 yields open sets $Q_1, Q_2, \dots$ such that $x_n \in Q_n$ and each $x \in X$ is in Q_n for at most finitely many $n \in \mathbb{N}$.

Case (a): *X is locally compact*. Put $n_1 = 2$. There is a compact neighborhood K_1 of x_{n_1} with $K_1 \subseteq Q_{n_1}$. Since x_{n_1} is not isolated and X is Hausdorff, there exists $n_2 > n_1$ with x_{n_2} in the interior $\mathring{K}_1$ of K_1. Local compactness provides a compact neighborhood K_2 of x_{n_2} such that $K_2 \subseteq \mathring{K}_1 \bigcap Q_{n_2}$. We continue inductively to find a sequence $n_1 < n_2 < \cdots$ of integers and a sequence $K_1 \supseteq K_2 \supseteq \cdots$ of compact sets such that

$$x_{n_j} \in K_j \subseteq Q_{n_j} \text{ for } j = 1, 2, \dots .$$

The finite intersection principle ensures there is some $x_0 \in \bigcap_{j=1}^{\infty} K_j$. Hence $x_0 \in \bigcap_{j=1}^{\infty} Q_{n_j}$. But this contradicts the fact that x_0 can be in Q_n for only finitely many values of n. We must conclude that $\Sigma(X) \nsubseteq M(X)$.

Case (b): *X is a complete metric space*. We repeat the induction argument of Case (a), replacing each K_j with a ball B_j centered at x_{n_j} of positive radius $r_j < 1/j$. Since the metric space X is complete and the radii tend to 0, the nested sequence $B_1 \supseteq B_2 \supseteq \cdots$ intersects at a single point x_0 (elementary). But then we have $x_0 \in B_j \subseteq Q_{n_j}$ for all $j \in \mathbb{N}$, a contradiction as in Case (a).

Therefore the theorem holds when $S = X$.

II. In the general case, since S is separable and has no relative isolated points, the same is true of $\overline{S}$, and furthermore, $\overline{S}$ is either locally compact or a complete metric space. Hence case I implies $\Sigma\left(\overline{S}\right) \nsubseteq M\left(\overline{S}\right)$. Then $\Sigma(X) \nsubseteq M(X)$ by fact (D) above. □

Corollary 21 *If X is separable, without isolated points, and either locally compact or complete metric, then $C_p(X)$ is not distinguished. Moreover, some linearly independent sequences in the feral space $L_\beta(X)$ are not bounded away from zero. Examples: $X = \mathbb{R}, \mathbb{S}$,* [0, 1]*, Cantor set, Helley space, $\beta\mathbb{Q}$, $\beta\mathbb{R}$ (see* [14] *and* [12, Theorem 5]*).*

5 Sufficient conditions on $Y \subseteq X$

Hahn–Banach enables the splitting Theorem 4, which enables

Theorem 22 *Assume $Y_1, \ldots, Y_n \subseteq X$ and each $L_X(Y_i)$ has the slctop. Then $L_X\left(\bigcup_{i=1}^n Y_i\right)$ also has the slctop, and* (B) *proves $C_p\left(\bigcup_{i=1}^n Y_i\right)$ is distinguished. Thus $C_p(X)$ is distinguished if $\bigcup_{i=1}^n Y_i = X$.*

Proof It suffices to prove $L_X(Y_1 \cup Y_2)$ has the slctop. Since $L_X(Y_i)$ has the slctop, so do subspaces $L_X(Y_i \backslash Y_{3-i})$ and $L_X(Y_1 \cap Y_2)$ $(i = 1, 2)$. The splitting Theorem 4 yields

$$L_\beta(X) = L_X(Y_1 \backslash Y_2) \oplus L_X(Y_1 \cap Y_2) \oplus L_X(Y_2 \backslash Y_1) \oplus L_X(X \backslash (Y_1 \cup Y_2)).$$

As the direct sum of the first three summands, $L_X(Y_1 \cup Y_2)$ also has the slctop. □

This abbreviates proof of

Corollary 23 [15, Theorem 3.16(3)] *Assume $X = \bigcup_{i=1}^n Y_i$ with each $C_p(Y_i)$ distinguished. If for each bounded set D in $C_p(Y_i)$ there is a bounded set B in $C_p(X)$ with $B|_{Y_i} = D$ (*e.g., *if each Y_i is l-embedded in X), then $C_p(X)$ is distinguished.*

Proof For bounded D in $C_p(Y_i)$ or bounded B in $C_p(X)$, given either one, we choose the other such that $B|_{Y_i} = D$. Thus $L_X(Y_i) = L_\beta(Y_i)$ since, in both spaces,

$$B^\circ \bigcap L(Y_i) = D^\bullet$$

is a basic 0-neighborhood, where B° is the polar in $L(X)$ and $D^\bullet$ is the polar in $L(Y_i)$. As each $C_p(Y_i)$ is distinguished, each $L_\beta(Y_i) = L_X(Y_i)$ has the slctop by [9, Corollary 3.4]. Hence $C_p(X)$ is distinguished by Theorem 22. □

