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PhD. Program in Statistics, Optimization and Applied Mathematics (EOMA)

Contributions to multi-issue bankruptcy problems with crossed claims

Philosophical Doctoral Thesis

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Dr./ Domingo Morales González, coordinator of the academic committee of the PhD program in Statistics, Optimization and Applied Mathematics,

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That Mr./ Rick Keevin Acosta Vega has carried out under the supervision of our Doctoral Program the thesis entitled "Contributions to multi-issue bankruptcy problems with crossed claims" in accordance with the terms and conditions defined in its Research Plan and in accordance with the Code of Good Practices of the Universidad Miguel Hernández de Elche, fulfilling the objectives satisfactorily for its public defense as a doctoral thesis.

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The coordinator of the Academic Committee of the PhD program in Statistics, Optimization and Applied Mathematics,

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Dedications

I dedicate this achievement to my parents, Arelis y Ricardo who with love and effort have accompanied me in this process, without hesitating at any moment of seeing my dreams come true, which are also their dreams.

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Chapter 1

Introduction

How to divide a resource among individuals who have claims on it, has been an interesting and relevant problem in many different contexts since ancient times. We can find references to fairness in treatment and distribution in the *Nicomachean Ethics* by Aristotle (4th Century BD) and several problems of this nature reported in the *Babylonian Talmud* (5th Century AD) (Rodkinson, 1918), which collected oral and written tradition of the Jewish religious and legal decisions since Moses time. The problem of dividing a resource among its claimants can be divided into four groups:

1. *Bankruptcy problems* (O'Neill, 1982; and Aumann and Maschler, 1985) describe situations in which there is a scarce and perfectly divisible resource (for example, money, water, greenhouse emissions, time...) and a finite set of agents who have claims or rights on it, but the resource is not sufficient to fully honor the agents' demands.

2. *Allocation problems* (Chun, 1988; and Herrero et al., 1999) deal with situations in which there is a perfectly divisible budget that has to be divided among a finite set of agents who have entitlements, but in this case the budget can be greater or less than the aggregate entitlement.

3. *Surplus problems* (Moulin, 1987) analyze situations of cooperative ventures that produce a non-negative surplus that must be distributed among the participating agents, taking only into account their opportunity costs.

4. *Loss problems* (Bergantiños and Vidal-Puga, 2004) illustrate situations mathematically similar to bankruptcy problems, but negative allocations are allowed.

In this thesis, we will focus on bankruptcy problems. These problems and their extensions have been widely studied in the literature (see Thomson 2003, 2015, 2019) and applied to very diverse situations, see, for example, Pulido et al. (2002, 2008), Niyato and Hossain (2006), Bergantiños et al. (2012), Casas-Mendez et al. (2011), Gozalvez et al. (2012), Hu et al. (2012), Lucas-Estañ et al. (2012), Giménez-Gómez et al. (2016), Sanchez-Soriano et al. (2016), among many others, which demonstrates its economic and managerial interest.

Roughly, speaking, solving a bankruptcy problem means finding a distribution of the estate among the claimants that meets three conditions: (1) no agent obtains more than his/her claim; (2) no agent obtains less than zero; and (3) the estate is completely distributed among the agents. Conditions (1) and (2) mean that there is no subsidization and Condition (3) implies Pareto efficiency. In general, there are two main goals in bankruptcy problems literature. Firstly, defining *allocation rules* applicable to all bankruptcy problems which meet the three previous conditions and some extra desirable properties which respond to different aspects of fairness. Secondly, finding axiomatic characterizations of the allocation rules or, its dual, which allocation rules satisfy a set of properties.

On the other hand, Calleja et al. (2005) introduced an extension of bankruptcy problems in which there is again a perfectly divisible amount of resource to be divided, but the claimants have different claims on different issues which must be satisfied by means of that amount of resource. Therefore, the new bankruptcy problems have two different characteristics from the original bankruptcy problems. First, the resource must be used to cover the demands on a number of issues. And, second, the claimants have a claim on each of those issues. These problems are called multi-issue bankruptcy problems. This kind of problems have been studied with different approaches in Borm et al. (2005) and Izquierdo and Timoner (2016). Of course, one-issue bankruptcy problems correspond to the well-known bankruptcy problems (B) (O'Neill, 1982; Aumann and Maschler, 1985).

As in the case of bankruptcy problems, solving a multi-issue bankruptcy problem means finding a distribution of the estate among the claimants that meets the following two conditions: (1) each agent receives an award for each issue that is non-negative and bounded above by her claim; and (2) the entire available amount must be allocated among the agents. The meaning of both conditions are similar to those for bankruptcy problems above. Two possible approaches to define rules are the following:

- Define rules that assign a matrix in a direct way, for example by solving a suitable optimization problem.

- Define two-stage rules. In the first stage, the total amount of resource is allocated to issues. In the second step, we solve as many one-issue bankruptcy problems as issues we have.

1.1 Motivation and description of the problem

Consider now that a certain authority is interested in reducing the emission of pollutants into the atmosphere (or water or soil). However, there are many pollutants, each with different effects and consequences. For example, there are pollutants that contribute to the greenhouse effect and thus to climate change, and others that are harmful to health because they are carcinogenic, cause respiratory problems or other diseases. Likewise, these pollutants can be grouped into families using different criteria, for example, effect or chemical composition. In the most general case, a pollutant could belong to two or more different families.

The entire system for the abatement of pollutants could be represented in a hierarchical structure of two levels (see Figure 1.1). In the first level, we would have the families of pollutants, and in the second level the pollutants themselves. The ultimate goal of that authority is for emissions of the different families of pollutants to be below certain levels in order to better control the pollution and their effects. In this sense, the authority fixes certain levels of emissions for each family of pollutants. However, pollutants could belong to more than one family. Thus, we consider the particular situation in which there are different amounts of emissions of different pollutants and the authority fixes maximum levels of emissions for each family of pollutants according, for example, to their harmful effects on health or contribution to climate change, in order to abate these emissions and keep them below certain levels. Therefore, we have a kind of multi-issue bankruptcy problem which is different from other multi-issue bankruptcy problems as the multi-issue problems as we will explain below. Of course, if we were only interested in a particular family of pollutants, we obtain a classical bankruptcy problem, where the estate is the amount fixed for the family and the claims are the current emissions of pollutants.

This problem remembers a multi-issue bankruptcy problem in which the issues are the families of pollutants. Multi-issue bankruptcy problems introduced by Calleja et al. (2005) describe situations in which there are a perfect divisible estate which can be divided between various issues, and a number

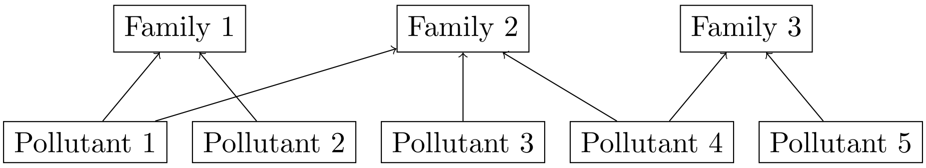

Figure 1.1: Example of a two level hierarchical structure.

of claimants that have claims on each of those issues[1]. Therefore, there are a perfect divisible estate, several issues, and claimants with vectors of claims with as many coordinates as issues, such that the total amount of claims is above the estate. The central question for these problems is how the estate should be allocated. This problem is solved by means of allocation rules and there are several approaches to it (see, for example, Calleja et al., 2005; Borm et al., 2005; Izquierdo and Timoner, 2016). However, we have ex ante one estate for each family, and the claimants are the pollutants, and each pollutant has just one claim which is the same for each of the estates of the families which it belongs to. This approach is different from the multi-issue bankruptcy problems in the literature.

In our case, we have several perfectly divisible estates, and claimants have exactly one claim which is used in all estates simultaneously. Now, again, the question is how the estate should be allocated in a reasonable way. As far as we know this approach is totally novel in the literature and then fill a gap in the bankruptcy problems that have been dealt until now.

1.2 Objectives

The objectives set out in this doctoral thesis are related to different aspects of the problem described in Section 1.1, in particular, they are the following:

1. Introduce a new bankruptcy model with multiple issues that is an extension of those already known in bankruptcy, enriching them with features that may be relevant in the management of scarce resources.

2. Solve the proposed bankruptcy model with multiple issues through suitable allocation rules that are extensions of bankruptcy rules known as can be the constrained equal awards rule, the proportional rule or the random arrival rule.

[1] "An issue constitutes a reason on the basis of which the estate is to be divided". (Calleja et al. 2005, page 731.)

3. Introduce significant properties for allocation rules in the context of the problems described in Section 1.1 and use them to characterize the proposed allocation rules.

4. Apply the new bankruptcy problem to real resource management problems.

Objectives 1 to 4 will be contextualized and detailed in greater depth in the corresponding chapters. In this way, the aim is to provide the reader with as much information as possible about the problem studied within each of the chapters.

1.3 Materials and methods

The materials used in the development of the research that is collected in this memory are the usual ones in Mathematics and, in particular, in Game Theory, namely, computers, specialized software, public data sets and, of course, the related literature.

The research was raised under a classic methodology in the field of data science, which evolved throughout the research period from the collection of information in the field of study to the proposal and testing of the solutions proposed to the problems studied, concluding with a qualitative and quantitative analysis of its characteristics, compared to existing methods. Schematically, the methodology followed the following steps:

1. Readings of the related literature (papers, book chapters, books, etc.).

2. Formulation of problems and models with their corresponding hypotheses.

3. Demonstration of the results and their possible application to real problems.

4. Publication and dissemination of the results.

5. Contact with other researchers to exchange ideas.

1.4 Structure of the thesis and main results

Sections 2.1/5.1 and 2.2/5.2 are dedicated to reviewing the main results in the literature related to bankruptcy problems and multi-issue bankruptcy problems, respectively. In particular, these problems are described in detail, the most relevant allocation rules, their most outstanding properties and their corresponding characterizations are presented. It is not intended, in any case, to make an exhaustive review but simply a review of some of the results that are related to those obtained in this doctoral thesis.

In this memory, we introduce a novel model of multi-issue bankruptcy problem. In our model, several perfectly divisible goods (estates) have to be allocated among certain set of agents (claimants) that have exactly one claim which is used in all estates simultaneously. In other words, unlike of the multi-issue bankruptcy problems already existent in the literature, this model study situations with multi-dimensional states, one for each issue and where each agent claims the same to the different issues in which participates. These problems are called multi-issue bankruptcy problems with crossed claims (see **Section 2.3/5.3**).

In this thesis, we theoretically extend 3 classical rules for solving bankruptcy problems to the case of multi-issue bankruptcy problems with crossed claims. These rules are the constrained equal-awards rule (CEA) (Maimonides, 12th Century AD), the proportional rule (PROP) (Aristotle, 4th Century BD) and the random arrival rule (RA) (O'Neill, 1982). These allocation rules can be found in **Section 2.3/5.3**.

On the one hand, we introduce the CEA rule for multi-issue bankruptcy problems with crossed claims by using two different iterative procedures. One of them is based on linear programming and the other is based on the CEA rule for one issue bankruptcy problems. On the other hand, the constrained proportional award rule for multi-issue bankruptcy problems with crossed claims is defined by means of an iterative procedure which is reminiscent of the CEA but applying the proportional rule at each step. This rule naturally extends the proportional rule for single issue bankruptcy problems. Finally, we define extensions of the family of sequential priority rules and the random arrival rule for multi-issue bankruptcy problems with crossed claims.

In **Section 2.4/5.4**, an analysis of the behavior of the allocation rules introduced for the context of multi-issue bankruptcy problems with crossed claims, according to principles like equity, efficiency, monotonicity or consistency is carried out. In fact, the study of all these properties is determinant to make a choice of a reasonable rule to different situations. As a result of the properties satisfied by an allocation rule, it can be deduced that it is a good choice when looking for equitable and consistent allocations. All properties introduced in **Section 2.4/5.4** are extensions of well-known properties used in bankruptcy problems.

Some of the properties that are introduced in this memory are the following. We start with a property related to efficiency, *Pareto efficiency.* A feasible allocation is Pareto efficient if there is no other feasible allocation in which some individual is better off and no individual is worse off. The following property is related to the non-discrimination of agents. *Equal treatment of equals* states that equal agents must receive the same. The next group of properties is related to what is the minimum that should be guaranteed to agents. *Conditional equal division* was introduced by Moulin (2000) for rationing problems. In our context, this property means that an agent should obtain her claim if this is less than any egalitarian distribution of the estates of the issues she claims, and in other case, at least the minimal egalitarian distribution of the estates of all issues she claims. *Guaranteed minimum award* means that a claimant should not receive less than what she would receive in the worst case, if the issues were distributed separately. *Conditional full compensation* means that if the claim of a claimant is so small that if all claimants with higher claims asked for the same amount as her, all claims would be fully honored, then it seems reasonable that said claimant receives her claim. An interesting group of properties are those that are related to monotonicity, that is, to changes in some of the parameters of the problem. *Claim monotonicity* means that if the claim of a claimant increasing she cannot receive less than she received in the previous situation. *Resource monotonicity* simply says that if the available resource increases, allocations to claimants do not decrease. And, *population monotonicity* which says that if all claimants agree that a claimant j will obtain her claim, then the remaining claimants should be worse off after claimant j is fully compensated. Three other properties relevant in the context of bankruptcy problems are the following. *Balanced contributions* requires that claimant j impacts to claimant k's allocation what claimant k impacts to claimant j's allocation. *Non-manipulability by splitting* means that it is not profitable to split one agent in several agents. And, finally, *consistency* which means that if a subset of claimants leave the problem respecting what had been assigned to those who remain, then what those players get in the new reduced problem is the same as what they got in the whole problem.

We characterize our CEA rule by means of *Pareto efficiency*, *conditional equal division* and *consistency*, showing that is uniquely determined by *conditional equal division* and *consistency*. We also provide another characterization of our CEA rule following the obtained in Yeh (2006) by using *Pareto efficiency*, *conditional full compensation*, *claim monotonicity* and *consistency*. The constrained proportional awards rule for multi-issue bankruptcy problems with crossed claims is characterized axiomatically by using five properties: *Pareto efficiency*, *equal treatment of equals*, *guaranteed minimum award*, *consistency*, and *non-manipulability by splitting*. However, we have not been able yet to find an axiomatic characterization of sequential priority

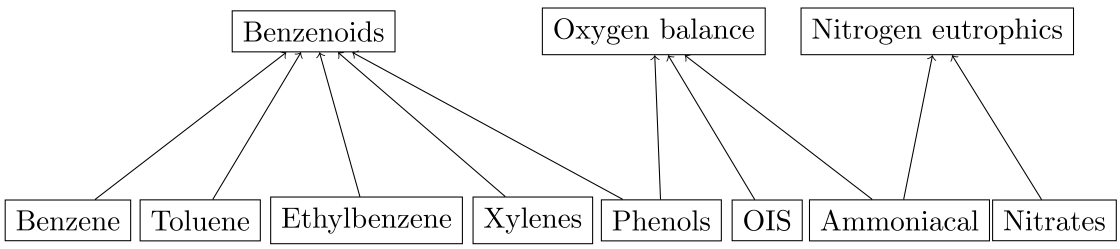

Figure 1.2: Relationship between families of pollutants and substances.

rules and the random arrival rule in the context of multi-issue bankruptcy problems with crossed claims. In fact, we have proven that the extensions, that we introduce in this memory, do not satisfy the properties usually used for their characterization in the case of bankruptcy problems or multi-issue bankruptcy problems, *resource monotonicity*, *population monotonicity* and *balanced contributions*. Therefore, these negative results show the complexity of finding properties that allow axiomatic characterizations of sequential priority rules and the random arrival rule in the context of multi-issue bankruptcy problems with crossed claims, so it is necessary to look for perhaps more specific properties (and likely more technical) to achieve it.

In **Section 2.5/5.5**, in order to illustrate how the allocation rules introduce in this memory work, we consider the following situation. A certain authority is interested in controlling the pollution in the waters of a river. In particular, they are interested in limiting the concentration (mg/l) of benzenoids, substances that influence the oxygen balance, and nitrogen eutrophics. For each of these families of pollutants certain levels of concentration are fixed in order to keep a reasonable quality of water. On the other hand, for each of these families several pollutants are monitored. For the first family: benzene, toluene, ethylbenzene, xylenes, and phenols. For the second family: ammoniacal, phenols, and oxidizable inorganic substances (OIS). Finally, for the third family: nitrates, and ammoniacal. For each of these substances there are maximum concentration limits (mg/l). The problem to be solved is how to allocate new limits to the substances taking into account the limits imposed by the authority to each family of pollutants. The relation between the families of pollutants and the substances in each family is shown in Figure 1.2. We observe that this problem is a multi-issue bankruptcy problem with crossed claims. The allocation rules introduced in this memory are applied to provide solutions to the problem.

Finally, **Chapter 4/7** concludes and presents further research. At the end of this memory the references used are listed.

Chapter 2

Extended Summary

2.1 Bankruptcy problems

In this section, some notation and basic concepts related to bankruptcy problems are introduced. In particular, we present the definition of bankruptcy problem, the main allocation rules, their properties and some characterizations in the literature.

2.1.1 Introduction

Allocation problems describe situations in which a resource (or resources) must be distributed among a set of agents. These problems are of great interest in many settings, for this reason the literature on the matter is extensive. A particular allocation problem arises in situations where there is a perfectly divisible resource over which there is a set of agents who have rights or demands, but the resource is not sufficient to honor them. This problem is known as bankruptcy problem and was first formally analyzed in O'Neill (1982) and Aumann and Maschler (1985). Since then it has been extensively studied in the literature and many allocation rules have been defined (see Thomson, 2003, 2015, 2019, for a detailed inventory of rules).

In the literature many applications of bankruptcy problems can be found. Some examples are the following. Pulido et al. (2002, 2008) study allocation problems in university management; Niyato and Hossain (2006), Gozalvez et al. (2012), and Lucas-Estañ et al. (2012) analyze radio resource allocation problems in telecommunications; Casas-Mendez et al. (2011) study the museum pass problem; Hu et al. (2012) analyze the airport problem; Gimínez-Gómez et al. (2016), Gutiérrez et al. (2018), and Duro et al. (2020) analyze the CO2 allocation problem; Sanchez-Soriano et al. (2016) study the apportionment problem in proportional electoral systems; and Wickramage et al. (2020) analyze water allocation problems in rivers.

2.1.2 Definition of bankruptcy problems

We consider a situation where there is a perfectly divisible resource E, called the estate or endowment, and there are a finite set of claimants $N = \{1, 2, ..., n\}$ that claim different amounts of it, $c_i > 0, i \in N$, with the same rights and with the condition that the estate is not sufficient for fully satisfying all the requests, i.e., $\sum_{i \in N} c_i > E$. Formally,

Definition 2.1. A *bankruptcy problem* (BP in short) is a triplet (N, E, c), where $N = \{1, 2, ..., n\}$ is the set of claimants; $E \in \mathbb{R}_{++}$ is the amount of available resources (assumed perfectly divisible); and $c = (c_1, c_2, ..., c_n) \in \mathbb{R}^N_+$ is the vector of claims, such that $C = \sum_{i \in N} c_i > E$. Moreover, the family of all these problems is denoted by $\mathcal{BP}$.

The main goal in a bankruptcy problem is to find an allocation which is as fair as possible, taking into account the demands of the claimants. Many solutions have been proposed depending on the principle(s) of fairness used.

Definition 2.2. Given a problem $(N, E, c) \in \mathcal{BP}$, a *feasible allocation* for it, it is a vector $x \in \mathbb{R}^N$ such that:

1. (Boundedness) $0 \leq x_i \leq c_i$, for all $i \in N$.
2. (Efficiency) $\sum_{i \in N : j \in \alpha(i)} x_i = E$,

and we denote by $A(N, E, c)$ the set of all its feasible allocations.

Requirement 1 means that each agent receives at most her claim but not less than nothing. Requirement 2 means that the estate must be fully distributed. Therefore, the vector $x \in \mathbb{R}^N$ represents a feasible allocation to the claimants.

Definition 2.3. An *allocation rule or simply a rule* for bankruptcy problems is a mapping R that associates with every $(N, E, c) \in \mathcal{BP}$ a unique feasible allocation $R(N, E, c) \in A(N, E, c)$.

2.1.3 Basic allocation rules for bankruptcy problems

In the literature there are many different bankruptcy rules (see Thomson, 2019), but we only consider five of them: the proportional rule, the constrained equal awards rule, the constrained equal losses rule, the Talmud rule and the random arrival rule. These five rules are considered basic solutions to bankruptcy problems. The proportional rule equates ratios between quantities obtained and quantities claimed. The constrained equal awards rule equals the quantities obtained, the constrained equal losses rule equals the losses obtained for each claimant, and the Talmud rule applies a different logic that consists in that anyone gets more than half of what they claim if the amount of the good available is less than half of the total amount claimed, and that nobody loses more than half of what that they claim if the available amount exceeds half of the overall claim. Finally, the random arrival rule is obtained as the average of the assignment vectors that result from dividing the value of the estate sequentially.

The proportional rule proposes that each creditor be distributed proportionally to their claim, treating all equally.

Definition 2.4. For every problem $(N, E, c) \in \mathcal{BP}$, the *proportional rule*, PROP, is defined as

$$PROP_i(N, E, c) = \frac{c_i}{C} E, i \in N.$$

The constrained equal-awards rule distributes the estate in such a way that each claimant is assigned the same amount as long as it is not exceed the amount claimed by the agent.

Definition 2.5. For every problem $(N, E, c) \in \mathcal{BP}$, the *constrained equal-awards rule*, CEA, is defined as

$$CEA_i(N, E, c) = \min\{c_i, \beta\}, i \in N,$$

where β is a positive real number satisfying $\sum_{i \in N} CEA_i(N, E, c) = E$.

The constrained equal-losses rule, unlike CEA, prioritizes the agents with the highest demands. The constrained equal-losses rule distributes loss relative to demand equally among all agents. This is that the amount they do not receive from the claim is the same for everyone as long as the condition is met that no claimant can lose a larger amount than she had claimed, that is, she cannot receive a negative payment.

Definition 2.6. For every problem $(N, E, c) \in \mathcal{BP}$, the *constrained equal-losses rule*, CEL, is defined as

$$CEL_i(N, E, c) = \max\{c_i - \alpha \ , 0\}, i \in N,$$

where α is a non-negative real number satisfying $\sum_{i \in N} CEL_i(N, E, c) = E$.

The Talmud rule (Aumann and Maschler, 1985) can be seen as a hybrid of CEA and CEL. Specifically using CEA if the estate is less than half the sum of the claims and CEL if it is greater. If it is the same, it is irrelevant which one applies because they both give the same result. The objective of this rule is that when the total loss is great nobody wins much and when the total loss is small nobody loses much.

Definition 2.7. For every problem $(N, E, c) \in \mathcal{BP}$, the *Talmud rule*, TAL, is defined as

$$TAL(N,E,c) = \begin{cases} CEA(N,E,c/2) & if \quad E \leq C/2 \\ c - CEA(N, C-E, c/2) & if \quad E > C/2 \end{cases}$$

or, by duality[1] relating CEL and CEA, i.e $CEL(N,E,c) = c - CEA(N, C - E, c)$:

$$TAL(N,E,c) = \begin{cases} CEA(N,E,c/2) & if \quad E \leq C/2 \\ c/2 + CEL(N, E - C/2, c/2) & if \quad E > C/2 \end{cases}$$

The random Arrival rule (O'Neil, 1982) is based on the assumption that the agents arrive at the payment center one by one and the first person who arrives receives the minimum between his demand and the estate, the same for the agent who arrives second but subtracting what has been given the first and so on until there is no estate. Since all order of arrival are assumed to be equiprobable, the average of all these allocations is taken.

Definition 2.8. For every problem $(N, E, c) \in \mathcal{BP}$, the *random arrival rule*, RA, is defined as

$$RA_i(N,E,c) = \frac{1}{|N|!}\left(\sum_{\sigma\in\Sigma(N)} \min\{c_i, \max\left\{E - \sum_{j\in N:\sigma(j)<\sigma(i)} c_j, 0\right\}\right), i \in N,$$

where $\Sigma(N)$ is the set of all possible orders of N, i.e., $\sigma \in \Sigma(N)$ is an one-to-one map from N to $\{1, 2, 3, \ldots, |N|\}$.

2.1.4 Some properties for allocation rules

In this subsection, we present several properties for allocation rules in bankruptcy problems. First of all, it should be noted that the efficiency property does not appear in the list below because it is part of the allocation rule definition itself.

[1]Two bankruptcy rules ρ and ρ^* are called one the dual of the other one if $\rho(N,E,c) = c - \rho^*(N, C-E, c)$, *i.e.* if we obtain the same solution dividing the estate E according to the claims c using the rule ρ or assigning to each agent her/his claim and subtracting a quota of $C - E$ computed using the rule ρ^* with the claims c.

Axiom 2.1 (Equal treatment of equals). *Given a rule R, it satisfies* equal treatment of equals*, if for every problem $(N, E, c) \in \mathcal{BP}$ and every pair of claimants $i, j \in N$, such that $c_i = c_j$, then $R_i(N, E, c) = R_j(N, E, c)$.*

This property requires equal treatment, meaning that players who demand the same amount must receive the same.

Axiom 2.2 (Anonymity). *Given a rule R, it satisfies* anonymity*, if for every problem $(N, E, c) \in \mathcal{BP}$, and each bijection $\sigma : N \to N$, $R_i(N, E, c) = R_{\sigma(i)}(N, E, c')$ for each $i \in N$, where $c' = (c_{\sigma(j)})_{j \in N}$.*

The property of anonymity dictates that the identity of the agents should not matter. The vector of payments chosen should depend only on the list of claims.

Axiom 2.3 (Balanced impact). *Given a rule R, it satisfies* balanced impact*, if for every problem $(N, E, c) \in \mathcal{BP}$, and every pair of claimants $i, j \in N$,*

$$R_i(N, E, c) - R_i(N \backslash \{j\}, \max\{E - c_j, 0\}, c_{-j}) = \\ R_j(N, E, c) - R_j(N \backslash \{i\}, \max\{E - c_i, 0\}, c_{-i}).$$

Balanced impact property requires that claimant j impacts to claimant k's allocation what claimant k impacts to claimant j's allocation.

Axiom 2.4 (Order preservation). *Given a rule R, it satisfies* Order preservation*, if for each problem $(N, E, c) \in \mathcal{BP}$ and each pair $i, j \in N$ such that $c_i \leq c_j$, then $R_i(N, E, c) \leq R_j(N, E, c)$, and $c_i - R_i(N, E, c) \leq c_j - R_j(N, E, c)$.*

A rule must respect the order of claims that is, if the demand of agent i is at least as great as that of agent j, he should receive at least as much as agent j. Also, the differences between claims and payment need to be sorted as well.

The following properties try to establish what amount a claimant should receive or what amount should be guaranteed to each claimant at least under certain reasonable conditions. In this sense, these properties are related to the guarantees that claimants receive when an allocation rule is applied.

Axiom 2.5 (Respect of minimal rights). *Given a rule R, it satisfies* respect of minimal rights*, if for every problem $(N, E, c) \in \mathcal{BP}$, for all $i \in N$,*

$$R_i(N, E, c) \geq \max\{0, E - \sum_{k \in N \backslash \{i\}} c_k\}.$$

Axiom 2.6 (Conditional equal division)**.** *Given a rule R, it satisfies* conditional equal division*, if for every problem $(N, E, c) \in \mathcal{BP}$, for all $i \in N$,*

$$R_i(N, E, c) \geq \min\left\{c_i, \frac{E}{|N|}\right\}.$$

Axiom 2.7 (Securement)**.** *Given a rule R, it satisfies* securement*, if for every problem $(N, E, c) \in \mathcal{BP}$, for all $i \in N$,*

$$R_i(N, E, c) \geq \min\left\{\frac{c_i}{|N|}, \frac{E}{|N|}\right\}.$$

Respect of minimal rights, conditional equal division and securement are related to the minimum amount that should reasonably be guaranteed to each claimant. The concept of minimal right was introduced by Tijs (1981) in the context of cooperative games to define the τ-value. Thus, respect of minimal rights says that a claimant should receive at least what is left when all the other claimants are completely satisfied in their claims. CED was introduced by Moulin (2000) for rationing problems and Herrero and Villar (2002) for bankruptcy problems with the name of exemption. Finally, securement was introduced for bankruptcy problems by Moreno-Ternero ad Villar (2004).

Axiom 2.8 (Conditional full compensation)**.** *Given a rule R, it satisfies* conditional full compensation*, if for every problem $(N, E, c) \in \mathcal{BP}$ and each $i \in N$, such that $\sum_{k\in N} \min\{c_k, c_i\} \leq E$, then $R_i(N, E, c) = c_i$.*

Conditional full compensation means that if the claim of a claimant is so small that if all claimants asked for the same amount as her, they all would receive their claims, then it seems reasonable that said claimant receives her claim. This property was introduced as sustainability by Herrero and Villar (2002).

The monotonic properties refer to what impact changes in some of the elements that define the problem have on the allocation, in particular, changes in the amount of resources available or in the demands of the claimants.

Axiom 2.9 (Resource monotonicity)**.** *Given a rule R, it satisfies* resource monotonicity*, if for every problem $(N, E, c) \in \mathcal{BP}$ and $E' \geq E$, $R_j(N, E', c) \geq R_j(N, E, c)$ for all $j \in N$.*

Axiom 2.10 (Claim monotonicity)**.** *Given a rule R, it satisfies* claim monotonicity*, if for every pair of problems $(N, E, c) \in \mathcal{BP}$ and $(N, E, c') \in \mathcal{BP}$, such that $c_i \geq c'_i$ and $c_j = c'_j$, for all $j \in N\backslash\{i\}$, then $R_i(N, E, c) \geq R_i(N, E, c')$.*

Claim monotonicity means that if the claim of a claimant increasing she cannot receive less than she received in the previous situation. In Kasajima and Thomson (2011) monotonicity properties are studied in the context of the adjudication of conflicting claims.

Another monotonicity property, which is satisfied by many bankruptcy rules, is population monotonicity which says that if all claimants agree that a claimant j will obtain her claim, then the remaining claimants should be worse off after claimant j is fully compensated.

Axiom 2.11 (Population monotonicity). *Given a rule R, it satisfies* population monotonicity*, if for every problem (N, E, c), and each $j \in N$,*

$$R_k(N, E, c) \geq R_k(N, E - c_j, c_{-j}), \text{ for all } k \in N \backslash \{j\},$$

where c_{-j} is the vector of claims from which $j-th$ coordinate has been deleted.

Axiom 2.12 (Composition up). *Given a rule R, it satisfies* composition up*, if for every problem $(N, E, c) \in \mathcal{BP}$, and $E' \in \mathbb{R}$, such that $0 \leq E < E'$, then:*

$$R(N, E', c) = R(N, E, c) + R(N, E' - E, c - R(N, E, c)).$$

The compensation up property assures us that this situation can be solved in two ways: You can cancel the initial delivery and apply the distribution rule to the modified problem, or you can keep the initial assignments of the agents and adjust the demands accordingly and apply the rule to distribute the difference $E' - E$. And therefore, the assignments do not vary if the distribution is carried out once or sequentially.

Axiom 2.13 (Composition down). *Given a rule R, it satisfies* composition down*, if for every problem $(N, E, c) \in \mathcal{BP}$, and $E' \in \mathbb{R}^{|P|}$, such that $0 \leq E' < E$, then:*

$$R(N, E', c) = R(N, E', R(N, E, c)).$$

This property implies that, when recalculating the distribution, it is irrelevant to take as claim the original claims or the distribution that resulted from the old estate.

Axiom 2.14 (Claims truncation invariance). *Given a rule R, it satisfies* claims truncation invariance*, if for every problem $(N, E, c) \in \mathcal{BP}$, when considering the problem $(N, E, c') \in \mathcal{BP}$ such that $c'_i = \min\{c_i, E\}$, for all $i \in N$; then $R(N, E, c) = R(N, E, c')$.*

Claims truncation invariance says that if the claims are truncated by the estate, then the final allocation does not change. This property appears in Curiel et al. (1987) and it is used to characterize the so-called game theoretical rules for bankruptcy problems. Dangan and Volij (1993) were the first to propose this property as an axiom.

Axiom 2.15 (Claim-consistency)**.** *Given a rule R, it satisfies* claim-consistency*, if for every problem $(N, E, c) \in \mathcal{BP}$, the following relation holds:*

$$R_i(N, E, c) = \frac{1}{|N|}\left[\min\{c_i, E\} + \sum_{j \in N\setminus\{i\}} R_i(N\setminus\{j\}, \max\{E - c_j, 0\}, (c_k)_{k\in N\setminus\{j\}})\right],$$

for all $i \in N$.

Claim-consistency was introduced by O'Neill (1982) to characterize the random arrival rule. This property says that the allocation can be viewed as an average of $|N|$ payoffs which are calculated by fixing a claimant and giving her as much as possible and then the remaining is shared among the other claimants.

The next property is a requirement of robustness when some agents leave the problem with their allocations (see Thomson, 2011, 2018). In particular, when a subset of claimants leave the problem respecting the allocations to those who remain, then it seems reasonable that claimants who leave will receive the same in the new problem as they did in the original. Before introducing the following property, we need to introduce the concept of reduced problem.

Definition 2.9. Given a problem $(N, E, c) \in \mathcal{BP}$, and $N' \subset N$, the *reduced problem* associated with N' is $(N', E', c|_{N'}) \in \mathcal{BP}$, where $E' = E - \sum_{i\in N\setminus N'} R_i(N, E, c)$ and $c|_{N'}$ is the vector whose coordinates correspond to the claimants in N'.

Axiom 2.16 (Consistency)**.** *Given a rule R, it satisfies* consistency*, if for every problem $(N, E, c) \in \mathcal{BP}$, and $N' \subset N$, it holds that*

$$R_i(N, E, c) = R_i(N', E', c|_{N'}), \text{ for all } i \in N'.$$

Consistency means that if a subset of claimants leave the problem respecting what had been assigned to those who remain, then what those players get in the new reduced problem is the same as what they got in the whole problem. Consistency properties have been used to characterize many bankruptcy rules, because they represents a requirement of robustness when some agents leave the problem with their allocations (see Thomson 2011, 2018) for surveys about the application of consistency properties and their principles behind.)

Axiom 2.17 (No advantageous transfer)**.** *Given a rule R, it satisfies* no advantageous transfer*, if for every problem $(N, E, c) \in \mathcal{BP}$, and $N' \subset N$, and each $c'|_{N'}$, if $\sum_{i \in N'} c'_i = \sum_{i \in N'} c_i$, then it holds that*

$$\sum_{i \in N'} R_i(N, E, c) = \sum_{i \in N'} R_i(N, E, (c'|_{N'}, c|_{N \setminus N'})).$$

No advantageous transfer means that no group of agents obtain more in the aggregate by transferring claims among themselves.

Axiom 2.18 (No advantageous merging or splitting)**.** *Given a rule R, it satisfies* no advantageous merging or splitting*, if for every problem $(N, E, c) \in \mathcal{BP}$, and $N' \subset N$, and each $c' \in \mathbb{R}_+^{N'}$, if there is $i \in N'$ such that $c'_i = c_i + \sum_{j \in N \setminus N'} c_j$ and for each $k \in N' \setminus \{i\}$, $c'_k = c_k$, then it holds that*

$$R_i(N', E, c') = R_i(N, E, c) + \sum_{j \in N \setminus N'} R_i(N, E, c).$$

No advantageous merging or splitting means that it is not advantageous for the claimants neither to divide themselves into several claimants nor to merge several claimants into one.

Axiom 2.19 (Self-duality)**.** *Given a rule R, it satisfies* self-duality*, if for every problem $(N, E, c) \in \mathcal{BP}$, then $R(N, E, c) = c - R(N, C - E, c)$.*

Self-duality was introduced by Aumann and Maschler (1985). Self-duality means that allocating rewards is the same as allocating losses (Herrero and Villar, 2001).

Axiom 2.20 (Homogeneity)**.** *Given a rule R, it satisfies* homogeneity*, if for every problem $(N, E, c) \in \mathcal{BP}$, and each $\lambda > 0$, such that $R_i(N, \lambda E, \lambda c) = \lambda R_i(N, E, c)$.*

Homogeneity means that if we require that claims and endowment are multiplied by the same positive number, then so should all awards. There should be no "scale" effect (Thomson, 2018).

2.1.5 Characterizations

We continue with a list of characterizations of the constrained equal awards rule, the proportional rule and the random arrival rule based on the properties we have presented in the previous section. They are grouped according to which solution comes out of the axioms (Thomson, 2019).

Theorem 2.1. *The constrained equal awards rule is the only rule satisfying:*

- *Equal treatment of equals, invariance under claims truncation, and composition up (Dagan, 1996);*
- *Conditional full compensation and composition down (Herrero and Villar, 2002);*
- *Securement, composition up and consistency (Chun, 2006);*
- *Consistency, conditional equal division and composition down (Herrero and Villar, 2002);*
- *Conditional full compensation, claim monotonicity and consistency (Yeh, 2006).*

Theorem 2.2. *The proportional rule is the only rule satisfying:*

- *Self-duality and composition up (Young, 1998);*
- *Self-duality and composition down (Young, 1998);*
- *Equal treatment of equals, composition, and self-duality (Herrero and Villar, 2001);*
- *For* $|N| \geq 3$*; no advantageous transfer (Moulin, 1985; Chun 1988; Ju and Miyagawa, 2002);*
- *No advantageous transfer and consistency;*
- *No advantageous merging or splitting (Chun, 1988; de Frutos, 1999; Ju and Miyagawa, 2002).*

Theorem 2.3. *The random arrival rule is the only rule satisfying:*

- *Claim-consistency (O'Neill, 1982);*
- *Balanced impact (Bergantiños and Mendez-Naya, 1997).*

2.2 Multi-issue bankruptcy problems

In this section, we present the definition of multi-issue bankruptcy problems, we study the rules for multi-issue allocation situations, the basic properties for this topic, and some characterizations are provided for the two-stage constrained equal awards and constrained equal losses rules, based on the properties of composition up and composition down among others.

2.2.1 Introduction

The multi-issue bankruptcy problems were studied by Calleja et al. (2005), where they presented an extension of the run-to-the-bank rule for bankruptcy situations, in which bankruptcy problems are explained in which the estate is divided not based on a single credit for each agent but taking into account several credits resulting from so-called problems. However, one difficulty with this approach is that although the payment is a formal idea to deal with the problems, the rules are difficult to compute. Some solutions to this problem are proposed by González-Alcón et al. (2007) where they solve this complex problem by constructing a different extension of the run-to-the-bank rule, and Borm et al. (2005), who study bankruptcy problems with a priori unions and use the same idea to first allocate money to the unions and then redivide the money within each union. Finally, Lorenzo-Freire et al. (2008), apply the two-stage idea to the constrained equal awards and constrained equal losses rules. They characterize the two-stage constrained equal awards rule and the two-stage constrained equal losses rules.

2.2.2 Multi-issue bankruptcy problems

We consider a situation where there is a perfectly divisible resource E, called the estate, which must be distributed among a finite number of issues M on which there is a finite set of claimants N that have claims in such a way that the sum of all the claims is greater than the available estate E.

Definition 2.10. A *multi-issue bankruptcy problem* (MB problem in short) is a 4-tuple (M, N, E, C), where $M = \{1, 2, ..., m\}$ is the set of issues, N is the set of claimants, $E \in \mathbb{R}_+$ is a perfectly divisible amount to be divided (the *estate*), and $C \in \mathcal{M}^+_{m \times n}$ is a matrix of claims. Every row in C represents the claims for an issue. A generic element of C, c_{ij}, denotes the amount of issue i that claimant j claims. Moreover, $\sum_{i \in M} \sum_{j \in N} c_{ij} > E$. The family of all these problems is denoted by $\mathcal{MB}$.

An example is presented below to illustrate what kinds of situations are modeled by multi-issue bankruptcy problems.

Example 2.1. Consider a country that has three regions A, B, and C. The government of the country has transferred the competences of education and health to the regions. The budget for education and health that the government makes available to the regions is 30 MU and each region has demands to spend on education and health of 5 MU and 7 MU, 7 MU and 9 MU, and 6 MU and 8 MU, respectively. In this situation, there are two issues:

education and health, three claimants the regions A, B and C, the estate is 30 and the matrix of claims is given by

$$\begin{pmatrix} 5 & 7 & 6 \\ 7 & 9 & 8 \end{pmatrix}$$

The total demand is 42 which exceeds the available budget of 30. Therefore, this situation describes a multi-issue bankruptcy problem. A possible allocation of the budget is given by the following matrix:

$$\begin{pmatrix} 3 & 5 & 4 \\ 5 & 7 & 6 \end{pmatrix}$$

This matrix indicates what each region receives to attend education and health. For example, region A receives 3 MU for education and 5 MU for health.

2.2.3 Allocation rules in multi-issue bankruptcy problems

As in bankruptcy problems, the main goal in a multi-issue bankruptcy problem is to find an allocation which is as fair as possible, taking into account the available estate and the demands of the claimants for each issue.

Definition 2.11. Given a problem $(M, N, E, C) \in \mathcal{MB}$, a *feasible allocation* for it, it is a matrix $X = (x_{ij})_{i \in M, j \in N} \in \mathbb{R}^{M \times N}$ such that:

1. $0 \leq x_{ij} \leq c_{ij}$, for all $i \in M$ and $j \in N$.
2. $\sum_{i \in M} \sum_{j \in N} x_{ij} = E$,

and we denote by $A(M, N, E, C)$ the set of all its feasible allocations.

A matrix $X \in \mathbb{R}^{M \times N}$ represents a desirable way of dividing E among the agents in N, according to the issues in M. Requirement (1) is that each agent receives an award for each issue that is non-negative and bounded above by her claim. Requirement (2) is that the entire available amount be allocated. These two requirements imply that $X = C$ whenever $\sum_{i \in M} \sum_{j \in N} c_{ij} = E$.

Definition 2.12. A *multi-issue bankruptcy rule* is a mapping R that associates with every $(M, N, E, C) \in \mathcal{MB}$ a unique matrix $R(M, N, E, C) \in A(M, N, E, C)$.

Three possible approaches to define allocation rules for multi-issue bankruptcy problems are the following:

- The first approach provides rules that assign a matrix in a direct way, for example by solving an optimization problem like this:

$$\begin{array}{ll} \min & \sum_{i\in M}\sum_{j\in N}(x_{ij}-c_{ij})^2 \\ s.a: & \sum_{i\in M}\sum_{j\in N}x_{ij}=E \\ & 0\leq x_{ij}\leq c_{ij} \end{array}$$

- The second approach provides one-stage rules. It simply consists of considering the problem as a bankruptcy problem with as many claims as the number of issues times the number of claimants.

- The third approach provides two-stage rules. In the first stage, the total amount of resource is allocated to issues. In the second step, we solve as many one-issue bankruptcy problems as issues we have.

The formal definition of the third approach to define allocation rules in multi-issue bankruptcy problems is given below.

Definition 2.13. Let R be a bankruptcy rule and let $(M,N,E,C)\in\mathcal{MB}$. The two-stage solution $R(M,N,E,C)$ is the allocation obtained from the following two-stage procedure:

Step 1: Consider the so-called *quotient bankruptcy problem* (M,E,c^M) where $c^M=(c_1^M,...,c_{|M|}^M)\in\mathbb{R}^M$ denotes the vector of total claims in the issues, i.e., $c_i^M=\sum_{j\in N}c_{ij}$ for all $i\in M$. Divide the amount E among the issues using the bankruptcy rule R. In this way, we obtain $R(M,E,c^M)\in\mathbb{R}^M$.

Step 2: For each $i\in M$, consider a new bankruptcy problem for the claimants $(N,R_i(M,E,c^M),(c_{ij})_{j\in N})\in\mathcal{BP}$. We again apply the bankruptcy rule R to each of those bankruptcy problems. Thus, for each $i\in M$, we obtain the allocation $R(N,R_i(M,E,c^M),(c_{ij})_{j\in N})$.

It is easy to check that the proposed allocation in Example 2.1 is the solution of the nonlinear program given in the first approach above. In the following subsections we illustrate how to apply the two-stage approach.

The two-stage CEA rule

The two-stage CEA rule, CEA^2, is given, for all $(M,N,E,C)\in\mathcal{MB}$ by

$$CEA^2_{ij}(M,N,E,C)=\min\{\lambda_i,c_{ij}\},\ \text{for all } i\in M, j\in N,$$

where $i \in M$, λ_i is such that $\sum_{j\in N} \min\{\lambda_i, c_{ij}\} = \min\left\{\beta, c_i^M\right\}$ and β is such that $\sum_{i\in M} \min\{\beta, c_i^M) = E$.

Note that CEA^2 differs from the one-stage CEA rule defined by

$$CEA^1_{ij}(M, N, E, C) = \min\{\beta, c_{ij}\},$$

for every $i \in M$ and $j \in N$ with β such that $\sum_{i\in M}\sum_{j\in N} \min\{\beta, c_{ij}\} = E$, i.e., the direct application of CEA by taking all claims as part of a single bankruptcy problem, i.e., a one-stage approach. A relevant distinction between the two extensions is that the latter does not satisfy the quotient property.

For the situation described in Example 2.1, first we solve the bankruptcy problem $(M = \{Ed, H\}, 30, (18, 24))$ whose CEA solution is $(15, 15)$. Then we solve the following two bankruptcy problems $(N = \{A, B, C\}, 15, (5, 7, 6))$ and $(N = \{A, B, C\}, 15, (7, 9, 8))$ whose CEA solutions are $(5, 5, 5)$ and $(5, 5, 5)$ respectively. In this particular case, if we consider that the one-stage approach we obtain the same allocation.

The two-stage CEL rule

The two-stage CEL rule, CEL^2, is given, for all $(M, N, E, C) \in \mathcal{MB}$ by

$$CEL^2_{ij}(M, N, E, C) = \max\{0, c_{ij} - \lambda_i\}, \text{ for all } i \in M, j \in N,$$

where $i \in M$, λ_i is such that $\sum_{j\in N} \max\{0, c_{ij} - \lambda_i\} = \max\left\{0, c_i^M - \beta,\right\}$ and β is such that $\sum_{i\in M} \max\left\{0, c_i^M - \beta,\right\} = E$.

Again, the two-stage CEL rule differs from the one-stage CEL rule. For the situation described in Example 2.1, first we solve the bankruptcy problem $(M = \{Ed, H\}, 30, (18, 24))$ whose CEL solution is $(12, 18)$. Then we solve the following two bankruptcy problems $(N = \{A, B, C\}, 12, (5, 7, 6))$ and $(N = \{A, B, C\}, 18, (7, 9, 8))$ whose CEL solutions are $(3, 5, 4)$ and $(5, 7, 6)$ respectively. In this particular case, if we consider the one-stage approach we obtain again the same allocation.

The two-stage proportional rule

Moreno-Ternero (2009) proves that the proportional rule is the unique bankruptcy rule such that the one-stage extension and two-stage extension for multi-issue bankruptcy problems coincide. Therefore, the proportional rule for multi-issue bankruptcy problems is defined for each $(M, N, E, C) \in \mathcal{MB}$ as follows:

$$P_{ij}(M, N, E, C) = \frac{E}{\sum_{k\in M}\sum_{h\in N} c_{kh}} c_{ij}, \quad i \in M, j \in N.$$

The two-stage RA rule

The two-stage RA rule, RA^2, is given, for all $(M, N, E, C) \in \mathcal{MB}$ by

$$RA^2_{ij}(M, N, E, C) = RA_j(N, RA_i(M, E, C^M), (c_{ij})_{j \in N}), \quad i \in M, j \in N.$$

Once again, the two-stage RA rule differs from the one-stage RA rule. For the situation described in Example 2.1, first we solve the bankruptcy problem $(M = \{Ed, H\}, 30, (18, 24))$ whose RA solution is $(12, 18)$. Then we solve the following two bankruptcy problems $(N = \{A, B, C\}, 12, (5, 7, 6))$ and $(N = \{A, B, C\}, 18, (7, 9, 8))$ whose RA solutions are $(3.33, 4.83, 3.83)$ and $(5, 7, 6)$ respectively.

Calleja et al. (2005) propose extensions of the random arrival rule for multi-issue bankruptcy problems considering the issues and the claimants combined. Thus, they define two extensions of the random arrival rule: the proportional random arrival rule and the queue random arrival rule. On the other hand, González-Alcón et al. (2007) propose a two-stage extension of the random arrival rule: first, they explicitly allocate the estate to the issues according to a marginal vector, and then, within each issue the corresponding part of the estate is divided among the claimants using the random arrival rule. This extension is in line with the two-stage extensions of the bankruptcy rules.

2.2.4 Properties for multi-issue bankruptcy rules

We now introduce several properties for multi-issue bankruptcy rules that are relevant to this thesis. All properties below can be found in Lorenzo-Freire et al. (2010) and Bergantiños et al. (2010).

Axiom 2.21 (Consistency in two stages). *Given a rule R, it satisfies* consistency in two stages*, if for every problem $(M, N, E, C) \in \mathcal{MB}$ and for all $i \in M$, $j \in N$,*

$$R_{ij}(M, N, E, C) = R_{1j}(\{1\}, N, R_{i1}(M, \{1\}, E, c^M), (c_{ik})_{k \in N}),$$

where $(M, \{1\}, E, c^M)$ is the quotient bankruptcy problem.

It is immediate that consistency in two stages is satisfied by all two-stage rules.

Axiom 2.22 (Quotient property). *Given a rule R, it satisfies* quotient property*, if for every problem $(M, N, E, C) \in \mathcal{MB}$ and for all $i \in M$,*

$$\sum_{j \in N} R_{ij}(M, N, E, C) = R_{i1}(M, \{1\}, E, c^M),$$

Quotient property means that the total quantity allocated to an issue in an MB problem is equal to the amount allocated to the same issue in the quotient bankruptcy problem, i.e., the final allocation to an issue is independent of the distribution of claims over that issue.

Axiom 2.23 (Consistency within the issues)**.** *Given a rule R, it satisfies* consistency within the issues*, if for every problem $(M, N, E, C) \in \mathcal{MB}$ and for all $i \in M$, $j \in N$,*

$$R_{ij}(M, N, E, C) = R_{1j}(\{1\}, N, \sum_{k \in N} R_{ik}(M, N, E, C), (c_{ik})_{k \in N}).$$

This property means that redividing the total amount within an issue using the same rule R should yield the same outcome.

Axiom 2.24 (Composition up)**.** *Given a rule R, it satisfies* composition up*, if for all problem $(M, N, E, C) \in \mathcal{MB}$ and for all $0 \leq E' \leq E$, we have*

$$R(M, N, E, C) = R(M, N, E', C) + R(M, N, E - E', C - R(M, N, E', C)).$$

This property has the same interpretation as its homologous axiom in bankruptcy problems (see Axiom 2.12).

Axiom 2.25 (Resource monotonicity)**.** *Given a rule R, it satisfies* resource monotonicity*, if for all $(M, N, E, C) \in \mathcal{MB}$ and $(M, N, E', C) \in \mathcal{MB}$ such that $0 \leq E' \leq E$,*

$$R(M, N, E', C) \leq R(M, N, E, C)$$

Axiom 2.26 (Claims truncation invariance)**.** *Given a rule R, it satisfies* claims truncation invariance*, if for all $(M, N, E, C) \in \mathcal{MB}$ we have*

$$R(M, N, E, C) = R(M, N, E, C_E),$$

where $C^E \in \mathbb{R}_+^{M \times N}$ is such that $c_{ij}^E = min\{c_{ij}, E\}$ for all $i \in M$ and $j \in N$.

See Axiom 2.14.

Axiom 2.27 (Equal treatment for the claimants within an issue)**.** *Given a rule R, it satisfies* equal treatment for the players within an issue*, if for each $(M, N, E, C) \in \mathcal{MB}$ and for all $i \in M$ and $j, k \in N$ such that $c_{ij} = c_{ik}$,*

$$R_{ij}(M, N, E, C) = R_{ik}(M, N, E, C).$$

This property says that if two claimants have the same claim for an issue they must receive the same for that issue.

Axiom 2.28 (Equal treatment for the issues)**.** *Given a rule R, it satisfies* equal treatment for the issues*, if for each $(M, N, E, C) \in \mathcal{MB}$ and for all $i, k \in M$ such that $c_i^M = c_k^M$,*

$$R_{i1}(M, \{1\}, E, c^M) = R_{k1}(M, \{1\}, E, c^M).$$

This property means that if two issues have the same total claim, then the total amount allocated to both issues must be the same independently of the distribution of claims over them.

Axiom 2.29 (Non-advantageous transfer across issues)**.** *Given a rule R, it satisfies* non-advantageous transfer across issues*, if for each pair of problems $(M, N, E, C), (M, N, E, C') \in \mathcal{MB}$ such that there is $S \subset M$ such that $\sum_{i \in S} \left(\sum_{j \in N} c_{ij} \right) = \sum_{i \in S} \left(\sum_{j \in N} c'_{ij} \right)$ and $c_{ij} = c'_{ij}$ when $(i, j) \in M \backslash S \times N$. Then, for each $i \in M \backslash S$,*

$$\sum_{j \in N} R_{ij}(M, N, E, C) = \sum_{j \in N} R_{ij}(M, N, E, C').$$

Axiom 2.30 (Non-advantageous transfer within issues)**.** *Given a rule R, it satisfies* non-advantageous transfer within issues*, if for each pair of problems $(M, N, E, C), (M, N, E, C') \in \mathcal{MB}$ such that there is an $i \in M$, and $S \subset N$ such that $\sum_{j \in S} c_{ij} = \sum_{j \in S} c'_{ij}$ and $c_{kj} = c'_{kj}$ when $k \in M \backslash \{i\}$ or $j \in N \backslash S$. Then, for each $j \in N \backslash S$,*

$$R_{ij}(M, N, E, C) = R_{ij}(M, N, E, C').$$

Non-advantageous transfer across issues and non-advantageous transfer within issues are related to the no advantageous transfer property in bankruptcy problems (see Axiom 2.17). However, while Axiom 2.17 says that the claimants who have redistributed their claims are not better off in the aggregate, non-advantageous transfer across issues and non-advantageous transfer within issues say that the agents who have not redistributed their claims are neither better off nor worse off.

2.2.5 Characterizations

In the following theorem we present the first characterization of the CEA rule. This theorem is inspired by a similar result for the CEA rule for bankruptcy situations in Dagan (1996).

Theorem 2.4 (Lorenzo-Freire et al., 2010)**.** *Let $(M, N, E, C) \in \mathcal{MB}$. The only rule which satisfies composition up, invariance under claims truncation, equal treatment for the players within an issue, equal treatment for issues, and the quotient property is the two-stage constrained equal awards rule CEA^2 .*

Theorem 2.5 (Bergantiños et al., 2010). *Let $(M, N, E, C) \in \mathcal{MB}$ such that $|N| \geq 3$ and $|M| \geq 3$. Then, the proportional rule is the unique rule satisfying non-advantageous transfer across issues and non-advantageous transfer within issues.*

Finally, Calleja et al. (2005) characterize the proportional random arrival rule and the queue random arrival rule by means of the so-called *P-consistency* and *Q-consistency* which are extensions of the property of claim-consistency introduced by O'Neill (1982) (see Axiom 2.15). González-Alcón et al. (2007) characterize the composite random arrival rule using issue consistency which is also an extension of claim-consistency. Lorenzo-Freire et al. (2007) use balanced impacts to characterize the random arrival rule for multi-issue bankruptcy problems. However, as far as we know the two-stage random arrival rule has not been characterized yet. In view of the axioms in Section 2.2.4 and the characterizations to which they give rise, an alternative to characterize the two-stage random arrival rule would be to define two new axioms: one related to balanced impacts between issues and another of balanced impacts within issues.

2.3 Multi-issue bankruptcy problems with crossed claims

In this section, we introduce the model of multi-issue bankruptcy problems with crossed claims which is an extension of bankruptcy problems. Moreover, we introduce four allocation rules for these problems, the constrained equal award rule, the constrained proportional rule, constrained sequential priority rules and the constrained random arrival rule, which are extensions of the corresponding allocation rules for bankruptcy problems. The content of this chapter is based on Acosta-Vega et al. (2021a, 2021b, 2022a, 2022b).

2.3.1 Introduction

In this section, we introduce a novel model of multi-issue bankruptcy problem inspired from a real problem of abatement of emissions of different families of pollutants in which pollutants can belong to more than one family. In our model, therefore, several perfectly divisible goods (estates) have to be allocated among certain set of agents (claimants) that have exactly one claim which is used in all estates simultaneously, this model study situations with multi-dimensional states, one for each issue and where each agent claims the same to the different issues in which participates.

To illustrate our model, consider now that a certain authority is interested in reducing the emission of pollutants into the atmosphere. However, there are many pollutants, each with different effects and consequences. There are pollutants that contribute to the greenhouse effect and thus to climate change, and others that are harmful to health because they are carcinogenic, cause respiratory problems or other diseases. On the one hand, water vapor (H2O), carbon dioxide (CO2), nitrous oxide (NO2), methane (CH4), and ozone (O3) are the primary greenhouse effect gases (GHG's), but also sulphur hexafluoride (SF6), hydrofluorocarbons (HFCs) and perfluorocarbons (PFCs) are relevant according to the Kyoto Protocol. On the other hand, carbon monoxide (CO), sulphur dioxide (SO2), nitrous oxide (NO2), ozone (O3), ammonia (NH3), particulate matter (PM), polycyclic aromatic hydrocarbon (PAH) and volatile organic compounds (VOCs) among others are considered very harmful to health. Thus, for example, the Gothenburg Protocol sets emission ceilings for SO2, NO2, VOCs and NH3. Therefore, we can find international, European, national and regional directives, laws, and regulations in order to control their emissions. Some examples are the well-known Paris Agreement or the Kyoto Protocol for the global reduction of GHGs or the Gothenburg Protocol to abate the acidification, eutrophication and ground-level ozone. Furthermore, we can observe that there are gases that contribute to both the greenhouse effect and the air pollution. For details on these and other topics related to the protection of the environment and health visit the webpage https://greenfacts.org.

