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Research Article

A Characterization of the Existence of a Fundamental Bounded Resolution for the Space $C_c(X)$ in Terms of X

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We characterize in terms of the topology of a Tychonoff space X the existence of a bounded resolution for $C_c(X)$ that swallows the bounded sets, where $C_c(X)$ is the space of real-valued continuous functions on X equipped with the compact-open topology.

1. Preliminaries

In the sequel, unless otherwise stated, X is a nonempty completely regular Hausdorff space. We represent by $C_p(X)$ the ring $C(X)$ of real-valued continuous functions defined on X equipped with the *pointwise* topology τ_p. As usual, we denote by $L_p(X)$ the weak* dual of $C_p(X)$. When $C(X)$ is equipped with the *compact-open* topology τ_c we write $C_c(X)$. As in [1], we denote by $C^*(X)$ the linear space of real-valued continuous and bounded functions defined on X. If $C^*(X)$ is regarded as a subspace of $C_c(X)$, we denote this space by $C_c^*(X)$. Since $C^*(X)$ is dense in $C_c(X)$, both $C_c^*(X)$ and $C_c(X)$ have the same dual. The Banach space $C^*(X)$ equipped with the supremum norm has recently been studied in [2]. Let us recall that a family $\{A_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ of subsets of a set X is called a *resolution* for X if it covers X and $A_\alpha \subseteq A_\beta$ whenever $\alpha \leq \beta$; i.e., $\alpha(i) \leq \beta(i)$ for every $i \in \mathbb{N}$ (see [3, Chapter 3]). If E is a topological vector space, a resolution $\{A_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ consisting of *bounded sets* (see [4, Definition 1.4.5]) is referred to as a *bounded resolution*. A bounded resolution $\{A_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ on E that *swallows the bounded sets*, i.e., such that for each bounded set Q in E there is $\gamma \in \mathbb{N}^\mathbb{N}$ with $Q \subseteq A_\gamma$, is referred to as a *fundamental bounded resolution*. The existence of such a resolution in a locally convex space E has shown to be equivalent to the existence of a so-called $\mathfrak{G}$-base of absolutely convex neighborhoods of the origin in the strong dual E'_β of E. Besides, fundamental bounded resolutions are essential in order to get a proper extension of the class of (DF)-spaces (see [5] for details). Fundamental compact resolutions, i.e., resolutions consisting of compact sets which swallow the compact sets, have been widely used, even in Banach space theory [6], since they were introduced in [7]. A well-known result of Christensen [8, Theorem 3.3] asserts that a metrizable space X is Polish if and only if X has a fundamental compact resolution. Moreover, it has been shown in [9, Theorem 2] that $C_c(X)$ has a $\mathfrak{G}$-base of neighborhoods of the origin if and only if X has a fundamental compact resolution. In this note we provide two characterizations, in terms of the domain space X, of the existence of a fundamental bounded resolution for $C_c(X)$, one by means of certain uniformity for X and the other purely topological. Our main motivation is the two following C_p-theoretic results.

Theorem 1 ([7, Theorem 3.7] or [10, Problem 216]). *The space $C_p(X)$ has a fundamental compact resolution if and only if X is countable and discrete.*

Theorem 2 ([11, Theorem 3.3]). *The space $C_p(X)$ has a fundamental bounded resolution if and only if X is countable.*

A space X is called K*-analytic* if there is an upper semicontinuous compact-valued map T from the product space $\mathbb{N}^\mathbb{N}$, where $\mathbb{N}$ is equipped with the discrete topology, into X such that $\bigcup\{T(\alpha) : \alpha \in \mathbb{N}^\mathbb{N}\} = X$. A family $\mathscr{F}$ of functions from a uniform space $(X, \mathscr{N})$ into a uniform space $(Y, \mathscr{M})$ is called *uniformly equicontinuous* [12, Chapter 7, Problem G] if for each $V \in \mathscr{M}$ there is $U \in \mathscr{N}$ such that $(f(x), f(y)) \in V$

whenever $f \in \mathcal{F}$ and $(x, y) \in U$. Let $\mathcal{N}$ be a uniformity for a (nonempty) set X and denote by $\tau_{\mathcal{N}}$ the uniform topology defined by $\mathcal{N}$. A base $\{U_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ of the uniformity $\mathcal{N}$ is called a $\mathfrak{G}$-base if $U_\beta \subseteq U_\alpha$ whenever $\alpha \leq \beta$. There is no loss of generality by assuming that each U_α is a symmetric vicinity.