A simpler version of Theorem 22 is

Theorem 24 *$C_p(X)$ is distinguished if both $L_X(Y)$ and $L_X(X \backslash Y)$ have the slctop for some $Y \subseteq X$.*

In light of (B), the hypothesis assumes both $C_p(Y)$ and $C_p(X \backslash Y)$ are distinguished. But distinguished $C_p(Y)$ and $C_p(X \backslash Y)$ alone may not suffice for $C_p(X)$, as demonstrated by $X = \mathbb{M}$ and $Y = \mathbb{Q}$ in example 15. What additional condition(s) on Y will ensure $C_p(X)$ is distinguished? Prior to our splitting theorem, a handful of answers appeared [12, 15, 18, 19].

The beautifully simple Kąkol-Leiderman answer [18, Proposition 2.3] clearly improves [15, Theorem 57, Corollary 58] and [12, (‡)]. It says *$C_p(X)$ is distinguished if $C_p(Y)$ is distinguished for some $Y \subseteq X$ with $|X \backslash Y| < \aleph_0$*. A second K-L answer is [18, Theorem 3.8]: *$C_p(X)$ is distinguished if some Y is an open F_σ subset of X and both $C_p(Y)$ and $C_p(X \backslash Y)$ are distinguished.*

Answers may well involve $L_X(Y)$. Evidently, $L_X(Y) \preceq L_\beta(Y)$ $(= L_\beta(Y|X)$ [11]). Thus $L_X(Y) = L_\beta(Y)$ precisely when $L_X(Y) \succeq L_\beta(Y)$, i.e., when $L_X(Y)$ has topology at least as fine as that of $L_\beta(Y)$.

Theorem 25 *Assume $Y \subseteq X$. The following three assertions are equivalent.*

1. $L_X(Y) = L_\beta(Y)$.
2. *Each bounded $S \subseteq C_p(Y)$ admits a bounded $T \subseteq C_p(X)$ with $S \subseteq \overline{T|_Y}^{C_p(Y)}$.*
3. *Each bounded $S \subseteq C_p(Y)$ admits a bounded $T \subseteq C_p(X)$ with $S \subseteq \overline{T|_Y}^{\mathbb{R}^Y}$.*

Proof If V is a set in either $C_p(X)$ or $L(X)$, we let V° denote the polar of V with respect to the dual pair $\langle C_p(X), L(X)\rangle$. If W is a set in either $C_p(Y)$ or $L(Y)$, let $W^\bullet$ denote the polar of W with respect to the dual pair $\langle C_p(Y), L(Y)\rangle$.

[(1) ⇒ (2)]. Given bounded $S \subseteq C_p(Y)$, the polar $S^\bullet$ is a 0 -neighborhood in $L_X(Y)$ by (1). The relative topology provides a 0-neighborhood U in $L_\beta(X)$ with $U \cap L(Y) \subseteq S^\bullet$. As $L_\beta(X)$ is the strong dual of $C_p(X)$, there is a bounded $T \subseteq C_p(X)$ with polar $T^\circ \subseteq U$. Hence

$$T^\circ \cap L(Y) \subseteq S^\bullet.$$

Clearly, $T^\circ \cap L(Y) = (T|_Y)^\bullet$, so the bipolar theorem yields

$$S \subseteq S^{\bullet\bullet} \subseteq \left(T^\circ \cap L(Y)\right)^\bullet = (T|_Y)^{\bullet\bullet} = \overline{\text{acx}}\,(T|_Y),$$

where $\overline{\text{acx}}\,(T|_Y)$ is the closure in $C_p(Y)$ of the absolutely convex hull $\text{acx}\,(T|_Y)$ of $T|_Y$. Without loss of generality, we may assume from the beginning that T is absolutely convex. Then so is $T|_Y$; *i.e.*, $T|_Y = \text{acx}\,(T|_Y)$, which means $\overline{\text{acx}}\,(T|_Y) = \overline{T|_Y}^{C_p(Y)}$. Therefore (2) follows.

[(2) ⇔ (3)]. If $U \subseteq C(Y)$ then $\overline{U}^{C_p(Y)} = \overline{U}^{\mathbb{R}^Y} \cap C(Y)$.