The entire system for the abatement of pollutants could be represented in a hierarchical structure of two levels (see Figure 2.1). In the first level, we would have the effect of pollutants, and in the second level the pollutants themselves. The ultimate goal of that authority is for emissions per year of the different pollutants to be below certain levels (for example, emitted tons per year) in order to better control the pollution and their effects. In this sense, the authority fixes certain levels of emissions (total tons per year) for each effect of pollutants. However, pollutants could contribute to more than one effect as we have shown above. Thus, we consider the particular situation in which there are different amounts of emissions of different pollutants and the authority fixes maximum levels of emissions (total tons per year) for each effect of pollutants according, for example, to their effects on air quality or contribution to climate change, in order to abate these emissions and keep them below certain levels (tons per year). The approach of setting a level of emissions per year is the usual one in the directives and protocols in this regard, so the particular impact of a pollutant in each effect in the atmosphere is not considered, it is simply a matter of reducing its emissions and with it, its negative impact on air quality or the greenhouse effect. Moreover,

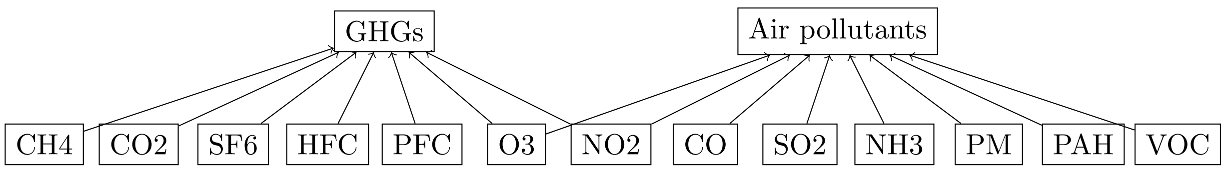

Figure 2.1: Example of a two level hierarchical structure for the abatement of GHGs and air pollutants.

in this context, if we set global emission levels for greenhouse gases and for air polluting gases separately, it seems reasonable that when distributing efforts to abate the different pollutants, more emphasis should be placed on those that are being emitted the most, for example, this is what happens with CO2 or NO2, and therefore, those that are being emitted least are less affected. Therefore, if we allocate the emission quotas among the different pollutants, this allocation would have to be as egalitarian as possible to the quantity claimed by the atmospheric pollutant that pollutes the least to keep controlled the pollution levels. This is reminiscent of the constrained equal award rule (CEA) in bankruptcy problems. Therefore, we have a kind of multi-issue bankruptcy problem which is different from other multi-issue bankruptcy problems as the multi-issue problems as we will explain below. Of course, if we were only interested in a particular effect of pollutants, we obtain a classical bankruptcy problem, where the estate is the amount fixed for the effect (total tons per year) and the claims are the current emission levels of pollutants (tons per year).

This problem remembers a multi-issue bankruptcy problem in which the issues are the effects of pollutants. Multi-issue bankruptcy problems introduced by Calleja et al. (2005) describe situations in which there are a perfect divisible estate which can be divided between various issues, and a number of claimants that have claims on each of those issues[2]. Therefore, there are a perfect divisible estate, several issues, and claimants with vectors of claims with as many coordinates as issues, such that the total amount of claims is above the estate. The central question for these problems is how the estate should be allocated. This problem is solved by means of allocation rules and there are several approaches to it (see, for example, Calleja et al. (2005), Borm et al. (2005), and Izquierdo and Timoner (2016)). However, we have ex ante one estate for each effect on atmosphere, and the claimants are the pollutants, and each pollutant has just one claim which is the same for each

[2] "An issue constitutes a reason on the basis of which the estate is to be divided". (Calleja et al. 2005, page 731.)

of the estates of the effects which it contributes to. This approach is different from the multi-issue bankruptcy problems in the literature.

In our case, we have several perfectly divisible estates, and claimants have exactly one claim which is used in all estates simultaneously. Now, again, the question is how the estate should be allocated in a reasonable way. As far as we know this approach is totally novel in the literature and then fills a gap in the bankruptcy problems that have been dealt until now.

In this context, we introduce three allocation rules and a family of allocation rules for multi-issue bankruptcy problems with crossed claims. We first introduce an allocation rule that generalizes the well-known constrained equal awards rule (CEA) from a procedure derived from analyzing the CEA rule for classical bankruptcy problems as the solution to a succession of linear programming problems. Moreover, we introduce the constrained proportional awards rule, which extends the classical proportional rule in the context of multi-issue bankruptcy problems with crossed claims. And, finally, we introduce constrained sequential priority rules (this is actually a family of rules) and the constrained random arrival rule for multi-issue bankruptcy problems with crossed claims.

2.3.2 Multi-issue bankruptcy problems with crossed claims

We consider a situation where there are a finite set of issues (effects of pollutants) $M = \{1, 2, ..., m\}$ such that each issue j has a perfectly divisible estate e_j (maximum level of emissions for that effect of pollutans). Let $E = (e_1, e_2, ..., e_m)$ be the vector of estates. There are a finite set of claimants (pollutants) $N = \{1, 2, ..., n\}$ such that each claimant i claims c_i (emissions of pollutant i) of those estates which belongs to. Let $c = (c_1, c_2, ..., c_n)$ be the vector of claims. Now, each claimant claims to different set of issues. Thus, α is a set-valued function that associates with every $i \in N$ a subset $\alpha(i) \subset M$. In fact, $\alpha(i)$ represents the issues to which claimant i asks for. Furthermore, $\sum_{i:j\in\alpha(i)} c_i > e_j$, for all $j \in M$, otherwise, those estates could be discarded from the problem because they do not impose any limitation, and so the allocation would be trivial.

Definition 2.14. A *multi-issue bankruptcy problem with crossed claims* (MBC in short) is a 5-tuple (M, N, E, c, α), where M is a finite set of issues, N is a finite set of agents, $E \in \mathbb{R}^M_{++}$ is the vector of estates, $c \in \mathbb{R}^N_{++}$ is the vector of claims, and $\alpha : N \to M$ is a set-valued function. Moreover, the family of all these problems is denoted by $\mathcal{MBC}$.

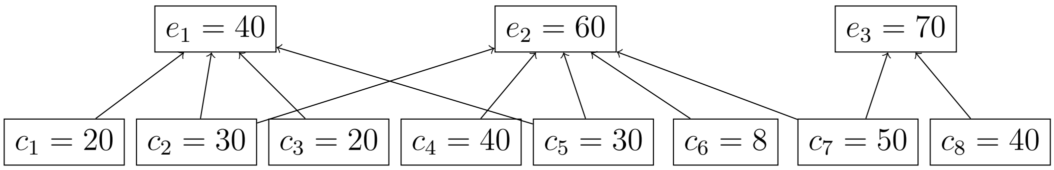

Figure 2.2: Example 2.2.

We illustrate how the structure of these problems is in the following example.

Example 2.2. Consider the following multi-issue bankruptcy problem with crossed claims (M, N, E, c, α) with $M = \{1, 2, 3\}$; $N = \{1, 2, 3, 4, 5, 6, 7, 8\}$; $E = (40, 60, 70)$; $c = (20, 30, 20, 40, 30, 8, 50, 40)$; and $\alpha(1) = \{1\}$, $\alpha(2) = \{1, 2\}$, $\alpha(3) = \{1\}$, $\alpha(4) = \{2\}$, $\alpha(5) = \{1, 2\}$, $\alpha(6) = \{2\}$, $\alpha(7) = \{2, 3\}$, and $\alpha(8) = \{3\}$. This situation is depicted in Figure 2.2.

At first sight, a simplistic approach could be to solve three bankruptcy problems, one for each issue, but this is not so simple, because there are claimants with claims to different issues and this could lead to unfeasible or incompatible allocations. Therefore, a more detailed analysis of this type of problem is necessary.

In Example 2.2, we can clearly observe the structure of our model of multi-issue bankruptcy problem. This model differs from other multi-issue models in three elements. First, there are several issues, each one with their own endowments (this feature would be similar to the approach in Izquierdo and Timoner (2016) to multi-issues bankruptcy problems). Second, we do not have a vector of claims for each issue as in all approaches to multi-issues bankruptcy problems have, but a single vector of claims for all issues simultaneously. And finally, this vector of claims can result in the same claim to be considered in several issues at the same time.

Definition 2.15. Given a problem $(M, N, E, c, \alpha) \in \mathcal{MBC}$, a *feasible allocation* for it, it is a vector $x \in \mathbb{R}^N$ such that:

1. $0 \leq x_i \leq c_i$, for all $i \in N$.

2. $\sum_{i \in N : j \in \alpha(i)} x_i \leq e_j$, for all $j \in M$,

and we denote by $A(M, N, E, c, \alpha)$ the set of all its feasible allocations.

These requirements are similar to those for MB problems. Requirement 1 means that each agent receives at most her claim but not less than nothing.

Requirement 2 means that no estates can be surpassed with the allocation. Therefore, the vector $x \in \mathbb{R}^N$ represents an allocation to the claimants which is simultaneously feasible for all issues.

Definition 2.16. An *allocation rule or simply a rule* for multi-issue bankruptcy problems with crossed claims is a mapping R that associates with every $(M, N, E, c, \alpha) \in \mathcal{MBC}$ a unique feasible allocation $R(M, N, E, c, \alpha) \in A(M, N, E, c, \alpha)$.

Example 2.3. Consider again the MBC problem in Example 2.2. Two possible allocations are the following:

- $R(M, N, E, c, \alpha) = (12.5, 7.5, 12.5, 7.5, 7.5, 7.5, 30, 40)$.
- $R(M, N, E, c, \alpha) = (13.75, 6.25, 13.75, 6.25, 6.5, 6.25, 35, 35)$.

Both allocations given in Example 2.3 satisfy the two requirements and, additionally, it is easy to check they are efficient for all issues, i.e., Requirement 2 is satisfied with equality. However, this is not possible in general as the following example shows.

Example 2.4. Consider again the MBC in Example 2.2 but only changing $c_7 = 65$ and $e_3 = 105$. In this situation, it is obvious that if an allocation is efficient for issue 3, then it is unfeasible for issue 2.

Therefore, in view of Example 2.4, we cannot make Requirement 2 more demanding if we want to achieve at least one possible allocation.

2.3.3 The constrained equal awards rule for MBC problems

As mentioned in previous sections, the constrained equal awards rule (CEA) is one of the main rules to solve bankruptcy problems (see Herrero and Villar, 2001). This rule simply divides as equally as possible the estate among the claimants. The question here is what as equally as possible means. In the context of one-issue bankruptcy problems, as equitably as possible means that no claimant can get more than those with smaller claims, except that the latter have already received their entire claim. As shown in Subsection 2.1.3, this can be formulated mathematically as follows:

For each $(N, E, c) \in \mathcal{B}$,

$$CEA_i(N, E, c) = \min\{c_i, \beta\}, \qquad i \in N, \tag{2.1}$$

where β is a positive real number satisfying $\sum_{i \in N} CEA_i(N, E, c) = E$.

The constrained equal awards rule (CEA) is one of the main rules to solve bankruptcy problems. This rule simply divides as equally as possible the estate among the claimants. The question here is how to extrapolate this to the MBC situations. To do this, in this thesis, we introduce the CEA rule as the result of the optimal solution of a succession of linear programs[3].

Given a problem $(N, E, c) \in \mathcal{B}$, in order to allocate E among the claimants according to CEA, we proceed as follows:

$$(P_1) \quad \begin{array}{ll} max & z^1 \\ s.a: & \sum_{i\in N} x_i \leq E \\ & x_i \leq c_i, \text{ for all } i \in N \\ & x_i \geq z^1, \text{ for all } i \in N \\ & x_i \geq 0, \text{ for all } i \in N, \text{ and } z^1 \geq 0 \end{array}$$

Let z^{*1} be the optimal value of the linear problem (P_1). If $nz^{*1} = E$, then $CEA_i(N, E, c) = z^{*1}$. Otherwise, the following linear problem must be solved:

$$(P_2) \quad \begin{array}{ll} max & z^2 \\ s.a: & \sum_{i\in N} x_i \leq E \\ & x_i \leq c_i, \text{ for all } i \in N \\ & x_i \geq z^{*1} + \mu^0(c_i - z^{*1})z^2, \text{ for all } i \in N \\ & x_i \geq 0, \text{ for all } i \in N, \text{ and } z^2 \geq 0 \end{array}$$

where for each $a \in \mathbb{R}$,

$$\mu^0(a) = \begin{cases} 0 & \textit{if } a \leq 0, \\ 1 & \textit{otherwise.} \end{cases}$$

Let z^{*2} be the optimal value of the linear problem (P_2). If $nz^{*1} + \sum_{i\in N} \mu^0(c_i - z^{*1})z^{*2} = E$, then $CEA_i(N, E, c) = z^{*1} + \mu^0(c_i - z^{*1})z^{*2}$. Otherwise, a new linear problem must be solved. In the general step k, we have the following linear problem:

$$(P_k) \quad \begin{array}{ll} max & z^k \\ s.a: & \sum_{i\in N} x_i \leq E \\ & x_i \leq c_i, \text{ for all } i \in N \\ & x_i \geq \sum_{h=1}^{k-1} \mu^0(c_i - \sum_{l=1}^{h-1} z^{*l})z^{*h} + \mu^0(c_i - \sum_{l=1}^{k-1} z^{*l})z^k, \text{ for all } i \in N \\ & x_i \geq 0, \text{ for all } i \in N, \text{ and } z^k \geq 0 \end{array}$$

[3]We can find in Izquierdo and Timoner (2016) another way to obtain CEA by using a quadratic optimization problem.

Again, let z^{*k} be the optimal value of the linear problem (P_k). If $\sum_{i\in N}\sum_{h=1}^{k}\delta^0(c_i-\sum_{l=1}^{h-1}z^{*l})z^{*h}=E$, then $CEA_i(N,E,c)=\sum_{h=1}^{k}\mu^0(c_i-\sum_{l=1}^{h-1}z^{*l})z^{*h}$. Otherwise, the linear problem (P_{k+1}) must be solved. And so on and so forth, until the estate is fully distributed or all claims are completely granted. It is obvious that this procedure ends in a finite number of steps and the final allocation is CEA.

Note that when the last linear problem of the procedure is solved, then we have that its optimal solution is $x^{*k}=CEA(N,E,c)$. We illustrate this procedure in the following example.

Example 2.5. Consider $(N,E,c)\in\mathcal{B}$ with $N=\{1,2,3,4,5,6,7,8\}$; $E=170$; $c=(20,30,20,40,30,8,50,40)$. We now apply the procedure described above to calculate the CEA rule of this problem.

1. First we solve (P_1). The optimal value of this linear problem is $z^{*1}=8$. Therefore, all claimants receive 8 units of the estate. In total 64 units of the estate have been distributed, therefore another round is necessary. Since claimant 6 has obtained her claim, this will not take part in the distribution of the estate in the next step.

2. In this step,we first guarantee claimants all what they have obtained until the previous step. The optimal value of (P_2) is $z^{*2}=12$. Thus, all claimants except claimant 6 are allocated 12 extra units of the estate. In total 148 units of the estate have been distributed, therefore another round is necessary. Since claimants 1 and 3 have already obtained their claims, these will not take part in the distribution of the estate in the next step.

3. Again, we first guarantee claimants all what they have obtained until the previous step. The optimal value of (P_3) is $z^{*3}=4.4$. Thus, all claimants except claimants 1, 3 and 6 are allocated 4.4 extra units of the estate. Since the estate has been fully distributed, the procedure ends and $CEA(N,E,c)=(20,24.4,20,24.4,24.4,8,24.4,24.4)$.

Note that this procedure perfectly fits to the following description of CEA in Thomson (2015):

At first, equal division takes place until each claimant receives an amount equal to the smallest claim. The smallest claimant drops out, and the next increments of the endowment are divided equally among the others until each of them receives an amount equal to the second smallest claim. The second smallest claimant drops out, and so on.

Therefore, following this same procedure we can define the *doubly constrained equal award rule*, CCEA, for MBC problems. Given a problem $(M, N, E, c, \alpha) \in \mathcal{MBC}$, in the general step k of the procedure, the linear problem to be solved is given by:

$$(P_k) \quad \begin{array}{ll} max & z^k \\ s.a: & \sum_{i \in N: j \in \alpha(i)} x_i \leq e_j, \text{ for all } j \in M \\ & x_i \leq c_i, \text{ for all } i \in N \\ & x_i \geq \sum_{h=1}^{k-1} \mu^0(a_i^h) z^{*h} + \mu^0(a_i^k) z^k, \text{ for all } i \in N \\ & x_i \geq 0, \text{ for all } i \in N, \text{ and } z^k \geq 0 \end{array}$$

where

$$a_i^h = \min \left\{ c_i - \sum_{l=1}^{h-1} \mu^0(a_i^l) z^{*l}, min_{j \in \alpha(i)} \left\{ e_j - \sum_{t \in N: j \in \alpha(t)} \sum_{s=1}^{h-1} \mu^0(a_t^s) z^{*s} \right\} \right\}.$$

Note that a_i^h measures whether claimant i can take part in the distribution of step h taking into account what she received before and whether there is still something to distribute in every issue she claims.

In this case, the procedure also ends in a finite number of steps, but not necessarily when all estates are fully distributed or all claims are completely granted. In this situation the procedure stops when $z^{*k} = 0$, and it holds that

$$CCEA_i(M, N, E, c, \alpha) = \sum_{h=1}^{k} \mu^0(a_i^h) z^{*h}.$$

Likewise, we also have that for the last linear problem solved its optimal solution x^{*k} is exactly $CCEA(M, N, E, c, \alpha)$.

It is important to emphasize that we have introduced CCEA rule for bankruptcy problems as the solution to a succession of linear programming problems and we have extended this procedure to MBC. So, it is easy to check that when we have exactly only one issue both coincide. The only difference is that we additionally take into account how much remains in each of the estates to which claimants ask for. However, this procedure for MBC problems does not guarantee that all estates are fully distributed, even when they could be. The next example illustrates this.

Example 2.6. Consider again the MBC problem in Example 2.2. We now calculate the CCEA rule of this problem by applying the procedure described above.

1. The optimal value of (P_1) is $z^{*1} = 8$ and this amount is allocated to each claimant. None of the estates have been fully distributed, so a next step is necessary. However, claimant 6 has obtained her claim, so this will not take part in the distribution of the estates in the next step.
2. In this step, we guarantee claimants all what they have obtained until the previous step. The optimal value of (P_2) is $z^{*2} = 2$, and this amount is allocated to all claimants except claimant 6. Now, the estate e_1 has been fully distributed, therefore, claimants requesting part of this estate cannot continue to receive anything else. Otherwise, the quantity available in that estate would be exceeded. However, the other two estates have not been fully distributed, hence another step is needed.
3. In this step, we guarantee claimants all what they have obtained until the previous step. The optimal value of (P_3) is $z^{*3} = 6$, and this amount is only allocated to claimants 4, 7, and 8. Now, the estate e_2 has been fully distributed, so claimants 4, and 7 cannot continue to receive anything else. Again, otherwise, the quantity available in that estate would be exceeded.
4. In this step, we again guarantee claimants all what they have obtained until the previous step, and claimant 8 is the only one that can receive something else in this step. The optimal value of (P_4) is $z^{*4} = 24$, and this amount is only allocated to claimant 8. However, the estate e_3 has not been fully distributed, in fact, there are still 14 units to be distributed. Furthermore $x^{*4} = (10, 10, 10, 16, 10, 8, 16, 40)$ that coincides with $CCEA(M, N, E, c, \alpha)$.

$CCEA(M, N, E, c, \alpha)$ is calculated in four steps, and the allocations in each step are the following:

Claimant	*Step* − 1	*Step* − 2	*Step* − 3	*Step* − 4	*Row* − *total*
1	8	2	−	−	10
2	8	2	−	−	10
3	8	2	−	−	10
4	8	2	6	−	16
5	8	2	−	−	10
6	8	−	−	−	8
7	8	2	6	−	16
8	8	2	6	24	40
Column − *total*	64	14	18	24	120

This allocation does not fully distribute all estates, but it is possible to obtain feasible allocations for this particular MBC problem which do. For example, the allocations given in Example 2.3 fully distribute all estates for this problem, but they are not as egalitarian as this. Furthermore, although the claimants have received 120 units in total, of the 170 units that make up the three estates, 156 have been actually distributed. This difference is because some claimants asked for in several issues simultaneously.

If we look carefully at the application of the procedure in Example 2.6, we observe that first a estate is fully distributed, then another estate is completely distributed, and finally the last estate cannot be distributed in its entirety. Therefore, we can design another procedure based on the CEA rule itself which follows this scheme. We illustrate it in the following example.

Example 2.7. Consider once again the MBC problem in Example 2.2. In order to calculate $CCEA(M, N, E, c, \alpha)$, we proceed as follows:

1. First we calculate the CEA rule for each of the three bankruptcy problems defined by each issue.
   - $(N^{1,1}, E^{1,1}, c^{1,1})$. $N^{1,1} = \{1, 2, 3, 5\}$, $E^{1,1} = 40$, and $c^{1,1} = (20, 30, 20, 30)$. $CEA(N^{1,1}, E^{1,1}, c^{1,1}) = (10, 10, 10, 10)$, and $\beta^{1,1} = 10$.
   - $(N^{2,1}, E^{2,1}, c^{2,1})$. $N^{2,1} = \{2, 4, 5, 6, 7\}$, $E^{2,1} = 60$, and $c^{2,1} = (30, 40, 30, 8, 50)$. $CEA(N^{2,1}, E^{2,1}, c^{2,1}) = (13, 13, 13, 8, 13)$, and $\beta^{2,1} = 13$.
   - $(N^{3,1}, E^{3,1}, c^{3,1})$. $N^{3,1} = \{7, 8\}$, $E^{3,1} = 70$, and $c^{3,1} = (50, 40)$. $CEA(N^{3,1}, E^{3,1}, c^{3,1}) = (35, 35)$, and $\beta^{3,1} = 35$.
2. Next, we take $\beta^{*1} = \min\{\beta^{1,1}, \beta^{2,1}, \beta^{3,1}\} = 10$, and we allocate each claimant i $\min\{c_i, \beta^{*1}\}$. Therefore, we obtain the allocation vector $(10, 10, 10, 10, 10, 8, 10, 10)$.

Now, it is obvious that estate e_1 has been fully distributed, and in the next step this bankruptcy problem and the claimants associated with it are excluded. Moreover, claimant 6 is also excluded because she has got her claim. The other two problems are updated in claimants, and decreasing estates and claims according to the allocation previously obtained.

1. We calculate the CEA rule for each of the two bankruptcy problems remaining.
   - $(N^{2,2}, E^{2,2}, c^{2,2})$. $N^{2,2} = \{4, 7\}$, $E^{2,2} = 12$, and $c^{2,2} = (30, 40)$. $CEA(N^{2,2}, E^{2,2}, c^{2,2}) = (6, 6)$, and $\beta^{2,2} = 6$.
   - $(N^{3,2}, E^{3,2}, c^{3,2})$. $N^{3,2} = \{7, 8\}$, $E^{3,2} = 50$, and $c^{3,2} = (40, 30)$. $CEA(N^{3,2}, E^{3,2}, c^{3,2}) = (25, 25)$, and $\beta^{3,2} = 25$.
2. Next, we take $\beta^{*2} = \min\{\beta^{2,2}, \beta^{3,2}\} = 6$, and we allocate each claimant i $\min\{c_i, \beta^{*2}\}$. Therefore, we obtain the allocation vector $(0, 0, 0, 6, 0, 0, 6, 6)$.

Now, it is obvious that estate e_2 has been fully distributed, and in the next step this bankruptcy problem and the claimants associated with it are excluded. The third problem is updated in claimants, and decreasing estates and claims according to the allocation previously obtained.

1. We calculate the CEA rule for each of the bankruptcy problem remaining.
   - $(N^{3,3}, E^{3,3}, c^{3,3})$. $N^{3,3} = \{8\}$, $E^{3,3} = 38$, and $c^{3,3} = (24)$. $CEA(N^{3,3}, E^{3,3}, c^{3,3}) = (24)$, and $\beta^{3,3} = 24$.
2. Next, we take $\beta^{*3} = \min\{\beta^{3,3}\} = 24$, and we allocate each claimant i $\min\{c_i, \beta^{*3}\}$. Therefore, we obtain the allocation vector $(0, 0, 0, 0, 0, 0, 0, 24)$.

The procedure stops because either estates have been completely distributed or claimants have obtained their claims. Finally, by adding the allocation vectors obtained in the procedure, we obtain that $CCEA(M, N, E, c, \alpha) = (10, 10, 10, 16, 10, 8, 16, 40)$.

The procedure based on the CEA rule is as follows. In general step k, we calculate the CEA rule for all bankruptcy problems defined by the available estates in the step k, for each we determine the values $\beta^{j,k}$, and take the minimum β^{*k} of all of them. We allocate β^{*k} to all active claimants in step k and nothing to the others. Next, we update the problem by revising downwards estates, claimants, and claims. After updating the problem, if no bankruptcy problem can be defined, we stops. Otherwise we go to the next step with the updated problem, and so on. Finally, the CCEA rule of the MBC is the sum of all allocation vectors obtained.

The following theorem states that both procedures introduced coincide for all MBC problems.

Theorem 2.6. *Given a problem $(M, N, E, c, \alpha) \in \mathcal{MBC}$, the allocation vectors obtained by the procedure based on linear programming and the procedure based on the CEA rule of classical bankruptcy problems coincide, and their outcome corresponds to the rule $CCEA(M, N, E, c, \alpha)$.*

An interesting property of CEA for bankruptcy problems is that it is the most egalitarian allocation in the sense of Lorenz. This result can be extended to the context of MBC problems. To do this, we first introduce the well-known concept of Lorenz dominance adapted to the context of multi-issue bankruptcy problems with crossed claims.

Given a problem $(M, N, E, c, \alpha) \in \mathcal{MBC}$, and two feasible vectors $x, y \in \mathbb{R}^N_+$, we say that x *Lorenz weakly dominates* y, $x \succeq_{wL} y$, if $\sum_{j=1}^{k} x_{(j)} \geq \sum_{j=1}^{k} y_{(j)}$, for all $k = 1, 2, ..., h$, $h \leq n$, where for a vector $z \in \mathbb{R}^N_+$, $z_{(1)}, \ldots, z_{(n)}$ represent its coordinates rewritten in increasing order.

Theorem 2.7. *Given a problem $(M, N, E, c, \alpha) \in \mathcal{MBC}$, the rule $CCEA(M, N, E, c, \alpha)$ Lorenz weakly dominates all feasible allocations.*

2.3.4 The constrained proportional rule for MBC problems

The proportional rule (PROP) is perhaps the most important rule to solve allocation problems in general, and bankruptcy problems in particular. This rule simply divides the resource in proportion to the claims. The question in MBC problems is what "in proportion to the claims" means. In the context of one-issue bankruptcy problems, "in proportion to the claims" means that all claimants receive the same amount for each unit of claim. How to extrapolate this to the MBC situations. To answer this question, we introduce the constrained proportional awards rule (CPA in short) as the result of an iterative process in which the available amount of at least one of the issues is fully distributed in each step and so on and so forth while possible. This rule is formally defined below.

Definition 2.17. Let $(M, N, E, c, \alpha) \in \mathcal{MBC}$, the *constrained proportional awards rule* for (M, N, E, c, α), $CPA(M, N, E, c, \alpha)$, is defined by means of the following iterative procedure:

Step 0.
1. $M^1 = \{j \in M : e_j^1 > 0\}$ is the set of active issues.
2. $\mathcal{N}^1 = \{i \in N : c_i^1 > 0$ and $e_j^1 > 0, \forall j \in \alpha(i)\}$ is the set of active claimants.
3. For each $j \in M$, $e_j^1 = e_j$, and for each $i \in N$, $c_i^1 = c_i$.

Step s.
1. $\mathcal{N}^s = \{i \in N : c_i^s > 0$ and $e_j^s > 0, \forall j \in \alpha(i)\}$. $M^s = \{j \in M : e_j^s > 0\}$.
2. For each $j \in M^s$, we calculate the greatest λ_j^s, so that $\lambda_j^s \sum_{i \in \mathcal{N}^s : j \in \alpha(i)} c_i^s \leq e_j^s$, and take $\lambda^s = \min\{\lambda_j^s : j \in M^s\}$.
3. Now, we allocate to each claimant $i \in M^s$, $a_i^s = \lambda^s c_i^s$, and $a_i^s = 0$ to the non-active claimants
4. We update the active issues, M^{s+1}, and the active claimants, $\mathcal{N}^{s+1}$. If $M^{s+1} = \varnothing$ or $\mathcal{N}^{s+1} = \varnothing$, then the process ends, and

$$CPA_i(M, N, E, c, \alpha) = \sum_{h=1}^{s} a_i^h, \forall i \in N.$$

Otherwise, the available amounts of issues and the claims are updated:

$$e_j^{s+1} = e_j^s - \lambda^s \sum_{i \in N : j \in \alpha(i)} c_i^s, \forall j \in M, \text{ and } \quad c_i^{s+1} = c_i^s - \lambda^s c_i^s, \forall i \in N,$$

and we go to Step $s+1$

The iterative procedure of CPA is well-defined and always leads to a single point. Moreover, since in each step at least the available amount of one issue is distributed in its entirety, except maybe in the last step, it ends in a finite number of steps, at most $|M|$. Finally, when we have a one-issue bankruptcy problem, then we obtain PROP. Therefore, this definition extends PROP to the context of MBC.

From the application of the iterative process to calculate CPA, we can consider the chains of active issues and active claimants in the application of the procedure to calculate $CPA(M, N, E, c, \alpha) \in \mathcal{MBC}$:

$$M^1 \supset M^2 \supset \ldots \supset M^r, \text{ and } \mathcal{N}^1 \supset \mathcal{N}^2 \supset \ldots \supset \mathcal{N}^r$$

From these chains, we can establish an order relationship between issues as follows. We say that issue j_1 strictly precedes issue j_2 in a chain of actives issues, $j_1 \prec j_2$, if there is M^s such that $j_1 \notin M^s$ and $j_2 \in M^s$, i.e., j_1 becomes non-active before than j_2. We write $j_1 \preceq j_2$ when j_1 becomes non-active before than j_2 or both issues become non-active at the same time. Finally, we write $j_1 \simeq j_2$ when both issues become non-active at the same time. Analogously, we can establish an order relationship between claimants.

Furthermore, we can associate with each pair of sets M^s and $\mathcal{N}^s$ a number ρ^s, $\rho^s \in [0,1]$, which represents the proportion of claims obtained by claimants in $\mathcal{N}^s$ but not in $\mathcal{N}^{s+1}$. Moreover, by construction $\rho^s < \rho^{s+1}$. Thus, we have that

$$0 < \rho^1 < \rho^2 < \ldots < \rho^r \leq 1$$

These $\rho' s$ represent the accumulative proportion of the claims allocated to the claimants, i.e., what part of their claims they have received up to a given step of the iterative procedure. In this way, this procedure is reminiscent of the constrained equal awards rule (CEA) in bankruptcy problems, but instead of using the principle of egalitarianism, the principle of proportionality is used, hence the name of *constrained proportional awards* rule. Therefore, not all claimants receive the same proportion of their claims, but the rule tries to keep the proportionality as much as possible restricted to (1) the relation between the available amounts of issues and the total claims to them, and (2) the goal of allocating as much as possible of all available amounts of issues.

Below some results about the iterative process that defines CPA are given.

Proposition 2.1. *The following statements hold*

1. *Given* $(M, N, E, c, \alpha) \in \mathcal{MBC}$*, if there are problems* $(M_1, N_1, E_1, c^1, \alpha_1)$, $(M_2, N_2, E_2, c^2, \alpha_2) \in \mathcal{MBC}$*, such that* $M_1 \cup M_2 = M$*,* $N_1 \cup N_2 = N$*,* $E_1 \oplus E_2 = E$*,* $c^1 \oplus c^2 = c$*, and* $\alpha_1(i) = \alpha(i), \forall i \in N_1$ *and* $\alpha_2(i) = \alpha(i), \forall i \in N_2$*, so that* $\left(\bigcup_{i \in N_1} \alpha(i)\right) \bigcap \left(\bigcup_{i \in N_2} \alpha(i)\right) = \varnothing$*, then*

   $$CPA(M, N, E, c, \alpha) = CPA(M_1, N_1, E_1, c^1, \alpha_1) \oplus CPA(M_2, N_2, E_2, c^2, \alpha_2).$$ [4]

2. *If* $\lambda^s < 1$*, then each* $h \in \arg\min\{\lambda^s_j : j \in M^s\} \subset M$ *becomes non-active in the next step.*

3. *If* $\lambda^s = 1$ *for some* s*, then the iterative procedure ends in that step.*

The first statement says that if a problem can be separated into two disjoint problems, then it is the same to calculate CPA for the whole problem as for each of them and then paste the results. The second states when an issue becomes non-active. Finally, the third provides another stopping criterion for the iterative procedure to calculate CPA. Moreover, when the procedure ends with $\lambda = 1$, then it means that there may be resources left over from some issues. Otherwise, all resources have been fully distributed.

2.3.5 Constrained sequential priority rules and the constrained random arrival rule for MBC problems

Sequential priority rules are defined when there is a priority order defined ex ante over the set of claimants according to some criterion, in such a way that if a claimant has a higher priority than another, the first must be satisfied first in her demand with as much of the resource, the second is satisfied with as much of the resource as is left, and so on until the resource is exhausted. Therefore, this rule simply allocates the resource according to the scheme of first come first served, where the order of arrival is given by the priority relationship. The question in MBC problems is what "be satisfied with as much of the resource as is left" means. In the context of one-issue bankruptcy problems, "be satisfied with as much of the resource as is left" is a simple idea since there is only one issue. How to extrapolate this to the MBC situations. To answer this question, we introduce the constrained sequential priority rule (CSP in short) which follows the same process as sequential priority rules but taking into account that there are several resources or issues. This rule is formally defined below.

[4] Given $a \in \mathbb{R}^n$ and $b \in \mathbb{R}^m$, $a \oplus b \in \mathbb{R}^{n+m}$, i.e., $\oplus$ is the concatenation operator of two vectors.

Definition 2.18. Let $(M, N, E, c, \alpha) \in \mathcal{MBC}$, and $\sigma \in \Sigma(N)$, the *constrained sequential priority rule* associated with $\sigma \in \Sigma(N)$ for (M, N, E, c, α), $CSP^{\sigma}(M, N, E, c, \alpha)$, is defined as follows:

$$CSP_j^{\sigma}(M,N,E,c,\alpha) = min\left\{c_i,\ max\left\{0,\ \min_{i\in\alpha(j)}\{e_i - \sum_{\substack{k\in N: i\in\alpha(k)\\ \sigma(k)<\sigma(j)}} c_k\}\right\}\right\}, \forall j \in N,$$

where $\Sigma(N)$ is the set of all possible orders of N.

For each $\sigma \in \Sigma(N)$, the iterative procedure of CSP^{σ} is well-defined and always leads to a single point. Moreover, by definition, it ends in a finite number of steps, at most $|N|$. Finally, when we have a one-issue bankruptcy problem, then we obtain SP^{σ}. Therefore, this definition extends sequential priority rules to the context of MBC. The following example illustrates how CSP works.

Example 2.8. Consider the following MBC situation with $M = \{1, 2, 3\}$, $N = \{1, 2, 3, 4, 5, 6, 7, 8\}$, $E = (9, 12, 9)$, $c = (3, 5, 4, 3, 5, 4, 3, 5)$, and $\alpha(1) = \{1\}, \alpha(2) = \{1\}, \alpha(3) = \{1, 2\}, \alpha(4) = \{1, 2\}, \alpha(5) = \{2\}, \alpha(6) = \{2, 3\}, \alpha(7) = \{2, 3\}, \alpha(8) = \{3\}$. If we take for example the priority order $\sigma = 1\,3\,5\,7\,2\,4\,6\,8$, CSP^{σ} is calculated sequentially as follows:

First, claimant 1 is attended:

$$CSP_1^{\sigma}(M, N, E, c, \alpha) = 3.$$

Second, the resources are updated down $E = (6, 12, 9)$ and claimant 3 is attended:

$$CSP_3^{\sigma}(M, N, E, c, \alpha) = 4.$$

Third, the resources are updated down $E = (2, 8, 9)$ and claimant 5 is attended:

$$CSP_5^{\sigma}(M, N, E, c, \alpha) = 5.$$

Fourth, the resources are updated down $E = (2, 3, 9)$ and claimant 7 is attended:

$$CSP_7^{\sigma}(M, N, E, c, \alpha) = 3.$$

Fifth, the resources are updated down $E = (2, 0, 6)$ and claimant 2 is attended:

$$CSP_2^{\sigma}(M, N, E, c, \alpha) = 2.$$

Sixth, the resources are updated down $E = (0, 0, 6)$ and claimant 4 cannot be attended because the resources she claims are exhausted, as a result she gets 0. Therefore, we move to the next in the priority order. Again, claimant 6 cannot be attended because one of the resources she claims is exhausted, as a result she gets 0. So, we move to the next. Claimant 8 is attended:

$$CSP_8^{\sigma}(M, N, E, c, \alpha) = 5.$$

Seventh, the resources are updated down $E = (0, 0, 1)$ and the sequential procedure ends. The final allocation is

$$CSP^{\sigma}(M, N, E, c, \alpha) = (3, 2, 4, 0, 5, 0, 3, 5).$$

Once constrained sequential priority rules associated with an order have been defined, the constrained random arrival rule is simply defined as their average.

Definition 2.19. Let $(M, N, E, c, \alpha) \in \mathcal{MBC}$, the *constrained random arrival rule* for (M, N, E, c, α), $CRA(M, N, E, c, \alpha)$, is defined as follows:

$$CRA_j(M, N, E, c, \alpha) = \frac{1}{n!} \sum_{\sigma \in \Sigma(N)} CSP_j^{\sigma}(M, N, E, c, \alpha),\ \forall j \in N,$$

where $\Sigma(N)$ is the set of all possible orders of N.

Although the definition of the rule is simple, it is immediately apparent that the biggest problem is in its computation. For Example 2.8, it would be necessary to calculate 8!=40320 sequential processes, which consumes a large amount of time.

2.4 Properties and characterizations of allocation rules for multi-issue bankruptcy problems with crossed claims

In this section we introduce a set of properties or axioms suitable for allocation rules in the context of bankruptcy problems. Each of the properties reflects a principle that we would like an allocation rule to satisfy. In general, the properties introduced here are suitable adaptations of the corresponding properties for allocation rules in bankruptcy problems which are fully accepted by the scientific community in the field of resource allocation and distribution problems. The allocation rules introduced in Chapter 4 will be studied axiomatically using these properties, obtaining characterizations for the constrained equal awards rule and the constrained proportional rule, but not for the sequential priority rules and the constrained random arrival rule. The content of this chapter is also based on Acosta-Vega et al. (2021a, 2021b, 2022a, 2022b).

2.4.1 Introduction

The problem of distributive justice has been treated from antiquity to the present day (see, for example, Fleischacker (2005) for a brief history of distributive justice). The question of whether or not a distribution is fair is difficult to answer from a sociopolitical perspective. However, if we analyze an allocation or distribution method from the perspective of the properties it satisfies, it is perhaps easier to conclude whether the method is better or worse in terms of those properties (or axioms). This type of analysis is commonly carried out from a mathematical perspective, in which the properties are presented under a mathematical formulation that formally describes some property or characteristic that is relevant to a distribution rule in a certain context. Two interesting books in this regard are Roemer (1996) and Binmore (2011).

Why is the axiomatic approach important in the analysis of a distribution, sharing or allocation problem? First, because it allows an analytical analysis of which properties satisfy the different distribution rules that are on the table and, therefore, which of them may be better in terms of the properties that are considered most relevant to the problem to be solved. Second, because from the properties that are considered relevant to the problem, a new rule can be defined that meets these properties and that was not initially on the table. Therefore, the axiomatic analysis of a problem can help to choose the best solution among the different alternatives or even to explicitly define which distribution rule should be the solution to the problem.

What properties are desirable to satisfy a distribution rule? The answer is that it depends on the problem. However, there are numerous properties in the literature that respond to different ideas and concepts. Five groups of properties have been considered in this chapter:

1. related to efficiency in the distribution of resources;
2. related to impartiality and equity;
3. related to minimal guarantees to claimants;
4. related to monotonicity;
5. and related to robustness.

Each of the rules introduced in the previous chapter are analyzed axiomatically using axioms from the previous groups. Axiomatic characterizations for the CCEA and CPA rules are presented, but a characterization for CRA has not yet been achieved.

2.4.2 Properties

In this subsection, we present several properties which are interesting in the context of MBC problems. These properties are related to efficiency, fairness, robustness or monotonicity, among others.

Before starting with the properties considered in this memory, we define two concepts related to the comparison between claimants. In our context, claimants are characterized by two features: their claims and the issues to which they claim. Therefore, both should be taken into account when we want to establish similarities between claimants. The following definition establish when two claimants are considered directly comparable and when equals.

Definition 2.20. Let $(M, N, E, c, \alpha) \in \mathcal{MBC}$, and two claimants $j, k \in N$, we say they are *homologous*, if $\alpha(j) = \alpha(k)$; and we say that they are *equal*, if they are homologous and $c_j = c_k$.

In the following subsections, we give a set of properties which are very natural and reasonable for an allocation rule in the context of multi-issue bankruptcy problems with crossed claims.

Properties related to efficiency in the distribution of resources

The properties in this subsection are related to how resources should be applied, but not to fairness or equity. Each of the axioms show when a distribution of resources is possible or more economically efficient from the point of view of the distribution itself, not of whether or not it is fair.

Axiom 2.31 (F)**.** *Given a rule R, it satisfies* feasibility*, if for every problem* $MBC = (M, N, E, c, \alpha) \in \mathcal{MBC}$*,* $\sum_{i:j\in\alpha(i)} R_i(M, N, E, c, \alpha) \leq e_j$*, for all* $j \in M$*.*

F means that no more than the estates can be distributed. In fact, this property is included in the definition of rule.

Axiom 2.32 (EFF)**.** *Given a rule R, it satisfies* efficiency*, if for every problem* $(M, N, E, c, \alpha) \in \mathcal{MBC}$*,* $\sum_{i:j\in\alpha(i)} R_i(M, N, E, c, \alpha) = e_j$*, for all* $j \in M$*.*

EFF simply says that the estates must be fully distributed, we know that this property is very demanding in the context of multi-issue bankruptcy problems with crossed claims as Example 2.4 shows. However, a weaker version of efficiency can be defined by considering that at least one estate to be fully distributed. This property can be defined as follows:

Axiom 2.33 (WEFF). *Given a rule R, it satisfies* weak efficiency*, if for every problem $(M, N, E, c, \alpha) \in \mathcal{MBC}$, $\sum_{i:j\in\alpha(i)} R_i(M, N, E, c, \alpha) = e_j$, for some $j \in M$.*

A weaker version of efficiency can be defined by considering Pareto efficiency. A feasible allocation is Pareto efficient if there is no other feasible allocation in which some individual is better off and no individual is worse off. Formally, given a problem $(M, N, E, c, \alpha) \in \mathcal{MBC}$, an allocation $x \in A(M, N, E, c, \alpha)$ is *Pareto efficient* if there is no other allocation $x' \in A(M, N, E, c, \alpha)$ such that $x'_i \geq x_i, \forall i \in N$, with at least one strict inequality. Now Pareto efficiency is defined as follows:

Axiom 2.34 (PEFF). *Given a rule R, it satisfies* Pareto efficiency*, if for every problem $(M, N, E, c, \alpha) \in \mathcal{MBC}$, $R(M, N, E, c, \alpha)$ is Pareto efficient.*

Note that $PEFF$ implies that at least the available amount of one issue is fully distributed, and no amount is left of an issue undistributed, if it is possible to do so. However, it does not require that all available amounts of the issues have to be fully distributed.

Properties related to impartiality and equity

The properties in this subsection are related to how should claimants be treated excluding differentiating between agents on the basis of characteristics exogenous to the problem itself, for example, names, gender, age, etc.

Axiom 2.35 (AN). *Given a rule R, it satisfies* anonymity*, if for every problem $(M, N, E, c, \alpha) \in \mathcal{MBC}$, $R_i(M, N, E, c, \alpha) = R_{\sigma(i)}(M, N, E, \sigma(c), \alpha \circ \sigma^{-1})$ for each $i \in N$, where σ is a permutation of N.*

AN means that the allocation does not depend on the name of the claimants but only on their characteristics with respect to the problem.

Axiom 2.36 (ETE). *Given a rule R, it satisfies* equal treatment of equals*, if for every problem $(M, N, E, c, \alpha) \in \mathcal{MBC}$ and every pair of equal claimants $i, j \in N$, $R_i(M, N, E, c, \alpha) = R_j(M, N, E, c, \alpha)$.*

ETE is related to impartiality and says that claimants with the same claims and the same set of issues must be treated equally in the final allocation. This is a basic requirement of fairness and non-arbitrariness.

On the other hand, a property that appears recurrently related to the Shapley value and, therefore, to the random arrival rule is that of balanced contributions (Myerson, 1980; Hart and Mas-Colell, 1989). In the context of bankruptcy problems this property was used by Bergantiños and Mendez-Naya (1997) to characterize the random arrival rule. Moreover, in the context of multi-issue bankruptcy problems was introduced by Lorenzo-Freire et al. (2007) and also used to characterize the random arrival rule.

Axiom 2.37 (BAL). *Given a rule R, it satisfies* balanced impact*, if for every problem $(M, N, E, c, \alpha)\mathcal{MBC}$, and every pair of claimants $j, k \in N$,*

$$R_j(M, N, E, c, \alpha) - R_j(M^{-k}, N, E^{-k}, c_{-k}, \alpha) = \\ R_k(M, N, E, c, \alpha) - R_k(M^{-j}, N, E^{-j}, c_{-j}, \alpha),$$

where M^{-h} is the set of issues for which claimants in $N\backslash\{h\}$ have claims; $E^{-h} = (e_1^{-h}, \ldots, e_m^{-h})$ so that $e_i^{-h} = e_i - c_h$ if $i \in \alpha(h)$ and $e_i^{-h} = e_i$ otherwise; and c_{-h} is the vector of claims from which $h-th$ coordinate has been deleted.

BAL requires that claimant j impacts to claimant k's allocation what claimant k impacts to claimant j's allocation.

Properties related to minimal guarantees to claimants

These properties try to establish what amount a claimant should receive or what amount should be guaranteed to each claimant at least under certain reasonable conditions. In this sense, these properties are related to the guarantees that claimants receive when an allocation rule is applied. Therefore, those allocation rules that satisfy this type of properties are usually quite protective of those claimants who have more modest demands on the resources. Some of these properties in the literature of bankruptcy problems adapted to multi-issue bankruptcy problems with crossed claims are the following.

Axiom 2.38 (RMR). *Given a rule R, it satisfies* respect of minimal rights*, if for every problem $(M, N, E, c, \alpha) \in \mathcal{MBC}$, for all $i \in N$,*

$$R_i(M, N, E, c, \alpha) \geq \min_{j:j\in\alpha(i)} \max\{0, e_j - \sum_{k\in N\backslash\{i\}} \delta(k, j)c_k\}.$$

Axiom 2.39 (CED)**.** *Given a rule R, it satisfies* conditional equal division*, if for every problem $(M, N, E, c, \alpha) \in \mathcal{MBC}$, for all $i \in N$,*

$$R_i(M, N, E, c, \alpha) \geq \min_{j:j \in \alpha(i)} \min \left\{ c_i, \frac{e_j}{|k : j \in \alpha(k)|} \right\}.$$

Axiom 2.40 (SEC)**.** *Given a rule R, it satisfies* securement*, if for every problem $(M, N, E, c, \alpha) \in \mathcal{MBC}$, for all $i \in N$,*

$$R_i(M, N, E, c, \alpha) \geq \min_{j:j \in \alpha(i)} \min \left\{ \frac{c_i}{|k : j \in \alpha(k)|}, \frac{e_j}{|k : j \in \alpha(k)|} \right\}.$$

RMR, CED and SEC are related to the minimum amount that should reasonably be guaranteed to each claimant. The concept of minimal right was introduced by Tijs (1981) in the context of cooperative games to define the τ-value. Thus, RMR says that a claimant should receive at least what is left when all the other claimants are completely satisfied in their claims. CED was introduced by Moulin (2000) for rationing problems. In our context, this property means that an agent should obtain her claim if this is less than any egalitarian distribution of the estates of the issues she claims, and in other case, at least the minimal egalitarian distribution of the estates of all issues she claims. Finally, SEC was introduced for bankruptcy problems by Moreno-Ternero ad Villar (2004). They use this property along with other properties to characterize the Talmud rule (Aumann and Maschler, 1985). In the environment of MBC problems, this property means that a rule should guarantee to agents at least the minimal egalitarian distribution of the estates of all issues they claims when they are feasible, and the minimal egalitarian distribution of the estates of all issues they claims otherwise.

Axiom 2.41 (GMA)**.** *Given a rule R, it satisfies* guaranteed minimum award*, if for every problem $(M, N, E, c, \alpha) \in \mathcal{MBC}$,*

$$R_i(M, N, E, c, \alpha) \geq \min \left\{ R_k \left(\{k\}, N_k, e_k, c|_{N_k} \right) : k \in \alpha(i) \right\}, \forall i \in N,$$

where $N_k = \{i \in N : j \in \alpha(i)\}$, and $c|_{N_k}$ is the vector whose coordinates correspond to the claimants in N_k.

GMA provides another way to guarantee a minimum amount to each claimant. These minimum amounts are determined from the analysis of the problems associated with each issue independently. In particular, the property states that a claimant should not receive less than what she would have received in the worst case, if the rule had been applied to each problem separately to each of the issues.

Axiom 2.42 (CFC)**.** *Given a rule R, it satisfies* conditional full compensation*, if for every problem $(M,N,E,c,\alpha) \in \mathcal{MBC}$ and each $i \in N$, such that $\sum_{k:j\in\alpha(k)} \min\{c_k, c_i\} \leq e_j$, for all $j \in M$, then $R_i(M,N,E,c,\alpha) = c_i$.*

CFC means that if the claim of a claimant is so small that if all claimants asked for the same amount as her, they all would receive their claims, then it seems reasonable that said claimant receives her claim. This property was introduced by Herrero and Villar (2002) and used to characterize the CEA rule.

Properties related to monotonicity

The monotonic properties refer to what impact changes in some of the elements that define the problem have on the allocation, in particular, changes in the amount of resources available or in the demands of the claimants. A property that is satisfied by most of the rules for bankruptcy problems is resource monotonicity. This property simply says that if the available resource increases, allocations to claimants do not decrease. In the particular context of multi-issue bankruptcy problems with crossed claims, this property reads as follows.

Axiom 2.43 (R-MON)**.** *Given a rule R, it satisfies* resource monotonicity*, if for every problem $(M,N,E,c,\alpha) \in \mathcal{MBC}$ and $E' \geq E$, $R_j(M,N,E',c,\alpha) \geq R_j(M,N,E,c,\alpha)$ for all $j \in N$.*

Axiom 2.44 (C-MON)**.** *Given a rule R, it satisfies* claim monotonicity*, if for every pair of problems $(M,N,E,c,\alpha) \in \mathcal{MBC}$ and $(M,N,E,c',\alpha) \in \mathcal{MBC}$, such that $c_i \geq c'_i$ and $c_j = c'_j$, for all $j \in N\backslash\{i\}$, then $R_i(M,N,E,c,\alpha) \geq R_i(M,N,E,c',\alpha)$.*

C-MON means that if the claim of a claimant increasing she cannot receive less than she received in the previous situation. In Kasajima and Thomson (2011) monotonicity properties are studied in the context of the adjudication of conflicting claims.

Another monotonicity property, which is satisfied by many bankruptcy rules, is population monotonicity which says that if all claimants agree that a claimant j will obtain her claim, then the remaining claimants should be worse off after claimant j is fully compensated.

Axiom 2.45 (P-MON)**.** *Given a rule R, it satisfies* population monotonicity*, if for every problem (M,N,E,c,α), and each $j \in N$,*

$$R_k(M,N,E,c,\alpha) \geq R_k(M^{-j},N,E^{-j},c_{-j},\alpha), \text{ for all } k \in N\backslash\{j\},$$

where M^{-j} *is the set of issues for which claimants in* $N\backslash\{j\}$ *have claims;* $E^{-j} = (e_1^{-j}, \ldots, e_m^{-j})$ *so that* $e_i^{-j} = e_i - c_j$ *if* $i \in \alpha(j)$ *and* $e_i^{-j} = e_i$ *otherwise; and* c_{-j} *is the vector of claims from which* $j-th$ *coordinate has been deleted.*

Properties related to robustness

The robustness properties also refer to what happens when changes occur in the problem but usually by actions carried out by the claimants themselves. Understanding robustness when these changes do not affect the final allocation, the allocation of those who have carried them out, or the allocation of those who have not done anything.

Axiom 2.46 (CTI). *Given a rule R, it satisfies* claims truncation invariance*, if for every problem* $(M, N, E, c, \alpha) \in \mathcal{MBC}$*, when considering the problem* $(M, N, E, c', \alpha) \in \mathcal{MBC}$ *such that* $c_i' = \min\{c_i, \min\{e_j | j \in \alpha(i)\}\}$*, for all* $i \in N$*; then* $R(M, N, E, c, \alpha) = R(M, N, E, c', \alpha)$.

CTI says that if the claims are truncated by the estates, then the final allocation does not change. This property appears in Curiel et al. (1987) and it is used to characterize the so-called game theoretical rules for one-issue bankruptcy problems. Dangan and Volij (1993) were the first to propose this property as an axiom.

The next property is a requirement of robustness when some agents leave the problem with their allocations (see Thomson, 2011, 2018). In particular, when a subset of claimants leave the problem respecting the allocations to those who remain, then it seems reasonable that claimants who leave will receive the same in the new problem as they did in the original. Before introducing the following property we need to introduce the concept of reduced problem.

Definition 2.21. Given a problem $(M, N, E, c, \alpha) \in \mathcal{MBC}$, and $N' \subset N$, the *reduced problem* associated with N', $MBC^{N'} = (M', N', E'^R, c|_{N'}, \alpha) \in \mathcal{MBC}$, where $M' = \{j \in M :$ there exists $i \in N'$ such that $j \in \alpha(i)\}$, $E'^R = \left(e_j'^R\right)_{j \in M'}$ with $e_j'^R = e_j - \sum_{i \in N \backslash N' : j \in \alpha(i)} R_i(M, N, E, c, \alpha)$, for all $j \in M'$, and $c|_{N'}$ is the vector whose coordinates correspond to the claimants in N'.

Axiom 2.47 (CONS). *Given a rule R, it satisfies* consistency*, if for every problem* $(M, N, E, c, \alpha) \in \mathcal{MBC}$*, and* $N' \subset N$*, it holds that*

$$R_i(M, N, E, c, \alpha) = R_i(M', N', E'^R, c|_{N'}, \alpha), \text{ for all } i \in N'.$$

$CONS$ means that if a subset of claimants leave the problem respecting what had been assigned to those who remain, then what those players get in the new reduced problem is the same as what they got in the whole problem. Consistency properties have been used to characterize many bankruptcy rules, because they represents a requirement of robustness when some agents leave the problem with their allocations (see Thomson (2011, 2018) for surveys about the application of consistency properties and their principles behind.)

The last two properties are related to claimants' ability to manipulate the final allocation by splitting their claims among several new claimants or merging their claims into a single claimant. It seems sensible that if the claimants do this they will not benefit and receive the same as they did in the original problem. These two possibilities are established in the following axioms.

Axiom 2.48 (NMS)**.** *Given a rule R, it satisfies* non-manipulability by splitting*, if for every pair of problems $(M, N, E, c, \alpha), (M, N', E, c', \alpha') \in \mathcal{MBC}$, such that:*

1. *$N \subset N'$, $S = \{i_1, \ldots, i_k\}$, such that $N' = (N \backslash S) \cup S_{i_1} \cup \ldots \cup S_{i_m}$, where S_{i_k} is the set of agents into which agent i_k has been divided.*
2. *$c'_j = c_j, \forall j \in N \backslash S$ and $\sum_{k \in S_{i_h}} c'_k = c_{i_h}, h = 1, \ldots, m$,*
3. *$\alpha'(j) = \alpha(j), \forall j \in N \backslash S$ and $\alpha'(j) = \alpha(i_h), \forall j \in S_{i_h}, h = 1, \ldots, m$,*

it holds

$$\sum_{j \in S_{i_h}} R_j(M, N', E, c', \alpha') = R_{i_h}(M, N, E, c, \alpha), h = 1, \ldots, m.$$

Axiom 2.49 (NMRM)**.** *Given a rule R, it satisfies* non-manipulability by restricted merging*, if for every pair of problems $(M, N, E, c, \alpha), (M, N', E, c', \alpha') \in \mathcal{MBC}$, such that:*

1. *$N \subset N'$,*
2. *$c_i = c'_i, \forall i \in N \backslash \{i_0\}$ and $c_{i_0} = \sum_{j \in (N' \backslash N) \cup \{i_0\}} c'_j$,*
3. *$\alpha(i) = \alpha'(i), \forall i \in N \backslash \{i_0\}$ and $\alpha(i) = \alpha'(i_0), \forall i \in (N' \backslash N) \cup \{i_0\}$,*

it holds

$$R_{i_0}(M, N, E, c, \alpha) = \sum_{i \in (N' \backslash N) \cup \{i_0\}} R_j(M, N', E, c', \alpha').$$

Note that in NMS we move from the allocation problem with set of claimants N to the problem with set of claimants N', i.e., some of the claimants are split into several new claimants, some of whom have the same names as in N. However, in $NMRM$ we move from the problem in N' to the problem in N, i.e., several claimants merge into one claimant who has the same name as in N', but all merged claimants are homologous. Thus, we are only considering the merging of homologous claimants. For this reason we call this axiom non-manipulability by "restricted" merging. Nevertheless, it seems reasonable from a perspective of symmetry of both properties, because when one claimant is split into several new claimants, these are homologous in the new problem.

2.4.3 Axiomatic analysis and characterization of the CCEA rule

The CCEA rule for MBC problems satisfies most of the properties above mentioned but efficiency as Example 2.6 shows. We establish this in the following theorem.