2. Fundamental Bounded Resolutions for $C_c(X)$

Although it can be easily seen that each metrizable locally convex space E has a fundamental bounded resolution, if the locally convex space $C_c(X)$ has a fundamental bounded resolution, unlike what happens with $C_p(X)$, the space X needs not be countable and moreover $C_c(X)$ needs not be metrizable.

Proposition 3. *Let X be a metrizable space. Then $C_c(X)$ has a fundamental bounded resolution if and only if X is σ-compact.*

Proof. If X is σ-compact then $C_c(X)$ is weakly K-analytic by [13, Proposition 2.2], or it has a compact resolution that swallows the compact sets by [14, Corollary 2.10]. In any case $C_c(X)$ has a bounded resolution. Since X is a $k_{\mathbb{R}}$-space then $C_c(X)$ is complete, hence locally complete. So, it follows from Valdivia's [3, Theorem 3.5] that there exists a resolution $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ of $C_c(X)$ consisting of Banach disks that swallows the bounded sets of $C_c(X)$, which shows that $C_c(X)$ has a fundamental bounded resolution. Conversely, if $C_c(X)$ has a fundamental bounded resolution then $C_p(X)$ has a bounded resolution, so [3, Corollary 9.2] shows that X is σ-compact. □

Example 4. According to the previous proposition $C_c(\mathbb{R})$ has a fundamental bounded resolution, but $\mathbb{R}$ is neither countable nor discrete. On the other hand, $C_c(\mathbb{Q})$ has also a fundamental bounded resolution, but $\mathbb{Q}$, although is countable, is not discrete. Observe that $C_c(\mathbb{R})$ is metrizable, but $C_c(\mathbb{Q})$ is not. Of course, if X is hemicompact, or even compact, then $C_c(X)$ is metrizable, or even a Banach space, and in this case $C_c(X)$ has obviously a fundamental bounded resolution.

Theorem 5. *The space $C_c(X)$ has a fundamental bounded resolution if and only if $(X, \mathcal{M})$, where $\mathcal{M}$ is the uniformity for X generated by the pseudometrics*

$$d_A(x, y) = \sup_{f \in A} |f(x) - f(y)| \tag{1}$$

for each bounded set A of $C_c(X)$, has a $\mathfrak{G}$-base.

Proof. Let E be the topological dual of $C_c(X)$ and denote by $\mathcal{B}$ the family of all bounded sets of $C_c(X)$ and by $\beta(E, C(X))$ the strong topology on E. By identifying X with its canonical homeomorphic embedding in $L_p(X)$, note that $X \subseteq L(X) \subseteq E$. The strong topology $\beta(E, C(X))$ generates a unique admissible translation-invariant uniformity $\mathcal{N}$ on E, so that $\tau_{\mathcal{N}} = \beta(E, C(X))$. By considering also $f \in C(X)$ as a linear functional on E, observe that for each $N \in \mathcal{N}$ there is $A \in \mathcal{B}$ such that

$$\left\{(u, v) \in E \times E : \sup_{f \in A} |\langle f, u - v\rangle| \leq 1\right\} \subseteq N. \tag{2}$$

Hence $M \subseteq X \times X$ belongs to the relative uniformity $\mathcal{M}$ of $\mathcal{N}$ to $X \times X$ if and only if there exists $A \in \mathcal{B}$ such that

$$\left\{(x, y) \in X \times X : \sup_{f \in A} |f(x) - f(y)| \leq 1\right\} \subseteq M. \tag{3}$$

If $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is a fundamental bounded resolution for $C_c(X)$, by setting

$$U_\alpha = \left\{(x, y) \in X \times X : \sup_{f \in A_\alpha} |f(x) - f(y)| \leq 1\right\} \tag{4}$$

we obtain a $\mathfrak{G}$-base $\{U_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ of $\mathcal{M}$. If $\alpha \leq \beta$ then $U_\beta \subseteq U_\alpha$, and if $M \in \mathcal{M}$ there is $A \in \mathcal{B}$ such that $(x, y) \in M$ whenever $\sup_{f \in A}|f(x) - f(y)| \leq 1$, so if $\gamma \in \mathbb{N}^{\mathbb{N}}$ is such that $A \subseteq A_\gamma$, clearly $U_\gamma \subseteq M$.