[(2) ⇒ (1)]. The polar $S^\bullet$ of an arbitrary bounded set $S \subseteq C_p(Y)$ is a basic 0 -neighborhood in $L_\beta(Y)$. From (2) there exists bounded $T \subseteq C_p(X)$ with $S \subseteq \overline{T|_Y}^{C_p(Y)} \subseteq (T|_Y)^{\bullet\bullet}$. Therefore

$$S^\bullet \supseteq (T|_Y)^{\bullet\bullet\bullet} = (T|_Y)^\bullet = T^\circ \cap L(Y).$$

Now T° is a 0-neighborhood in $L_\beta(X)$, so $T^\circ \cap L(Y)$ is a 0-neighborhood in the subspace $L_X(Y)$, and lies inside $S^\bullet$. Hence $L_X(Y) \succeq L_\beta(Y)$; assertion (1) holds. □

Very short proof of Corollary 23 Theorem 25 and [9, Corollary 3.4] imply each $L_X(Y_i) = L_\beta(Y_i)$ has the slctop, so Theorem 22 applies. □

Theorem 26 *If Y is a G_δ set in X, then $L_X(Y) = L_\beta(Y)$.*

Proof We assume $Y = \bigcap_{i=1}^\infty V_i$, with each V_i open in X and $V_{i+1} \subseteq V_i$. Let a bounded set S in $C_p(Y)$ be given. The fact [11] that $C_p(Y|X)$ is a large subspace of $C_p(Y)$ means there is some $B \subseteq C(X)$ such that $B|_Y$ is bounded in $C_p(Y)$ and $S \subseteq \overline{B|_Y}^{C_p(Y)}$. For each $m \in \mathbb{N}$ define

$$A_m = \{u \in C(X) : u \text{ vanishes off } V_m \text{ and } |u|_Y| \leq \mathbf{m}, |f|_Y| \text{ for some } f \in B\}$$

and let $T = \bigcup_{m=1}^\infty A_m$. If $x \in Y$ then $\sup_{u\in T} |u(x)| \leq \sup_{f\in B} |f(x)| < \infty$. If $x \in X\backslash V_n$ for some $n \in \mathbb{N}$, then $u(x) = 0$ for all $u \in \bigcup_{m=n+1}^\infty A_m$, and for all $u \in \bigcup_{m=1}^n A_m$ we have $|u(x)| \leq n$; thus $\sup_{u\in T} |u(x)| \leq n < \infty$. This proves T is bounded at all points $x \in X$; i.e., T is bounded in $C_p(X)$.

Let us be given $v \in S$, finite $\sigma \subseteq Y$, and $\varepsilon > 0$. Since $S \subseteq \overline{B|_Y}^{\mathbb{R}^Y}$, there is some $f \in B$ such that

$$|f(x) - v(x)| < \varepsilon \ \text{ for each } x \in \sigma.$$

Fix $m \in \mathbb{N}$ with $m \geq \max_{x\in\sigma} |f(x)|$. As m is suitably large, we easily find $u \in C(X)$ such that $u(x) = f(x)$ for each $x \in \sigma$ and $|u(x)| = \min\{m, |f(x)|\}$ for every $x \in X$. Hence $u \in A_m \subseteq T$ and $u|_Y$ is in the basic neighborhood of v in $\mathbb{R}^Y$ determined by σ and ε. Therefore each v in S is also in $\overline{T|_Y}^{\mathbb{R}^Y}$; i.e., assertion (3) of Theorem 25 holds, as must assertion (1): $L_X(Y) = L_\beta(Y)$. □

Theorem 26 and [9, Corollary 3.4] prove the promised quasi-converse of (B):

Theorem 27 *If Y is a G_δ set in X and $C_p(Y)$ is distinguished, then $L_X(Y)$ has the slctop.*

Corollary 28 *If $|X\backslash Y| \leq \aleph_0$ and $C_p(Y)$ is distinguished, then $L_X(Y)$ has the slctop.*

Proof The inequality assures Y is a G_δ set. □

This and Theorem 24 imply the first K-L answer, since $L_X(X\backslash Y)$ has the slctop if its dimension $|X\backslash Y|$ is finite. However, unlike co-finite Y, a co-countable Y with $C_p(Y)$ distinguished does not mean $L_X(X\backslash Y)$ has the slctop, as seen here:

Corollary 29 *The subspace $L_{\mathbb{M}}(\mathbb{P})$ of $L_\beta(\mathbb{M})$ has the slctop, $L_{\mathbb{M}}(\mathbb{Q})$ does not.*

Proof The first part is from Example 15 and Corollary 28. The second part follows from the first, Theorem 24, and the fact that $C_p(\mathbb{M})$ is not distinguished. □

Theorem 30 *If Y is both a G_δ and an F_σ set in X, and both $C_p(Y)$ and $C_p(X\backslash Y)$ are distinguished, then $C_p(X)$ is distinguished.*

Proof Apply Theorem 27 twice, then Theorem 24. □

An open F_σ set is also a G_δ set; K-L's second answer [18, Theorem 3.8] follows. Similar results in [12] are substantially improved by (C) above and by Theorem 32 and Corollary 33 below, which dismiss *C-embedded* from all of [12].

Corollary 31 *If Y is a co-countable F_σ set in X and $C_p(Y)$ is distinguished, then $C_p(X)$ is also distinguished.*

With G_δ in place of F_σ, the Corollary fails: take $X = \mathbb{M}$, $Y = \mathbb{P}$.