Theorem 2.8. *The CCEA rule for multi-issue bankruptcy problems with crossed claims satisfies F, $WEFF$, $PEFF$, ETE, CTI, RMR, CED, SEC, CFC, CM, and $CONS$.*

Next, the aim is to get a better knowledge of the CCEA rule for MBC by describing it in a unique way as a combination of some reasonable axioms. This combination of principles is very important to understand the behavior of this rule and be able to make a good choice. The characterization given uses three appealing axioms: Pareto efficiency, conditional equal division and consistency.

Theorem 2.9. *The CCEA rule for multi-issue bankruptcy problems with crossed claims is the only rule that satisfies $PEFF$, CED, and $CONS$.*

Note that from Theorem 2.9, we know that for $|N| = 2$, CCEA is the only rule satisfying $PEFF$ and CED for multi-issue bankruptcy problems with crossed claims. Next, in the following propositions, we show the role of each property in Theorem 2.9. We first prove that CED and $CONS$ imply $PEFF$, then, we show that CED and $CONS$ are necessary, and that any other combination of two properties do not imply the remaining third. Therefore, CCEA can also be characterized by only CED and $CONS$. This is established below in Corollary 2.1. However, we have preferred to keep the characterization with $PEFF$ because this way we obtain a different characterization of CCEA for the case of two agents as described above.

Proposition 2.2. *If a rule satisfies CED and $CONS$ then it satisfies $PEFF$.*

Corollary 2.1. The CCEA rule for multi-issue bankruptcy problems with crossed claims is the only rule that satisfies CED, and $CONS$.

Proposition 2.3. *The properties CED and $CONS$ in Theorem 2.9 (and Corollary 2.1) are necessary.*

Note that CED for MBC problems is the equivalent of conditional equal division lower bound (Moulin, 2000) for bankruptcy problems, but it is not the same as conditional equal division full compensation (this was called *exemption* by Herrero and Villar, 2001). In bankruptcy problems, for $|N| = 2$, CEA is the only rule satisfying the conditional equal division lower bound (Thomson, 2015), we do not mention efficiency because all bankruptcy rules satisfy it. Here, we obtain the same result for MBC problems by using $PEFF$ and CED (see the proof of the case $|N| = 2$ in Theorem 2.9). However, when using conditional equal division full compensation an extra property is necessary to characterize CEA in bankruptcy problems with $|N| = 2$. In addition, for bankruptcy problems when $|N| = 2$, conditional full compensation (this was called *sustainability* by Herrero and Villar, 2002) and conditional equal division full compensation coincide.

Proposition 2.4. *For multi-issue bankruptcy problems with crossed claims, CFC and CM imply CED.*

Corollary 2.2. The CCEA rule for multi-issue bankruptcy problems with crossed claims is the only rule that satisfies $PEFF$, CFC, CM and $CONS$.

Corollary 2.2 corresponds to the characterization of CCEA in MBC problems equivalent to the characterization of CEA in Yeh (2006) (see, Thomson 2015, Th. 4b and Th. 14). Of course, other characterizations of the classical constrained equal awards rule could try to be extended to this context, for example, the characterization of CEA in Herrero and Villar (2002). In the latter case, we would first have to define what *composition down* (Moulin, 2000) means in this context. Since we have many issues, it could be extended in different ways. In any case, this last characterization and others in the literature would be interesting for further research in this framework.

2.4.4 Axiomatic analysis and characterization of the CPA rule

CPA also satisfies most of the properties above mentioned. We establish this in the following theorem.

Theorem 2.10. *CPA for multi-issue bankruptcy problems with crossed claims satisfies F, $WEFF$, $PEFF$, ETE, GMA, $CONS$, NMS, and $NMRM$.*

Now, the aim is to achieve a better knowledge of the CPA rule for $\mathcal{MBC}$ by describing it in a unique way as a combination of some reasonable axioms. We characterize the CPA rule by means of $PEFF$, ETE, GMA, $CONS$, and NMS. Therefore, the CPA rule can be considered as a desirable way to distribute a set of issues among their claimants.

Theorem 2.11. *CPA is the only rule that satisfies $PEFF$, ETE, GMA, $CONS$, and NMS.*

Proposition 2.5. *Properties in Theorem 2.11 are logically independent.*

2.4.5 Axiomatic analysis of the CSP rules and the CRA rule

Before analyzing axiomatically the CSP rules and the CPA rule, we introduce a specific property related to the priority defined on the claimants.

Axiom 2.50 (PRI). *Given a rule R, it satisfies* priority*, if for every problem $(M, N, E, c, \alpha) \in \mathcal{MBC}$ and every pair of homologous claimants $j, k \in N$, if $\sigma(j) < \sigma(k)$, either $c_j - R_j(M, N, E, c, \alpha) \leq c_k - R_k(M, N, E, c, \alpha)$ or $R_k(M, N, E, c, \alpha) = 0$.*

Theorem 2.12. *CSP^{σ} for multi-issue bankruptcy problems with crossed claims satisfies F, $WEFF$, $PEFF$, $CONS$, and PRI.*

It is well-known that the random arrival rule does not satisfy $CONS$, so neither does the constrained random arrival rule.

Theorem 2.13. *CRA for multi-issue bankruptcy problems with crossed claims satisfies F, and ETE.*

In the context of multi-issue bankruptcy problems with crossed claims, the average operator does not preserve the property of Pareto efficiency and, therefore, CRA does not satisfy this basic property, as shown in the following proposition.

Proposition 2.6. *CRA rule does not satisfies $PEFF$.*

As for $WEFF$, in all the analyzed examples CRA satisfies this property, but we do not yet have a proof that this is the case, in general. The fact that CRA does not satisfy $PEFF$ is what makes us doubt the $WEFF$ property, because both have certain similarities.

A property that is satisfied by most of the rules for bankruptcy problems is endowment monotonicity which is also used in the characterization of sequential priority rules (see Thomson, 2019, Th.11.11). This property simply says that if the available resource increases, allocations to claimants do not decrease. Although this property seems very weak, constrained sequential priority rules do not satisfy it as the following proposition shows.

Proposition 2.7. *CSP^σ for multi-issue bankruptcy problems with crossed claims does not satisfy $R-MON$.*

Another monotonicity property is population monotonicity which says that if all claimants agree that a claimant j will obtain her claim, then the remaining claimants should be worse off after claimant j is fully compensated. This property is satisfied by many bankruptcy rules, including the random arrival rule. Moreover, the random arrival rule is characterized by using this property in Hwang and Wang (2009, Th.1). On the other hand, a property that appears recurrently related to the Shapley value and, therefore, to the random arrival rule is that of balanced contributions (Myerson, 1980; Hart and Mas-Colell, 1989). In the context of bankruptcy problems this property was used by Bergantiños and Mendez-Naya (1997) to characterize the random arrival rule. Moreover, in the context of multi-issue bankruptcy problems was introduced by Lorenzo-Freire et al. (2007) and also used to characterize the random arrival rule. This property requires that claimant j impacts to claimant k's allocation what claimant k impacts to claimant j's allocation. As mentioned above, this type of properties are used to characterize Shapley-like solutions as the random arrival is, but in the context of multi-issue bankruptcy problems with crossed claims, the constrained random arrival rule does not satisfy them as the following proposition shows.

Proposition 2.8. *CRA rule satisfies neither $P-MON$ nor BAL.*

The previous results show the complexity of finding properties that allow axiomatic characterizations of sequential priority rules and the random arrival rule in the context of multi-issue bankruptcy problems with crossed claims, so it is necessary to look for perhaps more specific properties (and likely more technical) to achieve it.

2.5 Application to the management of water pollution control

In this section we apply the bankruptcy model and the allocation rules to the case of the management of water pollution control. The content of this section is mainly based on Acosta-Vega et al. (2021b).

2.5.1 Introduction

Water is necessary for almost any form of life, which is why it is present in almost all the reports prepared by international institutions such as the United Nations (UN) and many of its specialized organizations, such as the World Health Organization (WHO) or Food and Agriculture Organization (FAO). The reason for this is that water is an essential good for economic development, health and the environment. Three of the great problems related to water are: access to fresh water, fresh water management and water pollution. In fact, the solution of these problems are directly or indirectly included in many of the *Sustainable Development Goals* (SDG's) promoted by the UN (https://www.un.org/sustainabledevelopment/). This paper deals with the third of these problems, water pollution, in particular, the design of water quality policies by means of water pollution control.

We would like to emphasize the importance of the water pollution control. In fact, water pollution is a serious threat to human health, to the survival of ecosystems and thus to the biodiversity of the planet. The contamination of fresh water causes numerous diseases, and reduces the availability of an already scarce resource that is essential for human consumption and for agriculture. Therefore, proper management of water pollution control in a certain region is imperative for the survival of the region and the development of its economic activity (Helmer and Hespanhol ,1997; Goel, 2009).

Goel (2009) defines a water pollutant as follows: "*A water pollutant can be defined as a physical, chemical or biological factor causing aesthetic or detrimental effects on aquatic life and on those who consume the water. Majority of the water pollutants are, however, in the form of chemicals which remain dissolved or suspended in water and give an environmental response which is often objectionable. Sometimes, physical and biological factors also act as pollutants. Among the physical factors, heat and radiations are important factors which have marked effects on organisms. Certain microorganisms present in water, especially pathogenic species, cause diseases to man and animals, and can be referred as biopollutants.*". Nesaratnam (2014) divides water pollutants into several categories: benzenoids, oxygen-demanding wastes, and eutrophing nutrients. Each of these groups of water pollutants come from different sources generally related to human activity and have different effects on water quality.

Toxic benzenoids as *benzene*, *ethylbenzene*, *toluene*, and *xylenes*, including also *fenols* are very poisonous to any living organisms, being able to cause serious diseases in humans. These aromatic hydrocarbons have a low boiling point and are abundant in petroleum representing its most dangerous fraction. Furthermore, hydrocarbons, once incorporated into a given organism, are very stable, being able to pass through many members of the food chain without being altered. They are therefore transferred to all trophic chain, a situation analogous to that of heavy metals and pesticides (see,Tomlinson (1971) for details about benzenoid compounds). Moreover, these substances, particularly fenols, can also affect negatively to the presence of dissolved oxygen (DO) in water which is essential for the development, inside it, of the life of animals and plants. A body of water is classified as contaminated when the DO concentration falls below the level necessary to maintain a normal biota for such water. The main cause of deoxygenation of water is the presence of substances that are called oxygen-demanding wastes. These are compounds that are easily degraded or decomposed due to bacterial activity in the presence of oxygen. Regarding the pollutants which are oxygen-demanding wastes, we can find the already mentioned *fenols*, *ammoniacal compounds*, *oxidizable inorganic substances* (OIS), and overall *biodegradable organic compounds* (BOC) (Riffat, 2013).

As for the last group of water pollutants, eutrophication is the process by which a body of water becomes excessively enriched with nutrients that induce excessive growth of aquatic plants and algae. The most evident effect of eutrophication is the creation of dense blooms of noxious and smelly phytoplankton that reduce water clarity and damage water quality. However, there are other more dangerous effects affecting life in the aquatic ecosystems. Apart from natural causes, eutrophication is caused by the action of man with the discharges of detergents, fertilizers or wastewater containing nitrates or phosphates in an aquatic system. Particularly relevant are nitrogen eutrophics as *nitrates* and *ammonical compounds* (see, for instance, Schindler (2006), Ansari et al. (2011), and Chislock et al. (2013) for details about the eutrophication problem).

Nowadays, there is a certain concern about other contaminants present in the water of which little is known, these contaminants are called emerging contaminants. Among these many products can be found such as pharmaceuticals and personal care products, nanomaterials, fire retardants, pesticides, plasticizers, surfactants, disinfection byproducts, antibiotic resistant bacteria, microplastics, and genes. Some recent works on these emerging pollutants are Geissen et al. (2015), Más-Plá (2018), Llamas et al. (2020), and Gomes et al. (2020), among others.

The European Union (EU) has promoted and implemented different environmental policies addressed to protect water quality. Thus, EU directives have specified emission limit values for water and set standards on how to monitor, report, and manage the water quality (see, for instance, Directive/2000/60/EC, Directive 2006/118/EC, Directive 2008/105/EC, and Directive 2013/39/EU). Steinebach (2019) analyzes the effectiveness of EU policies in the quality of the national water resources of the member states over a period of 23 years (1990-2012). In the case of Spain, there are also a legal body in order to control different aspects of water management, including the water quality (Royal Decree 9/2008). All the previous legal framework derive the responsibility for the management and control of water towards the regional and local authorities that are closest to the water resource. Consequently, on the one hand, the legislation establishes limits (of immission) to the parameters indicating contamination in body waters, whose values vary according to the use of body waters (bathing, purification, ...). On the other hand, the discharges are those that are carried out directly or indirectly to the body waters, whatever their nature. Finally, local and regional administrations can legislate on maximum concentrations (of emission) of pollutants in discharges made. Therefore, it is of great social and economic interest to generate tools that help local and regional authorities to design policies to control water quality.

Thus, in this chapter, we consider the situation in which a certain authority responsible of the water quality in her region is interested in controlling the water pollution. In particular, they are interested in limiting the concentration of the three categories of water pollutants mentioned above. On the one hand, for each of these categories of pollutants certain levels of concentration are fixed in order to keep a reasonable quality of water. On the other hand, the main substances mentioned above are monitored and there are also maximum concentration limits for them. The relation between the categories of pollutants and the substances in each category is shown in Figure 2.3. Thus, the problem to be solved by the authority is how to allocate new thresholds to the substances taking into account the limits fixed for each category of pollutants. Therefore, the authority faces an allocation problem with certain special characteristics. One way to solve the problem is to resort to solutions that can be found in the literature on allocation problems, or based on them to introduce new solutions adapted to the particular problem. However, the situation described in Figure 2.3 does not fit to a multi-issue bankruptcy problem, but to a multi-issue bankruptcy problem with crossed claims.

In the literature many other applications of bankruptcy problems can be found. Some examples are the following. Pulido et al. (2002, 2008) study allocation problems in university management; Niyato and Hossain (2006), Gozalvez et al. (2012), and Lucas-Estañ et al. (2012) analyze radio resource allocation problems in telecommunications; Casas-Mendez et al. (2011) study the museum pass problem; Hu et al. (2012) analyze the airport problem; Giménez-Gómez et al. (2016), Gutiérrez et al. (2018), and Duro et al. (2020) analyze the CO2 allocation problem; Sanchez-Soriano et al.

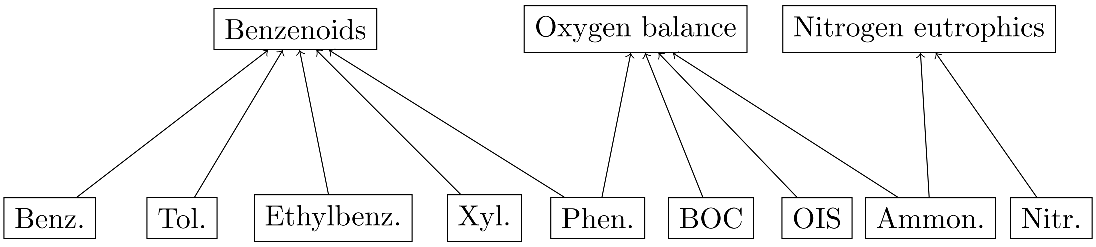

Figure 2.3: Relationship between families of pollutants and substances.

Pollutants	**Maximum value**
Benzene	0.05
Toluene	0.25
Ethylbenzene	0.15
Xylenes	0.15
Phenols	5
BOC	700
OIS	545
Ammoniacal compounds	150
Nitrate compounds	100

Table 2.1: Maximum allowed values (ppm) for some pollutants.

(2016) study the apportionment problem in proportional electoral systems; and Wickramage et al. (2020) analyze water allocation problems in rivers. However, as far as we know, there are no applications of bankruptcy problems to problems with the structure considered in this chapter.

2.5.2 A case study of management of water pollution control

As far as we know, there are no regulations for the emission of water pollutants at the state level in Spain (Regulation 606/2003) or EU level (Directives 2006/11/EC and 2000/60/EC), except for mercury and other minor pollutants. There exist only regulations at the local level, therefore, localities establish emission levels to authorize or not discharges.

In this section, we consider the applied limits corresponding to a representative local normative of spills to the sewage network established by the Honorary Granada City Council. (BOP Num. 129, 30/05/2000). These emission limits of the pollutants are shown in Table 2.1.

Now suppose that the city council wants to impose limitations on the concentrations (ppm) of each of the three categories of pollutants mentioned in Section 2.5.1 (benzenoids, oxygen-demanding wastes, and eutrophing nutrients), independently of the emission limits for each of the pollutants. In this sense, what the city council intends is to control emissions more by categories of pollutants than by each of the pollutants themselves, respecting, at the same time, the limitations on the emissions of each substance. Suppose the city council sets the following limits for each of the groups of pollutants: 4 ppm for benzenoids, 1000 ppm for oxygen-demanding wastes, and 150 ppm for nitrogen eutrophing nutrients. The emission limits situation is shown in Figure 2.4

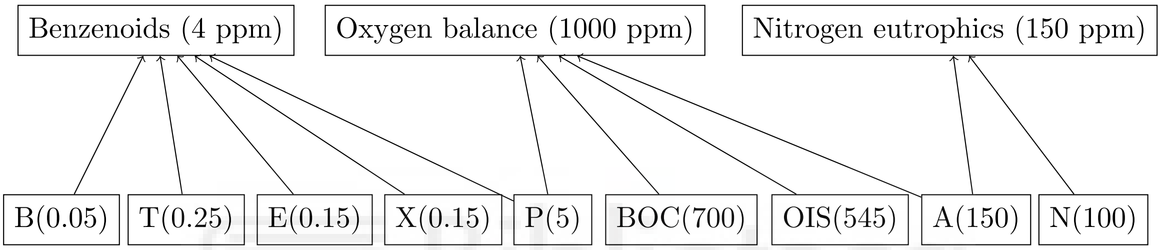

Figure 2.4: Relationship between families of pollutants and substances.

Associated with the situation described above we consider the following multi-issue bankruptcy problem with crossed claims $MBC = (M, N, E, c, \alpha)$ with

- $M = \{1, 2, 3\}$;
- $N = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$;
- $E = (4, 1000, 150)$;
- $c = (0.05, 0.25, 0.15, 0.15, 5, 700, 545, 150, 100)$; and
- $\alpha(1) = \{1\}$, $\alpha(2) = \{1\}$, $\alpha(3) = \{1\}$, $\alpha(4) = \{1\}$, $\alpha(5) = \{1, 2\}$, $\alpha(6) = \{2\}$, $\alpha(7) = \{2\}$, $\alpha(8) = \{2, 3\}$, and $\alpha(9) = \{3\}$.

It is obvious that the emission limits for each of the three pollutants categories are not sufficient to guarantee the emission limits for each of the substances, therefore, their limits must be recalculated down. To do this, we can now apply the allocation rules introduced in Chapter 2.3 for MBC problems.

Application of the CCEA rule

In this subsection, we apply the CCEA rule to the problem of determining the new thresholds. The calculation of the allocation is obtained in three steps using the iterative procedure based on the standard CEA:

1. First we calculate the CEA rule for each of the three bankruptcy problems defined by each issue (benzenoids, oxygen balance and nitrogen eutrophics).
   - $(N^{b,1}, E^{b,1}, c^{b,1})$. $N^{b,1} = \{B, T, E, X, P,\}$, $E^{b,1} = 4$, and $c^{b,1} = (0.05, 0.25, 0.15, 0.15, 5)$. $CEA(N^{b,1}, E^{b,1}, c^{b,1}) = (0.05, 0.25, 0.15, 0.15, 3.4)$, and $\beta^{b,1} = 3.4$.
   - $(N^{ob,1}, E^{ob,1}, c^{ob,1})$. $N^{ob,1} = \{P, BOC, OIS, A\}$, $E^{ob,1} = 1000$, and $c^{ob,1} = (5, 700, 545, 150)$. $CEA(N^{ob,1}, E^{ob,1}, c^{ob,1}) = (5, 422.5, 422.5, 150)$, and $\beta^{ob,1} = 422.5$.
   - $(N^{ne,1}, E^{ne,1}, c^{ne,1})$. $N^{ne,1} = \{A, N\}$, $E^{ne,1} = 150$, and $c^{ne,1} = (150, 100)$. $CEA(N^{ne,1}, E^{ne,1}, c^{ne,1}) = (75, 75)$, and $\beta^{ne,1} = 75$.
2. Next, we take $\beta^{*1} = \min\{\beta^{b,1}, \beta^{ob,1}, \beta^{ne,1}\} = 3.4$, and we allocate each claimant i, the quantity $\min\{c_i, \beta^{*1}\}$. Therefore, we obtain the allocation vector $(0.05, 0.25, 0.15, 0.15, 3.4, 3.4, 3.4, 3.4, 3.4)$.

Now, it is obvious that estate e_1 corresponding to benzenoids has been fully distributed, and in the next step this bankruptcy problem and the claimants associated with it are excluded. The other two problems are updated in claimants, and decreasing estates and claims according to the allocation previously obtained.

1. We calculate the CEA rule for each of the two bankruptcy problems remaining.
   - $(N^{ob,2}, E^{ob,2}, c^{ob,2})$. $N^{ob,2} = \{BOC, OIS, A\}$, $E^{ob,2} = 986.4$, and $c^{ob,2} = (696.6, 541.6, 146.6)$. $CEA(N^{ob,2}, E^{ob,2}, c^{ob,2}) = (419.9, 419.9, 146, 6)$, and $\beta^{ob,2} = 419.9$.
   - $(N^{ne,2}, E^{ne,2}, c^{ne,2})$. $N^{ne,2} = \{A, N\}$, $E^{ne,2} = 143.2$, and $c^{ne,2} = (146.6, 96.6)$. $CEA(N^{ne,2}, E^{ne,2}, c^{ne,2}) = (71.6, 71.6)$, and $\beta^{ne,2} = 71.6$.
2. Next, we take $\beta^{*2} = \min\{\beta^{ob,2}, \beta^{ne,2}\} = 71.6$, and we allocate each claimant i $\min\{c_i, \beta^{*2}\}$. Therefore, we obtain the allocation vector $(0, 0, 0, 0, 0, 71.6, 71.6, 71.6, 71.6)$.

Now, it is obvious that estate e_3 corresponding to nitrogen euthrophics has been fully distributed, and in the next step this bankruptcy problem and the claimants associated with it are excluded. The third problem is updated in claimants, and decreasing estates and claims according to the allocation previously obtained.

Pollutants	**Original maximum value**	**New maximum value**
Benzene	0.05	0.05
Toluene	0.25	0.25
Ethylbenzene	0.15	0.15
Xylenes	0.15	0.15
Phenols	5	3.4
BOC	700	460.8
OIS	545	460.8
Ammoniacal compounds	150	75
Nitrate compounds	100	75

Table 2.2: Maximum allowed values (ppm) for some pollutants after limiting the emissions of the three groups of pollutants using CCEA.

1. We calculate the CEA rule for each of the bankruptcy problem remaining.
   - $(N^{ob,3}, E^{ob,3}, c^{ob,3})$. $N^{ob,3} = \{BOC, OIS\}$, $E^{ob,3} = 771.6$, and $c^{ob,3} = (625, 470)$. $CEA(N^{ob,3}, E^{ob,3}, c^{ob,3}) = (385.8, 385.8)$, and $\beta^{ob,3} = 385.8$.
2. Next, we take $\beta^{*3} = \min\{\beta^{ob,3}\} = 385.8$, and we allocate each claimant i $\min\{c_i, \beta^{*3}\}$. Therefore, we obtain the allocation vector $(0, 0, 0, 0, 0, 385.5, 385.5, 0, 0)$.

The procedure stops because all estates have been completely distributed. Finally, by adding the allocation vectors obtained in the procedure, we obtain that $CCEA(M, N, E, c, \alpha) = (0.05, 0.25, 0.15, 0.15, 3.4, 460.8, 460.8, 75, 75)$.

Therefore, if the local government wants to limit the emissions of the three categories of pollutants below 4 ppm, 1000 ppm, and 150 ppm, respectively, but maintaining a fixed limit of emissions for each of the pollutant substances, an alternative would be to limit the emissions of the pollutant substances to the new values in the third column of Table 2.2.

Thus, we can observe how the CCEA rule for multi-issue problems with crossed claims can help authorities to design new water quality policies, in particular, how the limits of emissions of different substances can be established taking into account the categories of water pollutants which have different effects in the quality of water. We observe that the main reductions in the maximum level of emissions are assumed by those with the highest emissions limits.

Application of the constrained proportional rule

In this subsection, we apply the constrained proportional rule to the problem of determining the new thresholds. The calculation of the allocation is obtained in four steps:

Step 1. $\mathcal{N}^1 = N$, $M^1 = M$,

$E^1 = (4, 1000, 150)$, $c^1 = (0.05, 0.25, 0.15, 0.15, 5, 700, 545, 150, 100)$

- $\lambda_1^1 = 0.714$
- $\lambda_2^1 = 0.714$
- $\lambda_3^1 = 0.600$

$\lambda^1 = \min\{0.714, 0.714, 0.600\} = 0.600$

$$a_1^1 = 0.03, a_2^1 = 0.15, a_3^1 = 0.09, a_4^1 = 0.09, a_5^1 = 3, a_6^1 = 420, a_7^1 = 327, \\ a_8^1 = 90, a_9^1 = 60$$

Step 2. $\mathcal{N}^2 = \{1, 2, 3, 4, 5, 6, 7\}$, $M^2 = \{1, 2\}$,

$E^2 = (0.64, 160, 0)$, $c^2 = (0.02, 0.1, 0.06, 0.06, 2, 280, 218, 60, 40)$

- $\lambda_1^2 = 0.286$
- $\lambda_2^2 = 0.32$

$\lambda^2 = \min\{0.286, 0.32\} = 0.286$

$$a_1^2 = 0.01, a_2^2 = 0.03, a_3^2 = 0.02, a_4^2 = 0.02, a_5^2 = 0.57, a_6^2 = 80, a_7^2 = 62.29, \\ a_8^2 = 0, a_9^2 = 0$$

Step 3. $\mathcal{N}^3 = \{6, 7\}$, $M^2 = \{2\}$,

$E^3 = (0, 17.143, 0)$, $c^3 = (0.014, 0.071, 0.043, 0.043, 1.429, 200, 155.714, 60, 40)$

- $\lambda_2^3 = 0.048$

$\lambda^3 = \min\{0.048\} = 0.048$

$$a_1^2 = 0, a_2^2 = 0, a_3^2 = 0, a_4^2 = 0, a_5^2 = 0, a_6^2 = 9.64, a_7^2 = 7.50, a_8^2 = 0, a_9^2 = 0$$

Step 4. $\mathcal{N}^4 = \varnothing$, $M^2 = \varnothing$.

$$CPA(M, N, E, c, \alpha) = (0.036, 0.179, 0.107, 0.107, 3.571, 509.639, 396.79, 90, 60)\,.$$

Therefore, if the local government wants to limit the emissions of the three categories of pollutants below 4 ppm, 1000 ppm, and 150 ppm, respectively, but maintaining a fixed limit of emissions for each of the pollutant substances,

Pollutants	Original maximum value	New maximum value
Benzene	0.05	0.036
Toluene	0.25	0.179
Ethylbenzene	0.15	0.107
Xylenes	0.15	0.107
Phenols	5	3.571
BOC	700	509.639
OIS	545	396.79
Ammoniacal compounds	150	90
Nitrate compounds	100	60

Table 2.3: Maximum allowed values (ppm) for some pollutants after limiting the emissions of the three groups of pollutants using CPA.

an alternative would be to limit the emissions of the pollutant substances to the new values in the third column of Table 2.3.

Thus, we can observe how the constrained proportional awards rule for multi-issue problems with crossed claims can help authorities to design new water quality policies, in particular, how the limits of emissions of different substances can be established taking into account the categories of water pollutants which have different effects in the quality of water. We observe that, in this case, all pollutants are reduced in a certain amount in contrast to the results obtained when the CCEA rule is applied.

Application of an allocation based on arrival orders

In this subsection, we apply for a modification of the constrained random arrival rule to the problem in order to determine the new thresholds. The computation of the CRA requires considering 9! (=362880) which implies that its calculation is hard. For this reason, here we consider a modification of it, that we call CRA* which is a little bit simpler to compute and in this particular application it is computationally affordable. In particular, the CRA* consists of two levels of arrival orders. First, all the possible orders of the issues are taken. For the first issue in the order, we calculate the RA rule, then the estates and claims are updated down and the RA rule is calculated for the second issue in the order, and so on until the last issue in the order. Once we have obtained an allocation for each possible order of issues we take their average. Note that CRA* can be seen as a new and different allocation from CRA, but we use the same rationale behind[5]. Below an example of calculation for the following order of the categories nitrogen eutrophics, oxygen balance and benzenoids is given.

The category nitrogen eutrophics only has two pollutants associated with it, ammoniacal compounds (A) and nitrate compounds (N), therefore there are only two possible orders AN and NA. The estate to be allocated is 150 ppm. Then, the corresponding marginal contributions associated with those orders are $(150, 0)$ and $(50, 100)$ respectively. Taking the average, the allocation is $(100, 50)$.

The following category in the order is oxygen balance which has associated four pollutants: phenols (P), BOC (B), OIS (O) and ammoniacal compounds (A). Since 100 ppm have already been allocated to ammoniacal compounds, the available estate is 900 ppm, and the ammonical compounds are removed from the claimants of the oxygen balance category since it cannot receive more unless the estate of nitrogen eutrophics category is exceeded. Therefore, we have six possible orders of arrival for the remainder pollutants whose marginal contributions are the following:

Order	*P*	*BOC*	*OIS*
PBO	5	700	195
POB	5	350	545
BPO	5	700	195
BOP	0	700	200
OPB	5	350	545
OBP	0	355	545
Avg.	3.333	525.833	370.833

Therefore, the allocation is $(3.333, 525.833, 370.833)$.

Finally, the third category in the order is benzenoids which has five pollutants associated with it. However, phenols is discarded for the same reasons as ammoniacal compounds was discarded in the previous step. Moreover, the available estate is 0.667 ppm which exceeds the sum of the claims of the remaining pollutants, therefore, all of them receive their claim. As a result, the allocation is $(0.05, 0.25, 0.15, 0.15)$.

After finishing this process, the allocation associated with the order nitrogen eutrophics, oxygen balance and benzenoids is

$$(0.05, 0.25, 0.15, 0.15, 3.333, 525.833, 370.833, 100, 50).$$

[5]For this rule to be well defined, it is first necessary that the claims are truncated by the minimum of their associated endowments to avoid inconsistencies, i.e., we must take as claims $c_i' = \min\{c_i, \min_{k\in\alpha(i)}\{e_k\}\}, i \in N$.

Pollutants	**Original maximum value**	**New maximum value**
Benzene	0.05	0.0375
Toluene	0.25	0.1875
Ethylbenzene	0.15	0.1125
Xylenes	0.15	0.1125
Phenols	5	3.5165
BOC	700	525.7415
OIS	545	370.7415
Ammoniacal compounds	150	100
Nitrate compounds	100	50

Table 2.4: Maximum allowed values (ppm) for some pollutants after limiting the emissions of the three groups of pollutants using CRA*.

The allocation obtained for each of the possible orders of the pollutant categories is presented below.

Order	*B*	*T*	*E*	*X*	*P*	*BOC*	*O*	*A*	*N*
BON	0.025	0.125	0.075	0.075	3.7	525.65	370.65	100	50
BNO	0.025	0.125	0.075	0.075	3.7	525.65	370.65	100	50
OBN	0.05	0.25	0.15	0.15	3.333	525.833	370.833	100	50
ONB	0.05	0.25	0.15	0.15	3.333	525.833	370.833	100	50
NBO	0.025	0.125	0.075	0.075	3.7	525.65	370.65	100	50
NOB	0.05	0.25	0.15	0.15	3.333	525.833	370.833	100	50
Avg.	0.0375	0.1875	0.1125	0.1125	3.5165	525.7415	370.7415	100	50

Therefore, if the local government wants to limit the emissions of the three categories of pollutants below 4 ppm, 1000 ppm, and 150 ppm, respectively, but maintaining a fixed limit of emissions for each of the pollutant substances, an alternative would be to limit the emissions of the pollutant substances to the new values in the third column of Table 2.4.

Thus, we can observe how the CRA* rule for multi-issue problems with crossed claims can help authorities to design new water quality policies, in particular, how the limits of emissions of different substances can be established taking into account the categories of water pollutants which have different effects in the quality of water. Nevertheless, we observe that the CRA* allocation is not Pareto efficient, because the emission limits of benzene, toluene, ethylbenzene and xylenes could be slightly higher without exceeding any of the estates.

Summary

To sum up, the new thresholds of each pollutant with each rule can be seen below.

Pollutants	Original	CCEA	CPA	CRA*
Benzene	0.05	0.05	0.036	0.038
Toluene	0.25	0.25	0.179	0.188
Ethylbenzene	0.15	0.15	0.107	0.113
Xylenes	0.15	0.15	0.107	0.113
Phenols	5	3.4	3.571	3.517
BOC	700	460.8	509.639	525.742
OIS	545	460.8	396.79	370.742
Ammoniacal compounds	150	75	90	100
Nitrate compounds	100	75	60	50

Table 2.5: Summary of the application of each of the allocation rules.

Table 2.5 shows the differences obtained in the new emission limits when the three rules, CCEA, CPA and CRA*, are applied. Which of them is better? The answer will depend on which properties are considered more relevant in this type of problem.

At first glance, it can be seen that CCEA provides identical emission limits for those pollutants with very low original emission limits, and equals them for the rest. This is neither good nor bad, it simply means that the negative impact of all pollutants is implicitly considered to be similar. Here the only problem could be in the polluting companies, since those with pollutant emissions with higher original emission limits will have to make a greater effort to reduce emissions than other companies whose polluting emissions are concentrated in those with lower original emission limits.

The other two rules, CPA and CRA*, give very similar results and propose a more proportional review of emission limits. In this case, it would be implicitly considered that higher original limits will correspond to less harmful pollutants and that the impacts are proportional to their original limits. At the same time, this would mean that all companies would make efforts proportional to their polluting emissions regardless of whether the original emission limits were initially higher or lower.

Chapter 3

Discussion

The applications of operational research (OR) to environmental management problems have been increasing since the first works in the 1970s, see, for example, the review by Bloemhof-Ruwaarda et al. (1995), and the references therein. This review highlighted the potential of OR to solve environmental management problems or to include environmental elements in optimization problems. More recently, Mishra (2020) insists in the impact of OR in environmental management. In particular, many applications of game theory to environmental management problems can be found in the literature (see, for example, Hanley and Folmer, 1998; Dinar et al., 2008).

A particular allocation problem is arisen in situations where there is a perfectly divisible resource over which there is a set of agents who have rights or demands, but the resource is not sufficient to honor them. This problem is known as bankruptcy problem and was first formally analyzed in O'Neill (1982) and Aumann and Maschler (1985). Since then, it has been extensively studied in the literature and many allocation rules have been defined (see Thomson, 2019, for a detailed inventory of rules). In the literature we also find works that apply this bankruptcy problem model to study the water allocation problem (Wickramage et al., 2020) and the problem of allocation of pollution discharge permits in rivers (Aghasian et al., 2019; Moridi, 2019). But also to other environmental problems, see, for example, Giménez-Gómez et al. (2016), Gutiérrez et al. (2018), and Duro et al. (2020) which analyze the CO2 allocation problem.

However, the base bankruptcy model does not always fit all problems, which is the reason why there are different extensions of the classical bankruptcy model. Some of them are the following: Young (1994) and Moulin (2000) study bankruptcy problems in the indivisible goods case. An application of the discrete bankruptcy model to the apportionment problem in proportional electoral systems is given in Sánchez-Soriano et al. (2016). Pulido et al. (2002, 2008) introduce bankruptcy problems with references and claims to study allocation problems in university management. Gozalvez et al. (2012), and Lucas-Estañ et al. (2012) present bankruptcy problems with claims given by a discrete nonlinear function of the resource to analyze radio resource allocation problems in telecommunications. Habis and Herings (2013) and Kooster and Boonen (2019) study bankruptcy problems in which the estate and the claims are stochastic values. An interesting extension of bankruptcy problems are multi-issue allocation problems (Calleja et al., 2005). These describe situations in which there is a (perfect divisible) resource which can be distributed among several issues, and a (finite) number of agents that have claims on each of those issues, such that the total claim is above the available resource. This problem is also solved by means of allocation rules and there are different ways to do it (see, for example, Calleja et al., 2005; Borm et al., 2005; González-Alcón et al., 2007; Izquierdo and Timoner, 2016). However, the situation described in Figure 1.1 in Chapter 1 does not fit to a multi-issue allocation problem as referred in the previous paragraph, but to a multi-issue allocation problem with crossed claims as introduced in this doctoral thesis. These describe situations in which there are several (perfect divisible) resources and a (finite) set of agents who have claims on them, but only one claim (not a claim for each resource) with which one or more resources are requested. The total claim for each resource exceeds its availability. Therefore, in this thesis, we propose to use multi-issue allocation problems with crossed claims to allocate emission limits to pollutants. Therefore, once again real problems lead to the development of theoretical mathematical models that provide answers, as is the case presented in this doctoral thesis. As far as we know, there is no other similar bankruptcy model in the literature as the one proposed here. Consequently, this doctoral thesis contributes significantly to the development of the literature on (scarce) resource management problems, providing a new model of bankruptcy problems and their solutions.

As for the solutions that have been proposed in this doctoral thesis, three well-known bankruptcy rules have been extended: the constrained equal awards rule, the proportional rule and the random arrival rule. The first based on the principle of egalitarianism, the second based on the Aristotelian principle of treating equals equally and differently, and the third based on the principle of priority. In addition to extending these rules, they have been analyzed axiomatically to show their good properties as solutions to resource allocation problems, in the particular case of considering polluting emissions as resources. In this sense, the mathematical models proposed here, as well as their solutions, provide a methodology for the management of polluting emissions, in particular, in cases of emissions to the atmosphere or to water.

Chapter 4

Conclusions and Further Research

This thesis is related to one of the earliest problems arose in the economic literature. In fact, this problem already appeared in primal documents as the Talmud, or in essays of Aristotle or Maimonides. However, their mathematical modeling was first carried out by O'Neill (1982). The common and central question in these problems is how to divide when there is not enough. An extension of the classical bankruptcy problems appears with the introduction of multi-issue bankruptcy problems (Calleja et al. 2005) allowing that claims of agents can be referred to different issues.

Overall, we go beyond of it with the purpose of solving a real problem of abatement of emissions of different pollutants in which pollutants can contribute to more than one effect. To do this, we establish a new and original model based on multi-issues bankruptcy problems (MB) called multi-issue bankruptcy problems with crossed claims (MBC). This novel model presents a multi-dimensional state, one for each issue and each agent claims the same to the different issues in which participates, these are essential differences with respect to MB problems.

Similar as for MB problems, in this new framework, problems are solved through rules that assigns to each MB problem a distribution pointing out the amount obtained for each agent in each issue. First, we have allocated according to the CEA rule for bankruptcy problems introducing it as the solution to a succession of linear programming problems and extending this procedure to this framework. This new rule has been axiomatically analyzed and characterized using similar properties to those used to characterize CEA in bankruptcy problems. Moreover, this thesis is related to one of the most important concepts in allocation problems: proportionality. In allocation problems the concept of proportionality is put into practice with the well-known proportional rule. This rule has been extensively studied in the literature from many different point of views and for many allocation models. Focusing on bankruptcy models and their extensions to the multi-issue case, the proportional rule has been characterized in the context of bankruptcy problems in Chun (1988) and de Frutos (1999). In both papers, non-manipulability plays a central role in the axiomatic characterization of the proportional rule. For multi-issue bankruptcy problems, Ju et al. (2007) and Moreno-Ternero (2009) introduce two different definitions of proportional rule following two different approaches. Moreover, Ju et al. (2007) and Bergantiños et al. (2010) provide characterizations of both proportional rules. Again, in both approaches, non-manipulability is an essential property. In this thesis, we introduce a definition of proportional rule, that we call constrained proportional awards rule, for multi-issue bankruptcy problems with crossed claims and provide a characterization of it. Once again, non-manipulability is used. Therefore, we fill a gap in the literature of proportional distributions in allocation problems in line with the previous studies. Additionally, in many allocation problems the concept of priority is relevant, for example, in the legislation related to the liquidation of a company through bankruptcy in many countries, an order of priority is established to satisfy the claims. First, wage claims are settled, then taxes are paid. Creditor claims are then satisfied, and finally shareholder claims are addressed. Therefore, sequential priority rules, although simple, are not strange in real life. In this thesis extension of the sequential priority rules has been introduced in the context of multi-issue bankruptcy problems with crossed claims. Next, by means of the average of all these rules, the extension of the random arrival rule to this new context is defined. In the analysis of the properties satisfied by the sequential priority rules and the random arrival rule, it has been shown that the natural extensions of the properties used in the characterization of these rules in the context of bankruptcy problems and multi-issue bankruptcy problems are not satisfied by the constrained sequential priority rules and the constrained random arrival rule. Therefore, the characterizations of the rules introduced in this work will require properties that are perhaps too technical or very *ad hoc*, which could detract from a simple interpretation. The reason that these properties are not satisfied is that there is no efficiency in the context of multi-issue problems with crossed claims but rather a weaker concept such as Pareto efficiency. This means that the total quantity distributed can change from one problem to another, so the results from one problem to another are not easily comparable. This is a problem but it also tells us that these problems are of theoretical interest because they are not a mere and simple extension of bankruptcy problems but rather have a different structure that makes them interesting for further study.

In addition, we illustrate the interest and applicability of the model, and how the constrained equal awards rule, the constrained proportional rule and the constrained random arrival rule work by means of an application to the management of water pollution control.

In the literature of Operations Research (OR) there exist problems that could fit well in this theoretical model, for example, set covering problems. Bergantiños et al. (2020) study the problem of how to allocate costs in set covering problems when a reasonable cover is given in advance. These problems are described by a 4-tuple (N, M, c, A), where N is the set of agents, M is the set of facilities open, $c \in \mathbb{R}^M_+$ is the vector costs associated with the facilities, and $A = \{A_j\}_{j \in M}$ with $A_j \subset N$ for each $j \in M$ denotes the agents covered by each facility. The question to be answered is how to allocate the total costs among the agents. If we look carefully at the structure of the problem, we can observe a certain similarity with multi-issue bankruptcy problems with crossed claims in the following way. We first identify agents with pollutants and issues with facilities (regions). Thus, we have a set of pollutants that affect several regions, this is described by A that plays the role of function α. On the other hand, we consider that each region fixes a maximum level of pollution which is given by vector c that plays the role of vector E. Thus, the following problem arises: How to set pollutant emission levels when pollutants affect different regions? But one extra element is necessary in this problem: the pollutant emissions to be abated, i.e., the claims. Therefore, the set covering problem is the following. When we have a set of regions that impose limits on pollutant emissions, and these emissions come from several pollutants that can affect several of the regions simultaneously, the question to be answered is, how to set the emission limits of pollutants in such a way that the limits established by the regions are covered? This problem can be analyzed as a multi-issue bankruptcy problem with crossed claims. However, what happens if no reference on the ex-ante emissions of the pollutants are given? In this case, the problem have exactly four elements, (N, M, c, A), and the question to be answered is, how to set maximal limits of emissions of pollutants such that the regional limits are not exceeded? Therefore, these relationships between set covering problems and multi-issue bankruptcy problems with crossed claims would be interesting to study them in greater detail in further research.

Further research is pending to find a characterization of the constrained random arrival rule. Likewise, it would be interesting to find extensions of the CEL rule and the Talmud to the context of multi-issue bankruptcy problems with crossed claims and, of course, carry out its corresponding axiomatic analysis and find characterizations of those rules.

Finally, we also think that it would be interesting to try to define the cooperative games associated with these problems, which would also leave an open door for future research.

Chapter 5

Resumen extendido

5.1 Problemas de bancarrota

En esta sección, se introduce la notación y conceptos básicos relacionados con los problemas de bancarrota. En particular, se presenta la definición de problema de bancarrota, las principales reglas de reparto, sus propiedades y algunas caracterizaciones en la literatura.

5.1.1 Introducción

Los problemas de asignación describen situaciones en las que un recurso (o recursos) debe distribuirse entre un conjunto de agentes. Estos problemas son de gran interés en muchos escenarios, por lo que la literatura al respecto es extensa. Un problema particular de asignación surge en situaciones donde existe un recurso perfectamente divisible sobre el cual existe un conjunto de agentes que tienen derechos o demandas, pero el recurso no es suficiente para satisfacerlos. Este problema se conoce como problema de bancarrota y fue analizado formalmente por primera vez en O'Neill (1982) y Aumann y Maschler (1985). Desde entonces, se ha estudiado ampliamente en la literatura y se han definido muchas reglas de reparto o asignación (véase Thomson, 2003, 2015, 2019, para un inventario detallado de reglas).

En la literatura se pueden encontrar muchas aplicaciones de los problemas de bancarrota. Algunos ejemplos son los siguientes. Pulido et al. (2002, 2008) estudian los problemas de asignación en la gestión universitaria; Niyato y Hossain (2006), Gozalvez et al. (2012) y Lucas-Estañ et al. (2012) analizan problemas de asignación de recursos de radio en telecomunicaciones; Casas-Méndez et al. (2011) estudian el problema del pase de museo; Hu et al. (2012) analizan el problema aeroportuario; Giménez-Gómez et al. (2016), Gutiérrez et al. (2018) y Duro et al. (2020) analizan el problema de asignación de CO2;

Sánchez-Soriano et al. (2016) estudian el problema de la distribución en los sistemas electorales proporcionales; y Wickramage et al. (2020) analizan los problemas de asignación de agua en los ríos.

5.1.2 Definición del problema de bancarrota

Consideramos una situación en la que existe un recurso perfectamente divisible E, denominado estado, patrimonio o dotación, y existe un conjunto finito de reclamantes $N = \{1, 2, ..., n\}$ que demandan cantidades diferentes de ella, $c_i > 0, i \in N$, con los mismos derechos y con la condición de que el estado no sea suficiente para satisfacer íntegramente todas las demandas, es decir, $\sum_{i \in N} c_i > E$. Formalmente,

Definición 5.1. Un *problema de bancarrota* (BP en breve) es una terna (N, E, c), donde $N = \{1, 2, ..., n\}$ es el conjunto de reclamantes; $E \in \mathbb{R}_{++}$ es la cantidad de recurso disponible (asumida perfectamente divisible); y $c = (c_1, c_2, ..., c_n) \in \mathbb{R}^N_+$ es el vector de demandas, tal que $C = \sum_{i \in N} c_i > E$. Además, la familia de todos estos problemas se denotará por $\mathcal{BP}$.

El objetivo principal en un problema de bancarrota es encontrar una asignación o reparto que sea lo más justa posible, teniendo en cuenta las demandas de los reclamantes. En la literatura se han propuesto muchas soluciones dependiendo de los principios de equidad utilizados.

Definición 5.2. Dado un problema $(N, E, c) \in \mathcal{BP}$, un *reparto factible* para él, es un vector $x \in \mathbb{R}^N$ tal que:

1. (Acotado) $0 \leq x_i \leq c_i$, for all $i \in N$.
2. (Eficiencia) $\sum_{i \in N : j \in \alpha(i)} x_i = E$,

y se denotará por $A(N, E, c)$ el conjunto de todos los repartos factibles.

El requisito 1 significa que cada agente recibe como máximo su reclamación, pero no menos que nada. El requisito 2 significa que el estado debe distribuirse en su totalidad. Por lo tanto, el vector $x \in \mathbb{R}^N$ representa una asignación factible para los reclamantes.

Definición 5.3. Una *regla de reparto o simplemente una regla* para problemas de bancarrota es una aplicación R que asocia con cada $(N, E, c) \in \mathcal{BP}$ un único reparto factible $R(N, E, c) \in A(N, E, c)$.

5.1.3 Reglas básicas de reparto en problemas de bancarrota

En la literatura hay muchas reglas de reparto diferentes en problemas de bancarrota (véase Thomson, 2019), pero solo consideramos cinco de ellas: la regla proporcional, la regla igualitaria restringida, la regla de pérdidas iguales restringida, la regla del Talmud y la regla de llegada aleatoria. Estas cinco reglas se consideran soluciones básicas a los problemas de bancarrota. La regla proporcional iguala las proporciones entre las cantidades obtenidas y las cantidades reclamadas. La regla igualitaria restringida iguala las cantidades obtenidas, la regla de pérdidas iguales restringida iguala las pérdidas obtenidas por cada reclamante, y la regla del Talmud aplica una lógica diferente que consiste en que cualquiera obtiene más de la mitad de lo que reclama si la cantidad del bien disponible es menos de la mitad del monto total reclamado, y que nadie pierde más de la mitad de lo que reclama si el monto disponible excede la mitad de la demanda total. Finalmente, la regla de llegada aleatoria se obtiene como la media de los vectores de asignación que resultan de dividir secuencialmente el valor del estado.

La regla proporcional propone que cada acreedor reciba una cantidad proporcionalmente a su demanda, tratándose a todos por igual.

Definición 5.4. Para cada problema $(N, E, c) \in \mathcal{BP}$, la *regla proporcional*, PROP, se define como

$$PROP_i(N, E, c) = \frac{c_i}{C}E, i \in N.$$

La regla igualitaria restringida distribuye el estado de tal manera que a cada reclamante se le asigna la misma cantidad siempre que no exceda la cantidad reclamada por el agente.

Definición 5.5. Para cada problema $(N, E, c) \in \mathcal{BP}$, la *regla igualitaria restringida*, CEA, se define como

$$CEA_i(N, E, c) = \min\{c_i, \beta\}, i \in N,$$

donde β es un número real positivo que verifica que $\sum_{i \in N} CEA_i(N, E, c) = E$.

La regla de igualdad de pérdidas restringida, a diferencia de la CEA, prioriza a los agentes con las demandas más altas. La regla de pérdidas iguales restringidas distribuye las pérdidas relativas a la demanda por igual entre todos los agentes. Esto es, que la cantidad que no reciben de la reclamación es la misma para todos siempre que se cumpla la condición de que ningún reclamante puede perder una cantidad mayor a la que había reclamado, es decir, no puede recibir un pago negativo.

Definición 5.6. Para cada problema $(N, E, c) \in \mathcal{BP}$, la *regla de igualdad de pérdidas restringida*, CEL, se define como

$$CEL_i(N, E, c) = \max\{c_i - \alpha\ , 0\}, i \in N,$$

donde α es un número real positivo que verifica que $\sum_{i\in N} CEL_i(N, E, c) = E$.

La regla del Talmud (Aumann y Maschler, 1985) puede verse como un hbrido de la CEA y CEL. Concretamente utilizando la CEA si el estado es inferior a la mitad de la suma de las demandas y la CEL si es superior. Si es el mismo, es irrelevante cuál aplica porque ambos dan el mismo resultado. El objetivo de esta regla es que cuando la pérdida total es grande nadie gana mucho y cuando la pérdida total es pequeña nadie pierde mucho.

Definición 5.7. Para cada problema $(N, E, c) \in \mathcal{BP}$, la *regla Talmud*, TAL, se define como

$$TAL(N, E, c) = \begin{cases} CEA(N, E, c/2) & if \quad E \leq C/2 \\ c - CEA(N, C - E, c/2) & if \quad E > C/2 \end{cases}$$

o, por la dualidad[1] que relaciona las reglas CEL y CEA, es decir, $CEL(N, E, c) = c - CEA(N, C - E, c)$:

$$TAL(N, E, c) = \begin{cases} CEA(N, E, c/2) & if \quad E \leq C/2 \\ c/2 + CEL(N, E - C/2, c/2) & if \quad E > C/2 \end{cases}$$

La regla de llegada aleatoria (O'Neil, 1982) se basa en el supuesto de que los agentes llegan al centro de pago uno por uno y la primera persona que llega recibe el mínimo entre su demanda y el estado, lo mismo para el agente que llega el segundo pero restando lo que se ha dado al primero y así sucesivamente hasta que no quede estado. Dado que se supone que todos los órdenes de llegada son equiprobables, se toma el promedio de todas estas asignaciones.

Definición 5.8. Para cada problema $(N, E, c) \in \mathcal{BP}$, la *regla de llegada aleatoria*, RA, se define como

$$RA_i(N, E, c) = \frac{1}{|N|!}\left(\sum_{\sigma\in\Sigma(N)} \min\{c_i, \max\left\{E - \sum_{j\in N:\sigma(j)<\sigma(i)} c_j, 0\right\}\right), i \in N,$$

donde $\Sigma(N)$ es el conjunto de todos los órdenes de N, es decir, $\sigma \in \Sigma(N)$ es un aplicación biyectiva de N en $\{1, 2, 3, \ldots, |N|\}$.

[1]Dos reglas de reparto ρ and ρ^* se dicen duales si $\rho(N, E, c) = c - \rho^*(N, C - E, c)$, es decir, si se obtiene la misma solución repartiendo el estado E de acuerdo a las demandas c utilizando la regla ρ o asignando acada agente su demanda y sustrayéndole una parte de $C - E$ calculada utilizando la regla ρ^* con las demandas c.

5.1.4 Algunas propiedades para las reglas de reparto

En esta subsección, se presentan varias propiedades para las reglas de asignación en problemas de bancarrota. En primer lugar, cabe señalar que la propiedad de eficiencia no aparece en la lista siguiente porque forma parte de la propia definición de la regla de reparto.

Axioma 5.1 (Igual trato de iguales)**.** Dada una regla R, se dice que satisface *igual trato de iguales*, si para cada problema $(N, E, c) \in \mathcal{BP}$ y cada par de reclamantes $i, j \in N$, tal que $c_i = c_j$, entonces $R_i(N, E, c) = R_j(N, E, c)$.

Esta propiedad requiere un trato igualitario, lo que significa que los jugadores que exigen la misma cantidad deben recibir lo mismo.

Axioma 5.2 (Anonimato)**.** Dada una regla R, se dice que satisface *anonimato*, si para cada problema $(N, E, c) \in \mathcal{BP}$, y cada biyección $\sigma : N \to N$, $R_i(N, E, c) = R_{\sigma(i)}(N, E, c')$ para cada $i \in N$, donde $c' = (c_{\sigma(j)})_{j \in N}$.

La propiedad del anonimato dicta que la identidad de los agentes no debe importar. El vector de pagos elegido debe depender únicamente de la lista de demandas.

Axioma 5.3 (Impacto equilibrado)**.** Dada una regla R, se dice que satisface *impacto equilibrado*, si para cada problema $(N, E, c) \in \mathcal{BP}$, y cada par de reclamantes $i, j \in N$,

$$R_i(N, E, c) - R_i(N\backslash\{j\}, \max\{E - c_j, 0\}, c_{-j}) = \\ R_j(N, E, c) - R_j(N\backslash\{i\}, \max\{E - c_i, 0\}, c_{-i}).$$

La propiedad de impacto equilibrado requiere que el reclamante j impacte en la asignación del reclamante k lo que el reclamante k impacte en la asignación del reclamante j.

Axioma 5.4 (Preservación del orden)**.** Dada una regla R, se dice que satisface *preservación del orden*, si para cada problema $(N, E, c) \in \mathcal{BP}$, y cada par de reclamantes $i, j \in N$ tal que $c_i \leq c_j$, entonces $R_i(N, E, c) \leq R_j(N, E, c)$, y $c_i - R_i(N, E, c) \leq c_j - R_j(N, E, c)$.

Una regla de reparto debe respetar el orden de las demandas, es decir, si la demanda del agente i es al menos tan grande como la del agente j, debería recibir al menos tanto como el agente j. Además, las diferencias entre demandas y pagos también deben mantenerse.

Las siguientes propiedades intentan establecer qué cantidad debe recibir un reclamante o qué cantidad debe garantizarse a cada reclamante al menos bajo ciertas condiciones razonables. En este sentido, estas propiedades están relacionadas con las garantías que reciben los reclamantes cuando se aplica una regla de reparto.

Axioma 5.5 (Respeto de derechos mínimos)**.** Dada una regla R, se dice que satisface *respeto de derechos mínimos*, si para cada problema $(N, E, c) \in \mathcal{BP}$, para todo $i \in N$,

$$R_i(N, E, c) \geq \max\{0, E - \sum_{k \in N \setminus \{i\}} c_k\}.$$

Axioma 5.6 (División igualitaria condicionada)**.** Dada una regla R, se dice que satisface *división igualitaria condicionada*, si para cada problema $(N, E, c) \in \mathcal{BP}$, para todo $i \in N$,

$$R_i(N, E, c) \geq \min\left\{c_i, \frac{E}{|N|}\right\}.$$

Axioma 5.7 (Garantía)**.** Dada una regla R, se dice que satisface *garantía*, si para cada problema $(N, E, c) \in \mathcal{BP}$, para todo $i \in N$,

$$R_i(N, E, c) \geq \min\left\{\frac{c_i}{|N|}, \frac{E}{|N|}\right\}.$$

El respeto de los derechos mínimos, la división igualitaria condicionada y la de garantía están relacionados con el monto mínimo que razonablemente debe garantizarse a cada reclamante. El concepto de derecho mínimo fue introducido por Tijs (1981) en el contexto de los juegos cooperativos para definir el valor de τ. Por lo tanto, el respeto de los derechos mínimos dice que un reclamante debe recibir al menos lo que queda cuando todos los demás reclamantes están completamente satisfechos con sus demandas. El CED fue introducido por Moulin (2000) para problemas de racionamiento y Herrero y Villar (2002) para problemas de bancarrota con el nombre de exención. Finalmente, Moreno-Ternero ad Villar (2004) introdujeron la propiedad de garantía para problemas de bancarrota.

Axioma 5.8 (Compensación total condicionada)**.** Dada una regla R, se dice que satisface *compensación total condicionada*, si para cada problema $(N, E, c) \in \mathcal{BP}$ y cada $i \in N$, tal que $\sum_{k \in N} \min\{c_k, c_i\} \leq E$, entonces $R_i(N, E, c) = c_i$.