Conversely, if the uniform structure for X generated by the family of pseudometrics $\{d_A : A \in \mathcal{B}\}$, where

$$d_A(x, y) = \sup_{f \in A} |f(x) - f(y)|, \tag{5}$$

has a $\mathfrak{G}$-base $\{U_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$, this entails that for each $A \in \mathcal{B}$ there is $\delta \in \mathbb{N}^{\mathbb{N}}$ such that $\sup_{f \in A}|f(x) - f(y)| \leq 1$ for every $(x, y) \in U_\delta$. Setting

$$A_\alpha = \left\{f \in C(X) : \sup_{(x,y) \in U_\alpha} |f(x) - f(y)| \leq 1\right\}, \tag{6}$$

for each $\alpha \in \mathbb{N}^{\mathbb{N}}$, clearly $A_\alpha \subseteq A_\beta$ if $\alpha \leq \beta$ and $A \subseteq A_\delta$. Consequently $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is a fundamental bounded resolution for $C_c(X)$. □

Let X be a nonempty completely regular Hausdorff topological space and let $\mathcal{K}(X)$ denote the family of all compact sets of X.

Lemma 6. *A subset A of $C(X)$ is bounded for the compact-open topology τ_c if and only if there exists a sequence $\{\mathcal{F}_n(A) : n \in \mathbb{N}\}$ of subsets of $\mathcal{K}(X)$ such that*

(1) *for each $n \in \mathbb{N}$, if $K \in \mathcal{F}_n(A)$ then $\sup_{f \in A}\sup_{x \in K}|f(x)| \leq n$*
(2) *$\mathcal{F}_n(A) \subseteq \mathcal{F}_{n+1}(A)$ for each $n \in \mathbb{N}$*
(3) *$\bigcup_{n=1}^{\infty} \mathcal{F}_n(A) = \mathcal{K}(X)$*
(4) *If a set $Q \in \mathcal{K}(X)$ is such that $\sup_{x \in Q}|f(x)| \leq n$ for each $f \in C(X)$ with $\sup_{x \in K}|f(x)| \leq n$ for all $K \in \mathcal{F}_n(A)$, then $Q \in \mathcal{F}_n(A)$*

Proof. If A is τ_c-bounded and $n \in \mathbb{N}$, define

$$\mathcal{F}_n(A) = \left\{K \in \mathcal{K}(X) : \sup_{f \in A} \sup_{x \in K} |f(x)| \leq n\right\}. \tag{7}$$

Clearly $\mathcal{F}_n(A) \subseteq \mathcal{F}_{n+1}(A)$ for each $n \in \mathbb{N}$ and if $K \in \mathcal{K}(X)$ there is $m \in \mathbb{N}$ with

$$\sup_{f\in A}\sup_{x\in K}|f(x)| \leq m, \tag{8}$$

so that $K \in \mathcal{F}_m(A)$, which shows that $\bigcup_{n=1}^{\infty}\mathcal{F}_n(A) = \mathcal{K}(X)$. In addition, if $K \in \mathcal{F}_n(A)$ the relation $\sup_{f\in A}\sup_{x\in K}|f(x)| \leq n$ holds by construction. Finally, if we set

$$B_n = \left\{ f \in C(X) : \sup_{x\in K}|f(x)| \leq n \text{ for every } K \in \mathcal{F}_n(A) \right\} \tag{9}$$

then $A \subseteq B_n$. Therefore, if $Q \in \mathcal{K}(X)$ verifies that $\sup_{f\in B_n}\sup_{x\in Q}|f(x)| \leq n$ then

$$\sup_{f\in A}\sup_{x\in Q}|f(x)| \leq n, \tag{10}$$

so that $Q \in \mathcal{F}_n(A)$. Hence $\{\mathcal{F}_n(A) : n \in \mathbb{N}\}$ satisfies the required conditions.