Theorem 32 *The following three assertions are equivalent.*

1. *$C_p(X)$ is distinguished.*
2. *$C_p(Y)$ is distinguished for every $Y \subseteq X$ with $|X\backslash Y| = \aleph_0$.*
3. *(i) $C_p(Y)$ is distinguished for some $Y \subseteq X$ with $|X\backslash Y| = \aleph_0$, and*
   (ii) $\Sigma(X) \subseteq M(X)$.

Proof We already know (1) ⇒ (2), (3) [15].

[(2) ⇒ (1)]. Theorem 10 provides $Z \subseteq X$ with $|Z| = \aleph_0$ and $L_X(Z) \approx \varphi$, and hypothesis (2) says $C_p(Y)$ is distinguished for $Y = X\backslash Z$. By Corollary 28, $L_X(Y)$ has the slctop. Theorem 24 applies.

[(3) ⇒ (1)] We infer, again from (*i*) and Corollary 28, that $L_X(Y)$ has the slctop. And (*ii*) implies $L_X(X\backslash Y) \approx \varphi$ via Theorem 16 so that, just as before, (1) follows from Theorem 24. □

Example 15 showed we cannot omit part (*ii*) in (3). Nor can we omit part (*i*), as proved by Corollary 17 and Theorem 18. This negatively solves [12, Problem 17].

However, one may replace part (*i*) with (i') *Every closed* $\aleph_0$*-codimensional subspace is complemented in* $L_\beta(X)$. Indeed, $[(i') \wedge (ii)] \Leftrightarrow [L_\beta(X)$ *is* φ*-complemental*$] \Leftrightarrow (1)$.

In a P-space, countable intersections of open sets are open, making countable sets relatively discrete. If X is a P-space, then $\Sigma(X) \subseteq M(X)$ by (C) above. Therefore

Corollary 33 *Let X be a P-space. The space $C_p(X)$ is distinguished if $C_p(Y)$ is distinguished for some $Y \subseteq X$ with $|X \backslash Y| = \aleph_0$.*

We extend Theorem 27 to finite unions.

Theorem 34 *For G_δ sets $Y_1, \ldots, Y_n$ in X, the following three assertions are equivalent.*

1. *$C_p(Y_i)$ is distinguished for $i = 1, \ldots, n$.*
2. *$C_p\left(\bigcup_{i=1}^n Y_i\right)$ is distinguished.*
3. *$L_X\left(\bigcup_{i=1}^n Y_i\right)$ has the slctop.*

Proof Our (B) and (A) yield $[(3) \Rightarrow (2) \Rightarrow (1)]$.

$[(1) \Rightarrow (3)]$. Partition $\bigcup_{i=1}^n Y_i$ into finitely many disjoint sets $Z_1, \ldots, Z_k$ such that, for each $j \le k$, there is some $i \le n$ with $Z_j \subseteq Y_i$. By (1) and Theorem 27, each $L_X(Y_i)$ has the slctop. Each $L_X(Z_j)$, being a subspace of some $L_X(Y_i)$, also has the slctop. Splitting Theorem 4 yields $L_X\left(\bigcup_{i=1}^n Y_i\right) = L_X\left(\bigcup_{j=1}^k Z_j\right) = L_X(Z_1) \oplus \cdots \oplus L_X(Z_k)$, which has the slctop since each direct summand does. □

The theorem does not extend to countable unions. Indeed, $\mathbb{M}$ is a countable union of G_δ sets Y_i $(i \in \mathbb{N})$ consisting of the discrete open set $\mathbb{P}$ and the singleton sets $\{q\}$ $(q \in \mathbb{Q})$, with each $C_p(Y_i)$ obviously distinguished. (The singletons are evidently G_δ sets, since $\mathbb{M}$ refines a metric topology.) Yet we know $C_p\left(\bigcup_{i=1}^\infty Y_i\right) = C_p(\mathbb{M})$ is not distinguished.

Nor must (2) imply (3) for F_σ (or even closed) sets $Y_1, \ldots, Y_n$; e.g., take $X = \mathbb{M}$ and $Y_1 = \cdots = Y_n = \mathbb{Q}$, so that (2) holds and not (3) by Corollary 29. Nevertheless, equivalence of (1) and (2) persists for countable unions of F_σ sets [19, Proposition 2.2], as an independent analysis will show. We begin with

Lemma 35 *$L_X\left(Y_1 \bigcap Y_2\right)$ has the slctop if $X = Y_1 \bigcup Y_2$ with each $C_p(Y_i)$ distinguished.*