La propiedad de compensación total condicional significa que si la demanda de un reclamante es tan pequeño que si todos los reclamantes pidieran la misma cantidad que ella, todos recibirían sus demandas, entonces parece razonable que dicho reclamante reciba su demanda. Esta propiedad fue introducida como sostenibilidad por Herrero y Villar (2002).

Las propiedades de monotonía se refieren a qué impacto tienen sobre la asignación los cambios en algunos de los elementos que definen el problema, en particular, cambios en la cantidad de recursos disponibles o en las demandas de los reclamantes.

Axioma 5.9 (Monotonía en los recursos)**.** Dada una regla R, se dice que satisface *monotonía en los recursos*, si para cada problema $(N, E, c) \in \mathcal{BP}$ y $E' \geq E$, $R_j(N, E', c) \geq R_j(N, E, c)$ for all $j \in N$.

Axioma 5.10 (Monotonía en la demanda)**.** Dada una regla R, se dice que satisface *monotonía en la demanda*, si para cada par de problemas $(N, E, c) \in \mathcal{BP}$ y $(N, E, c') \in \mathcal{BP}$, tal que $c_i \geq c'_i$ y $c_j = c'_j$, para todo $j \in N \backslash \{i\}$, entonces $R_i(N, E, c) \geq R_i(N, E, c')$.

Monotonía en la demanda significa que si la demanda de un reclamante aumenta, no puede recibir menos de lo que recibió en la situación anterior. En Kasajima y Thomson (2011) las propiedades de monotonía se estudian en el contexto de la adjudicación de demandas en conflicto.

Otra propiedad de monotonía que es satisfecha por muchas reglas de reparto, es la monotonía en la población que dice que si todos los reclamantes están de acuerdo en que un reclamante j obtendrá su demanda, entonces los reclamantes restantes deberían estar peor después de que el reclamante j sea completamente compensado.

Axioma 5.11 (Monotonía en la población)**.** Dada una regla R, se dice que satisface *monotonía en la población*, si para cada problema (N, E, c), y cada $j \in N$,

$$R_k(N, E, c) \geq R_k(N, E - c_j, c_{-j}), \text{ para todo } k \in N \backslash \{j\},$$

donde c_{-j} es el vector de demandas de las cuales la $j-$ésima coordenada ha sido eliminada.

Axioma 5.12 (Compensación hacia arriba)**.** Dada una regla R, se dice que satisface *compensación hacia arriba*, si para cada problema $(N, E, c) \in \mathcal{BP}$, y $E' \in \mathbb{R}$,tal que $0 \leq E < E'$, entonces:

$$R(N, E', c) = R(N, E, c) + R(N, E' - E, c - R(N, E, c))$$

La propiedad de compensación hacia arriba nos asegura que esta situación se puede solucionar de dos formas: se puede cancelar el reparto inicial y aplicar la regla de reparto al problema modificado, o se pueden mantener las asignaciones iniciales de los agentes, ajustar las demandas en consecuencia y aplicar la regla para distribuir la diferencia $E' - E$. Y por tanto, las asignaciones no varían si la distribución se realiza de una sola vez o secuencialmente.

Axioma 5.13 (Compensación hacia abajo)**.** Dada una regla R, se dice que satisface *compensación hacia arriba*, si para cada problema $(N, E, c) \in \mathcal{BP}$, y $E' \in \mathbb{R}^{|P|}$, tal que $0 \leq E' < E$, entonces:

$$R(N, E', c) = R(N, E', R(N, E, c))$$

Esta propiedad implica que, al recalcular la distribución, es irrelevante tomar como pretensión las pretensiones originales o la distribución que resultó del antiguo estado.

Axioma 5.14 (Invarianza al truncar las demandas)**.** Dada una regla R, se dice que satisface *invarianza al truncar las demandas*, si para cada problema $(N, E, c) \in \mathcal{BP}$, cuando se considera el problema $(N, E, c') \in \mathcal{BP}$ tal que $c'_i = \min\{c_i, E\}$, para todo $i \in N$; entonces $R(N, E, c) = R(N, E, c')$.

La invarianza al truncar las demandas dice que si las demandas son truncadas por el estado, entonces la asignación final no cambia. Esta propiedad aparece en Curiel et al. (1987) y se utiliza para caracterizar las llamadas reglas de teoría de juegos para problemas de bancarrota. Dangan y Volij (1993) fueron los primeros en proponer esta propiedad como axioma.

Axioma 5.15 (Consistencia en las demandas)**.** Dada una regla R, se dice que satisface *consistencia en las demandas*, si para cada problema $(N, E, c) \in \mathcal{BP}$, se cumple la siguiente relación:

$$R_i(N, E, c) = \frac{1}{|N|}\left[\min\{c_i, E\} + \sum_{j \in N\setminus\{i\}} R_i(N\setminus\{j\}, \max\{E - c_j, 0\}, (c_k)_{k \in N\setminus\{j\}})\right],$$

para todo $i \in N$.

O'Neill (1982) introdujo la propiedad de consistencia en las demandas para caracterizar la regla de llegada aleatoria. Esta propiedad dice que la asignación se puede ver como un promedio de pagos de $|N|$ que se calculan fijando a un reclamante y dándole tanto como sea posible y luego el resto se comparte entre los otros reclamantes.

La siguiente propiedad es un requisito de robustez cuando algunos agentes dejan el problema con sus asignaciones (véase Thomson, 2011, 2018). En particular, cuando un subconjunto de reclamantes deja el problema respetando las asignaciones a los que quedan, entonces parece razonable que los reclamantes que se vayan reciban lo mismo en el nuevo problema que en el original. Antes de introducir la siguiente propiedad necesitamos introducir el concepto de problema reducido.

Definición 5.9. Dado un problema $(N, E, c) \in \mathcal{BP}$, y $N' \subset N$, el *problema reducido* asociado con N' es $(N', E', c|_{N'}) \in \mathcal{BP}$, donde $E' = E - \sum_{i \in N \setminus N'} R_i(N, E, c)$ y $c|_{N'}$ es el vector cuyas coordenadas corresponden a los reclamantes en N'.

Axioma 5.16 (Consistencia)**.** Dada una regla R, se dice que satisface *consistencia*, si para cada problema $(N, E, c) \in \mathcal{BP}$, y $N' \subset N$, se verifica que

$$R_i(N, E, c) = R_i(N', E', c|_{N'}), \text{ para todo } i \in N'.$$

La propiedad de consistencia significa que si un subconjunto de demandantes abandona el problema respetando lo que se les había asignado a los que permanecen, entonces lo que obtienen esos jugadores en el nuevo problema reducido es lo mismo que obtuvieron en el problema completo. Las propiedades de consistencia se han utilizado para caracterizar muchas reglas de reparto en problemas de bancarrota, porque representan un requisito de robustez cuando algunos agentes dejan el problema con sus asignaciones (véase Thomson (2011, 2018) para revisiones sobre la aplicación de las propiedades de consistencia y sus principios.)

Axioma 5.17 (Sin ventajas por transferencias)**.** Dada una regla R, se dice que satisface *sin ventajas por transferencias*, si para cada problema $(N, E, c) \in \mathcal{BP}$, y $N' \subset N$, y cada $c'|_{N'}$, si $\sum_{i \in N'} c'_i = \sum_{i \in N'} c_i$, entonces se verifica que

$$\sum_{i \in N'} R_i(N, E, c) = \sum_{i \in N'} R_i(N, E, (c'|_{N'}, c|_{N \setminus N'})).$$

La propiedad de sin ventajas por transferencias significa que ningún grupo de agentes obtiene más en conjunto transfiriendo derechos entre ellos.

Axioma 5.18 (Ninguna fusión o división ventajosa)**.** Dada una regla R, se dice que satisface *ninguna fusión o división ventajosa*, si para cada problema $(N, E, c) \in \mathcal{BP}$, y $N' \subset N$, y cada $c' \in \mathbb{R}_+^{N'}$, si existe $i \in N'$ tal que $c'_i = c_i + \sum_{j \in N \setminus N'} c_j$ y para cada $k \in N' \setminus \{i\}$, $c'_k = c_k$, entonces se verifica que

$$R_i(N', E, c') = R_i(N, E, c) + \sum_{j \in N \setminus N'} R_i(N, E, c).$$

La propiedad de ninguna fusión o división ventajosa significa que no es ventajoso para los demandantes ni dividirse en varios demandantes ni fusionar varios demandantes en uno solo.

Axioma 5.19 (Autodualidad)**.** . Dada una regla R, se dice que satisface *autodualidad*, si para cada problema $(N, E, c) \in \mathcal{BP}$, entonces $R(N, E, c) = c - R(N, C - E, c)$.

La autodualidad fue introducida por Aumann y Maschler (1985). La autodualidad significa que asignar recompensas es lo mismo que asignar pérdidas (Herrero y Villar, 2001).

Axioma 5.20 (Homogeneidad)**.** Dada una regla R, se dice que satisface *homogeneidad*, si para cada problema $(N, E, c) \in \mathcal{BP}$, y cada $\lambda > 0$, tal que $R_i(N, \lambda E, \lambda c) = \lambda R_i(N, E, c)$.

La homogeneidad significa que si requerimos que las demandas y el estado se multipliquen por el mismo número positivo, entonces también deberían hacerlo todas las asignaciones. No debe haber efecto de“escala” (Thomson, 2018).

5.1.5 Caracterizaciones

Continuamos con una lista de caracterizaciones de la regla igualitaria restringida, la regla proporcional y la regla de llegada aleatoria basada en las propiedades que hemos presentado en la sección anterior. Se agrupan según qué regla de reparto sale de los axiomas (Thomson, 2019).

Teorema 5.1. La regla igualitaria restringida es la única regla que satisface:

- Igual trato de iguales, invarianza al truncar las demandas, y composición hacia arriba (Dagan, 1996);
- Compensacioón total condicionada y composición hacia abajo (Herrero and Villar, 2002);
- Garantía, composición hacia arriba y consistencia (Chun, 2006);
- Consistencia, división igualitaria condicionada y composición hacia abajo (Herrero and Villar, 2002);
- Compensación total condicionada, monotonía en la demanda y consistencia (Yeh, 2006).

Teorema 5.2. La regla proporcional es la única regla que satisface:

- Autodualidad y composición hacia arriba (Young, 1998);

- Autodualidad y composición hacia abajo (Young, 1998);
- Igual trato de iguales, composición, y autodualidad (Herrero and Villar, 2001);
- Para $|N| \geq 3$; sin ventajas por transferencias (Moulin, 1985; Chun 1988; Ju and Miyagawa, 2002);
- sin ventajas por transferencias y consistencia;
- Ninguna fusioón o división ventajosa (Chun, 1988; de Frutos, 1999; Ju and Miyagawa, 2002).

Teorema 5.3. La regla de llegada alaeatoria es la única regla que satisface:

- Consistencia en las demandas (O'Neill, 1982);
- Impacto equilibrado (Bergantiños and Mendez-Naya, 1997).

5.2 Problemas de bancarrota con múltiples (sub)estados

En esta sección, presentamos la definición de problemas de bancarrota con múltiples (sub)estados, estudiamos las reglas de reparto para estas situaciones, las propiedades básicas para este tipo de problemas y se proporcionan algunas caracterizaciones para la regla de reparto igualitaria restringida en dos etapas y para la regla de reparto proporcional en dos etapas.

5.2.1 Introducción

Los problemas de bancarrota con múltiples (sub)estados fueron estudiados por Calleja et al. (2005), donde presentan una extensión de la regla "*run-to-the-bank*" (otro nombre que se suele dar a la regla de llegada aleatoria (Young, 1994)) para situaciones de bancarrota, en la que se explican problemas de bancarrota en los que se divide el estado no en base a un solo concepto para cada agente sino teniendo en cuenta varios conceptos, cada uno de los cuales puede representar un problema de bancarrota, de ahí que hablemos de "subestados" para nombrar a estos problemas. Sin embargo, una dificultad con este enfoque es que, aunque el pago es una idea formal para hacer frente a los problemas, las reglas son difíciles de calcular. Algunas soluciones a este problema son propuestas por González-Alcón et al. (2007) donde resuelven este problema de complejidad construyendo una extensión diferente de la regla "*run-to-the-bank*". Borm et al. (2005) estudian problemas de bancarrota con coaliciones a priori y usan la misma idea para primero asignar dinero a los coaliciones y luego redividir el dinero dentro de cada coalición. Finalmente, Lorenzo-Freire et al. (2008) aplican la idea de dos etapas a las reglas igualitaria restringida y de igualdad de pérdidas restringida, caracterizando ambas reglas de reparto.

5.2.2 Problemas de bancarrota con múltiples (sub)estados

Consideramos una situación en la que existe un recurso perfectamente divisible E, denominado estado, que debe distribuirse entre un número finito de conceptos M (subestados) sobre las que existe un conjunto finito de reclamantes N que tienen demandas de tal manera que la suma de todos los derechos sea mayor que el estado disponible E.

Definición 5.10. Un *problema de bancarrota con múltiples (sub)estados* (MB en breve) es una 4-tupla (M, N, E, C), donde $M = \{1, 2, ..., m\}$ es el conjunto de (sub)estados, N es el conjunto de reclamantes, $E \in \mathbb{R}_+$ es una cantidad perfectamente divisible que tiene que ser dividida (el *estado*), y $C \in \mathcal{M}^+_{m\times n}$ es una matriz de demandas. Cada fila en C representa las demandas para uno de los (sub)estados. Un elemento genérico de C, c_{ij}, denota la cantidad del (sub)estado i que el reclamante j demanda. Además, $\sum_{i\in M}\sum_{j\in N} c_{ij} > E$. La familia de todos estos problemas se denotará por $\mathcal{MB}$.

A continuación se presenta un ejemplo para ilustrar qué tipo de situaciones son modeladas por problemas de bancarrota de múltiples (sub)estados.

Ejemplo 5.1. Considérese un país que tiene tres regiones A, B y C. El gobierno del país ha transferido las competencias de educación y sanidad a las regiones. El presupuesto para educación y sanidad que el gobierno pone a disposición de las regiones es de 30 millones de euros y cada región tiene demandas para gastar en educación y sanidad de 5 y 7 millones, 7 y 9 millones, y 6 y 8 millones, respectivamente. En esta situación, hay dos sibestados o conceptos: educación y sanidad; tres reclamantes, las regiones A, B y C; el estado es 30 millones y la matriz de demandas en millones de euros está dada por

$$\begin{pmatrix} 5 & 7 & 6 \\ 7 & 9 & 8 \end{pmatrix}$$

La demanda total es de 42 millones, lo que supera el presupuesto disponible de 30 millones. Por lo tanto, esta situación describe un problema de bancarrota con múltiples (sub)estados. Un posible reparto del presupuesto viene dado por la siguiente matriz:

$$\begin{pmatrix} 3 & 5 & 4 \\ 5 & 7 & 6 \end{pmatrix}$$

Esta matriz indica lo que recibe cada región para atender educación y sanidad. Por ejemplo, la región A recibe 3 millones de euros para educación y 5 millones para sanidad.

5.2.3 Reglas de reparto para problemas de bancarrota con múltiples (sub)estados

Al igual que en los problemas de bancarrota, el objetivo principal en un problema de bancarrota con múltiples (sub)estados es encontrar una asignación que sea lo más justa posible, teniendo en cuenta el estado disponible y las demandas de los reclamantes para cada (sub)estado o concepto.

Definición 5.11. Dado un problema $(M, N, E, C) \in \mathcal{MB}$, un *reparto factible* para él es una matriz $X = (x_{ij})_{i \in M, j \in N} \in \mathbb{R}^{M \times N}$ tal que:

1. $0 \leq x_{ij} \leq c_{ij}$, para todo $i \in M$ y $j \in N$.
2. $\sum_{i \in M} \sum_{j \in N} x_{ij} = E$,

y se denotará por $A(M, N, E, C)$ el conjunto de todos los repartos factibles.

Una matriz $X \in \mathbb{R}^{M \times N}$ representa una forma deseable de dividir E entre los agentes en N, según los conceptos en M. El requisito (1) es que cada agente reciba una asignación por cada concepto que no sea negativo y esté limitado por su demanda. El requisito (2) es que se asigne la totalidad del monto disponible. Estos dos requisitos implican que $X = C$ siempre que $\sum_{i \in M} \sum_{j \in N} c_{ij} = E$.

Definición 5.12. Una *regla de reparto para problemas de bancarrota con múltiples (sub)estados* es una aplicación R que asocia con cada $(M, N, E, C) \in \mathcal{MB}$ una única matriz $R(M, N, E, C) \in A(M, N, E, C)$.

Tres enfoques posibles para definir las reglas de reparto para problemas de bancarrota con múltiples (sub)estados son los siguientes:

- El primer enfoque proporciona reglas que asignan una matriz de forma directa, por ejemplo, resolviendo un problema de optimización como este:

$$\begin{array}{rl} \min & \sum_{i \in M} \sum_{j \in N} (x_{ij} - c_{ij})^2 \\ s.a: & \sum_{i \in M} \sum_{j \in N} x_{ij} = E \\ & 0 \leq x_{ij} \leq c_{ij} \end{array}$$

- El segundo enfoque proporciona reglas en una etapa. Consiste simplemente en considerar el problema como un problema de bancarrota con tantas demandas como número de conceptos o (sub)estados multiplicado por el número de reclamantes.
- El tercer enfoque proporciona reglas de dos etapas. En la primera etapa, la cantidad total de recursos se asigna a los (sub)estados. En el segundo paso, resolvemos tantos problemas de bancarrota de un solo estado como problemas tengamos, esta es la razón por la que los hemos traducido de esta forma.

A continuación se da la definición formal del tercer enfoque para definir las reglas de reparto en los problemas de bancarrota con múltiples (sub)estados.

Definición 5.13. Sea R una regla de reparto para $\mathcal{BP}$ y sea $(M, N, E, C) \in \mathcal{MB}$. La regla de reparto en dos etapas $R(M, N, E, C)$ es el reparto obtenido del siguiente proceso en dos etapas:

Etapa 1: Considérese el así llamado *problema de bancarrota cociente* (M, E, c^M) donde $c^M = (c_1^M, ..., c_{|M|}^M) \in \mathbb{R}^M$ denota el vector de demandas totales de los (sub)estados, es decir, $c_i^M = \sum_{j \in N} c_{ij}$ para todo $i \in M$. Repártase E entre los (sub)estados utilizando la regla de reparto R. De este modo, se obtiene $R(M, E, c^M) \in \mathbb{R}^M$.

Etapa 2: Para cada $i \in M$, considérese un nuevo problema de bancarrota para los reclamantes $(N, R_i(M, E, c^M), (c_{ij})_{j \in N}) \in \mathcal{BP}$. Aplíquese de nuevo la regla de reparto R a cada uno de los problemas de bancarrota. Así, para cada $i \in M$, se obtiene el reparto $R(N, R_i(M, E, c^M), (c_{ij})_{j \in N})$.

Es fácil verificar que la asignación propuesta en el Ejemplo 5.1 es la solución del programa no lineal dado en el primer enfoque anterior. En las siguientes subsecciones ilustramos cómo aplicar el enfoque de dos etapas.

La regla CEA en dos etapas

La regla CEA en dos etapas, CEA^2, es definida, para todo $(M, N, E, C) \in \mathcal{MB}$ por

$$CEA_{ij}^2(M, N, E, C) = \min\{\lambda_i, c_{ij}\}, \text{ for all } i \in M, j \in N,$$

donde $i \in M$, λ_i es tal que $\sum_{j \in N} \min\{\lambda_i, c_{ij}\} = \min\left\{\beta, c_i^M\right\}$ y β es tal que $\sum_{i \in M} \min\{\beta, c_i^M\} = E$.

Obsérvese que CEA^2 difiere de la regla CEA en una etapa definida por

$$CEA_{ij}^1(M, N, E, C) = \min\{\beta, c_{ij}\},$$

para cada $i \in M$ y $j \in N$ con β tal que $\sum_{i\in M}\sum_{j\in N}\min\{\beta, c_{ij}\} = E$, es decir, la aplicación directa de la regla CEA tomando todas las demandas como parte de un único problema de bancarrota, es decir, un enfoque en una sola etapa. Una distinción relevante entre las dos extensiones es que la última no satisface la propiedad del cociente.

Para la situación descrita en el Ejemplo 5.1, primero resolvemos el problema de bancarrota $(M = \{E, H\}, 30, (18, 24))$ cuya solución CEA es $(15, 15)$. Luego resolvemos los siguientes dos problemas de bancarrota $(N = \{A, B, C\}, 15, (5, 7, 6))$ y $(N = \{A, B, C\}, 15, (7, 9, 8))$ cuyas soluciones CEA son $(5, 5, 5)$ y $(5, 5, 5)$ respectivamente. En este caso particular, si consideramos el enfoque de una etapa obtenemos la misma asignación.

La regla CEL en dos etapas

La regla CEL en dos etapas, CEL^2, es definida, para todo $(M, N, E, C) \in \mathcal{MB}$ por

$$CEL^2_{ij}(M, N, E, C) = \max\{0, c_{ij} - \lambda_i\}, \text{ for all } i \in M, j \in N,$$

donde $i \in M$, λ_i es tal que $\sum_{j\in N}\max\{0, c_{ij} - \lambda_i\} = \max\left\{0, c_i^M - \beta, \right\}$ y β es tal que $\sum_{i\in M}\max\left\{0, c_i^M - \beta, \right\} = E$.

Nuevamente, la regla CEL en dos etapas difiere de la regla CEL en una etapa. Para la situación descrita en el Ejemplo 5.1, primero resolvemos el problema de bancarrota $(M = \{E, H\}, 30, (18, 24)$ cuya solución CEL es $(12, 18)$. Luego resolvemos los siguientes dos problemas de bancarrota $(N = \{A, B, C\}, 12, (5, 7, 6))$ y $(N = \{A, B, C\}, 18, (7, 9, 8))$ cuyas soluciones CEL son $(3, 5, 4)$ y $(5, 7, 6)$ respectivamente, en este caso particular, si se considera el enfoque de una etapa obtenemos nuevamente la misma asignación.

La regla proporcional en dos etapas

Moreno-Ternero (2009) demuestra que la regla proporcional es la única regla concursal tal que coinciden la extensión en una etapa y la extensión en dos etapas para problemas de bancarrota con múltiples (sub)estados. Por lo tanto, la regla proporcional para los problemas de bancarrota con múltiples (sub)estados se define para cada $(M, N, E, C) \in \mathcal{MB}$ de la siguiente manera:

$$P_{ij}(M, N, E, C) = \frac{E}{\sum_{k\in M}\sum_{h\in N} c_{kh}} c_{ij}, \quad i \in M, j \in N.$$

La regla RA en dos etapas

La regla RA en dos etapas, RA^2, es definida, para todo $(M, N, E, C) \in \mathcal{MB}$ por

$$RA^2_{ij}(M, N, E, C) = RA_j(N, RA_i(M, E, C^M), (c_{ij})_{j\in N}), \quad i \in M, j \in N.$$

Una vez más, la regla RA en dos etapas difiere de la regla RA en una etapa. Para la situación descrita en el Ejemplo 5.1, primero resolvemos el problema de bancarrota $(M = \{E, H\}, 30, (18, 24)$ cuya solución RA es $(12, 18)$. Luego resolvemos los siguientes dos problemas de bancarrota $(N = \{A, B, C\}, 12, (5, 7, 6))$ y $(N = \{A, B, C\}, 18, (7, 9, 8))$ cuyas soluciones RA son $(3.33, 4.83, 3.83)$ y $(5, 7, 6)$ respectivamente.

Calleja et al. (2005) proponen extensiones de la regla de llegada aleatoria para problemas de bancarrota de múltiples problemas considerando los problemas y los demandantes combinados. Así, definen dos extensiones de la regla de llegadas aleatorias: la regla de llegadas aleatorias proporcionales y la regla de llegadas aleatorias en cola. Por otro lado, González-Alcón et al. (2007) proponen una extensión en dos etapas de la regla de llegada aleatoria: primero, asignan explícitamente el estado a los (sub)estados de acuerdo con un vector marginal, y luego, dentro de cada (sub)estado, la parte correspondiente del estado se divide entre los reclamantes utilizando el regla de llegada aleatoria. Esta extensión está en línea con las extensiones en dos etapas de las reglas de bancarrota.

5.2.4 Propiedades para reglas de reparto para problemas de bancarrota con múltiples (sub)estados

Ahora introducimos varias propiedades para las reglas de reparto para problemas de bancarrota con múltiples (sub)estados que son relevantes para esta tesis. Todas las propiedades a continuación se pueden encontrar en Lorenzo-Freire et al. (2010) y Bergantiños et al. (2010).

Axioma 5.21 (Consistencia en dos etapas)**.** Dada una regla R, se dice que satisface *consistencia en dos etapas*, si para cada problema $(M, N, E, C) \in \mathcal{MB}$ y para todo $i \in M$, $j \in N$,

$$R_{ij}(M, N, E, C) = R_{1j}(\{1\}, N, R_{i1}(M, \{1\}, E, c^M), (c_{ik})_{k\in N}),$$

donde $(M, \{1\}, E, c^M)$ es el problema de bancarrota cociente.

Es inmediato que todas las reglas de dos etapas satisfacen la consistencia en dos etapas.

Axioma 5.22 (Propiedad del cociente)**.** Dada una regla R, se dice que satisface *propiedad del cociente*, si para cada problema $(M, N, E, C) \in \mathcal{MB}$ y para todo $i \in M$,

$$\sum_{j \in N} R_{ij}(M, N, E, C) = R_{i1}(M, \{1\}, E, c^M).$$

La propiedad del cociente significa que la cantidad total asignada a un (sub)estado en un problema de MB es igual a la cantidad asignada al mismo (sub)estado en el problema de bancarrota cociente, es decir, la asignación final a un (sub)estado es independiente de la distribución de reclamaciones sobre ese (sub)estado.

Axioma 5.23 (Consistencia dentro de los (sub)estados)**.** Dada una regla R, se dice que satisface *Consistencia dentro de los (sub)estados*, si para cada problema $(M, N, E, C) \in \mathcal{MB}$ y para todo $i \in M$, $j \in N$,

$$R_{ij}(M, N, E, C) = R_{1j}(\{1\}, N, \sum_{k \in N} R_{ik}(M, N, E, C), (c_{ik})_{k \in N}).$$

Esta propiedad significa que al volver a dividir el monto total dentro de un (sub)estado usando la misma regla R debería arrojar el mismo resultado.

Axioma 5.24 (Composición hacia arriba)**.** Dada una regla R, se dice que satisface *composición hacia arriba*, si para todo problema $(M, N, E, C) \in \mathcal{MB}$ y para todo $0 \leq E' \leq E$, se tiene que

$$R(M, N, E, C) = R(M, N, E', C) + R(M, N, E - E', C - R(M, N, E', C)).$$

Esta propiedad tiene la misma interpretación que su axioma homólogo en problemas de bancarrota (véase el axioma 5.12).

Axioma 5.25 (Monotonía en los recursos)**.** Dada una regla R, se dice que satisface *monotonía en los recursos*, si para todo $(M, N, E, C) \in \mathcal{MB}$ y $(M, N, E', C) \in \mathcal{MB}$ tal que $0 \leq E' \leq E$,

$$R(M, N, E', C) \leq R(M, N, E, C).$$

Axioma 5.26 (Invarianza al truncar las demandas)**.** Dada una regla R, se dice que satisface *invarianza al truncar las demandas*, si para todo problema $(M, N, E, C) \in \mathcal{MB}$, se tiene que

$$R(M, N, E, C) = R(M, N, E, C_E),$$

donde $C^E \in \mathbb{R}_+^{M \times N}$ es tal que $c_{ij}^E = min\{c_{ij}, E\}$ para todo $i \in M$ y $j \in N$.

Véase el axioma 5.14.

Axioma 5.27 (Igual trato de iguales dentro de los (sub)estados)**.** Dada una regla R, se dice que satisface *igual trato de iguales dentro de los (sub)estados*, si para cada $(M, N, E, C) \in \mathcal{MB}$ y para todo $i \in M$ y $j, k \in N$ tal que $c_{ij} = c_{ik}$,

$$R_{ij}(M, N, E, C) = R_{ik}(M, N, E, C).$$

Esta propiedad dice que si dos reclamantes tienen la misma demanda para un (sub)estado, deben recibir lo mismo por ese (sub)estado.

Axioma 5.28 (Igual trato de los (sub)estados)**.** Dada una regla R, se dice que satisface *igual trato de los (sub)estados*, si para cada $(M, N, E, C) \in \mathcal{MB}$ y para todo $i, k \in M$ tal que $c_i^M = c_k^M$,

$$R_{i1}(M, \{1\}, E, c^M) = R_{k1}(M, \{1\}, E, c^M).$$

Esta propiedad significa que si dos (sub)estados tienen la misma demanda total, entonces el monto total asignado a ambos (sub)estados debe ser el mismo independientemente de la distribución de demandas sobre ellos.

Axioma 5.29 (Sin ventajas por transferencias entre (sub)estados)**.** Dada una regla R, se dice que satisface *sin ventajas por transferencias entre (sub)estados*, si para cada par de problemas $(M, N, E, C), (M, N, E, C') \in \mathcal{MB}$ tal que existe $S \subset M$ tal que $\sum_{i \in S} \left(\sum_{j \in N} c_{ij} \right) = \sum_{i \in S} \left(\sum_{j \in N} c'_{ij} \right)$ y $c_{ij} = c'_{ij}$ cuando $(i, j) \in M \backslash S \times N$. Entonces, para cada $i \in M \backslash S$,

$$\sum_{j \in N} R_{ij}(M, N, E, C) = \sum_{j \in N} R_{ij}(M, N, E, C').$$

Axioma 5.30 (Sin ventajas por transferencias dentro de los (sub)estados)**.** Dada una regla R, se dice que satisface *sin ventajas por transferencias dentro de los (sub)estados*, si para cada par de problemas $(M, N, E, C), (M, N, E, C') \in \mathcal{MB}$ tal que existe un $i \in M$, and $S \subset N$ tal que $\sum_{j \in S} c_{ij} = \sum_{j \in S} c'_{ij}$ y $c_{kj} = c'_{kj}$ cuando $k \in M \backslash \{i\}$ o $j \in N \backslash S$. Entonces, para cada $j \in N \backslash S$,

$$R_{ij}(M, N, E, C) = R_{ij}(M, N, E, C').$$

La propiedad de sin ventajas por transferencias entre (sub)estados y la de sin ventajas por transferencias dentro de los (sub)estados están relacionadas con la propiedad de sin ventajas por transferencias en problemas de bancarrota (véase el axioma 5.17). Sin embargo, mientras que el axioma 5.17 dice que los reclamantes que han redistribuido sus demandas no están mejor en conjunto, la propiedad de sin ventajas por transferencias entre (sub)estados y la de sin ventajas por transferencias dentro de los (sub)estados dicen que los agentes que no han redistribuido sus demandas no están ni mejor ni peor.

5.2.5 Caracterizaciones

En el siguiente teorema se presenta una caracterización de la regla CEA^2. Este teorema está inspirado en un resultado similar para la regla CEA para problemas de bancarrota en Dagan (1996).

Teorema 5.4 (Lorenzo-Freire et al., 2010)**.** Sea $(M, N, E, C) \in \mathcal{MB}$. La única regla de reparto que satisface composición hacia arriba, invarianza al truncar las demandas, igual trato de iguales dentro de los (sub)estados, igual trato de los (sub)estados, y la propiedad del cociente es la regla igualitaria restringida en dos etapas, CEA^2 .

En el siguiente teorema se proporciona una caracterización para la relga proporcional en problemas de bancarrota con múltiples (sub)estados.

Teorema 5.5 (Bergantiños et al., 2010)**.** Sea $(M, N, E, C) \in \mathcal{MB}$ tal que $|N| \geq 3$ y $|M| \geq 3$. Entonces, la regla proporcional es la única regla de reparto que satisface sin ventajas por transferencias entre (sub)estados y sin ventajas por transferencias dentro de los (sub)estados.

Finalmente, Calleja et al. (2005) caracterizan la regla de llegada aleatoria proporcional y la regla de llegada aleatoria de cola por medio de las denominadas *P-consistencia* y *Q-consistencia* que son extensiones de la propiedad de consistencia en las demandas introducida por O 'Neill (1982) (véase el axioma 5.15). González-Alcón et al. (2007) caracterizan la regla compuesta de llegadas aleatorias utilizando la consistencia en los (sub)estados, que también es una extensión de la consistencia en las demandas. Lorenzo-Freire et al. (2007) utilizan impactos equilibrados para caracterizar la regla de llegada aleatoria para problemas de bancarrota con múltiples (sub)estados. Sin embargo, hasta donde sabemos, la regla de llegada aleatoria de dos etapas aún no se ha caracterizado. En vista de los axiomas de la subsección 5.2.4 y las caracterizaciones a las que dan lugar, una alternativa para caracterizar la regla de llegada aleatoria en dos etapas sería definir dos nuevos axiomas: uno relacionado con impactos balanceados entre (sub)estados y otro de impactos equilibrados dentro de los (sub)estados.

5.3 Problemas de bancarrota con múltiples estados y demandas cruzadas

En esta sección, presentamos el modelo de problemas de bancarrota con múltiples estados y demandas cruzadas, que es una extensión de los problemas de bancarrota. Además, presentamos cuatro reglas de reparto para estos problemas, la regla igualitaria restringida, la regla proporcional restringida, las reglas de prioridad secuencial restringida y la regla de llegada aleatoria restringida, que son extensiones de las reglas de reparto correspondientes para problemas de bancarrota. El contenido de esta sección basa en Acosta-Vega et al. (2021a, 2021b, 2022a, 2022b).

5.3.1 Introducción

En esta sección presentamos un modelo novedoso de problema de bancarrota con múltiples estados inspirado en un problema real de reducción de emisiones de diferentes familias de contaminantes en el que los contaminantes pueden pertenecer a más de una familia. En nuestro modelo, por lo tanto, varios recursos perfectamente divisibles (estados) tienen que ser asignados entre cierto conjunto de agentes (demandantes) que tienen exactamente un derecho (reclamación o demanda) que se utiliza en todos y cada uno de los estados simultáneamente, este modelo estudia situaciones con estados multidimensionales, uno para cada estado y donde cada agente reclama lo mismo a los distintos estados en los que participa.

Para ilustrar este modelo, considérese ahora que cierta autoridad está interesada en reducir la emisión de contaminantes a la atmósfera. Sin embargo, existen muchos contaminantes, cada uno con diferentes efectos y consecuencias. Hay contaminantes que contribuyen al efecto invernadero y, por tanto, al cambio climático, y otros que son nocivos para la salud porque son cancerígenos, provocan problemas respiratorios u otras enfermedades. Por un lado, el vapor de agua (H2O), el dióxido de carbono (CO2), el óxido nitroso (NO2), el metano (CH4) y el ozono (O3) son los principales gases de efecto invernadero (GEI), pero también el hexafluoruro de azufre (SF6), los hidrofluorocarbonos (HFC) y los perfluorocarbonos (PFC) son relevantes según el Protocolo de Kioto. Por otro lado, el monóxido de carbono (CO), el dióxido de azufre (SO2), el óxido nitroso (NO2), el ozono (O3), el amoniaco (NH3), las partículas en suspensión (PM), los hidrocarburos aromáticos policíclicos (HAP) y los compuestos orgánicos volátiles (COV), entre otros, se consideran muy nocivos para la salud. Así, por ejemplo, el Protocolo de Gotemburgo establece techos de emisión de SO2, NO2, COV y NH3. Por ello, podemos encontrar directivas, leyes y reglamentos internacionales, europeos, nacionales y autonómicos para controlar sus emisiones. Algunos ejemplos son el conocido Acuerdo de París o el Protocolo de Kioto para la reducción global de GEI o el Protocolo de Gotemburgo para abatir la acidificación, la eutrofización y el ozono troposférico. Además, podemos observar que existen gases que contribuyen tanto al efecto invernadero como a la contaminación del aire. Para detalles sobre estos y otros temas relacionados con la protección

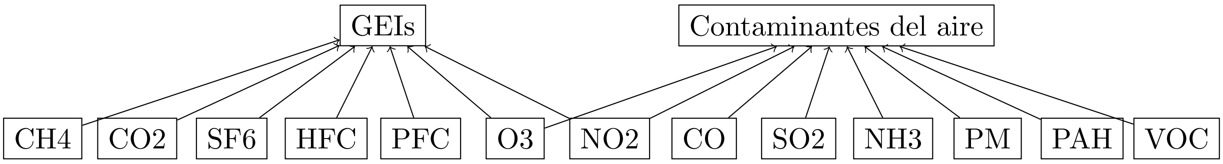

Figure 5.1: Ejemplo de una estructura jerárquica de dos niveles para la reducción de GEI y contaminantes del aire.

del medio ambiente y la salud visítese la página web https://greenfacts.org.

Todo el sistema de abatimiento de contaminantes podría representarse en una estructura jerárquica de dos niveles (véase la figura 5.3). En el primer nivel se tendría el efecto de los contaminantes, y en el segundo nivel los propios contaminantes. El objetivo último de dicha autoridad es que las emisiones anuales de los diferentes contaminantes estén por debajo de ciertos niveles (por ejemplo, toneladas anuales emitidas) para controlar mejor la contaminación y sus efectos. En este sentido, la autoridad fija ciertos niveles de emisiones (toneladas totales por año) para cada efecto de los contaminantes. Sin embargo, los contaminantes podrían contribuir a más de un efecto como se ha comentado con anterioridad. Así, se considera la situación particular en la que existen diferentes cantidades de emisiones de distintos contaminantes y la autoridad fija niveles máximos de emisión (toneladas anuales) para cada efecto de los contaminantes según, por ejemplo, sus efectos sobre la calidad del aire o su contribución al cambio climático, con el fin de abatir estas emisiones y mantenerlas por debajo de ciertos niveles (toneladas por año). El planteamiento de fijar un nivel de emisiones al año es el habitual en las directivas y protocolos al respecto, por lo que no se considera el impacto particular de un contaminante en cada efecto en la atmósfera, simplemente se trata de reducir sus emisiones y con ello, su impacto negativo en la calidad del aire o el efecto invernadero. Por lo tanto, tenemos un tipo de problema de bancarrota con múltiples estados que es diferente de otros problemas de bancarrota con múltiples (sub)estados como se explicará después. Por supuesto, si sólo se estuviera interesado en un efecto particular de los contaminantes, se obtiene un problema de bancarrota clásico, donde el estado es la cantidad fijada para el efecto (total de toneladas por año) y las demandas o derechos son los niveles actuales de emisión de contaminantes (toneladas). por año).

Estos problemas recuerdan a los problemas de bancarrota con múltiples (sub)estados en los cuales los (sub)estados son los efectos de los contaminantes. Los problemas de bancarrota con múltiples (sub)estados introducidos por Calleja et al. (2005) describen situaciones en las cuales hay un estado perfectamente divisible que puede ser dividido entre varios (sub)estados, y un determinado número de agentes que tienen demandas sobre cada uno de los (sub)estados[2]. Por tanto, hay un estado perfectamente divisible, varios (sub)estados, y agentes con vectores de demanda con tantas coordenadas como (sub)estados hay, tales que la demanda total es superior al estado. La pregunta fundamental para estos problemas es cómo debería ser repartido el estado. Este problema se ha resuelto mediante reglas de reparto y existen varios enfoques para su definición (véase, por ejemplo, Calleja et al., 2005; Borm et al., 2005; Izquierdo y Timoner, 2016). Sin embargo, en el enfoque del problema de esta tesis doctoral se tiene un estado ex ante para cada uno de los efectos en la atmósfera, y los agentes son los contaminantes, y cada contaminante tiene exactamente una demanda que es la misma para todos los estados de los efectos a los cuales él contribuye. Por tanto, este enfoque es diferente de los problemas de bancarrota con múltiples (sub)estados que existen en la literatura.

En nuestro caso, se tienen varios estados perfectamente divisibles, y agentes que tienen exactamente un valor de demanda que es utilizado en todos los estados simultáneamente. Ahora, de nuevo, la pregunta es cómo deberían ser los estados repartidos de un modo razonable. Hasta donde sabemos, este enfoque es totalmente novedoso en la literatura y cubre un hueco en los problemas de bancarrota que se han estudiado hasta ahora.

En este contexto, se introducen tres reglas de reparto y una familia de reglas de reparto para problemas de bancarrota con múltiples estados y demandas cruzadas. En primer lugar, se introduce una regla que generaliza la regla de igual reparto restringido (CEA) a partir del análisis de la regla CEA como solución de una sucesión de problemas lineales. Además, se introduce la regla proporcional restringida para problemas de bancarrota con múltiples estados y demandas cruzadas que generaliza la conocida regla proporcional. Y, finalmente, se introduce la familia de reglas con prioridad secuenciales restringidas y la regla de llegadas aleatorias restringida para problemas de bancarrota con múltiples estados y demandas cruzadas.

5.3.2 Problemas de bancarrota con múltiples estados y demandas cruzadas

Considérese una situación en la que hay un conjunto finito de estados (efectos de los contaminantes)$M = \{1, 2, ..., m\}$ tal que cada efecto j tiene un estado perfectamente divisible e_j (nivel máximo de emisiones para ese efecto de los contaminantes). Sea $E = (e_1, e_2, ..., e_m)$ el vector de estados. Hay

[2] "Un (sub)estado constituye una razón sobre la base de la cual el estado tiene que ser dividido". (Calleja et al. 2005, página 731.)

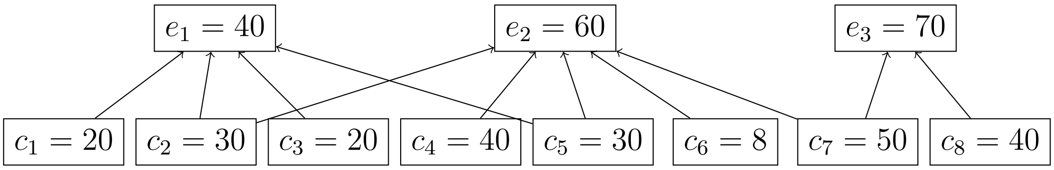

Figure 5.2: Ejemplo 5.2.

un conjunto finito de agentes (los contaminantes) $N = \{1, 2, ..., n\}$ tal que cada agente i demanda c_i (emisiones del contaminante i) de aquellos efectos a los que contribuye. Sea $c = (c_1, c_2, ..., c_n)$ el vector de demandas. Ahora, cada agente demanda a diferentes conjuntos de efectos. Así, α es una correspondencia que asocia con cada $i \in N$ un subconjunto $\alpha(i) \subset M$. De hecho, $\alpha(i)$ representa los efectos a los cuales el agente i contribuye. Además, $\sum_{i:j\in\alpha(i)} c_i > e_j$, para cada $j \in M$, en otro caso, esos estados podrían ser descartados del problema porque no imponen ninguna limitación, y de este modo la asignación sería trivial.

Definición 5.14. Un *problema de bancarrota con múltiples estados y demandas cruzadas* (MBC por sus siglas en ingles) es una 5-tupla (M, N, E, c, α), donde M es un conjunto finito de estados, N es un conjunto finito de agentes, $E \in \mathbb{R}^M_{++}$ es el vector de estados, $c \in \mathbb{R}^N_{++}$ es el vector de demandas, y $\alpha : N \to M$ es una correspondencia punto-conjunto. Además, la familia de todos estos problemas se denota por $\mathcal{MBC}$.

En el ejemplo siguiente se ilustra cuál es la estructura de estos problemas.

Ejemplo 5.2. Considérese el siguiente problema de bancarrota con múltiples estados y demandas cruzadas (M, N, E, c, α) with $M = \{1, 2, 3\}$; $N = \{1, 2, 3, 4, 5, 6, 7, 8\}$; $E = (40, 60, 70)$; $c = (20, 30, 20, 40, 30, 8, 50, 40)$; and $\alpha(1) = \{1\}$, $\alpha(2) = \{1, 2\}$, $\alpha(3) = \{1\}$, $\alpha(4) = \{2\}$, $\alpha(5) = \{1, 2\}$, $\alpha(6) = \{2\}$, $\alpha(7) = \{2, 3\}$, and $\alpha(8) = \{3\}$. Esta situación se muestra gráficamente en la figura 5.2.

A primera vista, un enfoque simplista podría ser solucionar tres problemas de bancarrota, uno para cada estado, pero esto no es tan simple, porque hay agentes con demandas sobre varios estados y esto prodría conducir a asignaciones infactibles o incompatibles. Por tanto, es necesario un análisis más detallado de este tipo de problemas.

En el ejemplo 5.2, se puede observar claramente la estructura de nuestro modelo de problema de bancarrota con múltiples estados. Este modelo difiere de otrod modelos con múltiples estados en tres elementos. Primero, hay varios estados, cada uno con su propia disponibilidad (esta característica sería similar al enfoque en Izquierdo y Timoner (2016) para problemas de bancarrota con múltiples (sub)estados). Segundo, en nuestro modelo no hay un vector de demandas con tantas coordenadas como estados haya, sino una simple demanda para todos los estados a los que reclama. Y finalmente, estas demandas son consideradas a la vez para varios estados.

Definición 5.15. Dado un problema $(M, N, E, c, \alpha) \in \mathcal{MBC}$, un *reparto factible* para él, es un vector $x \in \mathbb{R}^N$ tal que:

1. $0 \leq x_i \leq c_i$, para todo $i \in N$.
2. $\sum_{i \in N : j \in \alpha(i)} x_i \leq e_j$, para todo $j \in M$,

y se denotará por $A(M, N, E, c, \alpha)$ el conjunto de los repartos factibles.

Estos requisitos son similares a aquellos utilizados para problemas MB. El requisito 1 significa que cada agente recibe a lo sumo lo que demanda pero no menos que nada. El requisito 2 significa que no se puede repartir más de lo disponible en cada estado. Por tanto, el vector $x \in \mathbb{R}^N$ representa un reparto que es simultáneamente factible para todos los estados.

Definición 5.16. Una *regla de reparto o simplemente una regla* para problemas de bancarrota con múltiples estados y demandas cruzadas es una aplicación R que asocia a cada $(M, N, E, c, \alpha) \in \mathcal{MBC}$ un único reparto factible $R(M, N, E, c, \alpha) \in A(M, N, E, c, \alpha)$.

Ejemplo 5.3. Considérese de nuevo el problema en el ejemplo 5.2. Dos repartos posibles son los siguientes:

- $R(M, N, E, c, \alpha) = (12.5, 7.5, 12.5, 7.5, 7.5, 7.5, 30, 40)$.
- $R(M, N, E, c, \alpha) = (13.75, 6.25, 13.75, 6.25, 6.5, 6.25, 35, 35)$.

Ambos repartos en el ejemplo 5.3 satisfacen los dos requisitos y, además, es sencillo comprobar que son eficientes para todos los estados, es decir, el requisito 2 se satisface con igualdad. Sin embargo, esto no es posible en general como se muestra en el siguiente ejemplo.

Ejemplo 5.4. Considérese una vez más el problema en el ejemplo 5.2 pero cambiando $c_7 = 65$ y $e_3 = 105$. En esta situación, es obvio que si un reparto es eficiente para el estado 3, entonces es infactible para el estado 2.

Por tanto, a la vista del ejemplo 5.4, no se puede hacer el requisito 2 más exigente si se quiere alcanzar al menos un posible reparto.

5.3.3 La regla igualitaria restringida para problemas MBC

Como se mencionó en secciones previas, la regla igualitaria restringida (CEA) es una de las más importantes para resolver problemas de bancarrota (véase Herrero y Villar, 2001). Esta regla reparte el estado tan igualitariamente como sea posible entre los agentes. La cuestión aquí es qué se entiende por lo más igualitariamente posible. En el contexto de los problemas de bancarrota con un único estado, tan igualitariamente como sea posible significa que ningún agente puede conseguir más que aquellos con demandas más pequeñas, excepto que estos hayan recibido lo que demandaban. Como se mostró en la sección 5.1, esto puede formularse matemáticamente como:

Para cada $(N, E, c) \in \mathcal{B}$,

$$CEA_i(N, E, c) = \min\{c_i, \beta\}, \qquad i \in N, \tag{5.1}$$

donde β es un número real positivo que satsiface $\sum_{i \in N} CEA_i(N, E, c) = E$.

La cuestión aquí es cómo extrapolar esto al contexto de las situaciones MBC. Para ello, en esta tesis doctoral, introducimos la regla CEA como el resultado de la solución óptima de una sucesión de problemas lineales[3].

Dada un problema $(N, E, c) \in \mathcal{B}$, para repartir E entre los agentes de acuerdo con CEA, se procede como sigue:

$$(P_1) \quad \begin{array}{ll} max & z^1 \\ s.a: & \sum_{i \in N} x_i \leq E \\ & x_i \leq c_i, \text{ para todo } i \in N \\ & x_i \geq z^1, \text{ para todo } i \in N \\ & x_i \geq 0, \text{ para todo } i \in N, \text{ y } z^1 \geq 0 \end{array}$$

Sea z^{*1} el valor óptimo del problema lineal (P_1). Si $nz^{*1} = E$, entonces $CEA_i(N, E, c) = z^{*1}$. En otro caso, debe resolverse el siguiente problema lineal:

$$(P_2) \quad \begin{array}{ll} max & z^2 \\ s.a: & \sum_{i \in N} x_i \leq E \\ & x_i \leq c_i, \text{ para todo } i \in N \\ & x_i \geq z^{*1} + \mu^0 (c_i - z^{*1}) z^2, \text{ para todo } i \in N \\ & x_i \geq 0, \text{ para todo } i \in N, \text{ y } z^2 \geq 0 \end{array}$$

[3]En Izquierdo y Timoner (2016) puede verse otra manera de obtener la regla CEA utilizando problemas de optimización cuadrática.

donde para cada $a \in \mathbb{R}$,

$$\mu^0(a) = \begin{cases} 0 & si\ a \leq 0, \\ 1 & en\ otro\ caso. \end{cases}$$

Sea z^{*2} el valor óptimo del problema lineal (P_2). Si $nz^{*1} + \sum_{i \in N} \mu^0(c_i - z^{*1})z^{*2} = E$, entonces $CEA_i(N, E, c) = z^{*1} + \mu^0(c_i - z^{*1})z^{*2}$. En otro caso, un nuevo problema lineal debe ser resuelto. En el paso general k, se tiene el siguiente problema lineal:

$$(P_k) \quad \begin{array}{ll} max & z^k \\ s.a: & \sum_{i \in N} x_i \leq E \\ & x_i \leq c_i, \text{ para todo } i \in N \\ & x_i \geq \sum_{h=1}^{k-1} \mu^0(c_i - \sum_{l=1}^{h-1} z^{*l})z^{*h} + \mu^0(c_i - \sum_{l=1}^{k-1} z^{*l})z^k, \text{ para todo } i \in N \\ & x_i \geq 0, \text{ para todo } i \in N, \text{ y } z^k \geq 0 \end{array}$$

De nuevo, sea z^{*k} el valor óptimo del problema lineal (P_k). Si $\sum_{i \in N} \sum_{h=1}^{k} \delta^0(c_i - \sum_{l=1}^{h-1} z^{*l})z^{*h} = E$, entonces $CEA_i(N, E, c) = \sum_{h=1}^{k} \mu^0(c_i - \sum_{l=1}^{h-1} z^{*l})z^{*h}$. En otro caso, debe resolverse el problema lineal (P_{k+1}). Y así sucesivamente, hasta que el estado sea totalmente distribuido o todas las demandas satisfechas. Es obvio que este procedimiento finaliza en un número finito de pasos y el reparto final es la CEA.

Obsérvese que cuando se soluciona el último problema lineal, entonces su solución óptima es $x^{*k} = CEA(N, E, c)$. En el siguiente ejemplo se ilustra este procedimiento.

Ejemplo 5.5. Considérese $(N, E, c) \in \mathcal{B}$ con $N = \{1, 2, 3, 4, 5, 6, 7, 8\}$; $E = 170$; $c = (20, 30, 20, 40, 30, 8, 50, 40)$. Ahora se aplica el procedimiento descrito para calcular la regla CEA de este problema.

1. Primero se soluciona (P_1). El valor óptimo de este problema es $z^{*1} = 8$. Por tanto, todos los agentes reciben 8 unidades del estado. En total 64 unidades del estado han sido repartidas, por tanto, otra ronda es necesaria. Desde que el agente 6 ha obtenido su demanda, este no participará en el reparto del estado en el siguiente paso.

2. En este paso primero se garantiza a todos los agentes lo que ellos han obtenido hasta el paso anterior. El valor óptimo de (P_2) es $z^{*2} = 12$. De este modo, todos los agentes excepto el 6 reciben 12 unidades extra del estado. En total 148 unidades del estado han sido distribuidas, por tanto, otra ronda es necesaria. Dado que los agentes 1 y 3 han obtenido sus demandas, estos no tomarán parte en el reparto del estado en el siguiente paso.

3. Otra vez se garantiza a todos los agentes lo que han recibido hasta el paso anterior. El valor óptimo de (P_3) es $z^{*3} = 4.4$. Así, todos los agentes excepto 1, 3 y 6 reciben 4.4 unidades extra del estado. Puesto que el estado ha sido totalmente distribuidos, el procedimiento ha finalizado y $CEA(N, E, c) = (20, 24.4, 20, 24.4, 24.4, 8, 24.4, 24.4)$.

Obsérvese que este procedimiento se ajusta perfectamente a la siguiente descripción de la regla CEA en Thomson (2015):

Al principio, se realiza una división equitativa hasta que cada reclamante recibe una cantidad igual a la reclamación más pequeña. El reclamante más pequeño se retira, y los siguientes incrementos de la dotación se dividen por igual entre los demás hasta que cada uno de ellos recibe una cantidad igual al segundo reclamo más pequeño. El segundo reclamante más pequeño se retira, y así sucesivamente.

Por tanto, siguiendo el mismo procedimiento se puede definir la regla CEA para problemas MBC, que llamaremos *regla igualitaria dos veces restringida*, CCEA. Dado un problema $(M, N, E, c, \alpha) \in \mathcal{MBC}$, en el paso general k del procedimiento, el problema lineal que tiene que ser resuelto es el siguiente:

$$
\begin{array}{lll}
 & max & z^k \\
 & s.a: & \sum_{i\in N: j\in\alpha(i)} x_i \leq e_j, \text{ para todo } j \in M \\
(P_k) & & x_i \leq c_i, \text{ para todo } i \in N \\
 & & x_i \geq \sum_{h=1}^{k-1} \mu^0(a_i^h) z^{*h} + \mu^0(a_i^k) z^k, \text{ para todo } i \in N \\
 & & x_i \geq 0, \text{ para todo } i \in N, \text{ y } z^k \geq 0
\end{array}
$$

donde

$$
a_i^h = \min\left\{ c_i - \sum_{l=1}^{h-1} \mu^0(a_i^l) z^{*l}, min_{j\in\alpha(i)} \left\{ e_j - \sum_{t\in N: j\in\alpha(t)} \sum_{s=1}^{h-1} \mu^0(a_t^s) z^{*s} \right\} \right\}.
$$

Obsérvese que a_i^h mide si el agente i puede tomar parte en el reparto en el paso h teniendo en cuenta lo que ha recibido anteriormente y si hay todavía algo que repartir en cada estado en el que participa.

En este caso, el procedimiento también finaliza en un número finito de pasos, pero no necesariamente cuando todos los estados son totalmente distribuidos o todas las demandas son completamente satisfechas. En esta situación el procedimiento para cuando $z^{*k} = 0$, y

$$CCEA_i(M,N,E,c,\alpha) = \sum_{h=1}^{k} \mu^0(a_i^h) z^{*h}.$$

Asimismo, para el último problema lineal su solución óptima x^{*k} es exactamente $CCEA(M,N,E,c,\alpha)$.

Es importante enfatizar que la regla CEA se ha introducido para problemas de bancarrota como la solución de una sucesión de problemas lineales y se ha extendido este procedimiento al caso de problemas MBC. De este modo, es sencillo comprobar que cuando se tiene exactamente un estado ambos coinciden. La única diferencia es que se tiene que tener en cuenta lo que queda pendiente de repartir en cada uno de los estados para los cuales los agentes tienen demanda. Sin embargo, este procedimiento para problemas MBC no garantiza que todos los estados sean totalmente distribuidos, incluso cuando ellos pudieran serlo. El siguiente ejemplo ilustra tal situación.

Ejemplo 5.6. Considérese de nuevo el problema MBC en el ejemplo 5.2. Se calcula la regla CCEA del problema aplicando el procedimiento descrito anteriormente.

1. El valor óptimo de (P_1) es $z^{*1} = 8$ y esta cantidad es asignada a cada uno de los agentes. Ninguno de los estados ha sido completamente repartido, así que otro paso es necesario. Sin embargo, el agente 6 ha recibido su demanda, por lo que no participará del reparto en el siguiente paso.
2. En este paso, se garantiza a todos los agentes lo que ya han recibido en el paso previo. El valor óptimo de (P_2) es $z^{*2} = 2$, y esta cantidad es asignada a todos los agentes excepto el agente 6. Ahora resulta que el estado e_1 ha sido completamente repartido, por lo que todos los agentes vinculados a él no pueden recibir nada más. En otro caso, la cantidad repartido en ese estdo excedería lo disponible. Sin embargo, los otros dos estados todavía no han sido totalmente repartidos, por lo que otro paso es necesario.
3. En este paso, una vez más se garantiza a tdos los agentes lo que ya han recibido en los pasos anteriores. El valor óptimo de (P_3) es $z^{*3} = 6$, y esta cantidad es asignada a los agentes 4, 7, y 8. Ahora el estado e_2 ha sido reprtido en su totalidad, así pues los agentes 4 y 7 no pueden recibir nada más. De nuevo, en otro caso, la cantidad disponible en ese estado sería excedida.
4. En este paso, como siempre, en primer lugar se garantiza a todos los agentes lo que han recibido hasta el momento, siendo el agente 8 el único que puede recibir algo más. El valor óptimo de (P_4) es $z^{*4} = 24$, y esta cantidad es asignada al agente 8. No obstante, el estado e_3 no se

ha repartido completamente, de hecho, restan todavía 14 unidades por distribuir. Además $x^{*4} = (10, 10, 10, 16, 10, 8, 16, 40)$ que coincide con $CCEA(M, N, E, c, \alpha)$.