Conversely, if there exists a sequence $\{\mathcal{F}_n(A) : n \in \mathbb{N}\}$ of $\mathcal{K}(X)$ satisfying the four conditions of the statement of the lemma (actually, only the first and the third conditions are required) then clearly A is τ_c-bounded on X, for if $P \in \mathcal{K}(X)$ there is $k \in \mathbb{N}$ with $P \in \mathcal{F}_k(A)$ such that $\sup_{f\in A}\sup_{x\in P}|f(z)| \leq k$. □

In what follows the fourth condition above, which is independent of A, will be referred to as the *closure* condition of $\mathcal{F}_n(A)$, and we shall say that the family $\mathcal{F}_n(A)$ is *closed*.

Definition 7. A collection $\{\mathcal{F}_{\alpha,n} : (\alpha, n) \in \mathbb{N}^{\mathbb{N}} \times \mathbb{N}\}$ of closed subsets of $\mathcal{K}(X)$ will be called a *covering net* of $\mathcal{K}(X)$ if $\{\mathcal{F}_{\alpha,n} : n \in \mathbb{N}\}$ is an increasing covering of $\mathcal{K}(X)$ for each $\alpha \in \mathbb{N}^{\mathbb{N}}$ such that $\mathcal{F}_{\beta,n} \subseteq \mathcal{F}_{\alpha,n}$ whenever $\alpha \leq \beta$ for all $n \in \mathbb{N}$.

Theorem 8. *The space $C_c(X)$ has a fundamental bounded resolution if and only if there is a covering net $\{\mathcal{F}_{\alpha,n} : (\alpha, n) \in \mathbb{N}^{\mathbb{N}} \times \mathbb{N}\}$ such that if $\{\mathcal{F}_n : n \in \mathbb{N}\}$ is an increasing covering of $\mathcal{K}(X)$ by closed sets, there exists $\gamma \in \mathbb{N}^{\mathbb{N}}$ such that $\mathcal{F}_{\gamma,n} \subseteq \mathcal{F}_n$ for all $n \in \mathbb{N}$.*

Proof. If there exists a covering net $\{\mathcal{F}_{\alpha,n} : (\alpha, n) \in \mathbb{N}^{\mathbb{N}} \times \mathbb{N}\}$ for X which satisfies the property of the statement of the theorem, the sets

$$A_\alpha = \left\{ f \in C(X) : \sup_{x\in K}|f(x)| \leq n \text{ for every } K \in \mathcal{F}_{\alpha,n} \text{ and each } n \in \mathbb{N} \right\} \tag{11}$$

compose a fundamental bounded resolution for $C(X)$. Indeed, each set A_α is τ_c-bounded by virtue of the previous lemma, since $\{\mathcal{F}_{\alpha,n} : n \in \mathbb{N}\}$ is an increasing covering of $\mathcal{K}(X)$ consisting of closed sets such that $\sup_{f\in A_\alpha}\sup_{x\in K}|f(x)| \leq n$ for all $K \in \mathcal{F}_{\alpha,n}$. Besides $A_\alpha \subseteq A_\beta$ whenever $\alpha \leq \beta$ since $\mathcal{F}_{\beta,n} \subseteq \mathcal{F}_{\alpha,n}$, and if A is a τ_c-bounded subset of $C(X)$, according to the previous lemma there exists an increasing covering $\{\mathcal{F}_n(A) : n \in \mathbb{N}\}$ of $\mathcal{K}(X)$ consisting of closed sets. Therefore, by the condition in the statement of the theorem, there exists $\gamma \in \mathbb{N}^{\mathbb{N}}$ such that $\mathcal{F}_{\gamma,n} \subseteq \mathcal{F}_n(A)$ for all $n \in \mathbb{N}$. Now, if $f \in A$ then $\sup_{k\in K}|f(x)| \leq n$ for all $K \in \mathcal{F}_n(A)$ and $n \in \mathbb{N}$, in particular for each $K \in \mathcal{F}_{\gamma,n}$ and all $n \in \mathbb{N}$, which shows that $f \in A_\gamma$. Hence $A \subseteq A_\gamma$, which proves that $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is a fundamental bounded resolution for $C_c(X)$.