Proof Let $\{Z_n : n \in \mathbb{N}\}$ be a sequence of disjoint sets in $Y_1 \bigcap Y_2$. Since $C_p(Y_i)$ is distinguished $(i = 1, 2)$, Corollary 6 provides a sequence $\left\{V_{i,n} : n \in \mathbb{N}\right\}$ of relatively open sets in Y_i such that each $V_{i,n}$ contains Z_n and each $x \in Y_i$ is in $V_{i,n}$ for at most finitely many $n \in \mathbb{N}$. Let $U_{i,n}$ be an open set in X such that $U_{i,n} \bigcap Y_i = V_{i,n}$. Finally, let $Q_n := U_{1,n} \bigcap U_{2,n} \supseteq Z_n$. As each $x \in X$ is in Y_1 or Y_2, each x is in the open set Q_n for at most finitely many n. Therefore condition (6) of Theorem 5 holds for $Y = Y_1 \bigcap Y_2$, and so does equivalent condition (1), our conclusion. □

Lemma 36 *For F_σ sets $Y_1, \ldots, Y_n$ in X with each $C_p(Y_i)$ distinguished, $C_p\left(\bigcup_{i=1}^n Y_i\right)$ is distinguished.*

Proof First, reduce to the case $X = Y_1 \bigcup Y_2$. Then each $Y_i \backslash Y_{3-i}$ is a G_δ set $(i = 1, 2)$, the complement in X of an F_σ set. By (A) above, each $C_p(Y_i \backslash Y_{3-i})$ is distinguished. Theorems 4, 27, and Lemma 35 prove

$$L_\beta(X) = L_\beta(Y_1 \cup Y_2) = L_X(Y_1 \backslash Y_2) \oplus L_X(Y_1 \cap Y_2) \oplus L_X(Y_2 \backslash Y_1)$$

has the slctop. Hence $C_p(X)$ is distinguished by [9, Corollary 3.4]. □

Theorem 30 and [18, Proposition 5.7] follow immediately from Lemma 36. In fact, this section unifies/amplifies predecessors [15, Theorems 16, 57, 60, and Corollaries 58, 61, 62], [12, Theorems 22, 25, (‡)], [18, Propositions 2.3, 5.7, and Theorem 3.8], and [19, Proposition 2.2]. A quick summary already yields

Theorem 37 *If $X = Y_1 \cup \cdots \cup Y_n$ and each $C_p\left(Y_j\right)$ is distinguished, then so is $C_p(X)$ provided either (i) all Y_j are G_δ sets, or (ii) all are F_σ sets. Hence, provided (iii) all Y_j are open, or (iv) all are closed.*

It is insufficient that, separately, each Y_j be either open or closed: each of $\mathbb{P}$ and $\mathbb{Q}$ is either open or closed in Michael's line $\mathbb{M} = \mathbb{P} \cup \mathbb{Q}$, yet $C_p(\mathbb{M})$ is not distinguished.

We observed earlier that $X = \mathbb{M}$ disproves the countable version of (i). The countable versions of (ii) and (iv) are clearly equivalent. Thus both are proved by Kąkol-Leiderman [19, Proposition 2.2]; see our proof, below. The countable version of (iii) remains an open problem first posed by Professors Leiderman and Tkachuk [26, Question 4.10]:

Problem 38 *If $X = \bigcup_{j=1}^{\infty} Y_j$, with each Y_j open in X and each $C_p\left(Y_j\right)$ distinguished, must $C_p(X)$ also be distinguished?*

Since each closed $\overline{Y}_k$ is a countable union $\bigcup_{j=1}^{\infty}\left(\overline{Y}_k \cap Y_j\right)$ of relatively open sets, Theorems 34 and 40 reduce the problem to Y_j that are increasing, open and dense in X.

Expanding notation, if $Y \subseteq X$ and $g \in \mathbb{R}^Y$, define $P_g = \left\{f \in \mathbb{R}^Y : |f| \leq |g|\right\}$. Trivially, $C_p(Y|X)$ is large in $\mathbb{R}^Y$ if and only if, for each bounded set B in $\mathbb{R}^Y$, there is some $g \in \mathbb{R}^Y$ with $|g|$ large enough so that $B \subseteq \overline{P_g \cap C(Y|X)}^{\mathbb{R}^Y}$.

Lemma 39 *Suppose $h \in \mathbb{R}^X$ and $Y \subseteq X$ with $C_p(Y)$ distinguished. There exists $g \in \mathbb{R}^Y$ such that each finite $\sigma \subseteq Y$ admits $f \in C(X)$ with $f|_\sigma = h|_\sigma$ and $|f|_Y| \leq |g|$. If Y is closed, the statement holds for each finite $\sigma \subseteq X$.*