$CCEA(M, N, E, c, \alpha)$ se ha calculado en cuatro pasos, siendo las asignaciones en cada uno de ellos las siguientes:

$Agente$	$Paso - 1$	$Paso - 2$	$Paso - 3$	$Paso - 4$	$Fila - total$
1	8	2	–	–	10
2	8	2	–	–	10
3	8	2	–	–	10
4	8	2	6	–	16
5	8	2	–	–	10
6	8	–	–	–	8
7	8	2	6	–	16
8	8	2	6	24	40
$Columna - total$	64	14	18	24	120

Este reparto no distribuye todos los estados completamente, pero es posible para este problema en particular obtener repartos que sí lo hacen. Por ejemplo, los repartos dados en el ejemplo 5.3 distribuyen completamente todos los estados para este ejemplo, pero ellos no son tan igualitarios como el reparto que se da en este ejemplo. Además, aunque los agentes han recibido 120 unidades en total de las 170 unidades que sumaban los tres estados, 156 han sido realmente repartidas Esta diferencia se produce porque algunos agentes demandaban a varios estados simultáneamente.

Si observamos detenidamente la aplicación del procedimiento en el ejemplo 5.6, observamos que primero se reparte por completo un estado, luego se reparte por completo otro estado y finalmente el último estado no puede repartirse en su totalidad. Por lo tanto, podemos diseñar otro procedimiento basado en la propia regla CEA que siga este esquema. Lo ilustramos en el siguiente ejemplo.

Ejemplo 5.7. Considérese otra vez el problema del ejemplo 5.2. Para calcular $CCEA(M, N, E, c, \alpha)$, se procede como sigue:

1. Primero se calcula la regla CEA para cada uno de los tres problemas de bancarrota definidos por cada estado.
   - $(N^{1,1}, E^{1,1}, c^{1,1})$. $N^{1,1} = \{1, 2, 3, 5\}$, $E^{1,1} = 40$, y $c^{1,1} = (20, 30, 20, 30)$. $CEA(N^{1,1}, E^{1,1}, c^{1,1}) = (10, 10, 10, 10)$, y $\beta^{1,1} = 10$.
   - $(N^{2,1}, E^{2,1}, c^{2,1})$. $N^{2,1} = \{2, 4, 5, 6, 7\}$, $E^{2,1} = 60$, y $c^{2,1} = (30, 40, 30, 8, 50)$. $CEA(N^{2,1}, E^{2,1}, c^{2,1}) = (13, 13, 13, 8, 13)$, y $\beta^{2,1} = 13$.

- $(N^{3,1}, E^{3,1}, c^{3,1})$. $N^{3,1} = \{7, 8\}$, $E^{3,1} = 70$, y $c^{3,1} = (50, 40)$. $CEA(N^{3,1}, E^{3,1}, c^{3,1}) = (35, 35)$, y $\beta^{3,1} = 35$.

2. A continuación, se toma $\beta^{*1} = \min\{\beta^{1,1}, \beta^{2,1}, \beta^{3,1}\} = 10$, y se asigna a cada agente i $\min\{c_i, \beta^{*1}\}$. Por tanto, se obtiene el vector de reparto $(10, 10, 10, 10, 10, 8, 10, 10)$.

Es obvio que el estado e_1 se ha repartido completamente, y en el siguiente paro este problema de bancarrota y los agentes asociados a él son excluidos. Además, el agente 6 es también excluido porque ha obtenido su demanda. Los otros dos problemas son actualizados en agentes, y los estados y demandas reducidas de acuerdo con lo ya asignado.

1. Se calcula la regla CEA para cada uno de los dos problemas de bancarrota restantes.
   - $(N^{2,2}, E^{2,2}, c^{2,2})$. $N^{2,2} = \{4, 7\}$, $E^{2,2} = 12$, y $c^{2,2} = (30, 40)$. $CEA(N^{2,2}, E^{2,2}, c^{2,2}) = (6, 6)$, and $\beta^{2,2} = 6$.
   - $(N^{3,2}, E^{3,2}, c^{3,2})$. $N^{3,2} = \{7, 8\}$, $E^{3,2} = 50$, y $c^{3,2} = (40, 30)$. $CEA(N^{3,2}, E^{3,2}, c^{3,2}) = (25, 25)$, and $\beta^{3,2} = 25$.
2. A continuación, se toma $\beta^{*2} = \min\{\beta^{2,2}, \beta^{3,2}\} = 6$, y se asigna a cada agente i $\min\{c_i, \beta^{*2}\}$. Por tanto, se obtiene el vector de reparto $(0, 0, 0, 6, 0, 0, 6, 6)$.

De nuevo es obvio que el estado e_2 ha sido repartido completamente, y en el siguiente paso este problema de bancarrota y los agentes asociados a él son excluidos. El tercer problema es actualizado en agentes y los estados y demandas son reducidas de acuerdo a las asignaciones previas.

1. Se calcula la regla CEA para el problema de bancarrota restante.
   - $(N^{3,3}, E^{3,3}, c^{3,3})$. $N^{3,3} = \{8\}$, $E^{3,3} = 38$, y $c^{3,3} = (24)$. $CEA(N^{3,3}, E^{3,3}, c^{3,3}) = (24)$, y $\beta^{3,3} = 24$.
2. A continuación se toma $\beta^{*3} = \min\{\beta^{3,3}\} = 24$, y se asigna a cada agente i $\min\{c_i, \beta^{*3}\}$. Por tanto, se obtiene el vector de reparto $(0, 0, 0, 0, 0, 0, 0, 24)$.

El procedimiento finaliza porque o bien se han repartido completamente todos los estados o bien los agentes han obtenido sus demandas. Finalmente, sumando todos los vectores de reparto en el procedimiento, se obtiene que $CCEA(M, N, E, c, \alpha) = (10, 10, 10, 16, 10, 8, 16, 40)$.

El procedimiento basado en la regla CEA es como sigue. En el paso general k, se calcula la regla CEA para todos los problmas de bancarrota definidos por los estados disponibles en el paso k, para cada uno se determinan los valores $\beta^{j,k}$, y se toma el mínimo β^{*k} de todos ellos. Se asigna β^{*k} a todos los agentes activos en el paso k y nada al resto. A continuación se actualiza el problema revisando a la baja los estados, agentes y demandas. Después de actualizar el problema, si ningún problema de bancarrota puede ser definido, se para. En caso contrario, se va al siguiente paso con el problema actiualizado, y así sucesivamente. Finalmente, la regla CCEA del problema MBC es la suma de todos los vectores de reparto obtenidos a lo largo del procedimiento.

El teorema siguiente establece que los dos procedimientos descritos coinciden para todos los problemas MBC.

Teorema 5.6. Dada un problema $(M, N, E, c, \alpha) \in \mathcal{MBC}$, los vectores de reparto obtenidos por el procedimiento basado en programación lineal y el procedimiento basado en la regla CEA de los problemas de bancarrota coinciden, y su resultado se corresponde con la regla $CCEA(M, N, E, c, \alpha)$.

Una propiedad interesante de la regla CEA para los problemas de bancarrota es que es el reparto más igualitario posible en el sentido de Lorenz. Este resultado puede extenderse al contexto de los problemas MBC. Para ello, primero se introducirá el concepto de dominancia de Lorenz adaptado al contexto de los problemas de bancarrota con múltiples estados y demandas cruzadas.

Dado un problema $(M, N, E, c, \alpha) \in \mathcal{MBC}$, ly dos vectores factibles $x, y \in \mathbb{R}^N_+$, se dice que *x Lorenz domina débilmente a y*, $x \succeq_{wL} y$, si $\sum_{j=1}^{k} x_{(j)} \geq \sum_{j=1}^{k} y_{(j)}$, para todo $k = 1, 2, ..., h$, $h \leq n$, donde para un vector $z \in \mathbb{R}^N_+$, $z_{(1)}, \ldots, z_{(n)}$ representa sus coordendas reescrits en orden creciente.

Teorema 5.7. Dado un problema $(M, N, E, c, \alpha) \in \mathcal{MBC}$, la regla $CCEA(M, N, E, c, \alpha)$ Lorenz domina débilmente a todos los repartos factibles.

5.3.4 La regla proporcional restringida para problemas MBC

La regla proporcional (PROP) es quizás la regla más importante para resolver problemas de asignación en general y problemas de bancarrota en particular. Esta regla simplemente divide el recurso en proporción a las demandas. La pregunta en los problemas MBC es qué significa "en proporción a las demandas". En el contexto de los problemas de bancarrota con un solo estado, "en proporción a las demandas" significa que todos los reclamantes reciben la misma cantidad por cada unidad de demanda. ¿Cómo extrapolar esto a las situaciones MBC? Para responder a esta pregunta, se presenta la regla proporcional restringida (CPA, por sus siglas en inglés) como resultado de un proceso iterativo en el que la cantidad disponible de al menos uno de los estados se distribuye completamente en cada paso y así sucesivamente mientras sea posible. Esta regla se define formalmente a continuación.

Definición 5.17. Sea $(M, N, E, c, \alpha) \in \mathcal{MBC}$, la *regla proporcional restringida* para (M, N, E, c, α), $CPA(M, N, E, c, \alpha)$, se define mediante el siguiente procedimiento iterativo:

Paso 0.
1. $M^1 = \{j \in M : e^1_j > 0\}$ es el conjunto de estados activos.
2. $\mathcal{N}^1 = \{i \in N : c^1_i > 0 \text{ and } e^1_j > 0, \forall j \in \alpha(i)\}$ es el conjunto de agentes activos.
3. Para cada $j \in M$, $e^1_j = e_j$, y para cada $i \in N$, $c^1_i = c_i$.

Paso s.
1. $\mathcal{N}^s = \{i \in N : c^s_i > 0 \text{ and } e^s_j > 0, \forall j \in \alpha(i)\}$. $M^s = \{j \in M : e^s_j > 0\}$.
2. Para cada $j \in M^s$, se calcula el λ^s_j más grande, tal que $\lambda^s_j \sum_{i \in \mathcal{N}^s : j \in \alpha(i)} c^s_i \leq e^s_j$, y se toma $\lambda^s = \min\{\lambda^s_j : j \in M^s\}$.
3. A continuación, se asigna a cada agente $i \in M^s$, $a^s_i = \lambda^s c^s_i$, y $a^s_i = 0$ a los agentes no activos.
4. Se actualizan los estados activos, M^{s+1}, y los agentes activos, $\mathcal{N}^{s+1}$. Si $M^{s+1} = \varnothing$ or $\mathcal{N}^{s+1} = \varnothing$, entonces el proceso termina, y
$$CPA_i(M, N, E, c, \alpha) = \sum_{h=1}^{s} a^h_i, \forall i \in N.$$
En otro caso, las cantidades disponibles de los estados y las demandas son actualizadas:
$$e^{s+1}_j = e^s_j - \lambda^s \sum_{i \in N : j \in \alpha(i)} c^s_i, \forall j \in M, \text{ y } \quad c^{s+1}_i = c^s_i - \lambda^s c^s_i, \forall i \in N,$$
e ir al paso $s + 1$

El procedimiento iterativo de la regla CPA está bien definido y siempre da un único punto. Además, desde que en cada paso al menos un estado es distribuido completamente, excepto quizás en el último paso, el procedimiento finaliza en un número finito de pasos, a lo sumo $|M|$. Finalmente, cuando se tiene un problema de bancarrota con un solo estado, entonces se obtiene la regla PROP. Por tanto, esta definición generaliza la regla proporcional al contexto de los problemas MBC.

De la aplicación del proceso iterativo para calcular la regla CPA, se pueden definir las cadenas de estados activos y de agentes activos:
$$M^1 \supset M^2 \supset \ldots \supset M^r, \text{ y } \mathcal{N}^1 \supset \mathcal{N}^2 \supset \ldots \supset \mathcal{N}^r$$

A partir de estas cadenas, se puede establecer una relación d orden entre los estados de la forma siguiente. Se dice que el estado j_1 precede estrictamente al estado j_2 en una cadaena de estados activos, $j_1 \prec j_2$, si existe M^s

tal que $j_1 \notin M^s$ y $j_2 \in M^s$, es decir, j_1 llega a ser no activo antes que j_2. Se denotará por $j_1 \preceq j_2$ cuando j_1 llega a ser no activo antes que j_2 o ambos llegan a ser no activos a la vez. Finalmente, se denotará $j_1 \simeq j_2$ cuando ambos estados llegan a ser no activos a la vez. Análogamente, se puede establecer una relación de orden entre los agentes.

Ademś, con cada par de conjuntos M^s y $\mathcal{N}^s$ se puede asociar un número ρ^s, $\rho^s \in [0,1]$, que representa la proporción de las demandas obtenidas por los agentes en $\mathcal{N}^s$ pero no en $\mathcal{N}^{s+1}$. Además, por construcción $\rho^s < \rho^{s+1}$. De este modo, se tiene que

$$0 < \rho^1 < \rho^2 < \ldots < \rho^r \leq 1$$

Estos $\rho's$ representan la proporción acumulada de las demandas de los agentes que les han sido asignadas, es decir, qué parte de sus demandas han recibido hasta un paso dado en el procedimiento iterativo. En este sentido, este procedimiento tiene reminiscencias de la regla igualitaria restringida (CEA) en los problemas de bancarrota, pero en vez de utilizar el principio de igualitarismo se utiliza el principio de proporiconalidad, de ahél nombre de regla *proporcional restringida*. Por tanto, no todos los agentes reciben la misma proporción de sus demandas, pero la regla intenta mantener la proporcionalidad tanto como sea posible restringido a (1) la relación entre las cantidades disponibles de cada estado y las demandas que tienen, y (2) el objetivo de repartir tanto como sea posible de las cantidades disponibles de los estados.

A continuación se dan algunos resultados sobre el proceso iterativo que define la regla CPA.

Proposición 5.1. Los siguientes enunciados son ciertos.

1. Dado $(M, N, E, c, \alpha) \in \mathcal{MBC}$, si existen problemas $(M_1, N_1, E_1, c^1, \alpha_1)$, $(M_2, N_2, E_2, c^2, \alpha_2) \in \mathcal{MBC}$, tal que $M_1 \cup M_2 = M$, $N_1 \cup N_2 = N$, $E_1 \oplus E_2 = E$, $c^1 \oplus c^2 = c$, y $\alpha_1(i) = \alpha(i), \forall i \in N_1$ y $\alpha_2(i) = \alpha(i), \forall i \in N_2$, tal que $\left(\bigcup_{i\in N_1} \alpha(i)\right) \bigcap \left(\bigcup_{i\in N_2} \alpha(i)\right) = \varnothing$, entonces

   $$CPA(M, N, E, c, \alpha) = CPA(M_1, N_1, E_1, c^1, \alpha_1) \oplus CPA(M_2, N_2, E_2, c^2, \alpha_2).$$ [4]

2. Si $\lambda^s < 1$, entonces cada $h \in \arg\min\{\lambda^s_j : j \in M^s\} \subset M$ llega a ser no activo en el siguiente paso.
3. Si $\lambda^s = 1$ para algún s, entonces el proceso iterativo finaliza en ese paso.

[4]Dados $a \in \mathbb{R}^n$ y $b \in \mathbb{R}^m$, $a \oplus b \in \mathbb{R}^{n+m}$, es decir, $\oplus$ es el operador concatenación de dos vectores.

El primer enunciado dice que si un problema puede ser dividido en dos problemas disjuntos, entonces es lo mismo calcular la regla CPA para todo el problema o para cada parte por separado y luego pegarlas. El segundo enunciado establece cuándo un estado llega a ser no activo. Finalmente, el tercer enunciado proporciona otro criterio de parada para el procedimiento iterativo para calcular CPA. Además, cuando el procedimiento termina con $\lambda = 1$, eso significa que habrá algunos estados que no sus cantidades disponibles no son totalmente repartidas. En caso contrasio, todos los recursos disponibles se habrán repartido.

5.3.5 Reglas de reparto con prioridades y llegadas aleatorias para problemas MBC

Las reglas de prioridad secuencial se definen cuando existe un orden de prelación definido ex ante sobre el conjunto de agentes según algún criterio, de tal forma que si un agente tiene mayor prioridad que otro, el primero debe ser satisfecho primero en su demanda en todo lo que se pueda, el segundo recibe todo lo que se pueda con el recurso que queda, y así sucesivamente hasta que se agota el recurso. Por tanto, esta regla simplemente asigna el recurso según el esquema de orden de llegada, donde el orden de llegada viene dado por la relación de prioridad. La pregunta en los problemas MBC es qué significa "satisfacer todo lo que se pueda de la demanda con el recurso que queda". En el contexto de los problemas de bancarrota con un solo estado, "satisfacer todo lo que se pueda de la demanda con el recurso que queda" es una idea simple ya que solo hay un problema. ¿Cómo extrapolar esto a las situaciones MBC? Para responder a esta pregunta, presentamos la regla de prioridad secuencial restringida (CSP, por sus siglas en inglés) que sigue el mismo proceso que las reglas de prioridad secuencial pero teniendo en cuenta que hay varios recursos o problemas. Esta regla se define formalmente a continuación.

Definición 5.18. Sea $(M, N, E, c, \alpha) \in \mathcal{MBC}$, y $\sigma \in \Sigma(N)$, la *regla de prioridad secuencial restringida* asociada con $\sigma \in \Sigma(N)$ para (M, N, E, c, α), $CSP^{\sigma}(M, N, E, c, \alpha)$, se define como:

$$CSP_j^{\sigma}(M,N,E,c,\alpha) = min\left\{ c_i,\ max\left\{ 0,\ \min_{i\in\alpha(j)}\{e_i - \sum_{\substack{k\in N: i\in\alpha(k)\\ \sigma(k)<\sigma(j)}} c_k\} \right\} \right\}, \forall j \in N,$$

donde $\Sigma(N)$ es el conjunto de todas las posibles ordenaciones de N.

Para cada $\sigma \in \Sigma(N)$, el procedimiento iterativo de CSP^σ está bien definido y siempre da lugar a un solo punto. Además, por definición, termina en un número finito de pasos, a lo sumo en $|N|$. Finalmente, cuando se tiene un problema de bancarrota con un solo estado, se obtiene SP^σ. Por tanto, esta definición generaliza las reglas de prioridad secuencial al contexto de los problemas MBC. El siguiente ejemplo ilustra cómo se calcula CSP.

Ejemplo 5.8. Considérese la siguiente situación MBC con $M = \{1,2,3\}$, $N = \{1,2,3,4,5,6,7,8\}$, $E = (9,12,9)$, $c = (3,5,4,3,5,4,3,5)$, y $\alpha(1) = \{1\}, \alpha(2) = \{1\}, \alpha(3) = \{1,2\}, \alpha(4) = \{1,2\}, \alpha(5) = \{2\}, \alpha(6) = \{2,3\}, \alpha(7) = \{2,3\}, \alpha(8) = \{3\}$. Si se toma por ejemplo el orden de prioridad $\sigma = 1\,3\,5\,7\,2\,4\,6\,8$, CSP^σ se calcular secuencialmente como sigue:

El agente 1 es atendido en primer lugar:

$$CSP_1^\sigma(M,N,E,c,\alpha) = 3.$$

Se actualizan los estados $E = (6,12,9)$ y se atiende al agente 3:

$$CSP_3^\sigma(M,N,E,c,\alpha) = 4.$$

Se actualizan los estados $E = (2,8,9)$ y se atiende al agente 5:

$$CSP_5^\sigma(M,N,E,c,\alpha) = 5.$$

Se actualizan los estados $E = (2,3,9)$ y se atiende al agente 7:

$$CSP_7^\sigma(M,N,E,c,\alpha) = 3.$$

Se actualizan los estados $E = (2,0,6)$ y se atiende al agente 2:

$$CSP_2^\sigma(M,N,E,c,\alpha) = 2.$$

Se actualizan los estados $E = (0,0,6)$ y el agente 4 no puede ser atendido porque los estados a los que reclama están exhaustos, en consecuencia recibe 0. Por tanto, se continúa con el siguiente agente en el orden de prioridades. El agente 6 tampoco puede ser atendido porque uno de los estados a los que reclama está exhausto, por ello también recibe 0. Por ello, se continúa con el siguiente agente en el orden de prioridades. El agente 8 es atendido:

$$CSP_8^\sigma(M,N,E,c,\alpha) = 5.$$

Se actualizan los estados $E = (0,0,1)$ y el procedimiento secuencial termina. El reparto final es

$$CSP^\sigma(M,N,E,c,\alpha) = (3,2,4,0,5,0,3,5).$$

Una vez que se han definido las reglas de prioridad secuencial restringidas asociadas con un orden, la regla de llegada aleatoria restringida se define simplemente como su promedio.

Definición 5.19. Sea $(M, N, E, c, \alpha) \in \mathcal{MBC}$, la *regla de llegada aleatoria restringida* para (M, N, E, c, α), $CRA(M, N, E, c, \alpha)$, se define como:

$$CRA_j(M, N, E, c, \alpha) = \frac{1}{n!} \sum_{\sigma \in \Sigma(N)} CSP_j^{\sigma}(M, N, E, c, \alpha),\ \forall j \in N,$$

donde $\Sigma(N)$ es el conjunto de todas las posibles ordenaciones de N.

Aunque la definición de la regla es simple, es inmediatamente evidente que el mayor problema está en su cálculo. Para el ejemplo 5.8 sería necesario calcular 8!=40320 procesos secuenciales, lo que supone una gran cantidad de tiempo.

5.4 Propiedades y caracterizaciones de las reglas de reparto para problemas de bancarrota con múltiples estados y demadas cruzadas

En esta sección se presenta un conjunto de propiedades o axiomas adecuados para las reglas de reparto en el contexto de los problemas de bancarrota con múltiples estados y demandas cruzadas. Cada una de las propiedades refleja un principio que gustaría que satisficiera una regla de reparto. En general, las propiedades presentadas aquí son adaptaciones adecuadas de las propiedades correspondientes para las reglas de reparto en problemas de bancarrota que son plenamente aceptadas por la comunidad científica en el campo de los problemas de asignación y distribución de recursos. Las reglas de reparto introducidas en la sección 5.3 se estudiarán axiomáticamente utilizando estas propiedades, obteniendo caracterizaciones para la regla igualitaria doblemente restringida y la regla proporcional restringida, pero no para las reglas de prioridad secuencial y la regla de llegada aleatoria restringida. El contenido de esta sección también se basa en Acosta-Vega et al. (2021a, 2021b, 2022a, 2022b).

5.4.1 Introducción

El problema de la justicia distributiva ha sido tratado desde la antigüedad hasta nuestros días (véase, por ejemplo, Fleischacker (2005) para una breve historia de la justicia distributiva). La pregunta de si un reparto es justo o no es difícil de responder desde una perspectiva sociopolítica. Sin embargo, si analizamos un método de asignación o distribución desde la perspectiva de las propiedades que satisface, quizás sea más fácil concluir si el método es mejor o peor en términos de esas propiedades (o axiomas). Este tipo de análisis comúnmente se realiza desde una perspectiva matemática, en la que las propiedades se presentan bajo una formulación matemática que describe formalmente alguna propiedad o característica que es relevante para una regla de reparto o distribución en un contexto determinado. Dos libros interesantes al respecto son Roemer (1996) y Binmore (2011).

¿Por qué es importante el enfoque axiomático en el análisis de un problema de distribución, reparto o asignación? Primero, porque permite un análisis de qué propiedades satisfacen las diferentes reglas de distribución que están sobre la mesa y, por tanto, cuáles de ellas pueden ser mejores en cuanto a las propiedades que se consideran más relevantes para el problema a resolver. Segundo, porque a partir de las propiedades que se consideren relevantes para el problema, se puede definir una nueva regla que cumpla con estas propiedades y que inicialmente no estaba sobre la mesa. Por tanto, el análisis axiomático de un problema puede ayudar a elegir la mejor solución entre las diferentes alternativas o incluso a definir explícitamente qué regla de distribución debe ser la solución al problema.

¿Qué propiedades son deseables que satisfaga una regla de reparto? La respuesta es que depende del problema. Sin embargo, existen numerosas propiedades en la literatura que responden a diferentes ideas y conceptos. En esta sección se han considerado cinco grupos de propiedades:

1. relacionadas con la eficiencia en la distribución de los recursos;
2. relacionadas con la imparcialidad y la equidad;
3. relacionadas con las garantías mínimas a los demandantes;
4. relacionadas con la monotonía;
5. y relacionadas con la robustez.

Cada una de las reglas introducidas en la sección anterior se analizan axiomáticamente utilizando axiomas de los grupos anteriores. Se presentan caracterizaciones axiomáticas para las reglas CCEA y CPA, pero aún no se ha logrado una caracterización para CRA.

5.4.2 Propiedades

En esta subsección, se presentan varias propiedades que son interesantes en el contexto de los problemas MBC. Estas propiedades están relacionadas con la eficiencia, la equidad, la robustez o la monotonía, entre otras.

Antes de comenzar con las propiedades consideradas en esta memoria, se definen dos conceptos relacionados con la comparación entre agentes. En nuestro contexto, los demandantes se caracterizan por dos rasgos: sus demandas y los estados a los que reclaman. Por tanto, ambos deben tenerse en cuenta cuando se quieran establecer similitudes entre agentes. La siguiente definición establece cuándo dos agentes se consideran directamente comparables y cuándo iguales.

Definición 5.20. Sea $(M, N, E, c, \alpha) \in \mathcal{MBC}$, y dos agentes $j, k \in N$, se dice que son *homólogos*, si $\alpha(j) = \alpha(k)$; y se dice que son *iguales*, si son homólogos y $c_j = c_k$.

En las siguientes subsecciones, se da un conjunto de propiedades que son muy naturales y razonables para una regla de reparto en el contexto de problemas de bancarrota con múltiples estados y demandas cruzadas.

Propiedades relacionadas con la eficiencia de la distribución de los recursos

Las propiedades en esta subsección están relacionadas con cómo los recursos deberían ser aplicados, pero no en relación a su justicia o equidad. Cada uno de los axiomas que se presentan muestran cuándo un reparto de los recursos es posible o más eficiente económicamente desde el punto de vista del reparto en sí mismo, no si este es justo.

Axioma 5.31 (F)**.** Dada una regla R, se dice que satisface *factibilidad*, si para cada problema $(M, N, E, c, \alpha) \in \mathcal{MBC}$, $\sum_{i:j\in\alpha(i)} R_i(M, N, E, c, \alpha) \leq e_j$, para todo $j \in M$.

F significa que no se puede repartir más de lo que hay. De hecho, esta propiedad es tan básica que se incluye dentro de la propia definición de regla.

Axioma 5.32 (EFF)**.** Dada una regla R, se dice que satisface *eficiencia*, si para cada problema $(M, N, E, c, \alpha) \in \mathcal{MBC}$, $\sum_{i:j\in\alpha(i)} R_i(M, N, E, c, \alpha) = e_j$, para todo $j \in M$.

EFF simplemente dice que todos los estados deben ser repartidos completamente. Se ha mostrado con anterioridad que esta propiedad es muy exigente en el contexto de los problemas de bancarrota con múltiples estados y demandas cruzadas, véase el ejemplo 5.4. Sin embargo, una versión más débil puede considerase cuando sólo se exige que al menos un estado sea totalmente distribuido. Formalmente, esta propiedad se escrib como sigue:

Axioma 5.33 (WEFF)**.** Dada una regla R, se dice que satisface *eficiencia débil*, si para cada problema $(M, N, E, c, \alpha) \in \mathcal{MBC}$,

$$\sum_{i:j\in\alpha(i)} R_i(M, N, E, c, \alpha) = e_j,$$

para algún $j \in M$.

Una versión más débil de la propiedad de eficiencia es la de Pareto eficiencia. Un reparto factible es eficiente en el sentido de Pareto, si no hay ningún otro reparto para el cual alguno de los agentes está mejor y el resto no está peor. Formalmente, dado un problema $(M, N, E, c, \alpha) \in \mathcal{MBC}$, un reparto factible $x \in A(M, N, E, c, \alpha)$ es *Pareto eficiente* si no existe otro reparto factible $x' \in A(M, N, E, c, \alpha)$ tal que $x'_i \geq x_i, \forall i \in N$, con al menos una desigualdad estricta. Ahora, la propiedad de Pareto eficiencia se define como sigue:

Axioma 5.34 (PEFF)**.** Dada una regla R, se dice que satisface *Pareto eficiencia*, si para cada problema $(M, N, E, c, \alpha) \in \mathcal{MBC}$, $R(M, N, E, c, \alpha)$ es Pareto eficiente.

Obsérvese que $PEFF$ implica que al menos la cantidad disponible de un estado es completamente distribuida, y que no puede quedar ninguna cantidad de un estado sin distribuir si es posible hacerlo. Sin embargo, esta propiedad no persigue que todas las cantidades disponibles de los estados sean completamente distribuidas.

Propiedades relacionadas con la imparcialidad y la equidad

Las propiedades en esta subsección están relacionadas con cómo se debe tratar a los agentes, excluyendo la diferenciación entre agentes sobre la base de características exógenas al problema en sí, por ejemplo, nombres, género, edad, etc.

Axioma 5.35 (AN)**.** Dada una regla R, se dice que satisface *anonimato*, si para cada problema $(M, N, E, c, \alpha) \in \mathcal{MBC}$, $R_i(M, N, E, c, \alpha) = R_{\sigma(i)}(M, N, E, \sigma(c), \alpha \circ \sigma^{-1})$ para cada $i \in N$, donde σ es una permutación de N.

AN significa que la asignación no depende del nombre de los reclamantes sino únicamente de sus características con respecto al problema.

Axioma 5.36 (ETE)**.** Dada una regla R, se dice que satisface *igual trato de iguales*, si para cada problema $(M, N, E, c, \alpha) \in \mathcal{MBC}$ y cada par de agentes iguales $i, j \in N$, $R_i(M, N, E, c, \alpha) = R_j(M, N, E, c, \alpha)$.

La propiedad ETE está relacionada con la imparcialidad y dice que los agentes con las mismas demandas y el mismo conjunto de estados deben recibir el mismo trato en el reparto final. Este es un requisito básico de equidad y no arbitrariedad.

Por otra parte, una propiedad que aparece recurrentemente relacionada con el valor de Shapley y, por tanto, con la regla de llegada aleatoria es la de las contribuciones equilibradas (Myerson, 1980; Hart y Mas-Colell, 1989). En el contexto de problemas de bancarrota, esta propiedad fue utilizada por Bergantiños y Méndez-Naya (1997) para caracterizar la regla de llegada aleatoria. Además, en el contexto de los problemas de bancarrota con múltiples (sub)estados, Lorenzo-Freire et al. (2007) también la utilizan para caracterizar la regla de llegada aleatoria.

Axioma 5.37 (BAL)**.** Dada una regla R, se dice que satisface *impacto equilibrado*, si para cada problema $(M, N, E, c, \alpha)\mathcal{MBC}$, y cada par de agentes $j, k \in N$,

$$R_j(M, N, E, c, \alpha) - R_j(M^{-k}, N, E^{-k}, c_{-k}, \alpha) = R_k(M, N, E, c, \alpha) - R_k(M^{-j}, N, E^{-j}, c_{-j}, \alpha),$$

donde M^{-h} es el conjunto de estados para los cuales los agentes en $N\backslash\{h\}$ tienen demandas; $E^{-h} = (e_1^{-h}, \ldots, e_m^{-h})$ de modo que $e_i^{-h} = e_i - c_h$ si $i \in \alpha(h)$ y $e_i^{-h} = e_i$ en otro caso; y c_{-h} es el vector de demandas del cual la coordenada $h-$ésima ha sido eliminada.

BAL requiere que el agente j impacte en la asignación del agente k lo que el agente k impacte en la asignación del agente j.

Propiedades relacionadas con garantías mínimas

Estas propiedades intentan establecer qué cantidad debe recibir un agente o qué cantidad debe garantizarse a cada agente al menos bajo ciertas condiciones razonables. En este sentido, estas propiedades están relacionadas con las garantías que reciben los agentes cuando se aplica una regla de reparto. Por lo tanto, aquellas reglas de reparto que satisfacen este tipo de propiedades suelen ser bastante protectoras de aquellos agentes que tienen demandas más modestas sobre los recursos. Algunas de estas propiedades en la literatura de problemas de bancarrota adaptadas a problemas de bancarrota con múltiples estados y demandas cruzadas son las siguientes.

Axioma 5.38 (RMR)**.** Dada una regla R, se dice que satisface *respeto de derechos mínimos*, si para cada problema $(M, N, E, c, \alpha) \in \mathcal{MBC}$, para todo

$i \in N$,

$$R_i(M, N, E, c, \alpha) \geq \min_{j:j\in\alpha(i)} \max\{0, e_j - \sum_{k\in N\setminus\{i\}} \delta(k, j)c_k\}.$$

Axioma 5.39 (CED)**.** Dada una regla R, se dice que satisface *división igualitaria condicionada*, si para cada problema $(M, N, E, c, \alpha) \in \mathcal{MBC}$, para todo $i \in N$,

$$R_i(M, N, E, c, \alpha) \geq \min_{j:j\in\alpha(i)} \min \left\{ c_i, \frac{e_j}{|k : j \in \alpha(k)|} \right\}.$$

Axioma 5.40 (SEC)**.** Dada una regla R, se dice que satisface *garantía*, si para cada problema $(M, N, E, c, \alpha) \in \mathcal{MBC}$, para todo $i \in N$,

$$R_i(M, N, E, c, \alpha) \geq \min_{j:j\in\alpha(i)} \min \left\{ \frac{c_i}{|k : j \in \alpha(k)|}, \frac{e_j}{|k : j \in \alpha(k)|} \right\}.$$

Las propiedades RMR, CED y SEC están relacionadas con la mínima cantidad que deberá ser garantizada a cada agente de un modo razonable. El concepto de derecho mínimo fue introducido por Tijs (1981) en el contexto de los juegos coopertivos para definir el τ-valor. La propiedad RMR dice que un agente deberá recibir al menos lo que dejan los demás cuando todos ellos reciben sus demandas. La propidad CED fue introducida por Moulin (2000) para problemas de racionamiento. En el contexto de los problemas abordados en esta tesis doctoral, esta propiedad sisgnifica que unagente debería obtener lo que demanda si esta es más pequeña que cualquier distribución igualitaria de las cantidades disponibles en los estados a los que demanda, y en otro caso, al menos la distribución igualitaria mínima de dichas cantidades. Finalmente, SEC fue introducida para los problemas de bancarrota por Moreno-Ternero ad Villar (2004). Ellos utilizaron esta propiedad junto con otras para caracterizar la regla del Tamud (Aumann and Maschler, 1985). En el contexto de los problemas MBC, esta propiedad establece que una regla debería garantizar a los agentes al menos la distribuciíon igualitaria mínima de las cantidades disponibles en los estados que ellos reclaman, cuando sea factible, y en otro caso, la distribución igualitatria mínima de sus demandas.

Axioma 5.41 (GMA)**.** Dada una regla R, se dice que satisface *mínima recompensa garantizada*, si para cada problema $(M, N, E, c, \alpha) \in \mathcal{MBC}$,

$$R_i(M, N, E, c, \alpha) \geq \min \left\{ R_k\left(\{k\}, N_k, e_k, c|_{N_k}\right) : k \in \alpha(i) \right\}, \forall i \in N,$$

donde $N_k = \{i \in N : j \in \alpha(i)\}$, and $c|_{N_k}$ cuyas coordenadas se corresponden a los agentes en N_k.

La propiedad GMA ofrece otra forma de garantizar una cantidad mínima a cada reclamante. Estas cantidades mínimas se determinan a partir del análisis de cada problema asociado con cada estado de forma independiente. En particular, la propiedad establece que una reclamante no debe recibir menos de lo que habría recibido en el peor de los casos, si la regla se hubiera aplicado a cada problema por separado a cada uno de los estados.

Axioma 5.42 (CFC)**.** Dada una regla R, se dice que satisface *compensación total condicionada*, si para cada problema $(M, N, E, c, \alpha) \in \mathcal{MBC}$ y cada $i \in N$, tal que $\sum_{k:j\in\alpha(k)} \min\{c_k, c_i\} \leq e_j$, para todo $j \in M$, entonces $R_i(M, N, E, c, \alpha) = c_i$.

La propiedad CFC significa que si la demanda de un reclamante es tan pequeña que si todos los reclamantes pidieran la misma cantidad que ella, todos recibirían sus demandas, entonces parece razonable que dicho reclamante reciba la cantidad que demanda. Esta propiedad fue introducida por Herrero y Villar (2002) y utilizada para caracterizar la regla CEA.

Propiedades relacionadas con la monotonía

Las propiedades de monotonía se refieren a qué impacto tienen sobre el reparto los cambios en algunos de los elementos que definen el problema, en particular, cambios en la cantidad de recursos disponibles o en las demandas de los agentes. Una propiedad que cumple la mayoría de las reglas para los problemas de bancarrota es la monotonía en los recursos. Esta propiedad simplemente dice que si aumenta el recurso disponible, las asignaciones a los reclamantes no disminuyen. En el contexto particular de los problemas de bancarrota con múltiples estados y demandas cruzadas, esta propiedad dice lo siguiente.

Axioma 5.43 (R-MON)**.** Dada una regla R, se dice que satisface *monotonía en los recursos*, si para cada problema $(M, N, E, c, \alpha) \in \mathcal{MBC}$ y $E' \geq E$, $R_j(M, N, E', c, \alpha) \geq R_j(M, N, E, c, \alpha)$ para todo $j \in N$.

Axioma 5.44 (C-MON)**.** Dada una regla R, se dice que satisface *monotonía en la demanda*, si para cada par de problemas $(M, N, E, c, \alpha) \in \mathcal{MBC}$ y $(M, N, E, c', \alpha) \in \mathcal{MBC}$, tal que $c_i \geq c'_i$ y $c_j = c'_j$, para todo $j \in N\backslash\{i\}$, entonces $R_i(M, N, E, c, \alpha) \geq R_i(M, N, E, c', \alpha)$.

La propiedad *C-MON* significa que si la demanda de un reclamante aumenta, no puede recibir menos de lo que recibió en la situación anterior. En Kasajima y Thomson (2011) las propiedades de monotonía se estudian en el contexto de la adjudicación de demandas en conflicto.

Otra propiedad de monotonía que es satisfecha por muchas reglas de bancarrota, es la monotonía en la población, que dice que si todos los agentes están de acuerdo en que un reclamante j obtenga su demanda, entonces los reclamantes restantes deberían estar peor después de que el reclamante j sea completamente compensado.

Axioma 5.45 (P-MON)**.** Dada una regla R, se dice que satisface *monotonía en la población*, si para cada problema (M, N, E, c, α), y cada $j \in N$,

$$R_k(M, N, E, c, \alpha) \geq R_k(M^{-j}, N, E^{-j}, c_{-j}, \alpha), \text{ para todo } k \in N \backslash \{j\},$$

donde M^{-j} es el conjunto de estados para los cuales los agentes en $N \backslash \{j\}$ tienen demandas; $E^{-j} = (e_1^{-j}, \ldots, e_m^{-j})$ de modo que $e_i^{-j} = e_i - c_j$ si $i \in \alpha(j)$ y $e_i^{-j} = e_i$ en otro caso; y c_{-j} es el vector de demandas del cual la coordenada $j-$ésima ha sido eliminada.

Propiedades relacionadas con la robustez

Las propiedades de robustez también se refieren a lo que sucede cuando se producen cambios en el problema, pero generalmente por acciones realizadas por los propios demandantes. Entendiendo por robustez cuando estos cambios no afectan a la asignación final, a la asignación de quienes los han realizado, ni a la asignación de quienes no han hecho nada.

Axioma 5.46 (CTI)**.** Dada una regla R, se dice que satisface *invarianza al truncar las demandas*, si para cada problema $(M, N, E, c, \alpha) \in \mathcal{MBC}$, cuando se considera el problema $(M, N, E, c', \alpha) \in \mathcal{MBC}$ tal que $c'_i = \min\{c_i, \min\{e_j | j \in \alpha(i)\}\}$, para todo $i \in N$; entonces $R(M, N, E, c, \alpha) = R(M, N, E, c', \alpha)$.

La propiedad CTI dice que si las demandas son truncadas por las cantidades disponibles en los estados, entonces el reparto final no cambia. Esta propiedad aparece en Curiel et al. (1987) y se utiliza para caracterizar las llamadas reglas de teoría de juegos para problemas de bancarrota con un solo estado. Dangan y Volij (1993) fueron los primeros en proponer esta propiedad como axioma.

La siguiente propiedad es un requisito de robustez cuando algunos agentes dejan el problema con sus asignaciones (véase Thomson, 2011, 2018). En particular, cuando un subconjunto de reclamantes deja el problema respetando las asignaciones a los que quedan, entonces parece razonable que los reclamantes que se vayan reciban lo mismo en el nuevo problema que en el original. Antes de introducir la siguiente propiedad se necesita definir el concepto de problema reducido.

Definición 5.21. Dado un problema $(M, N, E, c, \alpha) \in \mathcal{MBC}$, y $N' \subset N$, el *problema reducido* asociado a N', $MBC^{N'} = (M', N', E'^R, c|_{N'}, \alpha) \in \mathcal{MBC}$, donde $M' = \{j \in M :\ \text{existe } i \in N' \text{ tal que } j \in \alpha(i)\}$, $E'^R = \left(e_j'^R\right)_{j \in M'}$ con $e_j'^R = e_j - \sum_{i \in N \setminus N' : j \in \alpha(i)} R_i(M, N, E, c, \alpha)$, para todo $j \in M'$, y $c|_{N'}$ es el vector cuyas coordenadas se corresponden a los agentes en N'.

Axioma 5.47 (CONS)**.** Dada una regla R, se dice que satisface *consistencia*, si para cada problema $(M, N, E, c, \alpha) \in \mathcal{MBC}$, y $N' \subset N$, se tiene que

$$R_i(M, N, E, c, \alpha) = R_i(M', N', E'^R, c|_{N'}, \alpha), \text{ para todo } i \in N'.$$

La propiedad $CONS$ significa que si un subconjunto de agentes abandona el problema respetando lo que se les había asignado a los que permanecen, entonces lo que obtienen esos agentes en el nuevo problema reducido es lo mismo que obtuvieron en el problema completo inicial. Las propiedades de consistencia se han utilizado para caracterizar muchas reglas de bancarrota, porque representan un requisito de robustez cuando algunos agentes dejan el problema con sus asignaciones (véase Thomson 2011, 2018) para revisiones sobre la aplicación de las propiedades de consistencia y los principios que hay detrás de ellas).

Las dos últimas propiedades están relacionadas con la capacidad de los reclamantes para manipular la asignación final dividiendo sus demandas entre varios nuevos reclamantes o fusionando sus demandas en un solo reclamante. Parece sensato que si los agentes hacen esto, no deberán beneficiarse y deberían por el contrario recibir lo mismo que recibieron en el problema original. Estas dos posibilidades se establecen en los siguientes axiomas.

Axioma 5.48 (NMS)**.** Dada una regla R, se dice que satisface *no manipulabilidad por división*, si para cada par de problemas $(M, N, E, c, \alpha), (M, N', E, c', \alpha') \in \mathcal{MBC}$, tal que:

1. $N \subset N'$, $S = \{i_1, \ldots, i_k\}$, tal que $N' = (N \setminus S) \cup S_{i_1} \cup \ldots \cup S_{i_m}$, donde S_{i_k} es el conjunto de agentes en los cuales el agente i_k se ha dividido.
2. $c'_j = c_j, \forall j \in N \setminus S$ y $\sum_{k \in S_{i_h}} c'_k = c_{i_h}, h = 1, \ldots, m$,
3. $\alpha'(j) = \alpha(j), \forall j \in N \setminus S$ y $\alpha'(j) = \alpha(i_h), \forall j \in S_{i_h}, h = 1, \ldots, m$,

se tiene que

$$\sum_{j \in S_{i_h}} R_j(M, N', E, c', \alpha') = R_{i_h}(M, N, E, c, \alpha), h = 1, \ldots, m.$$

Axioma 5.49 (NMRM)**.** Dada una regla R, se dice que satisface *no manipulabilidad por fusiones restringidas*, si para cada par de problemas (M, N, E, c, α), $(M, N', E, c', \alpha') \in \mathcal{MBC}$, tal que:

1. $N \subset N'$,
2. $c_i = c'_i, \forall i \in N \backslash \{i_0\}$ y $c_{i_0} = \sum_{j\in(N'\backslash N)\cup\{i_0\}} c'_j$,
3. $\alpha(i) = \alpha'(i), \forall i \in N\backslash\{i_0\}$ and $\alpha(i) = \alpha'(i_0), \forall i \in (N'\backslash N) \cup \{i_0\}$,

se tiene que

$$R_{i_0}(M, N, E, c, \alpha) = \sum_{i\in(N'\backslash N)\cup\{i_0\}} R_j(M, N', E, c', \alpha').$$

Obsérvese que en NMS pasamos del problema de reparto con un conjunto de reclamantes N a un problema con un conjunto de reclamantes N', es decir, algunos de los reclamantes se dividen en varios reclamantes nuevos, algunos de los cuales tienen los mismos nombres que en N. Sin embargo, en $NMRM$ pasamos del problema en N' al problema en N, es decir, varios reclamantes se fusionan en un único reclamante que tiene el mismo nombre que en N', pero todos los reclamantes fusionados son homólogos. Por lo tanto, solo estamos considerando la fusión de demandantes homólogos. Por esta razón llamamos a este axioma no manipulable por fusiones "restringidas". Sin embargo, parece razonable desde una perspectiva de simetría de ambas propiedades, ya que cuando un demandante se divide en varios nuevos demandantes, estos son homólogos en el nuevo problema.

5.4.3 Análisis axiomático y caracterización de la regla CCEA

La regla CCEA para problemas MBC satisface la mayoría de las propiedades mencionadas anteriormente, pero la eficiencia no la satisface como se mostró en el ejemplo 5.6. Esto lo establecemos en el siguiente teorema.

Teorema 5.8. *La regla CCEA para problemas de bancarrota con múltiples estados y demandas cruzadas satisface F, $WEFF$, $PEFF$, ETE, CTI, RMR, CED, SEC, CFC, CM, y $CONS$.*

A continuación, el objetivo es obtener un mejor conocimiento de la regla CCEA para MBC describiéndola de una manera única como una combinación de algunos axiomas razonables. Esta combinación de principios es muy importante para entender el comportamiento de esta regla y poder hacer una buena elección. La caracterización dada utiliza tres axiomas: eficiencia de Pareto, división igualitaria condicionada y consistencia.

Teorema 5.9. *La regla CCEA para problemas de bancarrota con múltiples estados y demandas cruzadas es la única que satisface $PEFF$, CED, y $CONS$.*

De la demostración del teorema 5.9, se sabe que para $|N| = 2$, la regla CCEA es la única que satisface $PEFF$ y CED para problemas de bancarrota con múltiples estados y demandas cruzadas. A continuación, en las siguientes proposiciones, se muestra el papel de cada propiedad en el teorema 5.9. Primero se demuestra que las propiedades CED y $CONS$ implican $PEFF$, luego, se demuestra que CED y $CONS$ son necesarias, y que cualquier otra combinación de dos propiedades no implica la tercera restante. Por lo tanto, la regla CCEA también puede caracterizarse haciendo uso sólo de las propiedades CED y $CONS$. Esto se establece a continuación en el corolario 5.1. Sin embargo, se ha preferido mantener la caracterización con $PEFF$ porque de esta manera se obtiene una caracterización diferente de la regla CCEA para el caso de dos agentes como se describe anteriormente.

Proposición 5.2. Si una regla satsiface las propiedades CED y $CONS$ entonces también satsiface $PEFF$.

Corolario 5.1. La regla CCEA para problemas de bancarrota con múltiples estados y demandas cruzadas es la única regla que satisface las propiedades CED, y $CONS$.

Proposición 5.3. Las propiedades CED y $CONS$ en el teorema 5.9 (y corolario 5.1) son necesarias.

Obsérvese que la propiedad CED para problemas MBC es el equivalente de la propiedad del límite inferior de la división igualitaria condicionada (Moulin, 2000) para problemas de bancarrota, pero no es la misma que la propiedad de compensación total de división igualitaria condicionada (esta fue llamada *exención* por Herrero y Villar (2001)). En problemas de bancarrota, para $|N| = 2$, la regla CEA es la única regla que satisface la propiedad del límite inferior de la división igualitaria condicionada (Thomson, 2015), no se menciona la eficiencia porque todas las reglas de bancarrota la cumplen. Aquí, se obtiene el mismo resultado para problemas MBC usando las propiedades $PEFF$ y CED (véase la demostración del caso $|N| = 2$ en el teorema 5.9). Sin embargo, cuando se utiliza la propiedad de compensación total de la división igualitaria condicionada, se necesita una propiedad adicional para caracterizar la regla CEA en problemas de bancarrota con $|N| = 2$. Además, para problemas de bancarrota cuando $|N| = 2$, coinciden las propiedades de compensación total condicionada (a esta la llamaron *sostenibilidad* Herrero y Villar (2002)) y de compensación total de división igualitaria condicionada.

Proposición 5.4. Para problemas de bancarrota con múltiples estados y demandas cruzadas, las propiedades CFC y CM implican CED.

Corolario 5.2. La regla CCEA para problemas de bancarrota con múltiples estados y demandas cruzadas es la única regla que satisface las propiedades $PEFF$, CFC, CM and $CONS$.

El corolario 5.2 corresponde a la caracterización de la regla CCEA en problemas MBC equivalente a la caracterización de la regla CEA en Yeh (2006) (véase, Thomson 2015, Th. 4b y Th. 14). Por supuesto, otras caracterizaciones de la regla CEA se podrían extender a este contexto, por ejemplo, la caracterización de CEA en Herrero y Villar (2002). En este último caso, primero tendríamos que definir qué significa *composición hacia abajo* (Moulin, 2000) en este contexto. Dado que se tienen muchos problemas, podría extenderse de diferentes formas. En cualquier caso, esta última caracterización y otras en la literatura serían interesantes de tratar en futuras investigaciones en este marco.

5.4.4 Análisis axiomático y caracterización de la regla CPA

La regla CPA también satisface la mayoría de las propiedades mencionadas anteriormente. Esto se establece en el siguiente teorema.

Teorema 5.10. La regla CPA para problemas de bancarrota con múltiples estados y demandas cruzadas satisface las propiedades F, $WEFF$, $PEFF$, ETE, GMA, $CONS$, NMS, y $NMRM$.

Ahora, el objetivo es lograr un mejor conocimiento de la regla CPA para $\mathcal{MBC}$ describiéndola de una manera única como una combinación de algunos axiomas razonables. Caracterizamos la regla CPA mediante $PEFF$, ETE, GMA, $CONS$ y NMS. Por lo tanto, la regla CPA puede considerarse como una forma deseable de distribuir un conjunto de estados entre sus reclamantes.

Teorema 5.11. La regla CPA para problemas de bancarrota con múltiples estados y demandas cruzadas es la única que satisface las propiedades $PEFF$, ETE, GMA, $CONS$, y NMS.

Proposición 5.5. Las propiedades en el teorema 5.11 son lógicamente independientes.

5.4.5 Análisis axiomático de las reglas CSP y CRA

Antes de analizar axiomáticamente las reglas CSP y la regla CPA, se introduce una propiedad específica relacionada con la prioridad definida sobre los reclamantes.

Axioma 5.50 (PRI)**.** Dada una regla R, se dice que satisface *prioridad*, si para cada problema $(M, N, E, c, \alpha) \in \mathcal{MBC}$ y para cada par de agentes homólogos $j, k \in N$, si $\sigma(j) < \sigma(k)$, o bien $c_j - R_j(M, N, E, c, \alpha) \leq c_k - R_k(M, N, E, c, \alpha)$ o bien $R_k(M, N, E, c, \alpha) = 0$.

Teorema 5.12. La regla CSP^σ para problemas de bancarrota con múltiples estados y demandas cruzadas satisface las propiedades F, $WEFF$, $PEFF$, $CONS$, y PRI.

Es bien sabido que la regla de llegada aleatoria no satisface $CONS$, por lo que tampoco lo hace la regla de llegada aleatoria restringida (CPA).

Teorema 5.13. La regla CRA para problemas de bancarrota con múltiples estados y demandas cruzadas satisface las propiedades F, y ETE.

En el contexto de problemas de bancarrota con múltiples estados y demandas cruzadas, el operador promedio no conserva la propiedad de eficiencia de Pareto y, por lo tanto, la regla CRA no satisface esta propiedad básica, como se muestra en la siguiente proposición.

Proposición 5.6. La regla CRA no satisface la propiedad $PEFF$.

En cuanto a la propiedad $WEFF$, en todos los ejemplos analizados la regla CRA cumple con esta propiedad, pero aún no tenemos una prueba de que así sea, en general. El hecho de que la regla CRA no satisfaga la propiedad $PEFF$ es lo que nos hace dudar de la propiedad del $WEFF$, ya que ambos tienen ciertas similitudes.

Una propiedad que cumplen la mayoría de las reglas para los problemas de bancarrota es la monotonía en los recursos, que también se utiliza en la caracterización de las reglas de prioridad secuencial (véase Thomson, 2019, Th.11.11). Esta propiedad simplemente dice que si aumenta el recurso disponible, las asignaciones a los reclamantes no disminuyen. Aunque esta propiedad parece muy débil, las reglas de prioridad secuencial restringida no la satisfacen, como muestra la siguiente proposición.

Proposición 5.7. La regla CSP^σ para problemas de bancarrota con múltiples estados y demandas cruzadas no satsiface la propiedad $R - MON$.

Otra propiedad de monotonía es la monotonía en la población, que dice que si todos los reclamantes están de acuerdo en que un reclamante j obtenga su demanda, entonces los reclamantes restantes deberían estar peor después de que el reclamante j sea completamente compensado. Esta propiedad se cumple con muchas reglas de bancarrota, incluida la regla de llegada aleatoria. Además, la regla de llegada aleatoria se caracteriza utilizando esta propiedad en Hwang y Wang (2009, Th.1). Por otra parte, una propiedad que aparece recurrentemente relacionada con el valor de Shapley y, por tanto, con la regla de la llegada aleatoria es la de las contribuciones equilibradas (Myerson, 1980; Hart y Mas-Colell, 1989). En el contexto de problemas de bancarrota, esta propiedad fue utilizada por Bergantiños y Méndez-Naya (1997) para caracterizar la regla de llegada aleatoria. Además, en el contexto de los problemas de bancarrota con múltiples (sub)estados, fue introducida por Lorenzo-Freire et al. (2007) y también se utiliza para caracterizar la regla de llegada aleatoria. Esta propiedad requiere que el reclamante j impacte en la asignación del reclamante k lo que el reclamante k impacte en la asignación del reclamante j. Como se mencionó anteriormente, este tipo de propiedades se utilizan para caracterizar soluciones tipo Shapley como lo es la regla de llegada aleatoria, pero en el contexto de problemas de bancarrota con múltiples estados y demandas cruzadas, la regla de llegada aleatoria restringida no los satisface como lo muestra la siguiente proposición .

Proposición 5.8. La regla CRA no satisface ni la propiedad $P - MON$ ni BAL.

Los resultados anteriores muestran la complejidad de encontrar propiedades que permitan caracterizaciones axiomáticas de las reglas de prioridad secuencial y la regla de llegada aleatoria en el contexto de problemas de bancarrota con múltiples estados y demandas cruzadas, por lo que es necesario buscar propiedades quizás más específicas (y probablemente más técnicas) para lograrlo.

5.5 Aplicación a la gestión del control de la contaminación de aguas

En este apartado aplicamos el modelo de bancarrota y las reglas de reparto al caso de la gestión de control de la contaminación del agua. El contenido de esta sección se basa principalmente en Acosta-Vega et al. (2021b).

5.5.1 Introducción

El agua es necesaria para casi cualquier forma de vida, por eso está presente en casi todos los informes elaborados por instituciones internacionales como la Organización de las Naciones Unidas (ONU) y muchos de sus organismos especializados, como la Organización Mundial de la Salud (OMS) o la Organización para la Agricultura y la Alimentación (FAO). La razó de esto es que el agua es un bien esencial para el desarrollo económico, la salud y el medio ambiente. Tres de los grandes problemas relacionados con el agua son: el acceso al agua dulce, la gestión del agua dulce y la contaminación del agua. De hecho, la solución de estos problemas está directa o indirectamente incluida en muchos de los *Objetivos de Desarrollo Sostenible* (ODS) promovidos por la ONU (https://www.un.org/sustainabledevelopment/). Esta sección trata del tercero de estos problemas, la contaminación del agua, en particular, el diseño de políticas de calidad del agua mediante el control de la contaminación del agua.

Nos gustaría enfatizar la importancia del control de la contaminación del agua. De hecho, la contaminación del agua es una grave amenaza para la salud humana, para la supervivencia de los ecosistemas y, por tanto, para la biodiversidad del planeta. La contaminación del agua dulce provoca numerosas enfermedades y reduce la disponibilidad de un recurso ya de por sí escaso e imprescindible para el consumo humano y para la agricultura. Por lo tanto, la gestión adecuada del control de la contaminación del agua en una determinada región es imprescindible para la supervivencia de la región y el desarrollo de su actividad económica (Helmer y Hespanhol, 1997; Goel, 2009).

Goel (2009) define un contaminante del agua de la siguiente manera: "*Un contaminante del agua se puede definir como un factor físico, químico o biológico que causa efectos estéticos o perjudiciales en la vida acuática y en quienes consumen el agua. Sin embargo, la mayoría de los contaminantes del agua se encuentran en forma de productos químicos que permanecen disueltos o suspendidos en el agua y dan una respuesta ambiental que a menudo es objetable. A veces, los factores físicos y biológicos también actúan como contaminantes. Entre los factores físicos, el calor y las radiaciones son factores importantes que tienen marcados efectos sobre los organismos. Ciertos microorganismos presentes en el agua, especialmente las especies patógenas, causan enfermedades al hombre y a los animales, y pueden denominarse biocontaminantes.*". Nesaratnam (2014) divide los contaminantes del agua en varias categorías: bencenoides, desechos que demandan oxígeno y nutrientes eutróficos. Cada uno de estos grupos de contaminantes del agua provienen de diferentes fuentes generalmente relacionadas con la actividad humana y tienen diferentes efectos sobre la calidad del agua.