Conversely, if $C_c(X)$ has a fundamental bounded resolution $\{B_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ and we set

$$\mathcal{F}_{\alpha,n} = \left\{ K \in \mathcal{K}(X) : \sup_{f\in B_\alpha}\sup_{x\in K}|f(x)| \leq n \right\} \tag{12}$$

then $\{\mathcal{F}_{\alpha,n} : n \in \mathbb{N}\}$ is an increasing covering of $\mathcal{K}(X)$ such that $\mathcal{F}_{\beta,n} \subseteq \mathcal{F}_{\alpha,n}$ whenever $\alpha \leq \beta$ for all $n \in \mathbb{N}$. In addition $\sup_{f\in B_\alpha}\sup_{x\in K}|f(x)| \leq n$ for $K \in \mathcal{F}_{\alpha,n}$ and $n \in \mathbb{N}$ by the definition of $\mathcal{F}_{\alpha,n}$. Moreover, $\mathcal{F}_{\alpha,n}$ satisfies the closure condition, for if

$$B_{\alpha,n} = \left\{ f \in C(X) : \sup_{Q\in\mathcal{F}_{\alpha,n}}\sup_{x\in Q}|f(x)| \leq n \right\} \tag{13}$$

then $B_\alpha \subseteq B_{\alpha,n}$, so if $\sup_{f\in B_{\alpha,n}}\sup_{x\in K}|f(x)| \leq n$ then $\sup_{f\in B_\alpha}\sup_{x\in K}|f(x)| \leq n$, which means that $K \in \mathcal{F}_{\alpha,n}$.

Now, if $\{\mathcal{F}_n : n \in \mathbb{N}\}$ is any increasing covering of $\mathcal{K}(X)$ consisting of closed sets, define

$$P = \left\{ f \in C(X) : \sup_{x\in K}|f(x)| \leq n \text{ for every } K \in \mathcal{F}_{\alpha,n} \text{ and each } n \in \mathbb{N} \right\} \tag{14}$$

and observe that if $K \in \mathcal{F}_n$ then $\sup_{f\in P}\sup_{x\in K}|f(x)| \leq n$, which according to the preceding lemma ensures that P is a τ_c-bounded set. Since $\{B_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is a fundamental bounded resolution for $C_c(X)$ there exists $\delta \in \mathbb{N}^{\mathbb{N}}$ such that $P \subseteq B_\delta$. Now if $Q \in \mathcal{F}_{\delta,n}$ then $\sup_{f\in B_\delta}\sup_{x\in Q}|f(x)| \leq n$ so that, in particular, $\sup_{f\in P}\sup_{x\in Q}|(x)| \leq n$. We claim that $Q \in \mathcal{F}_n$. Indeed, since $\sup_{x\in Q}|f(x)| \leq n$ for each $f \in P$, we have that $\sup_{x\in Q}|f(x)| \leq n$ holds for each $f \in C(X)$ such that $\sup_{x\in K}|f(x)| \leq n$ for every $K \in \mathcal{F}_n$ by virtue of the definition of P. Therefore, the closure property of $\mathcal{F}_n$ yields $K \in \mathcal{F}_n$. This shows that $\mathcal{F}_{\delta,n} \subseteq \mathcal{F}_n$ for every $n \in \mathbb{N}$, which, bearing in mind the properties of the family $\{\mathcal{F}_{\alpha,n} : (\alpha, n) \in \mathbb{N}^{\mathbb{N}} \times \mathbb{N}\}$ established before, guarantees that $\{\mathcal{F}_{\alpha,n} : (\alpha, n) \in \mathbb{N}^{\mathbb{N}} \times \mathbb{N}\}$ is a covering net for X consisting of closed sets that satisfies the required property. □

In what follows we shall refer to a Tychonoff space X satisfying the conditions of the statement of Theorem 8 as a *cn-space*. It is shown in [5, Proposition 3.2] that if X is a *cn*-space or, which is equivalent, if $C_c(X)$ has a fundamental bounded resolution, then $C_c(X)$ has a *countable cs*-network* at the origin [15, 16]. The next theorem shows that in order for X to be a *cn*-space it suffices that $C_c^*(X)$ have a fundamental bounded resolution.