Proof We know $C_p(Y|X)$ is always a large subspace of $C_p(Y)$, and $C_p(Y)$ *is distinguished (if and) only if* $C_p(Y|X)$ *is large in* $\mathbb{R}^Y$ [11, Theorem 10]. The latter yields $g \in \mathbb{R}^Y$ with

$$(|h| + \mathbf{1})\,|_Y \in \overline{P_g \cap C(Y|X)}^{\mathbb{R}^Y}.$$

Thus, given finite $\sigma \subseteq Y$, some $u \in P_g \cap C(Y|X)$ verifies $|u(y)| \geq |h(y)|$ for all $y \in \sigma$. By definition of $C(Y|X)$ there exists $\overline{u} \in C(X)$ with $\overline{u}|_Y = u$. The Tychonoff space X routinely admits $v \in P_{\overline{u}} \cap C(X)$ having the finitely many prescribed values

$$v(y) = \overline{u}(y) = u(y) \quad (y \in \sigma).$$

As $|v(y)| = |u(y)| \geq |h(y)|$, again X admits $f \in P_v \cap C(X)$ with prescribed values

$$f(y) = h(y) \quad (y \in \sigma).$$

But $f \in P_v$ and $v \in P_{\overline{u}}$, so $f \in P_{\overline{u}}$. Hence $f|_Y \in P_u$, and $u \in P_g$ ensures $f|_Y \in P_g$. Thus $|f|_Y| \leq |g|$, and we also have $f|_\sigma = h|_\sigma$, proving the initial result.

Suppose Y is closed. The initial result yields $g \in \mathbb{R}^Y$ such that, for each finite set $\sigma \subseteq X$, some function f' in $C(X)$ satisfies $\left|f'|_Y\right| \leq |g|$ and agrees with h on $\sigma' := \sigma \cap Y$. Let $\sigma'' = \sigma \backslash Y$. Since σ'' is finite in the open set $X \backslash Y$, some $f'' \in C(X)$ vanishes on Y and has the prescribed values $f''(x) = h(x) - f'(x)$ $(x \in \sigma'')$. Clearly, $f := f' + f'' \in C(X)$ with $f|_Y = f'|_Y \in P_g$, and f agrees with h on all of σ. □

Theorem 40 (KĄKOL- LEIDERMAN [19]) *If* $X = \bigcup_{n=1}^{\infty} Y_n$ *with each* Y_n *closed and each* $C_p(Y_n)$ *distinguished, then* $C_p(X)$ *is also distinguished.*

Proof Equivalently, we must have $M(X) = \mathbb{R}^X$, according to our recent [15, Theorem 14], *given fifty years ago* in Gulick's terminology [16, Theorem 9]. Since $M(X)$ always consists of points in the $\mathbb{R}^X$-closure of bounded sets in $C_p(X)$, we need only construct, for each $h \in \mathbb{R}^X$, a bounded set A in $C_p(X)$ with $h \in \overline{A}^{\mathbb{R}^X}$.

Let $h \in \mathbb{R}^X$ and finite $\sigma \subseteq X$ be given. For each $k \in \mathbb{N}$, Lemma 39 provides $g_k \in \mathbb{R}^{Y_k}$ and $f_k \in C(X)$ such that $\left|f_k|_{Y_k}\right| \leq |g_k|$ and $f_k|_\sigma = h|_\sigma$, with g_k independent of σ. Define each A_n $(n \in \mathbb{N})$ by

$$A_n = \left\{u \in C(X) : |u| \leq \mathbf{n} \text{ and } \left|u|_{Y_k}\right| \leq |g_k| \text{ for } k = 1, \ldots, n\right\}.$$

The set $A := \bigcup_{n=1}^{\infty} A_n$, independent of σ, is bounded at each point of $X = \bigcup_{n=1}^{\infty} Y_n$; in fact, if $x \in Y_n$ and $u \in A_m$, then, no matter the value of m,

$$|u(x)| \leq \max\{n, |g_n(x)|\}.$$

Indeed, if $m \leq n$ then $|u(x)| \leq m \leq n$; or, if $m > n$, then $|u(x)| = \left|u|_{Y_n}(x)\right| \leq |g_n(x)|$.

Finally, fix $n \in \mathbb{N}$ with $n > \max\{|h(x)| : x \in \sigma\}$. Since X is a Tychonoff space and n is sufficiently large, we may routinely choose some $u_0 \in C(X)$ such that

$$\begin{aligned}&\text{for each } x \in \sigma,\ u_0(x) = h(x) = f_1(x) = \cdots = f_n(x) \text{ and}\\ &\text{for each } x \in X,\ |u_0(x)| = \min\{n, |f_1(x)|, \ldots, |f_n(x)|\}.\end{aligned}$$

As $\left|f_k|_{Y_k}\right| \leq |g_k|$, the second line of the display assures that $u_0 \in A_n \subseteq A$; the first, that u_0 approximates h (exactly) on the arbitrary finite set σ in X. Therefore $h \in \overline{A}^{\mathbb{R}^X}$. □

Let $\mathcal{P}_Y$ be the canonical projection from $\mathbb{R}^X$ onto the subspace of functions whose support lies in $Y \subseteq X$. Replacing $C_p(Y)$ with $\mathbb{R}^Y$ in Theorem 25(3), we arrive at

Theorem 41 (ADDENDUM TO THEOREM 5) *The next four assertions are equivalent.*

1. *The subspace* $L_X(Y)$ *of* $L_\beta(X)$ *has the slctop.*
7. $L_X(Y) = L_\beta(Y)$ *and* $C_p(Y)$ *is distinguished.*
8. *Each bounded* $S \subseteq \mathbb{R}^Y$ *admits a bounded* $T \subseteq C_p(X)$ *with* $S \subseteq \overline{T|_Y}^{\mathbb{R}^Y}$.
9. *Each bounded* $A \subseteq \mathbb{R}^X$ *admits a bounded* $B \subseteq C_p(X)$ *with* $\mathcal{P}_Y(A) \subseteq \overline{\mathcal{P}_Y(B)}^{\mathbb{R}^X}$.