Los bencenoides tóxicos como *benceno*, *etilbenceno*, *tolueno* y *xilenos*, incluyendo también *fenoles* son muy venenosos para cualquier organismo vivo, pudiendo causar enfermedades graves en humanos. Estos hidrocarburos aromáticos tienen un punto de ebullición bajo y son abundantes en el petróleo y representan su fracción más peligrosa. Además, los hidrocarburos, una vez incorporados a un determinado organismo, son muy estables, pudiendo pasar a través de muchos miembros de la cadena alimentaria sin verse alterados. Por lo tanto, se transfieren a toda la cadena trófica, situación análoga a la de los metales pesados y los pesticidas (véase, Tomlinson (1971) para detalles sobre los compuestos bencenoides). Además, estas sustancias, en particular los fenoles, también pueden afectar negativamente a la presencia de oxígeno disuelto (OD) en el agua, imprescindible para el desarrollo, en su interior, de la vida de animales y plantas. Una masa de agua se clasifica como contaminada cuando la concentración de OD cae por debajo del nivel necesario para mantener una biota normal para dicha agua. La causa principal de la desoxigenación del agua es la presencia de sustancias que se denominan desechos demandantes de oxígeno. Estos son compuestos que se degradan o descomponen fácilmente debido a la actividad bacteriana en presencia de oxígeno. En cuanto a los contaminantes que son residuos demandantes de oxígeno, podemos encontrar los ya mencionados *fenoles*, *compuestos amoniacales*, *sustancias inorgánicas oxidables* (OIS), y en general *compuestos orgánicos biodegradables* (BOC) (Riffat, 2013).

En cuanto al último grupo de contaminantes del agua, la eutrofización es el proceso por el cual una masa de agua se enriquece excesivamente con nutrientes que inducen un crecimiento excesivo de plantas acuáticas y algas. El efecto más evidente de la eutrofización es la creación de densas floraciones de fitoplancton nocivo y maloliente que reducen la claridad del agua y dañan la calidad del agua. Sin embargo, existen otros efectos más peligrosos que afectan la vida en los ecosistemas acuáticos. Aparte de las causas naturales, la eutrofización se produce por la acción del hombre con los vertidos de detergentes, fertilizantes o aguas residuales que contienen nitratos o fosfatos en un sistema acuático. Particularmente relevantes son los eutróficos de nitrógeno como *nitratos* y *compuestos amónicos* (véase, por ejemplo, Schindler (2006), Ansari et al. (2011) y Chislock et al. (2013) para obtener detalles sobre el problema de la eutrofización).

En la actualidad existe cierta preocupación por otros contaminantes presentes en el agua de los cuales se sabe poco, a estos contaminantes se les denomina contaminantes emergentes. Entre estos muchos productos se pueden encontrar, como productos farmacéuticos y de cuidado personal, nanomateriales, retardantes de fuego, pesticidas, plastificantes, surfactantes, subproductos de desinfección, bacterias resistentes a los antibióticos, microplásticos y genes. Algunos trabajos recientes sobre estos contaminantes emergentes son Geissen et al. (2015), Más-Plá (2018), Llamas et al. (2020) y Gomes et al. (2020), entre otros.

La Unión Europea (UE) ha impulsado e implementado diferentes políticas ambientales dirigidas a proteger la calidad del agua. Por lo tanto, las directivas de la UE han especificado valores límite de emisión para el agua y establecen estándares sobre cómo monitorear, informar y gestionar la calidad del agua (véase, por ejemplo, Directiva/2000/60/EC, Directiva 2006/118/EC, Directiva 2008/ 105/CE y Directiva 2013/39/UE). Steinebach (2019) analiza la efectividad de las políticas de la UE en la calidad de los recursos hídricos nacionales de los estados miembros durante un período de 23 años (1990-2012). En el caso de España, también existe un cuerpo legal para controlar diferentes aspectos de la gestión del agua, incluida la calidad del agua (Real Decreto 9/2008). Todo el marco legal anterior deriva la responsabilidad de la gestión y control del agua hacia las autoridades regionales y locales más cercanas al recurso hídrico. En consecuencia, por un lado, la legislación establece límites (de inmisión) a los parámetros indicadores de contaminación en las aguas corporales, cuyos valores varían según el uso de las aguas corporales (baño, depuración,...). Por otra parte, los vertidos son aquellos que se realizan directa o indirectamente a los cuerpos de agua, cualquiera que sea su naturaleza. Finalmente, las administraciones locales y autonómicas pueden legislar sobre concentraciones máximas (de emisión) de contaminantes en los vertidos realizados. Por tanto, es de gran interés social y económico generar herramientas que ayuden a los entes locales y regionales a diseñar políticas para el control de la calidad del agua.

Así, en esta sección, consideramos la situación en la que cierta autoridad responsable de la calidad del agua en su región está interesada en controlar la contaminación del agua. En particular, están interesados en limitar la concentración de las tres categorías de contaminantes del agua mencionadas anteriormente. Por un lado, para cada una de estas categorías de contaminantes se fijan determinados niveles de concentración con el fin de mantener una calidad de agua razonable. Por otro lado, se monitorean las principales sustancias mencionadas anteriormente y también existen límites máximos de concentración para las mismas. La relación entre las categorías de contaminantes y las sustancias de cada categoría se muestra en la figura 5.3. Así, el problema a resolver por la autoridad es cómo asignar nuevos umbrales a las sustancias teniendo en cuenta los límites fijados para cada categoría de contaminantes. Por lo tanto, la autoridad enfrenta un problema de asignación con ciertas características especiales. Una forma de resolver el problema es recurrir a soluciones que se pueden encontrar en la literatura sobre problemas de asignación, o en base a ellas introducir nuevas soluciones adaptadas al problema en particular. Sin embargo, la situación descrita en la figura 5.3 no se ajusta a un problema de bancarrota con múltiples (sub)estados, sino a un problema de bancarrota con múltiples estados y demandas cruzadas.

En la literatura se pueden encontrar muchas otras aplicaciones de los problemas de bancarrota. Algunos ejemplos son los siguientes. Pulido et al. (2002, 2008) estudian los problemas de asignación en la gestión universitaria;

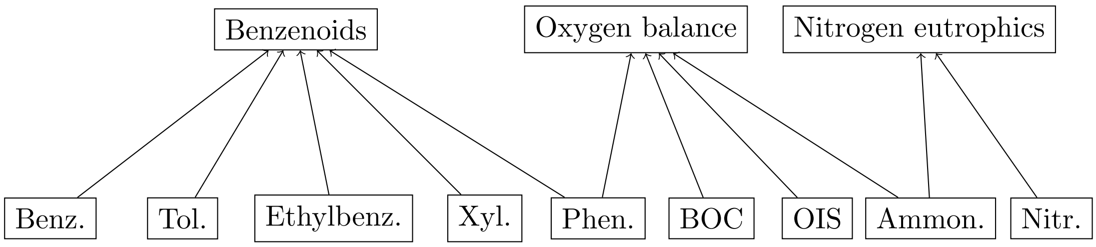

Figure 5.3: Relación entre familias de contaminantes y sustancias.

Niyato y Hossain (2006), Gozalvez et al. (2012) y Lucas-Estañ et al. (2012) analizan problemas de asignación de recursos de radio en telecomunicaciones; Casas-Méndez et al. (2011) estudian el problema del pase de museo; Hu et al. (2012) analizan el problema aeroportuario; Giménez-Gómez et al. (2016), Gutiérrez et al. (2018) y Duro et al. (2020) analizan el problema de asignación de CO2; Sánchez-Soriano et al. (2016) estudian el problema de la distribución en los sistemas electorales proporcionales; y Wickramage et al. (2020) analizan los problemas de asignación de agua en los ríos. Sin embargo, hasta donde sabemos, no hay aplicaciones de los problemas de bancarrota a los problemas con la estructura considerada en esta tesis doctoral.

5.5.2 Un caso estudio de gestión del control de la contaminación en aguas

Que sepamos, no existe normativa para la emisión de contaminantes del agua a nivel estatal en España (Reglamento 606/2003) o a nivel de la UE (Directivas 2006/11/CE y 2000/60/CE), excepto para el mercurio y otros contaminantes menores. Solo existen regulaciones a nivel local, por lo tanto, las localidades establecen niveles de emisión para autorizar o no las descargas.

En este apartado se consideran los límites aplicados correspondientes a una normativa local representativa de vertidos a la red de alcantarillado establecida por el Ayuntamiento de Granada. (BOP Núm. 129, 30/05/2000). Estos límites de emisión de los contaminantes se muestran en la tabla 5.1.

Supóngase ahora que el ayuntamiento quiere imponer limitaciones a las concentraciones (ppm) de cada una de las tres categorías de contaminantes mencionadas en la sección 5.5.1 (benzenoides, desechos demandantes de oxígeno y nutrientes eutrofizantes), independientemente de los límites de emisión para cada uno de los contaminantes. En este sentido, lo que pretende el ayuntamiento es controlar las emisiones más por categorías de contaminantes que por cada uno de los contaminantes en sí, respetando, al mismo tiempo, las

Contaminante	**Valor máximo**
Benzene	0.05
Toluene	0.25
Ethylbenzene	0.15
Xylenes	0.15
Phenols	5
BOC	700
OIS	545
Ammoniacal compounds	150
Nitrate compounds	100

Table 5.1: Valores máximos permitidos (ppm) para algunos contaminantes.

limitaciones de emisión de cada sustancia. Supongamos que el ayuntamiento establece los siguientes límites para cada uno de los grupos de contaminantes: 4 ppm para bencenoides, 1000 ppm para desechos que requieren oxígeno y 150 ppm para nutrientes eutrofizantes de nitrógeno. La situación de los límites de emisión se muestra en la figura 5.4.

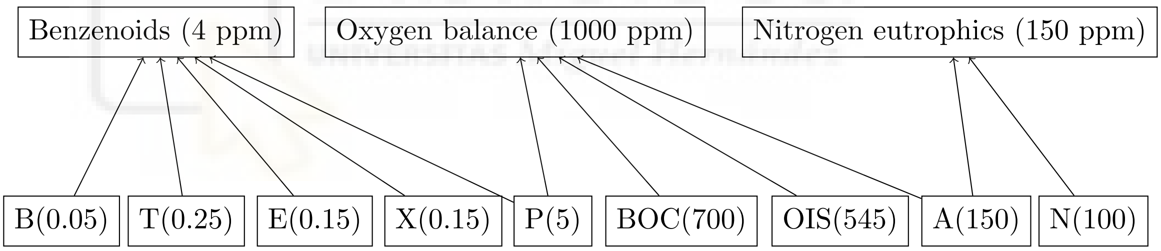

Figure 5.4: Relación entre familias de contaminantes y sustancias.

Asociado a la situación descrita anteriormente, consideramos el siguiente problema de bancarrota con múltiples estrados y demandas cruzadas $MBC = (M, N, E, c, \alpha)$:

- $M = \{1, 2, 3\}$;
- $N = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$;
- $E = (4, 1000, 150)$;
- $c = (0.05, 0.25, 0.15, 0.15, 5, 700, 545, 150, 100)$; y
- $\alpha(1) = \{1\}$, $\alpha(2) = \{1\}$, $\alpha(3) = \{1\}$, $\alpha(4) = \{1\}$, $\alpha(5) = \{1, 2\}$, $\alpha(6) = \{2\}$, $\alpha(7) = \{2\}$, $\alpha(8) = \{2, 3\}$, and $\alpha(9) = \{3\}$.

Es evidente que los límites de emisión para cada una de las tres categorías de contaminantes no son suficientes para garantizar los límites de emisión de cada una de las sustancias, por lo que se deben recalcular sus límites a la baja. Para hacer esto, ahora podemos aplicar las reglas de reparto presentadas en la sección 5.3 para problemas MBC.

Aplicación de la regla CCEA

En este apartado, aplicamos la regla CCEA al problema de determinar los nuevos umbrales. El cálculo de la asignación se obtiene en tres pasos mediante el procedimiento iterativo basado en la regla CEA:

1. Primero se calcula la regla CEA para cada uno de los tres problemas de bancarrota definidos por cada categoría (benzenoids, oxygen balance and nitrogen eutrophics).
   - $(N^{b,1}, E^{b,1}, c^{b,1})$. $N^{b,1} = \{B, T, E, X, P, \}$, $E^{b,1} = 4$, y $c^{b,1} = (0.05, 0.25, 0.15, 0.15, 5)$. $CEA(N^{b,1}, E^{b,1}, c^{b,1}) = (0.05, 0.25, 0.15, 0.15, 3.4)$, y $\beta^{b,1} = 3.4$.
   - $(N^{ob,1}, E^{ob,1}, c^{ob,1})$. $N^{ob,1} = \{P, BOC, OIS, A\}$, $E^{ob,1} = 1000$, y $c^{ob,1} = (5, 700, 545, 150)$. $CEA(N^{ob,1}, E^{ob,1}, c^{ob,1}) = (5, 422.5, 422.5, 150)$, y $\beta^{ob,1} = 422.5$.
   - $(N^{ne,1}, E^{ne,1}, c^{ne,1})$. $N^{ne,1} = \{A, N\}$, $E^{ne,1} = 150$, y $c^{ne,1} = (150, 100)$. $CEA(N^{ne,1}, E^{ne,1}, c^{ne,1}) = (75, 75)$, y $\beta^{ne,1} = 75$.
2. A continuación se calcula $\beta^{*1} = \min\{\beta^{b,1}, \beta^{ob,1}, \beta^{ne,1}\} = 3.4$, y se asigna a cada sustancia i, la cantidad $\min\{c_i, \beta^{*1}\}$. De este modo, se obtiene el vector de asignaciones $(0.05, 0.25, 0.15, 0.15, 3.4, 3.4, 3.4, 3.4, 3.4)$.

Ahora bien, es obvio que el estado e_1 correspondiente a los bencenoides se ha distribuido en su totalidad, y en el siguiente paso se excluye este problema de bancarrota y los reclamantes asociados con él. Los otros dos problemas se actualizan en demandantes, y se reducen los estados y demandas de acuerdo con la asignación previamente obtenida.

1. Se calcula la regla CEA para cada uno de los dos problemas de bancarrota restantes.
   - $(N^{ob,2}, E^{ob,2}, c^{ob,2})$. $N^{ob,2} = \{BOC, OIS, A\}$, $E^{ob,2} = 986.4$, y $c^{ob,2} = (696.6, 541.6, 146.6)$. $CEA(N^{ob,2}, E^{ob,2}, c^{ob,2}) = (419.9, 419.9, 146, 6)$, y $\beta^{ob,2} = 419.9$.
   - $(N^{ne,2}, E^{ne,2}, c^{ne,2})$. $N^{ne,2} = \{A, N\}$, $E^{ne,2} = 143.2$, y $c^{ne,2} = (146.6, 96.6)$. $CEA(N^{ne,2}, E^{ne,2}, c^{ne,2}) = (71.6, 71.6)$, y $\beta^{ne,2} = 71.6$.
2. A continuación se calcula $\beta^{*2} = \min\{\beta^{ob,2}, \beta^{ne,2}\} = 71.6$, y se asigna a cada sustancia i $\min\{c_i, \beta^{*2}\}$. De este modo, se obtiene el vector de asignaciones $(0, 0, 0, 0, 0, 71.6, 71.6, 71.6, 71.6)$.

Contaminante	Valor máximo original	Nuevo valor máximo
Benzene	0.05	0.05
Toluene	0.25	0.25
Ethylbenzene	0.15	0.15
Xylenes	0.15	0.15
Phenols	5	3.4
BOC	700	460.8
OIS	545	460.8
Ammoniacal compounds	150	75
Nitrate compounds	100	75

Table 5.2: Valores máximos permitidos (ppm) para algunos contaminantes tras limitar las emisiones de los tres grupos de contaminantes mediante CCEA.

Ahora, es obvio que el estado e_3 correspondiente a los eutróficos de nitrógeno se ha distribuido por completo, y en el siguiente paso se excluye este problema de bancarrota y los reclamantes asociados con él. El tercer problema se actualiza en reclamantes, y decreciendo estados y demandas según la asignación previamente obtenida.

1. Calculamos la regla CEA para cada uno de los problemas de bancarrota restantes.
   - $(N^{ob,3}, E^{ob,3}, c^{ob,3})$. $N^{ob,3} = \{BOC, OIS\}$, $E^{ob,3} = 771.6$, y $c^{ob,3} = (625, 470)$. $CEA(N^{ob,3}, E^{ob,3}, c^{ob,3}) = (385.8, 385.8)$, y $\beta^{ob,3} = 385.8$.
2. A continuación se calcula $\beta^{*3} = \min\{\beta^{ob,3}\} = 385.8$, y se asigna a cada sustancia i $\min\{c_i, \beta^{*3}\}$. De este modo, se obtiene el vector de asignaciones $(0, 0, 0, 0, 0, 385.5, 385.5, 0, 0)$.

El procedimiento se detiene porque todos los estados se han repartido por completo. Finalmente, sumando los vectores de asignación obtenidos en el procedimiento, obtenemos que $CCEA(M, N, E, c, alpha) = (0.05, 0.25, 0.15, 0.15, 3.4, 460.8, 460.8, 75, 75)$.

Por lo tanto, si el gobierno local quiere limitar las emisiones de las tres categorías de contaminantes por debajo de 4 ppm, 1000 ppm y 150 ppm, respectivamente, pero manteniendo un límite fijo de emisiones para cada una de las sustancias contaminantes, una alternativa sería limitar las emisiones de las sustancias contaminantes a los nuevos valores de la tercera columna de la tabla 5.2.

Así, se puede observar cómo la regla CCEA para problemas de bancarrota con múltiples estados y demandas cruzadas puede ayudar a las autoridades a diseñar nuevas políticas de calidad del agua, en particular, cómo se pueden establecer los límites de emisión de diferentes sustancias teniendo en cuenta las categorías de contaminantes del agua que tienen diferentes efectos en la calidad del agua. Se observa que las principales reducciones en el nivel máximo de emisiones son asumidas por aquellos con los límites de emisiones más altos.

Aplicación de la regla proporcional restringida

En este apartado, se aplica la regla proporcional restringida al problema de determinar los nuevos umbrales. El cálculo de la asignación se obtiene en cuatro pasos:

Step 1. $\mathcal{N}^1 = N$, $M^1 = M$,
$E^1 = (4, 1000, 150)$, $c^1 = (0.05, 0.25, 0.15, 0.15, 5, 700, 545, 150, 100)$
- $\lambda_1^1 = 0.714$
- $\lambda_2^1 = 0.714$
- $\lambda_3^1 = 0.600$

$\lambda^1 = \min\{0.714, 0.714, 0.600\} = 0.600$

$$a_1^1 = 0.03, a_2^1 = 0.15, a_3^1 = 0.09, a_4^1 = 0.09, a_5^1 = 3, a_6^1 = 420, a_7^1 = 327, a_8^1 = 90, a_9^1 = 60$$

Step 2. $\mathcal{N}^2 = \{1, 2, 3, 4, 5, 6, 7\}$, $M^2 = \{1, 2\}$,
$E^2 = (0.64, 160, 0)$, $c^2 = (0.02, 0.1, 0.06, 0.06, 2, 280, 218, 60, 40)$
- $\lambda_1^2 = 0.286$
- $\lambda_2^2 = 0.32$

$\lambda^2 = \min\{0.286, 0.32\} = 0.286$

$$a_1^2 = 0.01, a_2^2 = 0.03, a_3^2 = 0.02, a_4^2 = 0.02, a_5^2 = 0.57, a_6^2 = 80, a_7^2 = 62.29, a_8^2 = 0, a_9^2 = 0$$

Step 3. $\mathcal{N}^3 = \{6, 7\}$, $M^2 = \{2\}$,
$E^3 = (0, 17.143, 0)$, $c^3 = (0.014, 0.071, 0.043, 0.043, 1.429, 200, 155.714, 60, 40)$
- $\lambda_2^3 = 0.048$

$\lambda^3 = \min\{0.048\} = 0.048$

$$a_1^2 = 0, a_2^2 = 0, a_3^2 = 0, a_4^2 = 0, a_5^2 = 0, a_6^2 = 9.64, a_7^2 = 7.50, a_8^2 = 0, a_9^2 = 0$$

Step 4. $\mathcal{N}^4 = \varnothing$, $M^2 = \varnothing$.

$$CPA(M, N, E, c, \alpha) = (0.036, 0.179, 0.107, 0.107, 3.571, 509.639, 396.79, 90, 60)\,.$$

Contaminante	Valor máximo original	Nuevo valor máximo
Benzene	0.05	0.036
Toluene	0.25	0.179
Ethylbenzene	0.15	0.107
Xylenes	0.15	0.107
Phenols	5	3.571
BOC	700	509.639
OIS	545	396.79
Ammoniacal compounds	150	90
Nitrate compounds	100	60

Table 5.3: Valores máximos permitidos (ppm) para algunos contaminantes tras limitar las emisiones de los tres grupos de contaminantes mediante CPA.

Por lo tanto, si el gobierno local quiere limitar las emisiones de las tres categorías de contaminantes por debajo de 4 ppm, 1000 ppm y 150 ppm, respectivamente, pero manteniendo un límite fijo de emisiones para cada una de las sustancias contaminantes, una alternativa sería limitar las emisiones de las sustancias contaminantes a los nuevos valores de la tercera columna de la tabla 5.3.

Así, se puede observar cómo la regla de reparto proporcional restringida para problemas de bancarrota con múltiples estados y demandas cruzadas puede ayudar a las autoridades a diseñar nuevas políticas de calidad del agua, en particular, cómo se pueden establecer los límites de emisión de diferentes sustancias teniendo en cuenta las categorías de contaminantes del agua que tienen diferentes efectos en la calidad del agua. Observamos que, en este caso, todos los contaminantes se reducen en cierta cantidad en contraste con los resultados obtenidos cuando se aplica la regla CCEA.

Aplicación de una regla basada en llegadas aleatorias

En este apartado, se aplica una modificación de la regla de llegada aleatoria restringida al problema para determinar los nuevos umbrales. El cómputo de la CRA requiere considerar $9!(= 362880)$ lo que implica que su cálculo es muy laborioso. Por esta razón, aquí consideramos una modificación del mismo, que llamamos CRA*, que es un poco menos tedioso de calcular y en esta aplicación en particular es computacionalmente asequible. En particular, la CRA* consta de dos niveles de órdenes de llegada. En primer lugar, se toman todos los órdenes posibles de las categorías de contaminantes. Para la primera categoría en el orden, calculamos la regla RA, luego se actualizan los estados y las demandas y se calcula la regla RA para la segunda categoría en el orden, y así hasta la última categoría en el orden. Una vez que hemos obtenido una asignación para cada posible orden las categorías, tomamos su promedio. Téngase en cuenta que CRA* puede verse como una regla de reparto nueva y diferente de CRA, pero usamos el mismo razonamiento que hay detrás[5]. A continuación se da un ejemplo de cálculo para el siguiente orden de las categorías nitrógeno eutrófico, balance de oxígeno y bencenoides.

La categoría nitrógeno eutrófico solo tiene dos contaminantes asociados, compuestos amoniacales (A) y compuestos de nitrato (N), por lo tanto, solo hay dos órdenes posibles AN y NA. El estado a asignar es de 150 ppm. Entonces, las contribuciones marginales correspondientes asociadas con esos pedidos son $(150, 0)$ y $(50, 100)$ respectivamente. Tomando el promedio, la asignación es $(100, 50)$.

La siguiente categoría en el orden es el balance de oxígeno que tiene asociados cuatro contaminantes: fenoles (P), BOC (B), OIS (O) y compuestos amoniacales (A). Dado que ya se han asignado 100 ppm a compuestos amoniacales, el estado disponible es de 900 ppm, y los compuestos amoniacales se eliminan de los reclamantes de la categoría de balance de oxígeno, ya que no puede recibir más a menos que se supere el estado de la categoría de eutróficos nitrogenados. Por tanto, tenemos seis posibles órdenes de llegada para el resto de contaminantes cuyas contribuciones marginales son las siguientes:

Orden	*P*	*BOC*	*OIS*
PBO	5	700	195
POB	5	350	545
BPO	5	700	195
BOP	0	700	200
OPB	5	350	545
OBP	0	355	545
Promedio	3.333	525.833	370.833

Por lo tanto, la asignación es $(3.333, 525.833, 370.833)$.

Finalmente, la tercera categoría en el orden son los bencenoides, que tiene cinco contaminantes asociados. Sin embargo, los fenoles se descartan por las mismas razones que se descartaron los compuestos amoniacales en el paso anterior. Además, el estado disponible es de 0,667 ppm que supera la suma de las demandas de los restantes contaminantes, por lo que todos reciben su demanda. Como resultado, la asignación es $(0, 05, 0, 25, 0, 15, 0, 15)$.

[5]Para que esta regla esté bien definida, primero es necesario que las demandas se trunquen por el mínimo de sus estados asociados para evitar inconsistencias, es decir, debemos tomar como demandas $c'_i = \min\{c_i, \min_{k\in\alpha(i)}\{e_k\}\}, i \in N$.

Contaminante	Valor máximo original	Nuevo valor máximo
Benzene	0.05	0.0375
Toluene	0.25	0.1875
Ethylbenzene	0.15	0.1125
Xylenes	0.15	0.1125
Phenols	5	3.5165
BOC	700	525.7415
OIS	545	370.7415
Ammoniacal compounds	150	100
Nitrate compounds	100	50

Table 5.4: Valores máximos permitidos (ppm) para algunos contaminantes tras limitar las emisiones de los tres grupos de contaminantes mediante CRA*.

Una vez finalizado este proceso, la asignación asociada al orden nitrógeno eutrófico, balance de oxígeno y bencenoides es

$$(0.05, 0.25, 0.15, 0.15, 3.333, 525.833, 370.833, 100, 50).$$

A continuación se presenta la asignación obtenida para cada uno de los posibles órdenes de las categorías de contaminantes.

Orden	*B*	*T*	*E*	*X*	*P*	*BOC*	*O*	*A*	*N*
BON	0.025	0.125	0.075	0.075	3.7	525.65	370.65	100	50
BNO	0.025	0.125	0.075	0.075	3.7	525.65	370.65	100	50
OBN	0.05	0.25	0.15	0.15	3.333	525.833	370.833	100	50
ONB	0.05	0.25	0.15	0.15	3.333	525.833	370.833	100	50
NBO	0.025	0.125	0.075	0.075	3.7	525.65	370.65	100	50
NOB	0.05	0.25	0.15	0.15	3.333	525.833	370.833	100	50
Prom.	0.0375	0.1875	0.1125	0.1125	3.5165	525.7415	370.7415	100	50

Por lo tanto, si el gobierno local quiere limitar las emisiones de las tres categorías de contaminantes por debajo de 4 ppm, 1000 ppm y 150 ppm, respectivamente, pero manteniendo un límite fijo de emisiones para cada una de las sustancias contaminantes, una alternativa sería limitar las emisiones de las sustancias contaminantes a los nuevos valores de la tercera columna de la tabla 5.4.

Así, se puede observar cómo la regla CRA* para problemas de bancarrota con múltiples estados y demandas cruzadas puede ayudar a las autoridades a diseñar nuevas políticas de calidad del agua, en particular, cómo se pueden establecer los límites de emisión de diferentes sustancias teniendo en cuenta las categorías de contaminantes del agua que tienen diferentes efectos en la calidad del agua. Sin embargo, observamos que la asignación de CRA* no es eficiente en el sentido de Pareto, ya que los límites de emisión de benceno, tolueno, etilbenceno y xilenos podrían ser ligeramente superiores sin sobrepasar ninguno de los estados.

Resumen

A modo de resumen, a continuación se muestran los nuevos umbrales de cada contaminante con cada regla de reparto. La tabla 5.5 muestra las diferencias obtenidas en los nuevos límites de emisión cuando se aplican las tres reglas, CCEA, CPA y CRA*. ¿Cuál de ellos es mejor? La respuesta dependerá de qué propiedades se consideren más relevantes en este tipo de problemas.

Contaminante	**Original**	**CCEA**	**CPA**	**CRA***
Benzene	0.05	0.05	0.036	0.038
Toluene	0.25	0.25	0.179	0.188
Ethylbenzene	0.15	0.15	0.107	0.113
Xylenes	0.15	0.15	0.107	0.113
Phenols	5	3.4	3.571	3.517
BOC	700	460.8	509.639	525.742
OIS	545	460.8	396.79	370.742
Ammoniacal compounds	150	75	90	100
Nitrate compounds	100	75	60	50

Table 5.5: Resumen de la aplicación de cada una de las reglas de reparto.

A primera vista, se puede observar que la CCEA proporciona límites de emisión idénticos para aquellos contaminantes con límites de emisión originales muy bajos, y los iguala para el resto. Esto no es ni bueno ni malo, simplemente significa que el impacto negativo de todos los contaminantes se considera implícitamente similar. Aquí el único problema podría estar en las empresas contaminantes, ya que aquellas con emisiones contaminantes con límites originales de emisión más altos tendrán que hacer un mayor esfuerzo para reducir sus emisiones que otras empresas cuyas emisiones contaminantes se concentran en aquellas con límites originales de emisión más bajos.

Las otras dos reglas, CPA y CRA*, dan resultados muy similares y proponen una revisión más proporcional de los límites de emisión. En este caso, se consideraría implícitamente que los límites originales más altos corresponderán a contaminantes menos dañinos y que los impactos son proporcionales a sus límites originales. Al mismo tiempo, esto significaría que todas las empresas harían esfuerzos proporcionales a sus emisiones contaminantes independientemente de si los límites de emisión originales eran inicialmente más altos o más bajos.

Chapter 6

Discusión

Las aplicaciones de la investigación operativa (IO) a los problemas de gestión ambiental han ido en aumento desde los primeros trabajos en la década de 1970, véase, por ejemplo, la revisión de Bloemhof-Ruwaarda et al. (1995) y las referencias que contiene. Esta revisión destacó el potencial de OR para resolver problemas de gestión ambiental o para incluir elementos ambientales en problemas de optimización. Más recientemente, Mishra (2020) insiste en el impacto de la IO en la gestión ambiental. En particular, se pueden encontrar en la literatura muchas aplicaciones de la teoría de juegos a problemas de gestión ambiental (véase, por ejemplo, Hanley y Folmer, 1998; Dinar et al., 2008).

Un problema particular de reparto o asignación se presenta en situaciones donde existe un recurso perfectamente divisible sobre el cual existe un conjunto de agentes que tienen derechos o demandas, pero el recurso no es suficiente para satisfacerlos. Este problema se conoce como problema de bancarrota y fue analizado formalmente por primera vez en O'Neill (1982) y Aumann y Maschler (1985). Desde entonces, se ha estudiado ampliamente en la literatura y se han definido muchas reglas de reparto (ver Thomson, 2019, para un inventario detallado de reglas). En la literatura también encontramos trabajos que aplican este modelo de problema de bancarrota para estudiar el problema de asignación de agua (Wickramage et al., 2020) y el problema de asignación de permisos de descarga de contaminantes en ríos (Aghasian et al., 2019; Moridi, 2019) . Pero también a otros problemas ambientales, véase, por ejemplo, Giménez-Gómez et al. (2016), Gutiérrez et al. (2018) y Duro et al. (2020) que analizan el problema de asignación de CO2.

Sin embargo, el modelo de bancarrota básico no siempre se ajusta a todos los problemas, razón por la cual existen diferentes extensiones del modelo de bancarrota clásico. Algunas de ellas son los siguientes: Young (1994) y Moulin (2000) estudian los problemas de bancarrota en el caso de los bienes indivisibles. Una aplicación del modelo de bancarrota discreta al problema de asiganción de escaños en sistemas electorales proporcionales se da en Sánchez-Soriano et al. (2016). Pulido et al. (2002, 2008) introducen problemas de bancarrota con referencias y reclamaciones para estudiar los problemas de asignación en la gestión universitaria. Gozálvez et al. (2012) y Lucas-Estañ et al. (2012) presentan problemas de bancarrota con demandas dadas por una función no lineal discreta del recurso para analizar problemas de asignación de recursos de radio en telecomunicaciones. Habis y Herings (2013) y Kooster y Boonen (2019) estudian problemas de bancarrota en los que el estado y las demandas son valores estocásticos. Una extensión interesante de los problemas de bancarrota son los problemas de bancarrota con múltiples (sub)estados (Calleja et al., 2005). Estos describen situaciones en las que hay un recurso (perfectamente divisible) que se puede distribuir entre varios (sub)estados, y un número (finito) de agentes que tienen reclamaciones sobre cada uno de esos (sub)estados, de modo que la demanda total está por encima del recurso disponible. Este problema también se resuelve mediante reglas de reparto y existen diferentes formas de hacerlo (véase, por ejemplo, Calleja et al., 2005; Borm et al., 2005; González-Alcón et al., 2007; Izquierdo y Timoner, 2016). Sin embargo, la situación descrita en la Figura 1.1 del Capítulo 1 no se ajusta a un problema de bancarrota multiestado como el referido en el párrafo anterior, sino a un problema de bancarrota multiestado con demandas cruzadas como se introdujo en esta tesis doctoral. Estos describen situaciones en las que hay varios recursos (perfectamente divisibles) y un conjunto (finito) de agentes que tienen derechos o demandas sobre ellos, pero solo un derecho (no un derecho para cada recurso) con el que se solicitan uno o más recursos. La demanda total de cada recurso supera su disponibilidad. Por lo tanto, en esta tesis, proponemos utilizar problemas de bancarrota multiestado con demandas cruzadas para asignar límites de emisión a los contaminantes. Por tanto, una vez más los problemas reales conducen al desarrollo de modelos matemáticos teóricos que aporten respuestas, como es el caso que se presenta en esta tesis doctoral. Hasta donde sabemos, no existe en la literatura otro modelo de bancarrota similar al que aquí se propone. En consecuencia, esta tesis doctoral contribuye significativamente al desarrollo de la literatura sobre problemas de gestión de recursos (escasos), proporcionando un nuevo modelo de problemas de bancarrota y sus soluciones.

En cuanto a las soluciones que se han propuesto en esta tesis doctoral, se han extendido tres reglas de bancarrota muy conocidas: la regla igualitaria restringida, la regla proporcional y la regla de llegada aleatoria. El primero basado en el principio de igualitarismo, el segundo basado en el principio aristotélico de tratar igual a los iguales y diferentemente a los distintos, y el tercero basado en el principio de prioridad. Además de ampliar estas reglas, se han analizado axiomáticamente para mostrar sus buenas propiedades como soluciones a problemas de asignación de recursos, en el caso particular de considerar como recursos las emisiones contaminantes. En este sentido, los modelos matemáticos aquí propuestos, así como sus soluciones, proporcionan una metodología para la gestión de las emisiones contaminantes, en particular, en los casos de emisiones a la atmósfera o al agua.

Chapter 7

Conclusiones y futuras líneas de investigación

Esta tesis está relacionada con uno de los primeros problemas surgidos en la literatura económica. De hecho, este problema ya aparecá en documentos primitivos como el Talmud, o en ensayos de Aristóteles o Maimónides. Sin embargo, su modelización matemática fue realizada por primera vez por O'Neill (1982). La pregunta común y central en estos problemas es cómo dividir cuando no hay suficiente. Una extensión de los problemas de bancarrota clásicos aparece con la introducción de problemas de bancarrota con múltiples (sub)estados (Calleja et al. 2005) que permiten que las demandas de los agentes puedan referirse a diferentes temas.

En definitiva, vamos más allá con el fin de solucionar un problema real de reducción de emisiones de diferentes contaminantes en el que los contaminantes pueden contribuir a más de un efecto negativo. Para ello, establecemos un nuevo y original modelo basado en problemas de bancarrota con múltples estados (MB) denominados problemas de bancarrota con múltiples estados y demandas cruzadass (MBC). Este novedoso modelo presenta un estado multidimensional, uno para cada (sub)estado y cada agente reclama lo mismo a los diferentes (sub)estados en los que participa, estas son diferencias esenciales con respecto a los problemas MB.

De manera similar a los problemas MB, en este nuevo marco, los problemas se resuelven mediante reglas que asignan a cada problema MBC una distribución que indica la cantidad obtenida por cada agente en cada emisión. En primer lugar, hemos asignado según la regla CEA para problemas de bancarrota introduciéndola como solución a una sucesión de problemas de programación lineal y extendiendo este procedimiento a este marco. Esta nueva regla ha sido analizada y caracterizada axiomáticamente utilizando propiedades similares a las utilizadas para caracterizar la CEA en problemas de bancarrota. Además, esta tesis está relacionada con uno de los conceptos más importantes en los problemas de asignación: la proporcionalidad. En los problemas de asignación se pone en práctica el concepto de proporcionalidad con la conocida regla proporcional. Esta regla ha sido ampliamente estudiada en la literatura desde muchos puntos de vista diferentes y para muchos modelos de asignación. Centrándose en los modelos de bancarrota y sus extensiones al caso de múltiples (sub)estados, la regla proporcional se ha caracterizado en el contexto de los problemas de bancarrota en Chun (1988) y de Frutos (1999). En ambos artículos, la no manipulabilidad juega un papel central en la caracterización axiomática de la regla proporcional. Para problemas de bancarrota con múltiples (sub)estados, Ju et al. (2007) y Moreno-Ternero (2009) introducen dos definiciones diferentes de regla proporcional siguiendo dos enfoques diferentes. Además, Ju et al. (2007) y Bergantiños et al. (2010) proporcionan caracterizaciones de ambas reglas proporcionales. Nuevamente, en ambos enfoques, la no manipulabilidad es una propiedad esencial. En esta tesis, introducimos una definición de regla proporcional, que llamamos regla de adjudicación proporcional restringida, para problemas de bancarrota con múltiples (sub)estados y demandas cruzadas y proporcionamos una caracterización de la misma. Una vez más, se utiliza la no manipulabilidad. Por lo tanto, llenamos un vacío en la literatura de distribuciones proporcionales en problemas de asignación en línea con los estudios previos. Adicionalmente, en muchos problemas de asignación el concepto de prelación o prioridad es relevante, por ejemplo, en la legislación relacionada con la liquidación de una empresa por bancarrota en muchos países, se establece un orden de prelación para satisfacer las deudas. Primero, se liquidan las deudas salariales, luego se pagan los impuestos, después se satisfacen las deudas con los acreedores y, finalmente, se abordan las demandas de los accionistas. Por lo tanto, las reglas de prioridad secuencial, aunque simples, no son extrañas en la vida real. En esta tesis se ha introducido la extensión de las reglas de prelación secuencial en el contexto de problemas de bancarrota con múltiples (sub)estados y demandas cruzadas. A continuación, mediante la media de todas estas reglas, se define la extensión de la regla de llegada aleatoria a este nuevo contexto. En el análisis de las propiedades satisfechas por las reglas de prioridad secuencial y la regla de llegada aleatoria, se ha demostrado que las extensiones naturales de las propiedades utilizadas en la caracterización de estas reglas en el contexto de problemas de bancarrota y problemas de bancarrota con múltiples (sub)estados no son satisfechas por las reglas de prioridad secuencial restringida y la regla de llegada aleatoria restringida. Por lo tanto, las caracterizaciones de las reglas introducidas en este trabajo requerirán propiedades quizás demasiado técnicas o muy *ad hoc*, lo que podría restar valor a una interpretación simple. La razón por la que estas propiedades no se satisfacen es que no existe eficiencia en el contexto de problemas de bancarrota con múltiples (sub)estados y demandas cruzadas, ni tan siquiera un concepto más débil como la eficiencia de Pareto en el caso de la relga de llegadas aleatorias. Esto significa que la cantidad total distribuida puede cambiar de un problema a otro, por lo que los resultados de un problema a otro no son fácilmente comparables. Esto es un problema pero también nos dice que estos problemas son de interés teórico porque no son una mera y simple extensión de los problemas de bancarrota sino que tienen una estructura diferente que los hace interesantes para estudio adicional.

Además, ilustramos el interés y la aplicabilidad del modelo, y cómo funcionan la regla igualitaria restringida, la regla restringida proporcional y la regla restringida de llegada aleatoria por medio de una aplicación a la gestión del control de la contaminación del agua.

En la literatura de Investigación Operativa (IO) existen problemas que podrían encajar bien en este modelo teórico, por ejemplo, problemas de cobertura de conjuntos. Bergantiños et al. (2020) estudian el problema de cómo asignar costos en problemas de cobertura de conjuntos cuando se brinda una cobertura razonable por adelantado. Estos problemas se describen mediante una tupla de 4 (N, M, c, A), donde N es el conjunto de agentes, M es el conjunto de instalaciones abiertas, $c \in \mathbb{R}^M_+$ es el vector de costos asociado con las instalaciones, y $A = \{A_j\}_{j\in M}$ con $A_j \subset N$ para cada $j \in M$ denota los agentes cubiertos por cada instalación. La pregunta a responder es cómo distribuir los costos totales entre los agentes. Si observamos detenidamente la estructura del problema, podemos observar cierta similitud con los problemas de bancarrota con múltiples (sub)estados y demandas cruzadas de la siguiente forma. Primero identificamos agentes con contaminantes y problemas con instalaciones (regiones). Así, tenemos un conjunto de contaminantes que afectan a varias regiones, esto es descrito por A que cumple el rol de función α. Por otro lado, consideramos que cada región fija un nivel máximo de contaminación el cual viene dado por el vector c que hace el papel de vector E. Surge entonces el siguiente problema: ¿Cómo establecer los niveles de emisión de contaminantes cuando los contaminantes afectan a distintas regiones? Pero es necesario un elemento extra en este problema: las emisiones contaminantes a abatir, es decir, las demandas. Por lo tanto, el problema de cobertura de conjuntos es el siguiente. Cuando tenemos un conjunto de regiones que imponen límites a las emisiones contaminantes, y estas emisiones provienen de varios contaminantes que pueden afectar a varias de las regiones simultáneamente, la pregunta a responder es cómo establecer los límites de emisión de contaminantes de tal manera que se cubren los límites establecidos por las regiones. Este problema se puede analizar como un problema de bancarrota con múltiples estados y demandas cruzadas. Sin embargo,

¿qué sucede si no se dan referencias sobre las emisiones ex-ante de los contaminantes? En este caso, el problema tiene exactamente cuatro elementos, (N, M, c, A), y la pregunta a responder es, ¿cómo establecer límites máximos de emisión de contaminantes de manera que no se sobrepasen los límites regionales? Por lo tanto, estas relaciones entre los problemas de cobertura fija y los problemas de bancarrota con múltiples estados y demandas cruzadas sería interesante estudiarlas con mayor detalle en futuras investigaciones.

Está pendiente de investigación adicional para encontrar una caracterización de la regla de llegada aleatoria restringida. Asimismo, sería interesante encontrar extensiones de la regla CEL y del Talmud al contexto de problemas de bancarrota con múltiples (sub)estados y demandas cruzadas y, por supuesto, realizar su correspondiente análisis axiomático y encontrar caracterizaciones de dichas reglas.

Por último, también pensamos que sería interesante intentar definir los juegos cooperativos asociados a estos problemas, lo que también dejaría una puerta abierta a futuras investigaciones.

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Appendices

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ORIGINAL RESEARCH

Multi-issue bankruptcy problems with crossed claims

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Abstract
In this paper, we introduce a novel model of multi-issue bankruptcy problem inspired from a real problem of abatement of emissions of different pollutants in which pollutants can have more than one effect on atmosphere. In our model, therefore, several perfectly divisible goods (estates) have to be allocated among certain set of agents (claimants) that have exactly one claim which is used in all estates simultaneously. In other words, unlike of the multi-issue bankruptcy problems already existent in the literature, this model study situations with multi-dimensional states, one for each issue and where each agent claims the same to the different issues in which participates. In this context, we present an allocation rule that generalizes the well-known constrained equal awards rule from a procedure derived from analyzing this rule for classical bankruptcy problems as the solution to a sucession of linear programming problems. Next, we carry out an study of its main properties, and we characterize it using the well-known property of consistency.

Keywords Multi-issue bankruptcy problems · Allocation rules · Constrained equal awards rule

1 Introduction

A bankruptcy poblem describes a situation in which an endowment, perfectly divisible, must be distributed among a set of agents who have claims on it but the endowment is not enough to completely satisfy all of them. An allocation for such a problem should meet two reasonable conditions: (1) agents can neither receive more than they claim nor less than nothing, (2) the endowment should be fully distributed. These problems were first studied by O'Neill (1982)

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and Aumann and Maschler (1985) . Since then this problem and their extensions have been widely studied (see Thomson 2003, 2015, 2019) for an excellent analysis of bankruptcy problems from an axiomatic perspective) and many applications of them can be found in the literature (see, for example, Gallastegui et al. 2002; Pulido et al. 2008, 2002; Niyato and Hossain 2006; Casas-Méndez et al. 2011; Bergantiños et al. 2012; Gozálvez et al. 2012; Hu et al. 2012; Lucas-Estañ et al. 2012; Giménez-Gómez et al. 2016; Sánchez-Soriano et al. 2016; Gutiérrez et al. 2018; Bergantiños et al. 2018; Duro et al. 2020; Wickramage et al. 2020, among others).

To illustrate our model, consider now that a certain authority is interested in reducing the emission of pollutants into the atmosphere. However, there are many pollutants, each with different effects and consequences. There are pollutants that contribute to the greenhouse effect and thus to climate change, and others that are harmful to health because they are carcinogenic, cause respiratory problems or other diseases. On the one hand, water vapour (H2O), carbon dioxide (CO2), nitrous oxide (NO2), methane (CH4), and ozone (O3) are the primary greenhouse effect gases (GHG's), but also sulphur hexafluoride (SF6), hydrofluorocarbons (HFCs) and perfluorocarbons (PFCs) are relevant according to the Kyoto Protocol. On the other hand, carbon monoxide (CO), sulphur dioxide (SO2), nitrous oxide (NO2), ozone (O3), ammonia (NH3), particulate matter (PM), polycyclic aromatic hydrocarbon (PAH) and volatile organic compounds (VOCs) among others are considered very harmful to health. Thus, for example, the Gothenburg Protocol sets emission ceilings for SO2, NO2, VOCs and NH3. Therefore, we can find international, European, national and regional directives, laws, and regulations in order to control their emissions. Some examples are the well-known Paris Agreement or the Kyoto Protocol for the global reduction of GHGs or the Gothenburg Protocol to abate the acidification, eutrophication and ground-level ozone. Furthermore, we can observe that there are gases that contribute to both the greenhouse effect and the air pollution. For details on these and other topics related to the protection of the environment and health visit the webpage https://greenfacts.org.

The entire system for the abatement of pollutants could be represented in a hierarchical structure of two levels (see Fig. 1). In the first level, we would have the effect of pollutants, and in the second level the pollutants themselves. The ultimate goal of that authority is for emissions per year of the different pollutants to be below certain levels (for example, emitted tons per year) in order to better control the pollution and their effects. In this sense, the authority fixes certain levels of emissions (total tons per year) for each effect of pollutants. However, pollutants could contribute to more than one effect as we have shown above. Thus, we consider the particular situation in which there are different amounts of emissions of different pollutants and the authority fixes maximum levels of emissions (total tons per year) for each effect of pollutants according, for example, to their effects on air quality or contribution to climate change, in order to abate these emissions and keep them below certain levels (tons per year). The approach of setting a level of emissions per year is the usual one in the directives and protocols in this regard, so the particular impact of a pollutant in each effect in the atmosphere is not considered, it is simply a matter of reducing its emissions and with it, its negative impact on air quality or the greenhouse effect. Moreover, in this context, if we set global emission levels for greenhouse gases and for air polluting gases separately, it seems reasonable that when distributing efforts to abate the different pollutants, more emphasis should be placed on those that are being emitted the most, for example, this is what happens with CO2 or NO2, and therefore, those that are being emitted least are less affected. Therefore, if we allocate the emission quotas among the different pollutants, this allocation would have to be as egalitarian as possible to the quantity claimed by the atmospheric pollutant that pollutes the least to keep controlled the pollution levels. This is

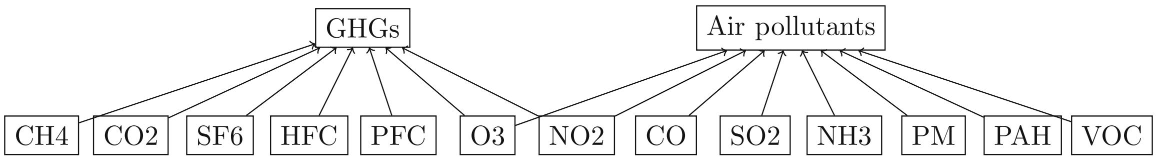

Fig. 1 Example of a two level hierarchical structure for the abatement of GHGs and air pollutants

reminiscent of the constrained equal award rule (CEA) in bankruptcy problems. Therefore, we have a kind of multi-issue bankruptcy problem which is different from other multi-issue bankruptcy problems as the multi-issue problems as we will explain below. Of course, if we were only interested in a particular effect of pollutants, we obtain a classical bankruptcy problem, where the estate is the amount fixed for the effect (total tons per year) and the claims are the current emission levels of pollutants (tons per year).

This problem remembers a multi-issue bankruptcy problem in which the issues are the effects of pollutants. Multi-issue bankruptcy problems introduced by Calleja et al. (2005) describe situations in which there are a perfect divisible estate which can be divided between various issues, and a number of claimants that have claims on each of those issues.[1] Therefore, there are a perfect divisible estate, several issues, and claimants with vectors of claims with as many coordinates as issues, such that the total amount of claims is above the estate. The central question for these problems is how the estate should be allocated. This problem is solved by means of allocation rules and there are several approaches to it (see, for example, Calleja et al. 2005; Borm et al. 2005; Izquierdo and Timoner 2016). However, we have ex ante one estate for each effect on atmosphere, and the claimants are the pollutants, and each pollutant has just one claim which is the same for each of the estates of the effects which it contributes to. This approach is different from the multi-issue bankruptcy problems in the literature.

In our case, we have several perfectly divisible estates, and claimants have exactly one claim which is used in all estates simultaneously. Now, again, the question is how the estate should be allocated in a reasonable way. As far as we know this approach is totally novel in the literature and then fills a gap in the bankruptcy problems that have been dealt until now. Moreover, we study a generalization of the constrained equal awards (CEA) rule (Maimonides, twelfth century) to this context, and provide an axiomatic analysis of it.

The CEA rule has been studied in multi-issue bankruptcy problems in different settings. Lorenzo-Freire et al. (2010) introduce the two-stage constrained equal awards rule. In the first stage, the estate is distributed among the issues taking into account the total claims for each issue and applying the CEA rule. In the second stage, the part of the estate assigned to each issue is distributed among the claimants by applying again the CEA rule. This rule is characterized by some known axioms, in this context, and a kind of consistency property that connects the first and second stage, among others. Bergantiños et al. (2011) provide three new characterizations of the two-stage CEA rule for multi-issue bankruptcy problems by adapting known properties for bankruptcy problems to the multi-issue case. Bergantiños et al. (2018) introduce and characterize a two-stage allocation rule for multi-issue bankruptcy problems by combining the CEA rule and the proportional rule. Izquierdo and Timoner (2016) propose the CEA rule for constrained multi-issue bankruptcy problems by using the so-called multi-issue reference systems and a quadratic optimization problem. They characterize their CEA rule by using consistency and Lorez-domination properties. In this paper, we introduce the CEA

[1] "An issue constitutes a reason on the basis of which the estate is to be divided". (Calleja et al. 2005, page 731.)

rule for multi-issue bankruptcy problems with crossed claims by using two different iterative procedures. One of them is based on linear programming and the other is related to the CEA rule for one issue bankrutcy problems. We characterize our CEA rule by means of Pareto efficiency, conditional equal division and consistency, showing that is uniquely determined by conditional equal division and consistency. Finally, we provide another characterization following the obtained in Yeh (2006).

The rest of the paper is organized as follows. In Sect. 2, multi-issue bankruptcy problems with crossed claims (MBC) are introduced and the concept of rule in this context is formally established. In Sect. 3, the constrained equal awards rule for multi-issue bankruptcy problems with crossed claims is defined as the optimal solution of a succession of linear programs and it is shown that satisfies the concept of Lorenz week dominance. In Sect. 4, an analysis of the behavior of the CEA rule in this context, according to principles like equity, efficiency monotonicity or consistency is carried out. In fact, the study of all these properties is determinant to make a choice of a reasonable rule to different situations. As a result of the properties satisfied by this rule, it can be deduced that it is a good choice when looking for allocations equitables or consistent, in particular, in the situation illustrated of abatement of emissions of different pollutants into the atmosphere. Additionally, in Sect. 5, we get a better insight of the CEA rule for multi-issue bankruptcy problems with crossed claims characterizing it by means of the combination of three of these principles, namely, Pareto efficiency, conditional equal division and consistency. Additionally, we obtain another characterization of CEA in this setting with similar properties to the characterization of CEA given in Yeh (2006). Section 6 concludes.

2 Multi-issue bankruptcy problems with crossed claims

We consider a situation where there are a finite set of issues (effects of pollutants) $M = \{1, 2, \ldots, m\}$ such that each issue j has a perfectly divisible estate e_j (maximum level of emissions for that effect of pollutans). Let $E = (e_1, e_2, \ldots, e_m)$ be the vector of estates. There are a finite set of claimants (pollutants) $N = \{1, 2, \ldots, n\}$ such that each claimant i claims c_i (emissions of pollutant i) of those estates which belongs to. Let $c = (c_1, c_2, \ldots, c_n)$ be the vector of claims. Now, each claimant claims to different set of issues. Thus, α is a set-valued function that associates with every $i \in N$ a subset $\alpha(i) \subset M$. In fact, $\alpha(i)$ represents the issues to which claimant i asks for. Furthermore, $\sum_{i:j\in\alpha(i)} c_i > e_j$, for all $j \in M$, otherwise, those estates could be discarded from the problem because they do not impose any limitation, and so the allocation would be trivial. Therefore, a *multi-issue bankruptcy problem with crossed claims* (MBC in short) is defined by a 5-tuple (M, N, E, c, α), and the family of all these problems is denoted by $\mathcal{MBC}$.

We illustrate the structure of these problems in the following example.

Example 1 Consider the following multi-issue bankruptcy problem with crossed claims $MBC = (M, N, E, c, \alpha)$ with $M = \{1, 2, 3\}$; $N = \{1, 2, 3, 4, 5, 6, 7, 8\}$; $E = (40, 60, 70)$; $c = (20, 30, 20, 40, 30, 8, 50, 40)$; and $\alpha(1) = \{1\}$, $\alpha(2) = \{1, 2\}$, $\alpha(3) = \{1\}$, $\alpha(4) = \{2\}$, $\alpha(5) = \{1, 2\}$, $\alpha(6) = \{2\}$, $\alpha(7) = \{2, 3\}$, and $\alpha(8) = \{3\}$. This situation is depicted in Fig. 2.

At first sight, a simplistic approach could be to solve three bankruptcy problems, one for each issue, but this is not so simple, because there are claimants with claims to different issues and this could lead to unfeasible or incompatible allocations. Therefore, a more detailed analysis of this type of problem is necessary.

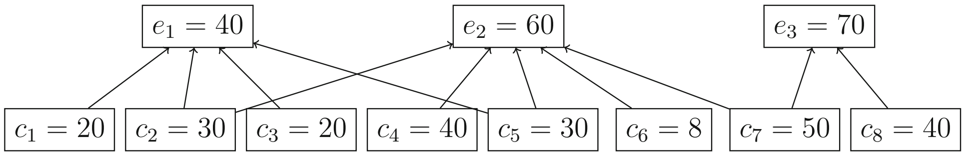

Fig. 2 Example 1

In Example 1, we can clearly observe the structure of our model of multi-issue bankruptcy problem. This model differs from other multi-issue models in three elements. First, there are several issues, each one with their own endowments (this feature would be similar to the approach in Izquierdo and Timoner (2016) to multi-issues bankruptcy problems), i.e., in our model the estate, E, is a vector while in other multi-issue models the estate E is a single number. Second, we do not have a vector of claims for each issue as in all approches to multi-issues bankruptcy problems have, but a single vector of claims for all issues simultaneously. And finally, this vector of claims can result in the same claim to be considered in several issues at the same time.

On the other hand, when there is a single issue in multi-issue bankruptcy problems with crossed claims correspond to the well-known bankruptcy problems ($\mathcal{B}$) (O'Neill 1982; Aumann and Maschler 1985). In this case, we have a triplet (N, E, c) instead of 5-tuple (M, N, E, c, α), because we only have a single issue and M and α become irrelevant for the analysis of this problem. Therefore, multi-issue bankruptcy problems are a possible extension of bankruptcy problems.

Given a problem $(M, N, E, c, \alpha) \in \mathcal{MBC}$, a *feasible allocation* for it, it is a vector $x \in \mathbb{R}^N$ such that:

1. $0 \leq x_i \leq c_i$, for all $i \in N$.
2. $\sum_{i \in N : j \in \alpha(i)} x_i \leq e_j$, for all $j \in M$,

and we denote by $A(M, N, E, c, \alpha)$ the set of all its feasible allocations.

Requirement 1 means that each agent receives at most her claim but not less than nothing. Requirement 2 means that no estates can be overpassed with the allocation. Therefore, a feasible allocation $x \in \mathbb{R}^N$ represents an allocation to the claimants which is simultaneously feasible for all issues.

A *rule* for multi-issue bankruptcy problems with crossed claims is a mapping R that associates with every $(M, N, E, c, \alpha) \in \mathcal{MBC}$ a unique feasible allocation $R(M, N, E, c, \alpha) \in A(M, N, E, c, \alpha)$.

Example 2 Consider again the MBC problem in Example 1. Two possible allocations are the following:

- $R(M, N, E, c, \alpha) = (12.5, 7.5, 12.5, 7.5, 7.5, 7.5, 30, 40)$.
- $R(M, N, E, c, \alpha) = (13.75, 6.25, 13.75, 6.25, 6.25, 6.25, 35, 35)$.

Both allocations given in Example 2 satisfy the two requirements and, additionally, it is easy to check that they are efficient for all issues, i.e., Requirement 2 is satisfied with equality. However, this is not possible, in general, as the following example shows.