Theorem 9. *Let X be completely regular. The space $C_c^*(X)$ has a fundamental bounded resolution if and only if X is a cn-space.*

Proof. If X is a *cn*-space, Theorem 8 ensures that $C_c(X)$ has a fundamental bounded resolution, which implies that $C_c^*(X)$, as a subspace of $C_c(X)$, also has a fundamental bounded resolution. Conversely, if $C_c^*(X)$ has a closed fundamental bounded resolution $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$, for each $\alpha \in \mathbb{N}^{\mathbb{N}}$ let B_α denote the closure of A_α in $C_c(X)$. Let us show that $\mathscr{B} = \{B_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is a fundamental bounded resolution for $C_c(X)$. Indeed, let us denote by $\mathscr{K}(X)$ the family of all compact sets of X and pick an arbitrary bounded subset B of $C_c(X)$. For each $K \in \mathscr{K}(X)$ and $f \in B$, set $M(f,K) = \sup\{|f(x)| : x \in K\}$ and define $f_K(x) = f(x)$ if $|f(x)| \le M(f,K)$ and $f_K(x) = \operatorname{sign}(f(x)) \cdot M(f,K)$ if $|f(x)| > M(f,K)$. Clearly $f_K \in C^*(X)$, besides $f_K(x) = f(x)$ for $x \in K$ and $|f_K(x)| \le |f(x)|$ for all $x \in X$. Define $P_B = \{f_K : f \in B, K \in \mathscr{K}(X)\}$ and note that f is an adherent point of P_B in $C_c(X)$. Therefore, B is contained in the closure of P_B in $C_c(X)$. If Q is any compact subset of X, the fact that B is τ_c-bounded guarantees that

$$\begin{aligned} \sup_{g \in P_B} \sup_{x \in Q} |g(x)| &= \sup_{f \in B, K \in \mathscr{K}(X)} \sup_{x \in Q} |f_K(x)| \\ &\le \sup_{f \in B} \sup_{x \in Q} |f(x)| < \infty, \end{aligned} \tag{15}$$

which shows that P_B is a τ_c-bounded subset of $C^*(X)$. Hence, there is $\gamma \in \mathbb{N}^{\mathbb{N}}$ such that $P_B \subseteq A_\gamma$, so that $B \subseteq B_\gamma$. Consequently $\mathscr{B}$ is a fundamental bounded resolution for $C_c(X)$, as stated. Another application of Theorem 8 shows that X is a *cn*-space. □

The existence of a bounded resolution on a locally convex space E does not imply the existence of a fundamental bounded resolution for E, as the following example shows.

Example 10. If X is an infinite Talagrand compact set, the weak* dual $L_p(C_p(X))$ of $C_p(C_p(X))$ has a bounded resolution but it has no fundamental bounded resolution.

Proof. Since $C_p(X)$ is K-analytic, $L_p(C_p(X))$ is also K-analytic by [17, Proposition 0.5.13]. So $L_p(C_p(X))$ has even a compact resolution by [3, Theorem 3.2]. Suppose by contradiction that $L_p(C_p(X))$ has a fundamental bounded resolution $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$. Identifying $C_p(X)$ with its homeomorphic copy in $L_p(C_p(X))$, for each $\alpha \in \mathbb{N}^{\mathbb{N}}$ set $B_\alpha = A_\alpha \cap C_p(X)$ and consider the family $\mathscr{B} = \{B_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$. We claim that B_α is a functionally bounded subset of $C_p(X)$. Indeed, if $F \in C(C_p(X))$ according to [17, Proposition 0.5.11] there exists a (unique) continuous functional u_F of $L_p(C_p(X))$ such that $u_F|_{C_p(X)} = F$. Since A_α is bounded, there is $C > 0$ such that $|u_F(a)| < C$ for all $a \in A_\alpha$. In particular, $|F(b)| = |u_F(b)| < C$ for every $b \in B_\alpha$. So $\mathscr{B}$ is a functionally bounded resolution in $C_p(X)$. If B is a functionally bounded subset of $C_p(X)$, then B, considered as a subset of $L_p(C_p(X))$, is bounded. Therefore, there is $\alpha \in \mathbb{N}^{\mathbb{N}}$ with $B \subseteq B_\alpha$. Hence $\mathscr{B}$ swallows the functionally bounded subsets of $C_p(X)$. Since $C_p(X)$ is Lindelöf and hence a μ-space, the family $\{\overline{B_\alpha}^{\tau_p} : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ consists of compact subsets and swallows the compact sets of $C_p(X)$. So $C_p(X)$ has a fundamental compact resolution. But according to Theorem 1, the space X should be countable and discrete. Therefore, X being compact is finite, a contradiction. □

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this paper.

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