Proof Here, if V is a set in either $\mathbb{R}^X$ or $L(X)$, we let V° denote the polar of V with respect to the dual pair $\langle\mathbb{R}^X, L(X)\rangle$. If W is a set in either $\mathbb{R}^Y$ or $L(Y)$, let $W^\bullet$ denote the polar of W with respect to the dual pair $\langle\mathbb{R}^Y, L(Y)\rangle$.

[(1) ⇒ (7)]. Certainly, $L_X(Y)$ has the slctop only when $L_X(Y) \succeq L_\beta(Y)$. Then $L_X(Y) = L_\beta(Y)$ also has the slctop. By [9, Corollary 3.4], $C_p(Y)$ is distinguished.

[(7) ⇒ (8)]. Suppose S is an arbitrary bounded set in $\mathbb{R}^Y$. Now $C_p(Y)$ being distinguished means $C_p(Y|X)$ is large in $\mathbb{R}^Y$ [11, Theorem 10], which implies some $U \subseteq C_p(X)$ such that $U|_Y$ is bounded in $C_p(Y)$ with $S \subseteq \overline{U|_Y}^{\mathbb{R}^Y}$. Also from (7), $L_X(Y) = L_\beta(Y)$. Theorem 25 provides bounded $T \subseteq C_p(X)$ with $U|_Y \subseteq \overline{T|_Y}^{\mathbb{R}^Y}$. So (8) is evident.

[(8) ⇒ (9)]. For bounded $A \subseteq \mathbb{R}^X$, the set $S := A|_Y$ is bounded in $\mathbb{R}^Y$. Now (8) provides bounded $T \subseteq C_p(X)$ with $A|_Y \subseteq \overline{T|_Y}^{\mathbb{R}^Y}$. Clearly, $\mathcal{P}_Y(A) \subseteq \overline{\mathcal{P}_Y(T)}^{\mathbb{R}^X}$. Setting $B = T$, we have (9).

[(9) ⇒ (1)]. The polar $S^\bullet$ of an arbitrary bounded set $S \subseteq \mathbb{R}^Y$ is a basic 0-neighborhood in the slctop $\beta\left(L(Y), \mathbb{R}^Y\right)$ on $L(Y)$. Let A be the unique bounded set in $\mathbb{R}^X$ for which $A|_Y = S$ and $\mathcal{P}_Y(A) = A$. By (9), there exists bounded $B \subseteq C_p(X)$ with $\mathcal{P}_Y(A) \subseteq \overline{\mathcal{P}_Y(B)}^{\mathbb{R}^X}$. Therefore

$$S = A|_Y = \mathcal{P}_Y(A)|_Y \subseteq \left(\overline{\mathcal{P}_Y(B)}^{\mathbb{R}^X}\right)|_Y = \overline{B|_Y}^{\mathbb{R}^Y} \subseteq (B|_Y)^{\bullet\bullet}.$$

Hence $S^\bullet \supseteq (B|_Y)^{\bullet\bullet\bullet} = (B|_Y)^\bullet = B^\circ \cap L(Y)$. The latter is a relative 0-neighborhood in the subspace $L_X(Y)$, whose topology must then be as fine as the slctop on $L(Y)$. Thus (1) holds. □

Several times in [15] we employed assertion (8) or (9) without knowing their equivalence to (1). Theorems 24 and 27 prove

Theorem 42 *If Y is an F_σ set in X and $C_p(X\backslash Y)$ is distinguished, then $C_p(X)$ is distinguished if (and only if) $L_X(Y)$ has the slctop.*

This yields Theorem 60 of [15], a theorem which, with clearly closed Y in X and clearly distinguished $C_p(X\backslash Y) = \mathbb{R}^{X\backslash Y}$, concludes that $C_p(X)$ is distinguished if and only if (9) holds. Another example: Theorem 22 essentially says that *$C_p(X)$ is distinguished if (and only if) X is covered by finitely many subsets Y for which $L_X(Y)$ has the slctop*. This is exactly what [15, Theorem 16(2)] says, more verbosely, in terms of condition (9). Also, from Theorems 22, 25, 41 follows [15, Theorem 16(3)]: it assumes $C_p(Y)$ is distinguished for each of finitely many Y covering X, each satisfying a condition clearly more stringent than (2) of Theorem 25. So (1) holds, then (7) of Theorem 41. Thus each $L_X(Y)$ has the slctop; Theorem 22 applies.