Remark 1 Consider again the MBC in Example 1 but only changing $c_7 = 65$ and $e_3 = 105$. In this situation, it is obvious that if an allocation is efficient for Issue 3, then it is unfeasible for Issue 2.

Therefore, in view of Remark 1, we cannot make Requirement 2 more demanding if we want to achieve at least one possible allocation.

3 The constrained equal awards rule for MBC problems

The constrained equal awards rule (CEA) is one of the main rules to solve bankruptcy problems (see Herrero and Villar 2001). This rule simply divides as equally as possible the estate among the claimants. The question here is what as equally as possible means. In the context of one-issue bankruptcy problems, as equitably as possible means that no claimant can get more than those with smaller claims, except that the latter have already received their entire claim. This can be formulated mathematically as follows:

For each $(N, E, c) \in \mathcal{B}$,

$$CEA_i(N, E, c) = \min\{c_i, \beta\}, \qquad i \in N, \tag{1}$$

where β is a positive real number satisfying $\sum_{i \in N} CEA_i(N, E, c) = E$.

How to extrapolate this to the MBC situations. To do this, in this paper, we introduce the CEA rule as the result of the optimal solution of a succession of linear programs.[2]

Given a problem $(N, E, c) \in \mathcal{B}$, in order to allocate E among the claimants according to CEA, we proceed as follows:

$$(P_1) \quad \begin{array}{l} max\ z^1 \\ s.a:\ \sum_{i \in N} x_i \le E \\ \qquad x_i \le c_i,\ \text{for all } i \in N \\ \qquad x_i \ge z^1,\ \text{for all } i \in N \\ \qquad x_i \ge 0,\ \text{for all } i \in N,\ \text{and } z^1 \ge 0 \end{array}$$

Let z^{*1} be the optimal value of the linear problem (P_1). If $nz^{*1} = E$, then $CEA_i(N, E, c) = z^{*1}$. Otherwise, the following linear problem must be solved:

$$(P_2) \quad \begin{array}{l} max\ z^2 \\ s.a:\ \sum_{i \in N} x_i \le E \\ \qquad x_i \le c_i,\ \text{for all } i \in N \\ \qquad x_i \ge z^{*1} + \mu^0(c_i - z^{*1})z^2,\ \text{for all } i \in N \\ \qquad x_i \ge 0,\ \text{for all } i \in N,\ \text{and } z^2 \ge 0 \end{array}$$

where for each $a \in \mathbb{R}$,

$$\mu^0(a) = \begin{cases} 0 & if\, a \le 0, \\ 1 & otherwise. \end{cases}$$

Let z^{*2} be the optimal value of the linear problem (P_2). If $nz^{*1} + \sum_{i \in N} \mu^0(c_i - z^{*1})z^{*2} = E$, then $CEA_i(N, E, c) = z^{*1} + \mu^0(c_i - z^{*1})z^{*2}$. Otherwise, a new linear problem must be solved. In the general step k, we have the following linear problem:

$$(P_k) \quad \begin{array}{l} max\ z^k \\ s.a:\ \sum_{i \in N} x_i \le E \\ \qquad x_i \le c_i,\ \text{for all } i \in N \\ \qquad x_i \ge \sum_{h=1}^{k-1} \mu^0(c_i - \sum_{l=1}^{h-1} z^{*l})z^{*h} + \mu^0(c_i - \sum_{l=1}^{k-1} z^{*l})z^k,\ \text{for all } i \in N \\ \qquad x_i \ge 0,\ \text{for all } i \in N,\ \text{and } z^k \ge 0 \end{array}$$

Again, let z^{*k} be the optimal value of the linear problem (P_k). If $\sum_{i \in N} \sum_{h=1}^{k} \mu^0(c_i - \sum_{l=1}^{h-1} z^{*l})z^{*h} = E$, then $CEA_i(N, E, c) = \sum_{h=1}^{k} \mu^0(c_i - \sum_{l=1}^{h-1} z^{*l})z^{*h}$. Otherwise, the

[2] We can find in Izquierdo and Timoner (2016) another way to obtain CEA by using a quadratic optimization problem.

linear problem (P_{k+1}) must be solved. And so on and so forth, until the estate is fully distributed or all claims are completely granted. It is obvious that this procedure ends in a finite number of steps and the final allocation is CEA.

Note that when the last linear problem of the procedure is solved, then we have that its optimal solution is $x^{*k} = CEA(N, E, c)$. We illustrate this procedure in the following example.

Example 3 Consider $(N, E, c) \in \mathcal{B}$ with $N = \{1, 2, 3, 4, 5, 6, 7, 8\}$; $E = 170$; $c = (20, 30, 20, 40, 30, 8, 50, 40)$. We now apply the procedure described above to calculate the CEA rule of this problem.

1. First we solve (P_1). The optimal value of this linear problem is $z^{*1} = 8$. Therefore, all claimants receive 8 units of the estate. In total 64 units of the estate have been distributed, therefore another round is necessary. Since claimant 6 has obtained her claim, this will not take part in the distribution of the estate in the next step.
2. In this step,we first guarantee claimants all what they have obtained until the previous step. The optimal value of (P_2) is $z^{*2} = 12$. Thus, all claimants except claimant 6 are allocated 12 extra units of the estate. In total 148 units of the estate have been distributed, therefore another round is necessary. Since claimants 1 and 3 have already obtained their claims, these will not take part in the distribution of the estate in the next step.
3. Again, we first guarantee claimants all what they have obtained until the previous step. The optimal value of (P_3) is $z^{*3} = 4.4$. Thus, all claimants except claimants 1, 3 and 6 are allocated 4.4 extra units of the estate. Since the estate has been fully distributed, the procedure ends and $CEA(N, E, c) = (20, 24.4, 20, 24.4, 24.4, 8, 24.4, 24.4)$.

Note that this procedure perfectly fits to the following description of CEA in Thomson (2015):

> At first, equal division takes place until each claimant receives an amount equal to the smallest claim. The smallest claimant drops out, and the next increments of the endowment are divided equally among the others until each of them receives an amount equal to the second smallest claim. The second smallest claimant drops out, and so on.

Therefore, following this same procedure we can define the CEA rule for MBC problems. Given a problem $(M, N, E, c, \alpha) \in \mathcal{MBC}$, in the general step k of the procedure, the linear problem to be solved is given by:

$$
(P_k) \qquad
\begin{aligned}
& max\ z^k \\
& s.a: \textstyle\sum_{i \in N: j \in \alpha(i)} x_i \leq e_j, \ \text{for all } j \in M \\
& \qquad x_i \leq c_i, \ \text{for all } i \in N \\
& \qquad x_i \geq \textstyle\sum_{h=1}^{k-1} \mu^0(a_i^h) z^{*h} + \mu^0(a_i^k) z^k, \ \text{for all } i \in N \\
& \qquad x_i \geq 0, \ \text{for all } i \in N, \ \text{and } z^k \geq 0
\end{aligned}
$$

where

$$
a_i^h = \min \left\{ c_i - \sum_{l=1}^{h-1} \mu^0(a_i^l) z^{*l}, min_{j \in \alpha(i)} \left\{ e_j - \sum_{t \in N: j \in \alpha(t)} \sum_{s=1}^{h-1} \mu^0(a_t^s) z^{*s} \right\} \right\}.
$$

Note that a_i^h measures whether claimant i can take part in the distribution of step h taking into account what she received before and whether there is still something to distribute in every issue she claims.

In this case, the procedure also ends in a finite number of steps, but not necessarily when all estates are fully distributed or all claims are completely granted. In this situation the procedure stops when $z^{*k} = 0$, and it holds that

$$CEA_i(M, N, E, c, \alpha) = \sum_{h=1}^{k} \mu^0(a_i^h) z^{*h}.$$

Likewise, we also have that for the last linear problem solved its optimal solution x^{*k} is exactly $CEA(M, N, E, c, \alpha)$.

It is important to emphasize that we have introduced CEA rule for bankruptcy problems as the solution to a sucession of linear programming problems and we have extended this procedure to MBC. So, it is easy to check that when we have exactly only one issue both coincide. The only difference is that we additionally take into account how much remains in each of the estates to which claimants ask for. However, this procedure for MBC problems does not guarantee that all estates are fully distributed, even when they could be. The next example illustrates this.

Example 4 Consider again the MBC problem in Example 1. We now calculate the CEA rule of this problem by applying the procedure described above.

1. The optimal value of (P_1) is $z^{*1} = 8$ and this amount is allocated to each claimant. None of the estates have been fully distributed, so a next step is necessary. However, claimant 6 has obtained her claim, so this will not take part in the distribution of the estates in the next step.
2. In this step, we guarantee claimants all what they have obtained until the previous step. The optimal value of (P_2) is $z^{*2} = 2$, and this amount is allocated to all claimants except claimant 6. Now, the estate e_1 has been fully distributed, therefore, claimants requesting part of this estate cannot continue to receive anything else. Otherwise, the quantity available in that estate would be exceeded. However, the other two estates have not been fully distributed, hence another step is needed.
3. In this step, we guarantee claimants all what they have obtained until the previous step. The optimal value of (P_3) is $z^{*3} = 6$, and this amount is only allocated to claimants 4, 7, and 8. Now, the estate e_2 has been fully distributed, so claimants 4, and 7 cannot continue to receive anything else. Again, otherwise, the quantity available in that estate would be exceeded.
4. In this step, we again guarantee claimants all what they have obtained until the previous step, and claimant 8 is the only one that can receive something else in this step. The optimal value of (P_4) is $z^{*4} = 24$, and this amount is only allocated to claimant 8. However, the estate e_3 has not been fully distributed, in fact, there are still 14 units to be distributed. Furthermore $x^{*4} = (10, 10, 10, 16, 10, 8, 16, 40)$ that coincides with $CEA(M, N, E, c, \alpha)$.

$CEA(M, N, E, c, \alpha)$ is calculated in four steps, and the allocations in each step are the following:

$Claimant$	$Step-1$	$Step-2$	$Step-3$	$Step-4$	$Row-total$
1	8	2	–	–	10
2	8	2	–	–	10
3	8	2	–	–	10
4	8	2	6	–	16
5	8	2	–	–	10
6	8	–	–	–	8
7	8	2	6	–	16
8	8	2	6	24	40
$Column-total$	64	14	18	24	120

This allocation does not fully distribute all estates, but it is possible to obtain feasible allocations for this particular MBC problem which do. For example, the allocations given in Example 2 fully distribute all estates for this problem, but they are not as egalitarian as this. Furthermore, although the claimants have received 120 units in total, of the 170 units that make up the three estates, 156 have been actually distributed. This difference is because some claimants asked for in several issues simultaneously.

If we look carefully at the application of the procedure in Example 4, we observe that first a estate is fully distributed, then another estate is completely distributed, and finally the last estate cannot be distributed in its entirety. Therefore, we can design another procedure based on the CEA rule itself which follows this scheme. We illustrate it in the following example.

Example 5 Consider once again the MBC problem in Example 1. In order to calculate $CEA(M, N, E, c, \alpha)$, we proceed as follows:

1. First we calculate the CEA rule for each of the three bankruptcy problems defined by each issue.
   - $(N^{1,1}, E^{1,1}, c^{1,1})$. $N^{1,1} = \{1, 2, 3, 5\}$, $E^{1,1} = 40$, and $c^{1,1} = (20, 30, 20, 30)$. $CEA(N^{1,1}, E^{1,1}, c^{1,1}) = (10, 10, 10, 10)$, and $\beta^{1,1} = 10$.
   - $(N^{2,1}, E^{2,1}, c^{2,1})$. $N^{2,1} = \{2, 4, 5, 6, 7\}$, $E^{2,1} = 60$, and $c^{2,1} = (30, 40, 30, 8, 50)$. $CEA(N^{2,1}, E^{2,1}, c^{2,1}) = (13, 13, 13, 8, 13)$, and $\beta^{2,1} = 13$.
   - $(N^{3,1}, E^{3,1}, c^{3,1})$. $N^{3,1} = \{7, 8\}$, $E^{3,1} = 70$, and $c^{3,1} = (50, 40)$. $CEA(N^{3,1}, E^{3,1}, c^{3,1}) = (35, 35)$, and $\beta^{3,1} = 35$.
2. Next, we take $\beta^{*1} = \min\{\beta^{1,1}, \beta^{2,1}, \beta^{3,1}\} = 10$, and we allocate each claimant i $\min\{c_i, \beta^{*1}\}$. Therefore, we obtain the allocation vector (10, 10, 10, 10, 10, 8, 10, 10).

Now, it is obvious that estate e_1 has been fully distributed, and in the next step this bankruptcy problem and the claimants associated with it are excluded. Moreover, claimant 6 is also excluded because she has got her claim. The other two problems are updated in claimants, and decreasing estates and claims according to the allocation previously obtained.

1. We calculate the CEA rule for each of the two bankruptcy problems remaining.
   - $(N^{2,2}, E^{2,2}, c^{2,2})$. $N^{2,2} = \{4, 7\}$, $E^{2,2} = 12$, and $c^{2,2} = (30, 40)$. $CEA(N^{2,2}, E^{2,2}, c^{2,2}) = (6, 6)$, and $\beta^{2,2} = 6$.
   - $(N^{3,2}, E^{3,2}, c^{3,2})$. $N^{3,2} = \{7, 8\}$, $E^{3,2} = 50$, and $c^{3,2} = (40, 30)$. $CEA(N^{3,2}, E^{3,2}, c^{3,2}) = (25, 25)$, and $\beta^{3,2} = 25$.

2. Next, we take $\beta^{*2} = \min\{\beta^{2,2}, \beta^{3,2}\} = 6$, and we allocate each claimant i $\min\{c_i, \beta^{*2}\}$. Therefore, we obtain the allocation vector $(0, 0, 0, 6, 0, 0, 6, 6)$.

Now, it is obvious that estate e_2 has been fully distributed, and in the next step this bankruptcy problem and the claimants associated with it are excluded. The third problem is updated in claimants, and decreasing estates and claims according to the allocation previously obtained.

1. We calculate the CEA rule for each of the bankruptcy problem remaining.
   - $(N^{3,3}, E^{3,3}, c^{3,3})$. $N^{3,3} = \{8\}$, $E^{3,3} = 38$, and $c^{3,3} = (24)$. $CEA(N^{3,3}, E^{3,3}, c^{3,3}) = (24)$, and $\beta^{3,3} = 24$.
2. Next, we take $\beta^{*3} = \min\{\beta^{3,3}\} = 24$, and we allocate each claimant i $\min\{c_i, \beta^{*3}\}$. Therefore, we obtain the allocation vector $(0, 0, 0, 0, 0, 0, 0, 24)$.

The procedure stops because either estates have been completely distributed or claimants have obtained their claims. Finally, by adding the allocation vectors obtained in the procedure, we obtain that $CEA(M, N, E, c, \alpha) = (10, 10, 10, 16, 10, 8, 16, 40)$.

The procedure based on the CEA rule is as follows. In general step k, we calculate the CEA rule for all bankruptcy problems defined by the available estates in the step k, for each we determine the values $\beta^{j,k}$, and take the minimum β^{*k} of all of them. We allocate β^{*k} to all active claimants in step k and nothing to the others. Next, we update the problem by revising downwards estates, claimants, and claims. After updating the problem, if no bankruptcy problem can be defined, we stop. Otherwise we go to the next step with the updated problem, and so on. Finally, the CEA rule of the MBC is the sum of all allocation vectors obtained.

The following theorem states that both procedures introduced coincide for all MBC problems.

Theorem 1 *Given a problem $MBC = (M, N, E, c, \alpha) \in \mathcal{MBC}$, the allocation vectors obtained by the procedure based on linear programming and the procedure based on the CEA rule of classical bankrupcty problems coincide, and their outcome corresponds to the rule $CEA(M, N, E, c, \alpha)$.*

Proof In order to prove this result, it suffices to demonstrate that there is a step in the procedure based on linear programming that coincides with the first step of the procedure based on the one-issue CEA rule, because after that step we can repeat the same reasoning again.

Given a problem $MBC = (M, N, E, c, \alpha)$, we consider, without loss of generality, that $c_1 \leq c_2 \leq c_3 \leq \ldots \leq c_n$. Let β^{*1} be the first value in the procedure based on the standard CEA rule. We distinguish three cases:

1. $\beta^{*1} \leq c_1$. In this case, $x_i = \beta^{*1}$, for all $i \in N$ and $z^1 = \beta^{*1}$ is a feasible solution of linear program (P_1). Moreover, it is optimal because if we increase z^1, some estate would be exceeded by the definition of β^{*1}.
2. $c_h \leq \beta^{*1} < c_{h+1}$. In this situation, $x_i = \min\{c_i, \beta^{*1}\}$, for all $i \in N$ and $z = \beta^{*1}$ is a feasible solution of some linear program (P_k), because β^{*1} determines the first estate which will be fully distributed. Indeed, after a number of steps of the procedure based on linear programming a first estate must have been distributed. Let k be that step, then the solution given above is a feasible one for linear problem (P_k) and also optimal by using the same reasoning as in the previous case.
3. $c_n \leq \beta^{*1}$. In this case, there is no problem and all claims are granted in both procedures.

For the remaining steps of the procedure based on the one-issue CEA rule the reasoning is the same as above by only taking into account that in the procedure based on linear programming we guarantee claimants all what they have obtained until the previous step, and in the procedure based on the one-issue CEA rule we update the problem. Thus, for obtaining feasible solutions for the linear programs from the procedure based on CEA, we just have to accumulate the allocations. □

An interesting property of CEA for bankruptcy problems is that it is the most egalitarian allocation in the sense of Lorenz. This result can be extended to the context of MBC problems. To do this, we first introduce the well-known concept of Lorenz dominance adapted to the context of multi-issue bankruptcy problems with crossed claims.

Given a problem $MBC = (M, N, E, c, \alpha) \in \mathcal{MBC}$, and two feasible vectors $x, y \in \mathbb{R}^N_+$, we say that x *Lorenz weakly dominates* y, $x \succeq_{wL} y$, if $\sum_{j=1}^{k} x_{(j)} \geq \sum_{j=1}^{k} y_{(j)}$, for all $k = 1, 2, \ldots, h$, $h \leq n$, where for a vector $z \in \mathbb{R}^N_+$, $z_{(1)}, \ldots, z_{(n)}$ represent its coordinates rewritten in increasing order.

Theorem 2 *Given a problem $MBC = (M, N, E, c, \alpha) \in \mathcal{MBC}$, the rule CEA$(M, N, E, c, \alpha)$ Lorenz weakly dominates all feasible allocations.*

Proof The proof follows from the procedure based on linear programing to calculate $CEA(M, N, E, c, \alpha)$. Indeed, since the minimum amount that all active claimants have to receive is maximized in each step, and the estates are successively exhausted during the application of the procedure, no other allocation can improve the equality of the distribution. If so, in some of the steps the solution would not be optimal, and this is a contradiction. □

4 Properties

In this section, we present several properties which are interesting in the context of MBC problems. These properties are related to efficiency, fairness, consistency or monotonicity, among others.

First, we define when two claimants are considered equal. In our context, claimants are characterized by two features: their claims and the issues to which they claim. Therefore, both should be taken into account in the definition of equal agents. Given a problem $MBC = (M, N, E, c, \alpha) \in \mathcal{MBC}$, and two claimants $i, j \in N$, we say they are *equal*, if $c_i = c_j$ and $\alpha(i) = \alpha(j)$.

Next, we give a set of properties which are very natural and reasonable for an allocation rule.

Property (EFF) *Given a rule R, it satisfies efficiency, if for every problem $MBC = (M, N, E, c, \alpha) \in \mathcal{MBC}$, $\sum_{i:j\in\alpha(i)} R_i(M, N, E, c, \alpha) = e_j$, for all $j \in M$.*

EFF simply says that the estates must be fully distributed, we know that this property is very demanding in the context of multi-issue bankruptcy problems with crossed claims as Remark 1 shows. However, a weaker version of efficiency can be defined by considering Pareto efficiency. A feasible allocation is Pareto efficient if there is no other feasible allocation in which some individual is better off and no individual is worse off. Formally, given a problem $MBC = (M, N, E, c, \alpha) \in \mathcal{MBC}$, an allocation $x \in A(M, N, E, c, \alpha)$ is *Pareto efficient* if there is no other allocation $x' \in A(M, N, E, c, \alpha)$ such that $x'_i \geq x_i, \forall i \in N$, with at least one strict inequality. Now Pareto efficiency is defined as follows:

Property (PEFF) *Given a rule R, it satisfies Pareto efficiency, if for every problem* $MBC = (M, N, E, c, \alpha) \in \mathcal{MBC}$, $R(M, N, E, c, \alpha)$ *is Pareto efficient.*

Note that $PEFF$ implies that at least one estate is fully distributed, but not EFF. However, EFF implies $PEFF$. Therefore, $PEFF$ is a weaker property than EFF.

Property (ETE) *Given a rule R, it satisfies equal treatment of equals, if for every problem* $MBC = (M, N, E, c, \alpha) \in \mathcal{MBC}$ *and every pair of equal claimants* $i, j \in N$, $R_i(M, N, E, c, \alpha) = R_j(M, N, E, c, \alpha)$.

ETE is related to impartiality and says that claimants with the same claims and the same set of issues must be treated equally in the final allocation.

Property *(CTI) Given a rule R, it satisfies claims truncation invariance, if for every problem* $MBC = (M, N, E, c, \alpha) \in \mathcal{MBC}$, *when considering the problem* $MBC' = (M, N, E, c', \alpha) \in \mathcal{MBC}$ *such that* $c'_i = \min\left\{c_i, \min\{e_j | j \in \alpha(i)\}\right\}$, *for all* $i \in N$; *then* $R(M, N, E, c, \alpha) = R(M, N, E, c', \alpha)$.

CTI says that if the claims are truncated by the estates, then the final allocation does not change. This property appears in Curiel et al. (1987) and it is used to characterize the so-called game theoretical rules for one-issue bankruptcy problems. Dagan and Volij (1993) were the first to propose this property as an axiom.

Property (RMR) *Given a rule R, it satisfies respect of minimal rights, if for every problem* $MBC = (M, N, E, c, \alpha) \in \mathcal{MBC}$, *for all* $i \in N$,

$$R_i(M, N, E, c, \alpha) \geq \min_{j : j \in \alpha(i)} \max\{0, e_j - \sum_{k \in N \setminus \{i\} : j \in \alpha(k)} c_k\}.$$

Property (CED) *Given a rule R, it satisfies conditional equal division, if for every problem* $MBC = (M, N, E, c, \alpha) \in \mathcal{MBC}$, *for all* $i \in N$,

$$R_i(M, N, E, c, \alpha) \geq \min_{j : j \in \alpha(i)} \min\left\{c_i, \frac{e_j}{|k : j \in \alpha(k)|}\right\}.$$

Property (SEC) *Given a rule R, it satisfies securement, if for every problem* $MBC = (M, N, E, c, \alpha) \in \mathcal{MBC}$, *for all* $i \in N$,

$$R_i(M, N, E, c, \alpha) \geq \min_{j : j \in \alpha(i)} \min\left\{\frac{c_i}{|k : j \in \alpha(k)|}, \frac{e_j}{|k : j \in \alpha(k)|}\right\}.$$

RMR, CED and SEC are related to the minimum amount that should reasonably be guaranteed to each claimant. The concept of minimal right was introduced by Tijs (1981) in the context of cooperative games to define the τ-value. Thus, RMR says that a claimant should receive at least what is left when all the other claimants are completely satisfied in their claims. CED was introduced by Moulin (2000) for rationing problems. In our context, this property means that an agent should obtain her claim if this is less than any egalitarian distribution of the estates of the issues she claims, and in other case, at least the minimal egalitarian distribution of the estates of all issues she claims. Finally, SEC was introduced for bankruptcy problems by Moreno-Ternero and Villar (2004). They use this property along with other properties to characterize the Talmud rule (Aumann and Maschler 1985). In the environment of MBC problems, this property means that a rule should guarantee to agents at least the minimal egalitarian distribution of the estates of all issues they claims when they are feasible, and the minimal egalitarian distribution of the estates of all issues they claims otherwise.

Property (CFC) *Given a rule R, it satisfies conditional full compensation, if for every problem* $MBC = (M, N, E, c, \alpha) \in \mathcal{MBC}$ *and each* $i \in N$, *such that* $\sum_{k:j\in\alpha(k)} \min\{c_k, c_i\} \leq e_j$, *for all* $j \in M$, *then* $R_i(M, N, E, c, \alpha) = c_i$.

CFC means that if the claim of a claimant is so small that if all claimants with higher claims asked for the same amount as her, all claims would be fully honored, then it seems reasonable that said claimant receives her claim. This property was introduced by Herrero and Villar (2002) and used to characterize the CEA rule.

Property (CM) *Given a rule R, it satisfies claim monotonicity, if for every pair of problems* $MBC = (M, N, E, c, \alpha) \in \mathcal{MBC}$ *and* $MBC' = (M, N, E, c', \alpha) \in \mathcal{MBC}$, *such that* $c_i \geq c'_i$ *and* $c_j = c'_j$, *for all* $j \in N\backslash\{i\}$, *then* $R_i(M, N, E, c, \alpha) \geq R_i(M, N, E, c', \alpha)$.

CM means that if the claim of a claimant increasing she cannot receive less than she received in the previous situation. In Kasajima and Thomson (2011) monotonicity properties are studied in the context of the adjudication of conflicting claims.

Before introducing the last property we need to introduce the concept of reduced problem. Given a problem $MBC = (M, N, E, c, \alpha) \in \mathcal{MBC}$, and $N' \subset N$, the reduced problem associated with N', $MBC^{N'} = (M', N', E'^R, c|_{N'}, \alpha) \in \mathcal{MBC}$, where $M' = \{j \in M :$ there exists $i \in N'$ such that $j \in \alpha(i)\}$, $E'^R = \left(e'^R_j\right)_{j\in M'}$ with $e'^R_j = e_j - \sum_{i\in N\backslash N':j\in\alpha(i)} R_i(M, N, E, c, \alpha)$, for all $j \in M'$, and $c|_{N'}$ is the vector whose coordinates correspond to the claimants in N'.

Property (CONS) *Given a rule R, it satisfies consistency, if for every problem* $MBC = (M, N, E, c, \alpha) \in \mathcal{MBC}$, *and* $N' \subset N$, *it holds that*

$$R_i(M, N, E, c, \alpha) = R_i(M', N', E'^R, c|_{N'}, \alpha), \text{ for all } i \in N'.$$

$CONS$ means that if a subset of claimants leave the problem respecting what had been assigned to those who remain, then what those players get in the new reduced problem is the same as what they got in the whole problem. Consistency properties have been used to characterize many bankruptcy rules, because they represents a requirement of robustness when some agents leave the problem with their allocations (see Thomson (2011, 2018) for surveys about the application of consistency properties and their principles behind.)

The CEA rule for MBC problems satisfies all properties above mentioned but efficiency as Example 4 shows. We establish this in the following theorem.

Theorem 3 *The CEA rule for multi-issue bankruptcy problems with crossed claims satisfies PEFF, ETE, CTI, RMR, CED, SEC, CFC, CM, and CONS.*

Proof We prove the result property by property.

- CEA satisfies $PEFF$ by definition.
- ETE. If two claimants are symmetric, then CEA allocates both the same, since the procedure to calculate the rule treats, in each step, all active claimants egalitarianly, so if two claimants are symmetric, they stop receiving at the same step.
- CTI. Since the claims are used as upperbounds in the procedure based on linear programming and the estates cannot be exceeded, the CEA rule satisfies CTI because upperbounds are not relevant for solving the linear programs when they are above the estates.
- RMR. CEA satisfies this property by definition of the rule.

- CED. Given a problem $MBC = (M, N, E, c, \alpha) \in \mathcal{MBC}$, and $i \in N$. We distinguish three cases:
  1. $R_i(M, N, E, c, \alpha) = c_i \geq \min_{j:j\in\alpha(i)} \min\left\{c_i, \frac{e_j}{|k:j\in\alpha(k)|}\right\}$.
  2. $R_i(M, N, E, c, \alpha) < c_i$ and there exists $j \in \alpha(i)$ which is fully distributed. Consider that the first issue in $\alpha(i)$ whose associated estate was fully distributed was j^* in the k-th step of the procedure based on the standard CEA rule. Until step $(k-1)$-th claimant i has received the same as those active claimants in the step k-th and more than those inactive claimants in step $(k-1)$-th. Now in the step k-th, claimant i is going to receive exactly the same as the other active claimants in the issue j^*, and they can be less than the beginning, therefore, $R_i(M, N, E, c, \alpha) \geq \min\left\{c_i, \frac{e_j}{|k:j^*\in\alpha(k)|}\right\}$.
  3. $R_i(M, N, E, c, \alpha) < c_i$ and there does not exist $j \in \alpha(i)$ which is fully distributed. This case is not possible because if there does not exist $j \in \alpha(i)$ which is fully distributed, then claimant i gets her claim.

- SEC. CED implies SEC.
- CFC. This property follows from the structure of the linear programs in the procedure to calculate CEA.
- CM. This property also follows from the structure of the linear programs in the procedure to calculate CEA.
- CONS. Given $MBC = (M, N, E, c, \alpha) \in \mathcal{MBC}$ and $MBC^{N'} = (M', N', E'^{CEA}, c|_{N'}, \alpha) \in \mathcal{MBC}$ the reduced game associated with $N' \subset N$, let r be the number of steps needed to calculate CEA of the problem $MBC = (M, N, E, c, \alpha)$ by applying the procedure based on the standard CEA rule. We define the following two sequences of sets:

$$N = N_1 \supset N_2 \supset \cdots \supset N_r, \text{ and } N' = N'_1 \supset N'_2 \supset \cdots \supset N'_r,$$

where N_k is the subset of claimants of N who were active in step k, and, N'_k is the subset of claimants of N' who were active in step k. Obviously, $N'_k \subset N_k$, for all $k = 1, 2, \ldots, r$. Note that, in the application of the procedure based on the standard CEA rule, at least one estate is fully distributed in each step, and if at the end of the procedure some estates are not fully distributed, then if feasible the last active claimants will receive as much as possible until the limit of their claims.
Now, let f be the last step in which at least one estate was fully distributed, $f \leq r$. We distinguish three cases:

  1. $f < 1$. In this case, all claimants get their claims in the problem $MBC = (M, N, E, c, \alpha)$, and, obviously, claimants in N' also get their claims in $MBC^{N'} = (M', N', E'^{CEA}, c|_{N'}, \alpha)$. In fact, this case is not considered as a problem according to the definition of MBC problems.
  2. $1 \leq f < r$. Since the CEA rule for standard bankruptcy problems is consistent, and $N'_k \subset N_k$, for all $k = 1, 2, \ldots, f$, claimants in N' will get until that step the same in the application of the procedure based on the standard CEA rule in both problems, $MBC = (M, N, E, c, \alpha)$ and $MBC^{N'} = (M', N', E'^{CEA}, c|_{N'}, \alpha)$. Thus, in the case of some N'_k, $k \leq f$, to be empty, then the result immediately follows. Therefore, we consider now that $N'_k \neq \varnothing$ for some $k > f$. In this case, active claimants in N'_{f+1} will get their claims both in MBC^N and $MBC^{N'}$. Consequently, $CEA_i(M, N, E, c, \alpha) = CEA_i(M', N', E'^{CEA}, c|_{N'}, \alpha)$, for all $i \in N'$.

3. $f = r$. Since the CEA rule for standard bankruptcy problems is consistent, and $N'_k \subset N_k$, for all $k = 1, 2, \ldots, f = r$, claimants in N' will get the same in both problems when CEA is applied.

$\square$

5 Characterization

In this section, the aim is to get a better knowledge of the CEA rule for MBC by describing it in a unique way as a combination of some reasonable axioms. This combination of principles is very important to understand the behavior of this rule and be able to make a good choice. The characterization given uses three appealing axioms: Pareto efficiency, conditional equal division and consistency. In particular, consistency is a robust axiom that describes the invariance of a rule with respect to any change in the number of agents. In fact, this symbolic principle focus on the reduction in the number of agents and reflects not only fairness but also stability and its role in characterizations is significant, an interesting introduction to the literature on the "consistency principle" and its"converse"can be found in Thomson (2011).

Lemma 1 *For each problem $(M, N, E, c, \alpha) \in \mathcal{MBC}$, and each Pareto efficient allocation $x \in A(M, N, E, c, \alpha)$, if for each $N' \subset N$ with $|N'| = |N| - 1$, we have $x_i = CEA_i(M', N', E'^x, c|_{N'}, \alpha)$ for all $i \in N'$, then $x = CEA(M, N, E, c, \alpha)$.*

Proof We first prove that if there is $x_i = CEA_i(M, N, E, c, \alpha)$, then the result holds. Indeed, let us consider x in the conditions of the statement, and $x_i = CEA_i(M, N, E, c, \alpha)$. We now consider $N' = N \backslash \{i\}$, since $x_i = CEA_i(M, N, E, c, \alpha)$,

$$(M', N', E'^x, c|_{N'}, \alpha) = (M', N', E'^{CEA}, c|_{N'}, \alpha).$$

By hypothesis, we have that

$$x_k = CEA_k(M', N', E'^x, c|_{N'}, \alpha) \text{ for all } k \in N'.$$

Moreover, since CEA satisfies consistency,

$$CEA_k(M, N, E, c, \alpha) = CEA_k(M', N', E'^{CEA}, c|_{N'}, \alpha) \text{ for all } k \in N'.$$

Therefore, $CEA_k(M', N', E'^{CEA}, c|_{N'}, \alpha) = x_k$ for all $k \in N'$.

Let us consider x in the conditions of the statement and we assume without loss of generality that $x_1 \leq x_2 \ldots \leq x_{|N|}$. We are going to prove that $x_1 = CEA_1(M, N, E, c, \alpha)$ always holds. We distinguish two cases:

- $x_1 = c_1$. We take any $N' \subset N$ with $1 \in N'$ and $|N'| = |N| - 1$. By definition of CEA, in the first step the following linear program has to be solved:

$$\begin{aligned} & max\ z^1 \\ & s.a: \textstyle\sum_{i \in N': j \in \alpha(i)} x_i \leq e_j'^x, \text{ for all } j \in M' \\ & \qquad x_i \leq c_i, \text{ for all } i \in N' \\ & \qquad x_i \geq z^1, \text{ for all } i \in N' \\ & \qquad x_i \geq 0, \text{ for all } i \in N', \text{ and } z^1 \geq 0 \end{aligned}$$

  By hypothesis, we know that $c_1 = CEA_1(M', N', E'^x, c|_{N'}, \alpha)$. Therefore, since c_1 is the minimum value of the allocation x, the solution $x_i = c_1$ for all $i \in N'$ is a feasible solution of the linear program above, and also optimal because $x_1 \leq c_1$. Therefore $z^{*1} = c_1$.

On the other hand, for the original problem (M, N, E, c, α), in the first step the linear program to be solved is given by

$$\begin{aligned} & max\ z^1 \\ & s.a: \textstyle\sum_{i\in N: j\in\alpha(i)} x_i \le e_j,\ \text{for all } j \in M \\ & \qquad x_i \le c_i,\ \text{for all } i \in N \\ & \qquad x_i \ge z^1,\ \text{for all } i \in N \\ & \qquad x_i \ge 0,\ \text{for all } i \in N,\ \text{and } z^1 \ge 0 \end{aligned}$$

Now, since $e_j'^x \le e_j$ for all $j \in M'$ and $c_1 \le x_j$ for all $j \in N$, $z^{*1} = c_1$ is a feasible solution of this linear program and also optimal for the same reasons as before. Therefore, $CEA_1(M, N, E, c, \alpha) = c_1$.

- $x_1 < c_1$. We take any $N' \subset N$ with $1 \in N'$ and $|N'| = |N| - 1$. By definition of CEA, in the first step the following linear program has to be solved:

$$\begin{aligned} & max\ z^1 \\ & s.a: \textstyle\sum_{i\in N': j\in\alpha(i)} x_i \le e_j'^x,\ \text{for all } j \in M' \\ & \qquad x_i \le c_i,\ \text{for all } i \in N' \\ & \qquad x_i \ge z^1,\ \text{for all } i \in N' \\ & \qquad x_i \ge 0,\ \text{for all } i \in N',\ \text{and } z^1 \ge 0 \end{aligned}$$

By hypothesis, we know that $x_1 = CEA_1(M', N', E'^x, c|_{N'}, \alpha)$. Therefore, since x_1 is the minimum value of the allocation x, the solution $x_i = x_1$ for all $i \in N'$ is a feasible solution of the linear program above. Moreover, it must be optimal because otherwise, since $x_1 < c_1$, $CEA_1(M', N', E'^x, c|_{N'}, \alpha) > x_1$ which leads to a contradiction with the hypothesis. Furthermore, for the same reason, at least one of the inequalities $\sum_{i\in N': j\in\alpha(i)} x_i \le e_j'^x$, $j \in M'$, must be saturated in the optimal solution $x_i = x_1$ for all $i \in N'$.
Eventually, for each $N' \subset N$ with $1 \in N'$ and $|N'| = |N| - 1$, the saturated issue could be different. Note that all claimants i in N' such that the saturated issue belongs to $\alpha(i)$ are allocated the same, x_1. Now, we take N' such that $1 \in \{i \in N : j^* \in \alpha(i)\} \subset N'$ such that j^* is a saturated item in the first step of the calculation of CEA. Note that each claimant i in N' such that $j^* \in \alpha(i)$ receive exactly x_1 in $CEA(M', N', E'^x, c|_{N'}, \alpha)$.
On the other hand, for the original problem (M, N, E, c, α), in the first step the linear program to be solved is given by

$$\begin{aligned} & max\ z^1 \\ & s.a: \textstyle\sum_{i\in N: j\in\alpha(i)} x_i \le e_j,\ \text{for all } j \in M \\ & \qquad x_i \le c_i,\ \text{for all } i \in N \\ & \qquad x_i \ge z^1,\ \text{for all } i \in N \\ & \qquad x_i \ge 0,\ \text{for all } i \in N,\ \text{and } z^1 \ge 0 \end{aligned}$$

Now, since $e_{j^*}'^x = e_{j^*}$, $e_j'^x \le e_j$ for all $j \in M\backslash\{j^*\}$ and $x_1 \le x_j$ for all $j \in N$, $z^{*1} = x_1$ is a feasible solution of this linear program and also optimal for the same reasons as before. Therefore, $CEA_1(M, N, E, c, \alpha) = x_1$.
If N' such that $1 \in \{i \in N : j^* \in \alpha(i)\} \subset N'$ with j^* a saturated item in the first step of the calculation of CEA does not exist, then there will be $j' \in M$ such that $j' \in \alpha(i)$ for all $i \in N$ which will be saturated for all $N' \subset N$ with $1 \in N'$ and $|N'| = |N| - 1$. Therefore, taking into account that each claimant i in N' such that $j^* \in \alpha(i)$ receive

exactly x_1 in $CEA(M', N', E'^x, c|_{N'}, \alpha)$, we obtain that $x_i = x_1$ for all $i \in N$. Once again, following similar arguments as before, we have that $CEA_1(M, N, E, c, \alpha) = x_1$.

□

Theorem 4 *The CEA rule for multi-issue bankruptcy problems with crossed claims is the only rule that satisfies PEFF, CED, and CONS.*

Proof We proceed by induction in the number of claimants in the problem.

1. $|N| = 1$. If a rule R satisfies CED then, it is obvious that for all problem (M, N, E, c, α), such that $|N| = 1$,

$$R_1(M, N, E, c, \alpha) = \min_{j:j\in\alpha(1)} \min\left\{c_1, e_j\right\}.$$

2. $|N| = 2$. We distinguish two cases:

   a. $\alpha(1) \cap \alpha(2) = \varnothing$. In this case, by the definition of rule and $PEFF$,

$$R_1(M, N, E, c, \alpha) = \min_{j:j\in\alpha(1)} \min\left\{c_1, e_j\right\} \text{ and}$$
$$R_2(M, N, E, c, \alpha) = \min_{j:j\in\alpha(2)} \min\left\{c_2, e_j\right\}.$$

   b. $\alpha(1) \cap \alpha(2) \neq \varnothing$. By the definition itself of allocation rule,

$$R_1(M, N, E, c, \alpha) \leq \min_{j:j\in\alpha(1)\setminus\alpha(2)} \min\left\{c_1, e_j\right\} = c_1',$$
$$R_2(M, N, E, c, \alpha) \leq \min_{j:j\in\alpha(2)\setminus\alpha(1)} \min\left\{c_1, e_j\right\} = c_2'.$$

   We now take

$$e^* = \min\left\{e_j : j \in \alpha(1) \cap \alpha(2)\right\}.$$

   Next, two cases are distinguished:

   i. $c_1' + c_2' \leq e^*$. Since $R_1(M, N, E, c, \alpha) \leq c_1'$, $R_2(M, N, E, c, \alpha) \leq c_2'$, and R satisfies $PEFF$,

$$R_1(M, N, E, c, \alpha) = c_1' \text{ and } R_2(M, N, E, c, \alpha) = c_2'.$$

   .

   ii. $c_1' + c_2' > e^*$. If a rule satisfies CED, then

$$\begin{aligned} R_i(M, N, E, c, \alpha) &\geq \min\left\{c_i, \min_{j:j\in\alpha(i)}\left\{\frac{e_j}{|\{k : j \in \alpha(k)\}|}\right\}\right\} \\ &= \min\left\{c_i', \frac{e^*}{2}\right\}, i = 1, 2. \end{aligned}$$

   Finally, we assume, without loss of generality, that $c_1' \leq c_2'$ and distinguish three cases:

   A. $\frac{e^*}{2} \leq c_1' \leq c_2'$. Then,

$$R_i(M, N, E, c, \alpha) \geq \frac{e^*}{2}, i = 1, 2.$$

   By the definition of rule,

$$R_i(M, N, E, c, \alpha) = \frac{e^*}{2}, i = 1, 2.$$

B. $c'_1 \leq \frac{e^*}{2} \leq c'_2$. Then,

$$R_1(M, N, E, c, \alpha) \geq c'_1 \text{ and } R_2(M, N, E, c, \alpha) \geq \frac{e^*}{2}.$$

Now, since $R_1(M, N, E, c, \alpha) \leq c'_1$, $R_1(M, N, E, c, \alpha) = c'_1$; and by $PEFF$,

$$R_2(M, N, E, c, \alpha) = \min\{c'_2, e^* - c'_1\} = e^* - c'_1.$$

C. $c'_1 \leq c'_2 \leq \frac{e^*}{2}$. Then $R_i(M, N, E, c, \alpha) \geq c'_i, i = 1, 2$. Hence,

$$R_i(M, N, E, c, \alpha) = c'_i, i = 1, 2.$$

Therefore, if a rule satisfies $PEFF$ and CED is also completely determined when $|N| = 2$. Since CEA satisfies $PEFF$ and CED, any other rule that satisfies these properties coincides with CEA when $|N| \leq 2$

3. $|N| = 3$. Let R be a rule that satisfies $PEFF$, CED and $CONS$, and let $(M, N, E, c, \alpha) \in \mathcal{MBC}$, then we have that

$$R(M, N, E, c, \alpha) = CEA(M, N, E, c, \alpha).$$

Indeed, for each $N' = \{i_1, i_2\} \subset N$ such that $|N'| = 2$, since R satisfies CONS,

$$R_{i_k}(M', N', E'^R, c|_{N'}, \alpha) = R_{i_k}(M, N, E, c, \alpha), k = 1, 2,$$

and since $|N'| = 2$, we have that

$$R_{i_k}(M', N', E'^R, c|_{N'}, \alpha) = CEA_{i_k}(M', N', E'^R, c|_{N'}, \alpha), k = 1, 2.$$

Since we can take all possible $N' = \{i_1, i_2\} \subset N$, by Lemma 1

$$R(M, N, E, c, \alpha) = CEA(M, N, E, c, \alpha).$$

4. $|N| \leq k$. Let us suppose that for each (M, N, E, c, α) with $|N| \leq k$, $R(M, N, E, c, \alpha) = CEA(M, N, E, c, \alpha)$.
5. $|N| = k + 1$. For each $N' \subset N$ such that $|N'| = k$, since R satisfies CONS,

$$R_i(M', N', E'^R, c|_{N'}, \alpha) = R_i(M, N, E, c, \alpha), i \in N'.$$

and since $|N'| \leq k$, we have that

$$R_i(M', N', E'^R, c|_{N'}, \alpha) = CEA_i(M', N', E'^R, c|_{N'}, \alpha), i \in N'.$$

Finally, since we can take all possible $N' \subset N$ with $|N'| = k$, by Lemma 1,

$$R(M, N, E, c, \alpha) = CEA(M, N, E, c, \alpha).$$

□

Note that from Theorem 4, we know that for $|N| = 2$, CEA is the only rule satisfying $PEFF$ and CED for multi-issue bankruptcy problems with crossed claims. Next, in the following propositions, we show the role of each property in Theorem 4. We first prove that CED and $CONS$ imply $PEFF$, then, we show that CED and $CONS$ are necessary, and that any other combination of two properties do not imply the remaining third. Therefore, CEA can also be characterized by only CED and $CONS$. This is established below in Corollary 1. However, we have preferred to keep the characterization with $PEFF$ because this way we obtain a different characterization of CEA for the case of two agents as described above.

Proposition 1 *If a rule satisfies CED and CONS then it satisfies PEFF.*

Proof Let us suppose by contradiction that a rule R satisfies CED and $CONS$ but not $PEFF$. Therefore, there is a problem $(M, N, E, c, \alpha) \in \mathcal{MBC}$ such that $R(M, N, E, c, \alpha)$ is not Pareto efficient. Since $R(M, N, E, c, \alpha)$ is not Pareto effcient, there exists $x \in A(M, N, E, c, \alpha)$ such that $x_i \geq R_i(M, N, E, c, \alpha), \forall i \in N$, with at least one strict inequality. Le i_0 be such that $x_{i_0} > R_{i_0}(M, N, E, c, \alpha)$. We now take $N' = \{i_0\}$ and its associated reduced problem $(M', N', E', c|_{N'}, \alpha)$. For each $j \in M'$,

$$e'_j = e_j - \sum_{i \in N \setminus \{i_0\} : j \in \alpha(i)} R_i(M, N, E, c, \alpha).$$

Since $x \in A(M, N, E, c, \alpha)$ and satisfies that $x_{i_0} > R_{i_0}(M, N, E, c, \alpha)$, then $x_{i_0} \leq e'_j, \forall j \in \alpha(x_{i_0})$. Moreover, by CED,

$$R_{i_0}(M', N', E', c, \alpha) \geq \min \left\{ c_{i_0}, \min_{j : j \in \alpha(i_0)} \left\{ e'_j \right\} \right\}.$$

Therefore, we have the following chain of inequalities

$$R_{i_0}(M, N, E, c|_{N'}, \alpha) \geq \min \left\{ c_{i_0}, \min_{j : j \in \alpha(i_0)} \left\{ e'_j \right\} \right\} \geq x_{i_0} > R_{i_0}(M, N, E, c, \alpha),$$

which is a contradiction with the fact that R satisfies $CONS$. □

Corollary 1 *The CEA rule for multi-issue bankruptcy problems with crossed claims is the only rule that satisfies CED, and $CONS$.*

Proposition 2 *The properties CED and CONS in Theorem 4 (and Corollary 1) are necessary.*

Proof We consider the two possible situations:

- For each problem $(M, N, E, c, \alpha) \in \mathcal{MBC}$, we define the following rule:

$$R^*_i(M, N, E, c, \alpha) = \min \left\{ c_i, \min_{j \in \alpha(i)} \left\{ e_j - \sum_{k : k < i; j \in \alpha(k)} R*_k(M, N, E, c, \alpha) \right\} \right\}, \forall i \in N,$$

  which is calculated recursively allocating from the agent with the lowest number to the agent with the highest number.
  By definition this rule satisfies $PEFF$ and $CONS$, but not CED.
- For each problem $(M, N, E, c, \alpha) \in \mathcal{MBC}$, we define the following rule in two steps. We first allocate to each agent the following:

$$F_i(M, N, E, c, \alpha) = \min \left\{ c_i, \min_{j : j \in \alpha(i)} \left\{ \frac{e_j}{|\{k : j \in \alpha(k)\}|} \right\} \right\}, \forall i \in N.$$

  Then we consider the following problem (M, N, E', c', α):

$$e'_j = e_j - \sum_{i : j \in \alpha(i)} F_i(M, N, E, c, \alpha), \forall j \in M,$$

$$c'_i = c_i - F_i(M, N, E, c, \alpha), \forall i \in N.$$

  Finally, the rule is given by

$$R_i(M, N, E, c, \alpha) = F_i(M, N, E, c, \alpha) + R^*_i(M, N, E', c', \alpha), \forall i \in N.$$

By definition this rule satisfies CED and $PEFF$, but not $CONS$ as the following example shows. We consider the problem $M = \{1\}, N = \{1, 2, 3\}, E = (12), c = (3, 6, 6), \alpha(i) = 1, \forall i \in N$. We have that

$$F_1(M, N, E, c, \alpha) = 3, F_2(M, N, E, c, \alpha) = 4, F_3(M, N, E, c, \alpha) = 4,$$

and

$$R_1(M, N, E, c, \alpha) = 3, R_2(M, N, E, c, \alpha) = 5, R_3(M, N, E, c, \alpha) = 4.$$

Let us consider $N' = \{2, 3\}$, it is easy to check that

$$R_2(M', N', E', c|_{N'}, \alpha) = 4.5, \text{ and } R_3(M', N', E', c|_{N'}, \alpha) = 4.5.$$

□

Note that CED for MBC problems is the equivalent of conditional equal division lower bound (Moulin 2000) for bankruptcy problems, but it is not the same as conditional equal division full compensation (this was called *exemption* by Herrero and Villar (2001)). In bankruptcy problems, for $|N| = 2$, CEA is the only rule satisfying the conditional equal division lower bound (Thomson 2015), we do not mention efficiency because all bankruptcy rules satisfy it. Here, we obtain the same result for MBC problems by using $PEFF$ and CED (see the proof of the case $|N| = 2$ in Theorem 4). However, when using conditional equal division full compensation an extra property is necessary to characterize CEA in bankruptcy problems with $|N| = 2$. In addition, for bankruptcy problems when $|N| = 2$, conditional full compensation (this was called *sustainability* by Herrero and Villar (2002)) and conditional equal division full compensation coincide.

Proposition 3 *For multi-issue bankruptcy problems with crossed claims, CFC and CM imply CED.*

Proof Let R be a rule satisfying CFC and CM, and $(M, N, E, c, \alpha) \in \mathcal{MBC}$. For each $i \in N$, we consider the following problem (M, N, E, c'^i, α) ,

$$c_i'^i = \min\left\{c_i, \min_{j:j\in\alpha(i)}\left\{\frac{e_j}{|\{k : j \in \alpha(k)\}|}\right\}\right\}, c_k'^i = c_k, \forall k \in N\backslash\{i\}.$$

By CFC, it holds that $R_i(M, N, E, c'^i, \alpha) = c_i'^i, \forall i \in N$. Since R satisfies CM,

$$R_i(M, N, E, c, \alpha) \geq R_i(M, N, E, c'^i, \alpha), \forall i \in N.$$

Therefore, R satisties CED. □

Corollary 2 *The CEA rule for multi-issue bankruptcy problems with crossed claims is the only rule that satisfies $PEFF$, CFC, CM and $CONS$.*

Corollary 2 corresponds to the characterization of CEA in MBC problems equivalent to the characterization of CEA in Yeh (2006) (see, Thomson 2015, Th. 4b and Th. 14). Of course, other characterizations of the classical constrained equal awards rule could try to be extended to this context, for example the characterization of CEA in Herrero and Villar (2002). In the latter case, we would first have to define what *composition down* (Moulin 2000) means in this context. Since we have many issues, it could be extended in different ways. In any case, this last characterization and others in the literature would be interesting for further research in this framework.

6 Conclusions and further research

This paper is related to one of the earliest problems arised in the economic literature. In fact, this problem already appeared in primal documents as the Talmud, or in essays of Aristotle or Maimonides. However, their mathematical modelization was first carried out by O'Neill (1982). The common and central question in these problems is how to divide when there is not enough. An extension of the classical bankcruptcy problems appears with the introductin of multi-issue bankruptcy problems (Calleja et al. 2005) allowing that claims of agents can be referred to different issues.

In this paper, we go beyong of it with the purpose of solving a real problem of abatement of emissions of different pollutants in which pollutants can contribute to more than one effect. To do this, we establish a new and original model based on multi-issues bankrupcy problem (MB) called multi-issue bankruptcy problems with crossed claims (MBC). This novel model presents a multi-dimensional state, one for each issue and each agent claims the same to the different issues in which participates, these are essential differences with respect to MB problems.

Similar as for MB problems, in this new framework, problems are solved through rules that assigns to each MB problem a distribution pointing out the amount obtained for each agent in each issue. In this paper, we have allocated according to the CEA rule for bankruptcy problems introducing it as the solution to a sucession of linear programming problems and extending this procedure to this framework. Currently, we are working to solve this problem through other possible allocations or rules. In particular, futher research will include to extent to this context other rules already studied for MB problems as the proportional rule[3] (Moreno-Ternero 2009; Bergantiños et al. 2010), the constrained equal losses (CEL) analyzed for MB problems in Lorenzo-Freire et al. (2010), the ramdon arrival rule (O'Neill 1982) based on the Shapley value (Shapley 1953) (see Algaba et al. 2019b for an updating on theoretical and applied aspects about this outstanding value), or the Talmud rule studied in the setting of bankrupcy problems (see, for instance, Moreno-Ternero and Villar 2006), among others.

Finally, in the literature of Operations Research (OR) there exist problems that could fit well in this theoretical model, for example, set covering problems. Bergantiños et al. (2020) study the problem of how to allocate costs in set covering problems when a reasonable cover is given in advance. These problems are described by a 4-tuple (N, M, c, A), where N is the set of agents, M is the set of facilities open, $c \in \mathbb{R}^M_+$ is the vector costs associated with the facilities, and $A = \{A_j\}_{j \in M}$ with $A_j \subset N$ for each $j \in M$ denotes the agents covered by each facility. The question to be answered is how to allocate the total costs among the agents. If we look carefully at the structure of the problem, we can observe a certain similarity with multi-issue bankruptcy problems with crossed claims in the following way. We first identify agents with pollutants and issues with facilities (regions). Thus, we have a set of pollutants that affect several regions, this is described by A that plays the role of function α. On the other hand, we consider that each region fixes a maximum level of pollution which is given by vector c that plays the role of vector E. Thus, the following problem arises: How to set pollutant emission levels when pollutants affect different regions? But one extra element is necessary in this problem: the pollutant emissions to be abated, i.e., the claims. Therefore, the set covering problem is the following. When we have a set of regions that impose limits on pollutant emissions, and these emissions come from several pollutants that can affect

[3] The proportional rule is one of the most relevant and popular to deal with allocation problems in general, see, for instance, Algaba et al. (2019a) who introduce two solutions belonging to the family of proportional solutions for the problem of sharing the profit of a combined ticket for a transport system in the setting of coloured graphs.

several of the regions simultaneously, the question to be answered is, how to set the emission limits of pollutants in such a way that the limits established by the regions are covered? This problem can be analyzed as a multi-issue bankruptcy problem with crossed claims. However, what happens if no reference on the ex-ante emissions of the pollutants are given? In this case, the problem have exactly four elements, (N, M, c, A), and the question to be answered is, how to set maximal limits of emissions of pollutants such that the regional limits are not exceeded? Therefore, these relationships between set covering problems and multi-issue bankruptcy problems with crossed claims would be interesting to study them in greater detail in further research.

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On proportionality in multi-issue problems with crossed claims

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Abstract

In this paper, we analyze the problem of how to adapt the concept of proportionality to situations where several perfectly divisible resources have to be allocated among certain set of agents that have exactly one claim which is used for all resources. In particular, we introduce the constrained proportional awards rule, which extend the classical proportional rule to these situations. Moreover, we provide an axiomatic characterization of this rule.

Keywords: Game theory, multi-issue allocation problems, proportional rule

1 Introduction

Allocation problems describe situations in which a resource (or resources) must be distributed among a set of agents. These problems are of great interest in many settings, for this reason the literature on the matter is extensive. A particular allocation problem is arisen in situations where there is a perfectly divisible resource over which there is a set of agents who have rights or demands, but the resource is not sufficient to satisfy them. This problem is known as bankruptcy problem and was first formally analyzed in O'Neill (1982) and Aumann and Maschler (1985). Since then it has been extensively studied in the literature and many allocation rules have been defined (see Thomson, 2003, 2015, 2019, for a detailed inventory of rules). In the literature many applications of bankruptcy problems can be found. Some examples are the following. Pulido et al. (2002, 2008) study allocation problems in university management; Niyato and Hossain (2006), Gozalvez et al. (2012), and Lucas-Estañ et al. (2012) analyze radio resource allocation problems in telecommunications; Casas-Mendez et al. (2011) study the musseum pass problem; Hu et al.

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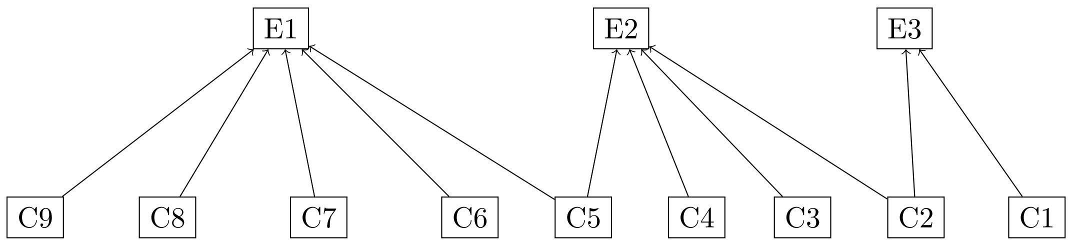

Figure 1: Multi-issue problem with crossed claims.

(2012) analyze the airport problem; Giménez-Gómez et al. (2016), Gutiérrez et al. (2018), and Duro et al. (2020) analyze the CO2 allocation problem; Sanchez-Soriano et al. (2016) study the apportionment problem in proportional electoral systems; and Wickramage et al. (2020) analyze water allocation problems in rivers.