Theorem 42 retains conclusion, relaxes hypothesis of [15, Corollary 61].

Proof of relaxed hypothesis Assume X, Y, Z fulfill the hypothesis of [15, Corollary 61]. We prove X, Y must fulfill the hypothesis of Theorem 42. The hypothesis in [15] distinguishes both $C_p(Y)$ and $C_p(X\backslash Y) = \mathbb{R}^{X\backslash Y}$, and makes Y an F_σ set (closed, even) in X. There only remains to show $L_X(Y)$ has the slctop under assumption (from [15]) that either (*i*) Y is dense and C-embedded in Z, or (*ii*) Y is l -embedded in Z, where Z is a Tychonoff space suitably refined by X ($X \succeq Z$) so that X and Z induce the same topology on the subset Y. Recall that Y is *l-embedded* in Z if $Y \subseteq Z$ and there is a continuous linear extender $\phi : C_p(Y) \to C_p(Z)$ satisfying $\phi(f)|_Y = f$ for each $f \in C(Y)$.

In case (*i*) each bounded $S \subseteq C_p(Y)$ admits a bounded $T \subseteq C_p(Z)$ with $S = T|_Y$ [15, Lemma 26 and Proposition 24]. Likewise in case (*ii*), since the continuous linear image of a bounded set is bounded. Thus in both cases, condition (2) of Theorem 25 holds with X replaced by Z, and so does (1); i.e., $L_Z(Y) = L_\beta(Y)$. Now $X \succeq Z$ implies $L_\beta(X) \succeq L_\beta(Z)$. So the induced topologies on $L(Y)$ verify $L_X(Y) \succeq L_Z(Y) = L_\beta(Y)$. Always, $L_\beta(Y) \succeq L_X(Y)$. Therefore $L_X(Y) = L_\beta(Y)$. As $C_p(Y)$ is distinguished, $L_\beta(Y) = L_X(Y)$ has the slctop. □

Hence Theorem 42 also yields [15, Corollary 62], a special case of [15, Corollary 61(*i*)]. To simplify [15, Corollary 61(*ii*)], omit Z, obtaining

Theorem 43 *Suppose an F_σ set Y is l-embedded in X. The space $C_p(X)$ is distinguished if (and only if) both $C_p(Y)$ and $C_p(X\backslash Y)$ are distinguished.*

Proof $L_X(X \backslash Y)$ has the slctop (Theorem 27), and so does $L_X(Y) = L_\beta(Y)$ via [9, Corollary 3.4], Theorem 25, and the fact that the continuous linear image of a bounded set is bounded. Apply Theorem 24. □

The simplified Theorem 43 dispatches the original [15, Corollary 61(*ii*)]. Indeed, if Y is l-embedded in some $Z \preceq X$ and if Z and X induce the same topology on Y, then Y is also l -embedded in X: any continuous linear extender from $C_p(Y)$ into $C_p(Z)$ is also continuous into $C_p(X)$, a superspace of $C_p(Z)$.

One may likewise simplify [15, Corollary 61(*i*)] by replacing *l-embedded* in Theorem 43 with *dense and C -embedded.*

6 A note on $\Sigma(X)$

We remark two well-known properties of the space $\Sigma(X)$.

Theorem 44 *Always,* $\Sigma(X)$ *is an ultrabornological Baire subspace of* $\mathbb{R}^X$.

Proof The case $X = [0, 1]$ is stated as an exercise in [3, 3.12.66]. Now $\mathbb{R}^X$ is always a Baire space (Bourbaki) in which $\Sigma(X)$ is G_δ-dense ($\Leftrightarrow$ [38, 4.9] by De Morgan's Laws). Therefore the subspace $\Sigma(X)$ is also Baire, as noted in [38, 4.12].

If $|X| = \aleph_0$ then $\Sigma(X)$ is metrizable, hence bornological. If $|X| > \aleph_0$, let Y be the one-point Lindelöfication of the space X equipped with its discrete topology. That is, $Y = X \cup \{\xi\}$ where $\xi \notin X$, the points of X are declared to be discrete and a base of neighborhoods of ξ consists of all sets of the form $N \cup \{\xi\}$, where $|X \backslash N| \le \aleph_0$. Let $E = \{f \in C(Y) : f(\xi) = 0\}$, a closed hyperplane in $C_p(Y)$. One routinely sees that for $f \in E$, the restriction $Rf := f|_X$ belongs to $\Sigma(X)$. Indeed, the restriction map R is a linear homeomorphism from E onto $\Sigma(X)$.

As Y is a Lindelöf space, $C_p(Y)$ is bornological by the Buchwalter-Schmets theorem. Its one-codimensional subspace E must also be bornological [7] (or see [17, 13.5.2]). The bornological and sequentially complete space $\Sigma(X)$ must be ultrabornological. □

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