An extension of bankruptcy problems are multi-issue allocation problems (Calleja et al., 2005). These describe situations in which there is a (perfect divisible) resource which can be distributed between several issues, and a (finite) number of agents that have claims on each of those issues, such that the total claim is above the available resource. This problem is also solved by means of allocation rules and there are different ways to do it (see, for example, Calleja et al. (2005), Borm et al. (2005), and Izquierdo and Timoner (2016)). Ju et al. (2007), Moreno-Ternero (2009) and Bergantiños et al. (2010) study the proportional rule for multi-issue allocation problems.

However, the situation described in Figure 1 does not fit to a multi-issue allocation problem as referred in the previous paragraph, but to a multi-issue allocation problem with crossed claims introduced by Acosta-Vega et al. (2021). These describe situations in which there are several (perfect divisible) resources and a (finite) set of agents who have claims on them, but only one claim (not a claim for each resource) with which one or more resources are requested. The total claim for each resource exceeds its availability. This problem is solved by means of allocation rules (Acosta-Vega et al., 2021).

In this paper, in order to solve allocation problems as the described in Figure 1, we introduce the constrained proportional awards rule for multi-issue allocation problems with crossed claims that naturally extends the proportional rule for single issue allocation problems. This rule is characterized axiomatically by using five properties: *Pareto efficiency*, *equal treatment of equals*, *guaranteed minimum award*, *consistency*, and *non-manipulability by splitting*. The first one says that there is no a feasible allocation in which at least one of the claimants receive more. *Equal treatment of equals* states that equal agents must receive the same. *Guaranteed minimum award* means that a claimant should not receive less than what she would receive in the worst case, if the issues were distributed separately. *Consistency* requieres that if a subset of agents leave the problem respecting what had been allocated to those who remain, then what those agents receive in the new problem is the same as what they received in the original problem. Finally, *non-manipulability by splitting* means that it is not profitable to split one agent in several agents. Moreover, although it is not necessary in characterization, the constrained proportional awards rule also satisfies *non-manipulability by restricted merging* which guarantees that it is not profitable to merge several "homologous" agents into one.

The rest of the paper is organized as follows. Section 2 presents multi-issue allocation problems with crossed claims (MAC), and the concept of rule for these problems. In Section 3, the constrained proportional awards rule for multi-issue bankruptcy problems with crossed claims is defined. In Section 4, we present several properties which are interesting in the context of MAC problems. In Section 5, we characterize the constrained proportional awards rule. Section 6 concludes.

2 Multi-issue allocation problems with crossed claims

A one-issue allocation problem is given by a triplet (N, E, c), where N is the set of claimants, $E \in \mathbb{R}_+$ is a perfectly divisible amount of resource (the *issue* or *estate*) to be divided, and c is the vector of demands, such that $C = \sum_{j\in N} c_j > E$. One of the most relevant ways to allocate the resource among the claimants in one-issue allocation problem is the proportional rule (PROP) (Aristotle, 4th Century BD), which is defined as follows:

$$PROP_j(N, E, c) = \frac{c_j}{C}E, \quad j \in N. \tag{1}$$

We now consider a situation where there are a finite set of issues $\mathcal{I} = \{1, 2, ..., m\}$ such that there is a perfectly divisible amount e_i of each issue i. Let $E = (e_1, e_2, ..., e_m)$ be the vector of available amounts of issues. There are a finite set of claimants $N = \{1, 2, ..., n\}$ such that each claimant j claims c_j. Let $c = (c_1, c_2, ..., c_n)$ be the vector of claims. However, each claimant claims to different sets of issues, in general. Thus, let α be a set-valued function that associates with every $j \in N$ a set $\alpha(j) \subset \mathcal{I}$. In fact, $\alpha(j)$ represents the issues to which claimant j asks for. Furthermore, $\sum_{j:i\in\alpha(j)} c_j > e_i$, for all $i \in \mathcal{I}$, otherwise, those issues could be discarded from the problem because they do not impose any limitation, and so the allocation would be trivial. Therefore, a multi-issue allocation problem with crossed claims (MAC in short) is defined by a 5-tuple $(\mathcal{I}, N, E, c, \alpha)$, and the family of all these problems is denoted by $\mathcal{MAC}$.

Given a problem $(\mathcal{I}, N, E, c, \alpha) \in \mathcal{MAC}$, a *feasible allocation* for it, it is a vector $x \in \mathbb{R}^N$ such that:

1. $0 \leq x_i \leq c_i$, for all $i \in N$.

2. $\sum_{i\in N:j\in\alpha(i)} x_i \leq e_j$, for all $j \in M$,

and we denote by $A(\mathcal{I}, N, E, c, \alpha)$ the set of all its feasible allocations.

These two requirements are standard in the literature of allocation problems.

A *rule* for multi-issue bankruptcy problems with crossed claims is a mapping R that associates with every $(\mathcal{I}, N, E, c, \alpha) \in \mathcal{MBC}$ a unique feasible allocation $R(\mathcal{I}, N, E, c, \alpha) \in A(\mathcal{I}, N, E, c, \alpha)$.

3 The constrained proportional awards rule for $\mathcal{MAC}$ problems

The proportional rule (PROP) is perhaps the most important rule to solve allocation problems in general,[1] and bankruptcy problems in particular. This rule simply divides the resource in proportion to the claims. The question in $\mathcal{MAC}$ problems is what "in proportion to the claims" means. In the context of one-issue allocation problems, "in proportion to the claims" means that all claimants receive the same amount for each unit of claim. How to extrapolate this to the MAC situations. To answer this question, we introduce the constrained proportional awards rule (CPA in short) as the result of an iterative process in which the available amount of at least one of the issues is fully distributed in each step and so on and so forth while possible. This rule is formally defined below.

Definition 1. *Let* $(\mathcal{I}, N, E, c, \alpha) \in \mathcal{MAC}$, *the* constrained proportional awards rule *for* $(\mathcal{I}, N, E, c, \alpha)$, $CPA(\mathcal{I}, N, E, c, \alpha)$, *is defined by means of the following iterative procedure:*

Step 0.

1. $\mathcal{I}^1 = \{i \in \mathcal{I} : e_i^1 > 0\}$ *is the set of active issues.*
2. $\mathcal{N}^1 = \{j \in N : c_j^1 > 0$ *and* $e_i^1 > 0, \forall i \in \alpha(j)\}$ *is the set of active claimants.*
3. *For each* $i \in \mathcal{I}$, $e_i^1 = e_i$, *and for each* $j \in N$, $c_j^1 = c_j$.

Step s.

1. $\mathcal{N}^s = \{j \in N : c_j^s > 0$ *and* $e_i^s > 0, \forall i \in \alpha(j)\}$. $\mathcal{I}^s = \{i \in \mathcal{I} : e_i^s > 0\}$.
2. *For each* $i \in \mathcal{I}^s$, *we calculate the greatest* λ_i^s, *so that* $\lambda_i^s \sum_{j \in \mathcal{N}^s : i \in \alpha(j)} c_j^s \leq e_i^s$, *and take* $\lambda^s = \min\{\lambda_i^s : i \in \mathcal{I}^s\}$.
3. *Now, we allocate to each claimant* $j \in \mathcal{N}^s$, $a_j^s = \lambda^s c_j^s$, *and* $a_j^s = 0$ *to the non-active claimants.*
4. *We update the active issues,* $\mathcal{I}^{s+1}$, *and the active claimants,* $\mathcal{N}^{s+1}$. *If* $\mathcal{I}^{s+1} = \varnothing$ *or* $\mathcal{N}^{s+1} = \varnothing$, *then the process ends, and*

$$CPA_j(\mathcal{I}, N, E, c, \alpha) = \sum_{h=1}^{s} a_j^h, \forall j \in N.$$

Otherwise, the available amounts of issues and the claims are updated:

$$e_i^{s+1} = e_i^s - \lambda^s \sum_{j \in N : i \in \alpha(j)} c_j^s, \forall i \in \mathcal{I}, \text{ and } \; c_j^{s+1} = c_j^s - \lambda^s c_j^s, \forall j \in N,$$

and we go to Step $s + 1$.

The iterative procedure of CPA is well-defined and always leads to a single point. Moreover, since in each step at least the available amount of one issue is distributed in its entirety, except maybe in the last step, it ends in a finite number of steps,

[1]See, for instance, Algaba et al. (2019) who present two solutions belonging to the family of proportional solutions for the problem of sharing the profit of a combined ticket for a transport system.

at most $|\mathcal{I}|$. Finally, when we have a one-issue allocation problem, then we obtain PROP. Therefore, this definition extends PROP to the context of MAC.

From the application of the iterative process to calculate CPA, we can consider the chains of active issues and active claimants in the application of the procedure to calculate $CPA(\mathcal{I}, N, E, c, \alpha) \in \mathcal{MAC}$:

$$\mathcal{I}^1 \supset \mathcal{I}^2 \supset \ldots \supset \mathcal{I}^r, \text{ and } \mathcal{N}^1 \supset \mathcal{N}^2 \supset \ldots \supset \mathcal{N}^r$$

From these chains, we can establish an order relationship between issues as follows. We say that issue i_1 strictly precedes issue i_2 in a chain of actives issues, $i_1 \prec i_2$, if there is $\mathcal{I}^s$ such that $i_1 \notin \mathcal{I}^s$ and $i_2 \in \mathcal{I}^s$, i.e., i_1 becomes non-active before than i_2. We write $i_1 \preceq i_2$ when i_1 becomes non-active before than i_2 or both issues become non-active at the same time. Finally, we write $i_1 \simeq i_2$ when both issues become non-active at the same time. Analogously, we can establish an order relationship between claimants.

Furthermore, we can associate with each pair of sets $\mathcal{I}^s$ and $\mathcal{N}^s$ a number ρ^s, $\rho^s \in [0, 1]$, which represents the proportion of claims obtained by claimants in $\mathcal{N}^s$ but not in $\mathcal{N}^{s+1}$. Moreover, by construction $\rho^s < \rho^{s+1}$. Thus, we have that

$$0 < \rho^1 < \rho^2 < \ldots < \rho^r \leq 1.$$

These $\rho' s$ represent the accumulative proportion of the claims allocated to the claimants, i.e., what part of their claims they have received up to a given step of the iterative procedure. In this way, this procedure is reminiscent of the constrained equal awards rule (CEA) in bankuptcy problems, but instead of using the principle of egalitarianism, the principle of proportionality is used, hence the name of *constrained proportional awards* rule. Therefore, not all claimants receive the same proportion of their claims, but the rule tries to keep the proportionality as much as possible restricted to (1) the relation between the available amounts of issues and the total claims to them, and (2) the goal of allocating as much as possible of all available amounts of issues.

Below some results about the iterative process that defines CPA are given.

Proposition 1. *The following statements hold*

1. *Given* $(\mathcal{I}, N, E, c, \alpha) \in \mathcal{MAC}$, *if there are problems* $(\mathcal{I}_1, N_1, E_1, c^1, \alpha_1), (\mathcal{I}_2, N_2, E_2, c^2, \alpha_2) \in \mathcal{MAC}$, *such that* $\mathcal{I}_1 \cup \mathcal{I}_2 = \mathcal{I}$, $N_1 \cup N_2 = N$, $E_1 \oplus E_2 = E$, $c^1 \oplus c^2 = c$, *and* $\alpha_1(j) = \alpha(j), \forall j \in N_1$ *and* $\alpha_2(j) = \alpha(j), \forall j \in N_2$, *so that* $\left(\bigcup_{j \in N_1} \alpha(j)\right) \bigcap \left(\bigcup_{j \in N_2} \alpha(j)\right) = \varnothing$, *then*

$$CPA(\mathcal{I}, N, E, c, \alpha) = CPA(\mathcal{I}_1, N_1, E_1, c^1, \alpha_1) \oplus CPA(\mathcal{I}_2, N_2, E_2, c^2, \alpha_2).^2$$

2. *If* $\lambda^s < 1$, *then each* $h \in \arg\min\{\lambda_i^s : i \in \mathcal{I}^s\} \subset \mathcal{I}$ *becomes non-active in the next step.*

3. *If* $\lambda^s = 1$ *for some* s, *then the iterative procedure ends in that step.*

[2] Given $a \in \mathbb{R}^n$ and $b \in \mathbb{R}^m$, $a \oplus b \in \mathbb{R}^{n+m}$, i.e., $\oplus$ is the concatenation operator of two vectors.

Proof. These statements follow from the own structure of the iterative procedure to calculate CPA. We next provide a brief outline of each one.

1. This follows from the fact that since there are no crossed demands between the two subproblems, they do not affect each other and, therefore, the results are independent of each other.

2. $\lambda^s < 1$ implies that there are some issues in $\mathcal{I}^s$ for which their available amounts are not enough to satisfy the total claims to them in step s. Thus, $\lambda_h^s \sum_{j\in\mathcal{N}^s:h\in\alpha(j)} c_j^s = e_h^s$, $\forall h \in \arg\min\{\lambda_i^s : i \in \mathcal{I}^s\}$. Therefore, when λ^s is applied, all those issues become non-active in the next step.

3. $\lambda^s = 1$ implies that all active claimants in $\mathcal{N}^s$ receive their pendent claims, then they become non-active in the next step, i.e., $\mathcal{N}^{s+1} = \varnothing$. Therefore, the procedure ends.

□

The first statement says that if a problem can be separated into two disjoint problems, then it is the same to calculate CPA for the whole problem as for each of them and then paste the results. The second states when an issue becomes non-active. Finally, the third provides another stopping criterion for the iterative procedure to calculate CPA. Moreover, when the procedure ends with $\lambda = 1$, then it means that there may be resources left over from some issues. Otherwise, all resources have been fully distributed.

4 Properties

In this section, we present several properties which are interesting in the context of MAC problems. These properties are related to efficiency, fairness, consistency, and manipulability.

First, we introduce two concepts related to two claimants comparisons. In MAC situations claimants are characterized by two elements: their claims and the issues to which they claim. Therefore, both should be taken into account when establishing comparisons among them.

Definition 2. *Let $(\mathcal{I}, N, E, c, \alpha) \in \mathcal{MAC}$, and two claimants $j, k \in N$, we say they are* homologous*, if $\alpha(j) = \alpha(k)$; and we say that they are* equal*, if they are homologous and $c_j = c_k$.*

Next, we give a set of properties which are very natural and reasonable for an allocation rule in MAC situations.

The first property relates to efficiency. In allocation problems is desirable that the resources to be fully distributed, but in MAC situations this is not always possible (see Acosta-Vega, 2021). Therefore, a weaker version of that is considered in which only is required that there is no a feasible allocation in which at least one of the claimants receive more. This is established in the following axiom.

Axiom 1 (PEFF)**.** *Given a rule R, it satisfies* Pareto efficiency*, if for every problem $(\mathcal{I}, N, E, c, \alpha) \in \mathcal{MAC}$, there is no a feasible allocation $a \in \mathbb{R}_+^N$ such that $a_j \geq R_j(\mathcal{I}, N, E, c, \alpha), \forall j \in N$, with at least one strict inequality.*

Note that $PEFF$ implies that at least the available amount of one issue is fully distributed, and no amount is left of an issue undistributed, if it is possible to do so. However, it does not require that all available amounts of the issues have to be fully distributed. On the other hand, a feasible allocation that satisfies the condition in Axiom 1 is called Pareto efficient.

The second property states that equal claimants should receive the same in the final allocation. This is a basic requirement of fairness and non-arbitrariness. This is defined in the following axiom.

Axiom 2 (ETE)**.** *Given a rule R, it satisfies* equal treatment of equals*, if for every problem $(\mathcal{I}, N, E, c, \alpha) \in \mathcal{MAC}$ and every pair of equal claimants $j, k \in N$, $R_j(\mathcal{I}, N, E, c, \alpha) = R_k(\mathcal{I}, N, E, c, \alpha)$.*

The third property assures the minimum that should be guaranteed to each claimant. In our case, these minimum amounts are determined from the analysis of the problems associated with each issue independently. In particular, the property states that a claimant should not receive less than what she would have received in the worst case, if the rule had been applied to each problem separately to each of the issues. This is established in the following property.

Axiom 3 (GMA)**.** *Given a rule R, it satisfies* guaranteed minimum award*, if for every problem $(\mathcal{I}, N, E, c, \alpha) \in \mathcal{MAC}$,*

$$R_j(\mathcal{I}, N, E, c, \alpha) \geq \min \left\{ R_j\left(\{i\}, N_i, e_i, c|_{N_i}\right) : i \in \alpha(j) \right\}, \forall j \in N,$$

where $N_i = \{k \in N : i \in \alpha(k)\}$, and $c|_{N_i}$ is the vector whose coordinates correspond to the claimants in N_i.

The fourth property is a requirement of robustness when some agents leave the problem with their allocations (see Thomson, 2011, 2018). In particular, when a subset of claimants leave the problem respecting the allocations to those who remain, then it seems reasonable that claimants who leave will receive the same in the new problem as they did in the original. This is formally given in the following axiom.

Axiom 4 (CONS)**.** *Given a rule R, it satisfies* consistency*, if for every problem $(\mathcal{I}, N, E, c, \alpha) \in \mathcal{MAC}$, and $N' \subset N$, it holds that*

$$R_j(\mathcal{I}, N, E, c, \alpha) = R_j(\mathcal{I}', N', E'^R, c|_{N'}, \alpha), \text{ for all } j \in N',$$

where $(\mathcal{I}', N', E', c|_{N'}, \alpha) \in \mathcal{MAC}$, called the reduced problem associated with N'*, $\mathcal{I}' = \{i \in \mathcal{I} :$ there exists $k \in N'$such that $i \in \alpha(k)\}$, $E'^R = (e_1'^R, \ldots, e_m'^R)$ so that $e_i'^R = e_i - \sum_{j \in N \setminus N' : i \in \alpha(j)} R_j(\mathcal{I}, N, E, c, \alpha)$, for all $i \in \mathcal{I}'$, and $c|_{N'}$ is the vector whose coordinates correspond to the claimants in N'.*

The last two properties are related to claimants' ability to manipulate the final allocation by splitting their claims among several new claimants or merging their claims into a single claimant. It seems sensible that if the claimants do this, they will not benefit and receive the same as they did in the original problem. These two possibilities are established in the following axioms.

Axiom 5 (NMS)**.** *Given a rule R, it satisfies* non-manipulability by splitting, *if for every pair of problems* $(\mathcal{I}, N, E, c, \alpha), (\mathcal{I}, N', E, c', \alpha') \in \mathcal{MAC}$*, such that:*

1. $N \subset N'$*,* $S = \{i_1, \ldots, i_k\}$*, such that* $N' = (N \backslash S) \cup S_{i_1} \cup \ldots \cup S_{i_m}$*, where* S_{i_k} *is the set of agents into which agent* i_k *has been divided.*
2. $c'_j = c_j, \forall j \in N \backslash S$ *and* $\sum_{k \in S_{i_h}} c'_k = c_{i_h}, h = 1, \ldots, m,$
3. $\alpha'(j) = \alpha(j), \forall j \in N \backslash S$ *and* $\alpha'(j) = \alpha(i_h), \forall j \in S_{i_h}, h = 1, \ldots, m,$

it holds

$$\sum_{j \in S_{i_h}} R_j(\mathcal{I}, N', E, c', \alpha') = R_{i_h}(\mathcal{I}, N, E, c, \alpha), h = 1, \ldots, m.$$

Axiom 6 (NMRM)**.** *Given a rule R, it satisfies* non-manipulability by restricted merging*, if for every pair of problems* $(\mathcal{I}, N, E, c, \alpha), (\mathcal{I}, N', E, c', \alpha') \in \mathcal{MAC}$*, such that:*

1. $N \subset N'$*,*
2. $c_j = c'_j, \forall j \in N \backslash \{j_0\}$ *and* $c_{j_0} = \sum_{k \in (N' \backslash N) \cup \{j_0\}} c'_k$*,*
3. $\alpha(j) = \alpha'(j), \forall j \in N \backslash \{j_0\}$ *and* $\alpha(j) = \alpha'(j_0), \forall j \in (N' \backslash N) \cup \{j_0\}$*,*

it holds

$$R_{j_0}(\mathcal{I}, N, E, c, \alpha) = \sum_{j \in (N' \backslash N) \cup \{j_0\}} R_j(\mathcal{I}, N', E, c', \alpha').$$

Note that in NMS we move from the allocation problem with set of claimants N to the problem with set of claimants N', i.e., one of the claimants is splitted into several new claimants, one of whom has the same name as in N. However, in $NMRM$ we move from the problem in N' to the problem in N, i.e., several claimants merge into one claimant who has the same name as in N', but all merged claimants are homologous. Thus, we are only considering the merging of homologous claimants. For this reason we call this axiom non-manipulability by "restricted" merging. Nevertheless, it seems reasonable from a perspective of symmetry of both properties, because when one claimant is splitted into several new claimants, these are homologous in the new problem.

CPA satisfies all properties above mentioned. We establish this in the following theorem.

Theorem 1. *CPA for multi-issue bankruptcy problems with crossed claims satisfies* $PEFF$*,* ETE*,* GMA*,* $CONS$*,* NMS*, and* $NMRM$*.*

Proof. We go axiom by axiom.

- CPA satisfies $PEFF$ and GMA by definition.
- If two claimants are equal, then CPA allocates both the same, since the procedure to calculate the rule treats, in each step, all active equal claimants egalitarianly, so if two claimants are equal, they stop receiving at the same step. Therefore, CPA satisfies ETE.

- Given $(\mathcal{I}, N, E, c, \alpha) \in \mathcal{MAC}$ and $(\mathcal{I}', N', E'^{CPA}, c|_{N'}, \alpha) \in \mathcal{MAC}$ the reduced problem associated with $N' \subset N$. Let us consider the following sets obtained from the application of CPA to $(\mathcal{I}, N, E, c, \alpha)$:

$$\mathcal{A}^s = \mathcal{N}^s \setminus \mathcal{N}^{s+1}, s = 1, \ldots, r, \text{ and } \mathcal{B}^s = \mathcal{I}^s \setminus \mathcal{I}^{s+1}, s = 1, \ldots, r.$$

  We now consider the following sets: $N' \cap \mathcal{A}^s$, $s = 1, \ldots, r$. Taking into account the definitions of E'^{CPA}, $\mathcal{N}^s$, and $\mathcal{I}^s$, it is evident that claimants in $N' \cap \mathcal{A}^s$ cannot receive more than ρ^s times their claims, because otherwise, the available amounts of issues $e_i'^{CPA}$, $i \in \mathcal{B}^s$, would be exceeded. Thus, from the definition of CPA, claimants in $N' \cap \mathcal{A}^s$ have to receive exactly ρ^s times their claims. Therefore, the claimants in $(\mathcal{I}', N', E'^{CPA}, c|_{N'}, \alpha)$ receive the same as in $(\mathcal{I}, N, E, c, \alpha)$. Hence, CPA satisfies consistency.

- Note that when one claimant splits into several new claimants, CPA for the new problem will have the same number of iterations as in the original one, since the claim for each issue will be obviously the same in each step. Therefore, all split claimants will receive the same proportion of their claims which coincides with the proportion obtained by the split claimant in the original problem. Thus, the aggregate allocation of the split claimants in the new problem coincides with the allocation of the split claimant in the original problem.

- When two homologous claimants merge into a new one claimant, we can make a completely analogous reasoning as in the case of a claimant splits into several new claimants. Therefore, CPA also satisfies NMRM.

□

5 Characterization

In this section, the aim is to achieve a better knowledge of the CPA rule for $\mathcal{MAC}$ by describing it in a unique way as a combination of some reasonable axioms. We characterize the CPA rule by means of $PEFF$, ETE, GMA, $CONS$, and NMS. Therefore, the CPA rule can be considered as a desirable way to distribute a set of issues among their claimants. Before giving the characterization of CPA, we need the following lemmas.

Lemma 1. *Let $(\mathcal{I}, N, E, c, \alpha) \in \mathcal{MAC}$, such that $|\mathcal{I}| = 1$. If a rule R satisfies $PEFF$, ETE, and NMS, then*

$$R_i(\mathcal{I}, N, E, c, \alpha) = \frac{c_1}{\sum_{j \in N} c_j} e, \text{ for all } i \in N.$$

Proof. First note that in this case the function α is irrelevant. Let $R_1, R_2, \ldots, R_n$ be the allocations for claimants in N, respectively. By $PEFF$, and the definition of rule, we know that there are $\beta_i \in [0, 1]$, $i \in N$, such that $R_i = \beta_i c_i, i \in N$, and $\sum_{i \in N} \beta_i c_i = e$.

Consider the following chain of problems:

$$(\mathcal{I}, N, E, c, \alpha) \longrightarrow (\mathcal{I}, N(q), E, c(q), \alpha)$$

where the first problem is the original, the second is the problem in which each claimant i is split into a number of identical claimants k_i, $k_i \in \mathbb{N}_+$, with claims exactly equal to $q \in \mathbb{R}_+$. We now distinguish two cases:

1. $c_1, c_2, \ldots, c_n \in \mathbb{Q}_+$. In this case, there exists $q \in \mathbb{Q}_+$ such that $c_i = k_i q, k_i \in \mathbb{N}_+, i \in N$. Now, by $PEFF$ and ETE, we have that

$$R_j(\mathcal{I}, N(q), E, c(q), \alpha) = \beta q, j \in N(q).$$

On the other hand, by NMS, it holds for every $i \in N$ that

$$\beta_i c_i = k_i \beta q = \beta c_i \Rightarrow \beta_i = \beta$$

2. $c_1, c_2, \ldots, c_n \in \mathbb{R}_+$. In this case, for each $\varepsilon > 0$, there exists $q \in \mathbb{R}_+$ such that $c_i = k_i(q)q + \varepsilon_i(q), k_i(q) \in \mathbb{N}_+$, and $\varepsilon_i(q) < \frac{\varepsilon}{n}$, for all $i \in N$.

Now, by ETE, we have the following equality for the second problem:

$$\left(\sum_{i \in N} k_i(q)\right) \beta(q) q + \sum_{i \in N} \delta_i(q) \varepsilon_i(q) = e,$$

where $\beta(q) \in [0,1]$, and $\delta_i(q) \in [0,1]$ for all $i \in N$. This equality can be written as follows:

$$\beta(q) \sum_{i \in N} (c_i - \varepsilon_i(q)) + \sum_{i \in N} \delta_i(q) \varepsilon_i(q) = e,$$

or equivalently,

$$\frac{e}{\sum_{i \in N} c_i} - \beta(q) = \frac{\sum_{i \in N} \{(\delta_i(q) - \beta(q)) \varepsilon_i(q)\}}{\sum_{i \in N} c_i},$$

taking limits on both sides when q goes to zero, we obtain that $\lim_{q \to 0^+} \beta(q) = \frac{e}{\sum_{i \in N} c_i}$.

On the other hand, by NMS, for each q and for each $i \in N$,

$$\beta_i c_i = k_i(q) \beta(q) q + \delta_i(q) \varepsilon_i(q) = \beta(q) c_i + (\delta_i(q) - \beta(q)) \varepsilon_i(q).$$

Since $\lim_{q \to 0^+} \beta(q) = \frac{e}{\sum_{i \in N} c_i}$, $\beta_i = \frac{e}{\sum_{i \in N} c_i}$, for each $i \in N$.

$\square$

Lemma 2. *For each problem $(\mathcal{I}, N, E, c, \alpha) \in \mathcal{MAC}$, and each rule R that satisfies $PEFF$, ETE, GMA and NMS, if for each $N' \subset N$ with $|N'| = |N| - 1$, we have $R_i(\mathcal{I}, N, E, c, \alpha) = CPA_i(\mathcal{I}', N', E'^R, c|_{N'}, \alpha)$ for all $i \in N'$, then $R(\mathcal{I}, N, E, c, \alpha) = CPA(\mathcal{I}, N, E, c, \alpha)$.*

Proof. We first prove that if there is $R_i = R_i(\mathcal{I}, N, E, c, \alpha) = CPA_i(\mathcal{I}, N, E, c, \alpha)$, then the result holds. Indeed, let us consider R in the conditions of the statement, and $R_i = CPA_i(\mathcal{I}, N, E, c, \alpha)$. We now consider $N' = N \backslash \{i\}$, since $R_i = CPA_i(\mathcal{I}, N, E, c, \alpha)$,

$$(\mathcal{I}', N', E'^{R}, c|_{N'}, \alpha) = (\mathcal{I}', N', E'^{CPA}, c|_{N'}, \alpha).$$

By hypothesis, we have that for all $k \in N'$,

$$R_k(\mathcal{I}, N, E, c, \alpha) = CPA_k(\mathcal{I}', N', E'^{R}, c|_{N'}, \alpha) = CPA_k(\mathcal{I}', N', E'^{CPA}, c|_{N'}, \alpha).$$

Moreover, since CPA satisfies consistency,

$$CPA_k(\mathcal{I}, N, E, c, \alpha) = CPA_k(\mathcal{I}', N', E'^{CPA}, c|_{N'}, \alpha) \text{ for all } k \in N'.$$

Therefore, $CPA_k(\mathcal{I}, N, E, c, \alpha) = R_k(\mathcal{I}, N, E, c, \alpha)$ for all $k \in N'$.

Let us consider R in the conditions of the statement and we assume without loss of generality that $\beta_1 = \frac{R_1}{c_1} \leq \beta_2 = \frac{R_2}{c_2} \leq \ldots \leq \beta_{|N|} = \frac{R_{|N|}}{c_{|N|}}$, where for the sake of simplicicty we denote $R_k(\mathcal{I}, N, E, c, \alpha)$ by R_k for each $k \in N$.

First, for $\alpha(1)$, for every $i \in \alpha(1)$, we take $\gamma_i > 0$ such that $\gamma_i \sum_{j \in N : i \in \alpha(j)} c_j = e_i$. We now define $\gamma_1 = \min\{\gamma_i : i \in \alpha(1)\}$, and we assume that γ_1 is without loss of generality obtained for issue 1.

Second, $\beta_1 \leq \gamma_1$, otherwise we would have that

$$\sum_{j : 1 \in \alpha(j)} \beta_j c_j \geq \beta_1 \sum_{j : 1 \in \alpha(j)} c_j > \gamma_1 \sum_{j : 1 \in \alpha(j)} c_j = e_1,$$

which is a contradiction.

Third, by Lemma 1 and GMA, $R_1 \geq \min\{\frac{c_1}{\sum_{j : i \in \alpha(j)} c_j} e_i : i \in \alpha(1)\} = \gamma_1 c_1$. Therefore, $\beta_1 = \gamma_1$. Now, by $PEFF$, $\beta_j = \gamma_1$ for all $j \in N$ such that $1 \in \alpha(j)$.

Fourth, for each $N' \subset N$ with $1 \in N'$ and $|N'| = |N| - 1$, by hypothesis and the definition of CPA, we have that β_1 coincides with the λ^1's of the iterative procedures for calculating each $CPA(\mathcal{I}', N', E'^{R}, c|_{N'}, \alpha)$. This implies that $\beta_1 = \lambda^1 = \min\{\lambda_i^1 : i \in \mathcal{I}'^1\}$, for each $N' = N \backslash \{k\}, k \in N \backslash \{1\}$, where

$$\lambda_i^1 \sum_{j \in \mathcal{N}'^1 : i \in \alpha(j)} c_j^1 = e_i^1 - \delta(i, k) \beta_k c_k, \forall i \in \mathcal{I}',$$

where $\delta(i, k) = 1$ if $i \in \alpha(k)$, and 0 otherwise. Since $R_1 = \beta_1 c_1 = CPA_i(\mathcal{I}', N', E'^{R}, c|_{N'}, \alpha)$, $\arg\min\{\lambda_i^1 : i \in \mathcal{I}'^1\} \in \alpha(1)$, otherwise claimant 1 would obtain more than $\beta_1 c_1$ which is a contradiction. In particular, the minimum will be attained, although possibly among others, at issue 1, because $\beta_j = \gamma_1$ for all $j \in N$ such that $1 \in \alpha(j)$.

On the other hand, by definition of CPA, we have that $\lambda^1 = \min\{\lambda_i^1 : i \in \mathcal{I}^1\}$, where

$$\lambda_i^1 \sum_{j \in \mathcal{N}^1 : i \in \alpha(j)} c_j^1 = e_i^1, \forall i \in \mathcal{I}.$$

This λ^1 can be also obtain by solving the following simple linear program:

$$\begin{array}{lll} \lambda^1 = & \max & \lambda \\ & \text{subject to} & \lambda \sum_{j\in\mathcal{N}^1: i\in\alpha(j)} c_j^1 \leq e_i^1, \forall i \in \mathcal{I} \\ & & \lambda \geq 0 \end{array}$$

It is obvious that $\lambda^1 \leq \gamma_1$, because of the definition of γ_1. Now, since R satisfies $PEFF$, γ_1 is a feasible solution for the linear program above and $\lambda^1 \leq \gamma_1$, γ_1 is an optimal solution of the problem. Therefore, the inequality associated with issue 1 is saturated in the optimal solution and by definition of CPA claimant 1 will obtain $\gamma_1 c_1 = \beta_1 c_1 = R_1$, i.e., $R_1(\mathcal{I}, N, E, c, \alpha) = CPA_1(\mathcal{I}, N, E, c, \alpha)$. □

Theorem 2. *CPA is the only rule that satisfies $PEFF$, ETE, GMA, $CONS$, and NMS.*

Proof. We distinguish three cases, depending on the number of claimants in the problem.

1. $|N| = 1$. In this case, all rules that satisfy $PEFF$ coincide with CPA.
2. $|N| = 2$. We distinguish two cases:
   (a) $\alpha(1) \cap \alpha(2) = \varnothing$. In this situation, since the rule satisfies $PEFF$ we can consider two separate problems of only one claimant each. Now by applying the case $|N| = 1$, all rules that satisfy $PEFF$ coincide with CPA.
   (b) $\alpha(1) \cap \alpha(2) \neq \varnothing$. We consider other two cases:
      i. $\alpha(1) = \alpha(2)$. By GMA, Lemma 1, and the definition of rule,

$$R_1 = \frac{c_1}{c_1 + c_2} e, \text{ and } R_2 = \frac{c_2}{c_1 + c_2} e,$$

      where e is the minimum of the available amounts of the issues.
      ii. $\alpha(1) \neq \alpha(2)$. First note that by Lemma 1, we know that for every single issue we obtain the proportional distribution of the available amount among the corresponding claimants. Therefore, in order to apply GMA, we can consider without loss of generality the two situations shown in Figure 2.

      We next analyze the two situations in Figure 2:

      A. By GMA, $c_1 \geq e_1$, $c_2 \geq e_3$, and $c_1 + c_2 \geq e_2$,

$$R_1 \geq \min\{e_1, \frac{c_1}{c_1 + c_2} e_2\}, \text{ and } R_2 \geq \min\{\frac{c_2}{c_1 + c_2} e_2, e_3\}$$

      If $\min\{e_1, \frac{c_1}{c_1+c_2} e_2\} = e_1$, then $R_1 = e_1$, and by $PEFF$ $R_2 = \min\{c_2, e_2 - e_1, e_3\}$. If $\min\{e_1, \frac{c_1}{c_1+c_2} e_2\} = \frac{c_1}{c_1+c_2} e_2$, then we have two possibilities:

      - $\min\{\frac{c_2}{c_1+c_2} e_2, e_3\} = e_3$, then $R_2 = e_3$, and by $PEFF$ $R_1 = \min\{c_1, e_2 - e_3, e_1\}$.

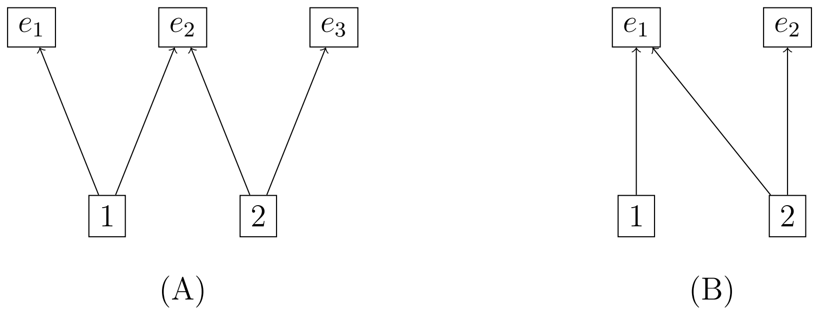

Figure 2: Basic 2-claimants situations when $\alpha(1) \cap \alpha(2) \neq \varnothing$ and $\alpha(1) \neq \alpha(2)$.

- $\min\{\frac{c_2}{c_1+c_2}e_2, e_3\} = \frac{c_2}{c_1+c_2}e_2$, then $R_1 = \frac{c_1}{c_1+c_2}e_2$ and $R_2 = \frac{c_2}{c_1+c_2}e_2$.

B. By GMA, $c_1 \geq e_1$, $c_2 \geq e_3$, and $c_1 + c_2 \geq e_2$,

$$R_1 \geq \frac{c_1}{c_1+c_2}e_1, \text{ and } R_2 \geq \min\{\frac{c_2}{c_1+c_2}e_1, e_2\}$$

Now reasoning as in the previous case,

$$R_1 = \frac{c_1}{c_1+c_2}e_1, \text{ and } R_2 = \frac{c_2}{c_1+c_2}e_1$$

or

$$R_1 = \min\{c_1, e_1 - e_2\}, \text{ and } R_2 = e_2$$

3. $|N| = 3$. Let R be a rule that satisfies $PEFF$, ETE, GMA, $CONS$, and NMS, and let $(\mathcal{I}, N, E, c, \alpha) \in \mathcal{MAC}$, then we have that

$$R(\mathcal{I}, N, E, c, \alpha) = CPA(\mathcal{I}, N, E, c, \alpha).$$

Indeed, for each $N' = \{i_1, i_2\} \subset N$ such that $|N'| = 2$, since R satisfies CONS,

$$R_{i_k}(\mathcal{I}', N', E'^R, c|_{N'}, \alpha) = R_{i_k}(\mathcal{I}, N, E, c, \alpha), k = 1, 2,$$

and since $|N'| = 2$, we have that

$$R_{i_k}(\mathcal{I}', N', E'^R, c|_{N'}, \alpha) = CPA_{i_k}(\mathcal{I}', N', E'^R, c|_{N'}, \alpha), k = 1, 2.$$

Since we can take all possible $N' = \{i_1, i_2\} \subset N$, by Lemma 2

$$R(\mathcal{I}, N, E, c, \alpha) = CPA(\mathcal{I}, N, E, c, \alpha).$$

4. $|N| \leq k$. Let us suppose that for each $(\mathcal{I}, N, E, c, \alpha)$ with $|N| \leq k$, $R(\mathcal{I}, N, E, c, \alpha) = CPA(\mathcal{I}, N, E, c, \alpha)$.

5. $|N| = k + 1$. For each $N' \subset N$ such that $|N'| = k$, since R satisfies CONS,

$$R_i(\mathcal{I}', N', E'^R, c|_{N'}, \alpha) = R_i(\mathcal{I}, N, E, c, \alpha), i \in N'.$$

and since $|N'| \leq k$, we have that

$$R_i(\mathcal{I}', N', E'^R, c|_{N'}, \alpha) = CPA_i(\mathcal{I}', N', E'^R, c|_{N'}, \alpha), i \in N'.$$

Finally, since we can take all possible $N' \subset N$ with $|N'| = k$, by Lemma 2,

$$R(\mathcal{I}, N, E, c, \alpha) = CPA(\mathcal{I}, N, E, c, \alpha).$$

□

Proposition 2. *Properties in Theorem 2 are logically independent.*

Proof. We consider the four posibilities:

- (No $PEFF$) The null rule satisfies all properties but $PEFF$.
- (No ETE) Consider an order on the set of claimants and a rule which reimburses each claimant all that can be, in that order, until it is not possible to do it. If we assume that when a claimant splits into several new claimants or some claimants leave, the order in which the claims are attended is preserved, then this rule satisfies $PEFF$, GMA, $CONS$, and NMS, but not ETE.
- (No GMA) Consider a rule that has two phases. In the first phase, each issue is distributed proportionally, but only among those claimants that only demand the corresponding issue. In the second phase, the amounts of each issue are updated down accordingly, and distributed among the rest of the claimants applying CPA. This rule satisfies $PEFF$, ETE, $CONS$, and NMS, but not GMA.
- (No $CONS$) For every problem $(\mathcal{I}, N, E, c, \alpha) \in \mathcal{MAC}$, consider the following rule defined in two steps:

  1. First, we allocate to each claimant j, $\min \{R_j(\{i\}, N_i, e_i, c|_{N_i}) : i \in \alpha(j)\}$.
  2. Next, we revise down the available amounts of issues and the claims, and we assume without loss of generality that $e'_1 \leq e'_2 \leq \ldots \leq e'_m$. Then we begin to distribute each state proportionally among the claimants, starting from the smallest to the largest quantity available. It is not until one state has been fully distributed or the claimants fully satisfied that we move on to the next updating the claims. We continue until all the states have been distributed as much as possible.

  The allocation to each claimant is the sum of everything that she has obtained in each of the steps of the procedure described.

  By definition this rule satisfies GMA, $PEFF$, and ETE. Moreover, using arguments similar to those used in Theorem 1, it can be shown that this rule satisfies NMS. However, it does not satisfies $CONS$ since this rule does not coincide with CPA, and if we consider reduced problems with $|N'| = 2$, by $PEFF$, ETE, GMA, and NMS, we obtain the allocations prescribed by CPA.

- (No NMS) The CEA rule for MAC satisfies all properties but NMS (Acosta-Vega et al., 2021).

□

6 Conclusions

In allocation problems the concept of proportionality is put into practice with the well-known proportional rule. This rule has been extensively studied in the literature from many different point of views and for many allocation models. Focusing on bankruptcy models and their extensions to the multi-issue case, the proportional rule has been characterized in the context of bankruptcy problems in Chun (1988) and de Frutos (1999). In both papers, non-manipulability plays a central role in the axiomatic characterization of the proportional rule. For multi-issue allocation problems, Ju et al. (2007) and Moreno-Ternero (2009) introduce two different definitions of proportional rule following two different approaches. Moreover, Ju et al. (2007) and Bergantiños et al. (2010) provides charaterizations of both proportional rules. Again, in both approaches, non-manipulability is an essential property. In this paper, we introduce a definition of proportional rule, that we call constrained proportional awards rule, for multi-issue allocation problems with crossed claims and provide a characterization of it. Once again, non-manipulability is used. Therefore, we fill a gap in the literature of proportional distributions in allocation problems in line with the previous studies.

Futher research will include the introduction and analysis, in this context, of the ramdon arrival rule (O'Neill, 1982). This rule is related to the well-known Shapley value (Shapley, 1953), see Algaba et al. (2019b) for an updating on theoretical and applied aspects about this value. The stydy of the Talmud rule introduced by Aumann and Maschler (1985) would be also of interest in this setting.

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On priority in multi-issue bankruptcy problems with crossed claims

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Abstract

In this paper, we analyze the problem of how to adapt the concept of priority to situations where several perfectly divisible resources have to be allocated among certain set of agents that have exactly one claim which is used for all resources. In particular, we introduce constrained sequential priority rules and two constrained random arrival rules, which extend the classical sequential priority rules and the random arrival rule to these situations. Moreover, we provide an axiomatic analysis of these rules.

Keywords: Allocation problem, multi-issue bankruptcy problems, sequential priority rule, random arrival rule

1 Introduction

There is a vast literature on problems in which a resource must be allocated among a set of claimants. A relevant allocation problem arises when there is a (perfectly divisible) resource (for example, money, water, time...) over which there is a set of claimants who have rights or demands, but the resource is scarce. This problem is known as bankruptcy problem (O'Neill, 1982; Aumann and Maschler, 1985). Many allocation rules have been defined to provide a solution to this problem (see Thomson (2003, 2015, 2019) for a detailed inventory of rules and their axiomatic analysis). One of the most prominent allocation rules is the random arrival rule (O'Neill, 1982) also known as the run-to-the-bank rule (Young, 1994). For this rule the following dynamic interpretation can be given. The problem is solved by holding a race in such a way that the first claimant who arrives is satisfied with as much of the resource, the second is satisfied with as much of the resource as is left, and so on until the

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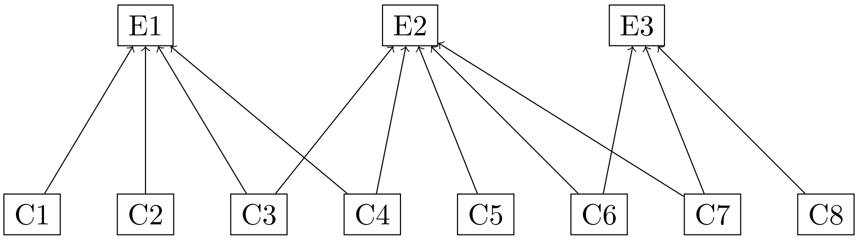

Figure 1: Multi-issue problem with crossed claims.

resource is exhausted. If we consider that all orders of arrival are equally likely, the random arrival rule is the expected value. Likewise, we can consider that each of these orders of arrival is a priority relationship defined ex ante over the claimants according to some criterion, and thus we would have a rule for each of these orders, these rules are called sequential priority rules. Moreover, Sánchez-Soriano (2021) extends the sequential priority rules and, their average, the random arrival rule to the case in which withdrawals are bounded from above.

An extension of bankruptcy problems are multi-issue allocation problems (Calleja et al., 2005). In these situations there is also a (perfectly divisible) resource but it can be distributed between several issues, and a (finite) number of agents have claims on each of those issues, so that the total claim is higher than the available resource. This problem is also solved by means of allocation rules and there are different ways to do it. Calleja et al. (2005) introduce two extensions of the random arrival rule to the context of multi-issue allocation problems. González-Alcón et al. (2007) propose a new extension of the random arrival rule to this context. An alternative approach to multi-issue bankruptcy problems is when the resource has already been divided a priori into the issues according to exogenous criteria but claimants continue having a claim for each issue (Izquierdo and Timoner, 2016).

However, the situation described in Figure 1 does not fit to any multi-issue allocation bankruptcy problem as referred in the previous paragraph, but to a multi-issue bankruptcy problem with crossed claims introduced by Acosta-Vega et al. (2021, 2022). These describe situations in which there are several (perfect divisible) resources and a (finite) set of agents who have claims on them, but only one claim (not a claim for each resource) with which one or more resources are requested. The total claim for each resource exceeds its availability.

In this paper, in order to solve allocation problems as the described in Figure 1, we introduce sequential priority rules and, their average, the random arrival rule for multi-issue allocation problems with crossed claims that naturally extend the sequential priority rule and the random arrival rule for single issue allocation problems.

The rest of the paper is organized as follows. Section 2 presents multi-issue allocation problems with crossed claims (MBC), and the concept of rule for these problems. In Section 3, constrained sequential priority rules and, their average, the constrained random arrival rule for multi-issue allocation problems with crossed claims are defined. In Section 4, we present several properties which are interesting in the context of MBC problems. Section 6 concludes.

2 Multi-issue bankruptcy problems with crossed claims

Bankruptcy problems concern allocating the available resource (estate) to a number of individuals (claimants) according to their demands/claims when all these demands add up to more than the available resource. Mathematically, a bankruptcy problem is given by a triplet (N, E, c), where N is the set of claimants, $E \in \mathbb{R}_+$ is a perfectly divisible amount of resource (the *issue* or *estate*) to be divided, and c is the vector of demands, such that $C = \sum_{j \in N} c_j > E$. The set of all bankruptcy problems with set of agents N is denoted by BP^N.

The main goal in a bankruptcy problem is to find an allocation which is as fair as possible, taking into account the demands of the claimants. An allocation rule R is a function defined for each N from BP^N to $\mathbb{R}_+^N$ such that for each $(N, E, c) \in BP^N$, $R(N, E, c) \in \mathbb{R}_+^N$ satisfies the following two conditions:

1. $\sum_{i \in N} R_i(N, E, c) = E$.
2. $0 \leq R_i(N, E, c) \leq c_i$, $\forall i \in N$.

Many allocation rules have been proposed depending on the principle(s) of fairness or rationale used. Allocation rules defined on the basis of the existence of a priority order on the claimants and that satisfies their demands sequentially according to that priority order are the following:

The *sequential priority* rule associated with $\sigma \in \Sigma(N)$ (SP^σ) is defined by setting

$$SP_i^\sigma(E, c) = min\left\{ c_i,\ max\{0,\ E - \sum_{j \in N : \sigma(j) < \sigma(i)} c_j\} \right\},\ \forall i \in N,$$

where $\Sigma(N)$ is the set of all possible orders of N. The family of sequential priority rules, SP-family, consists of all these rules.

The *random arrival* rule (RA) (O'Neill, 1982) is defined by setting

$$RA_i(E, c) = \frac{1}{n!} \sum_{\sigma \in \Sigma(N)} SP_i^\sigma(E, c),\ \forall i \in N.$$

The random arrival rule coincides with the Shapley value (Shapley, 1953) of the pessimistic bankruptcy game (O'Neill, 1982). This is a relevant fact, because the Shapley value is one of the most prominent solution concepts in cooperative game theory (Roth, 1988; Algaba et al., 2019a, 2019b).

We now consider a situation where there is a finite set of issues $\mathcal{I} = \{1, 2, ..., m\}$ such that there is a perfectly divisible amount e_i of each issue i. Let $E = (e_1, e_2, ..., e_m)$ be the vector of available amounts of issues. There is a finite set of claimants $N = \{1, 2, ..., n\}$ such that each claimant j claims c_j. Let $c = (c_1, c_2, ..., c_n)$ be the vector of claims. However, each claimant claims to different sets of issues, in general. Thus, let α be a set-valued function that associates with every $j \in N$ a set $\alpha(j) \subset \mathcal{I}$. In fact, $\alpha(j)$ represents the issues to which claimant j asks for. Furthermore, $\sum_{j : i \in \alpha(j)} c_j > e_i$, for all $i \in \mathcal{I}$, otherwise, those issues could be discarded from the problem because they do not impose any limitation, and so the allocation would be trivial. Therefore, a multi-issue bankruptcy problem with crossed claims (MBC in short) is defined by a 5-tuple $(\mathcal{I}, N, E, c, \alpha)$, and the family of all these problems is denoted by $\mathcal{MBC}$.

Given a problem $(\mathcal{I}, N, E, c, \alpha) \in \mathcal{MBC}$, a *feasible allocation* for it, it is a vector $x \in \mathbb{R}^N$ such that:

1. $0 \leq x_i \leq c_i$, for all $i \in N$.
2. $\sum_{i \in N: j \in \alpha(i)} x_i \leq e_j$, for all $j \in M$,

and we denote by $A(\mathcal{I}, N, E, c, \alpha)$ the set of all its feasible allocations.

An *allocation rule* for multi-issue bankruptcy problems with crossed claims is a mapping R that associates with every $(\mathcal{I}, N, E, c, \alpha) \in \mathcal{MBC}$ a unique feasible allocation $R(\mathcal{I}, N, E, c, \alpha) \in A(\mathcal{I}, N, E, c, \alpha)$.

3 Constrained sequential priority rules for $\mathcal{MBC}$ problems

Sequential priority rules are defined when there is a priority order defined over the set of claimants, in such a way that if a claimant has a higher priority than another, the first must be satisfied first in her demand. This rule simply allocates the resource according to the scheme of first come first served, where the order of arrival is given by the priority relationship. The question in $\mathcal{MBC}$ problems is what "be satisfied with as much of the resource as is left" means. In the context of one-issue allocation problems, "be satisfied with as much of the resource as is left" is a simple idea since there is only one issue. How to extrapolate this to the MBC situations. To answer this question, we introduce the constrained sequential priority rule (CSP in short) which follows the same process as sequential priority rules but taking into account that there are several resources or issues. This rule is formally defined below.

Definition 1. *Let* $(\mathcal{I}, N, E, c, \alpha) \in \mathcal{MBC}$*, and* $\sigma \in \Sigma(N)$*, the* constrained sequential priority rule *associated with* $\sigma \in \Sigma(N)$ *for* $(\mathcal{I}, N, E, c, \alpha)$*,* $CSP^\sigma(\mathcal{I}, N, E, c, \alpha)$*, is defined as follows:*

$$CSP_j^\sigma(\mathcal{I}, N, E, c, \alpha) = min\left\{ c_j,\ max\left\{ 0,\ \min_{i \in \alpha(j)} \{e_i - \sum_{\substack{k \in N: i \in \alpha(k) \\ \sigma(k) < \sigma(j)}} c_k\} \right\} \right\},\ \forall j \in N,$$

where $\Sigma(N)$ *is the set of all possible orders of* N*.*

For each $\sigma \in \Sigma(N)$, the iterative procedure of CSP^σ is well-defined and always leads to a single point. Moreover, by definition, it ends in a finite number of steps, at most $|N|$. Finally, when we have a one-issue allocation problem, then we obtain SP^σ. Therefore, this definition extends sequential priority rules to the context of MBC. The following example illustrates how CSP works.

Example 1. *Consider the situation described by Figure 1 with* $\mathcal{I} = \{1, 2, 3\}$, $N = \{1, 2, 3, 4, 5, 6, 7, 8\}$, $E = (9, 12, 9)$, $c = (3, 5, 4, 3, 5, 4, 3, 5)$, *and* $\alpha(1) = \{1\}, \alpha(2) = \{1\}, \alpha(3) = \{1, 2\}, \alpha(4) = \{1, 2\}, \alpha(5) = \{2\}, \alpha(6) = \{2, 3\}, \alpha(7) = \{2, 3\}, \alpha(8) = \{3\}$. *If we take for example the priority order* $\sigma = 1\,3\,5\,7\,2\,4\,6\,8$, CSP^σ *is calculated sequentially as follows:*

First, claimant 1 is attended:

$$CSP_1^\sigma(\mathcal{I}, N, E, c, \alpha) = 3.$$

Second, the resources are updated down $E = (6, 12, 9)$ *and claimant 3 is attended:*

$$CSP_3^\sigma(\mathcal{I}, N, E, c, \alpha) = 4.$$

Third, the resources are updated down $E = (2, 8, 9)$ *and claimant 5 is attended:*

$$CSP_5^\sigma(\mathcal{I}, N, E, c, \alpha) = 5.$$

Fourth, the resources are updated down $E = (2, 3, 9)$ *and claimant 7 is attended:*

$$CSP_7^\sigma(\mathcal{I}, N, E, c, \alpha) = 3.$$

Fifth, the resources are updated down $E = (2, 0, 6)$ *and claimant 2 is attended:*

$$CSP_2^\sigma(\mathcal{I}, N, E, c, \alpha) = 2.$$

Sixth, the resources are updated down $E = (0, 0, 6)$ *and claimant 4 cannot be attended because the resources she claims are exhausted, as a result she gets 0. Therefore, we move to the next in the priority order. Again, claimant 6 cannot be attended because one of the resources she claims is exhausted, as a result she gets 0. So, we move to the next. Claimant 8 is attended:*

$$CSP_8^\sigma(\mathcal{I}, N, E, c, \alpha) = 5.$$

Seventh, the resources are updated down $E = (0, 0, 1)$ *and the sequential procedure ends. The final allocation is*

$$CSP^\sigma(\mathcal{I}, N, E, c, \alpha) = (3, 2, 4, 0, 5, 0, 3, 5).$$

Once constrained sequential priority rules associated with an order have been defined, the constrained random arrival rule is simply defined as their average.

Definition 2. *Let* $(\mathcal{I}, N, E, c, \alpha) \in \mathcal{MBC}$, *the* constrained random arrival rule *for* $(\mathcal{I}, N, E, c, \alpha)$, $CRA(\mathcal{I}, N, E, c, \alpha)$, *is defined as follows:*

$$CRA_j(\mathcal{I}, N, E, c, \alpha) = \frac{1}{n!} \sum_{\sigma \in \Sigma(N)} CSP_j^\sigma(\mathcal{I}, N, E, c, \alpha),\ \forall j \in N,$$

where $\Sigma(N)$ *is the set of all possible orders of* N.

Although the definition of the rule is simple, it is immediately apparent that the biggest problem is in its computation. For Example 1, it would be necessary to calculate 8!=40320 sequential processes, which consumes a large amount of time.

4 Properties

In this section, we present several properties which are interesting in the context of MBC problems. These properties are related to efficiency, fairness, priority, monotonicity and consistency.

First, we introduce two concepts related to two claimants comparisons. In MBC situations claimants are characterized by two elements: their claims and the issues to which they claim. Therefore, both should be taken into account when establishing comparisons among them.

Definition 3. *Let* $(\mathcal{I}, N, E, c, \alpha) \in \mathcal{MBC}$*, and two claimants* $j, k \in N$*, we say they are* homologous*, if* $\alpha(j) = \alpha(k)$*; and they are called* equal*, if they are homologous and* $c_j = c_k$*.*

Next, we give a set of properties which are very natural and reasonable for an allocation rule in MBC situations.

The first property relates to efficiency. In allocation problems is desirable that the resources to be fully distributed, but in MBC situations this is not always possible (see Acosta-Vega, 2021). Therefore, a weaker version of that is considered in which only is required that there is no a feasible allocation in which at least one of the claimants receive more. This is established in the following axiom.

Axiom 1 (PEFF)**.** *Given a rule* R*, it satisfies* Pareto efficiency*, if for every problem* $(\mathcal{I}, N, E, c, \alpha) \in \mathcal{MBC}$*, there is no a feasible allocation* $a \in \mathbb{R}^N_+$ *such that* $a_j \geq R_j(\mathcal{I}, N, E, c, \alpha), \forall j \in N$*, with at least one strict inequality.*

Note that $PEFF$ implies that at least the available amount of one issue is fully distributed, and no amount is left of an issue undistributed, if it is possible to do so. However, it does not require that all available amounts of the issues have to be fully distributed. On the other hand, a feasible allocation that satisfies the condition in Axiom 1 is called Pareto efficient.

The second property states that equal claimants should receive the same in the final allocation. This is a basic requirement of fairness and non-arbitrariness. This is defined in the following axiom.

Axiom 2 (ETE)**.** *Given a rule* R*, it satisfies* equal treatment of equals*, if for every problem* $(\mathcal{I}, N, E, c, \alpha) \in \mathcal{MBC}$ *and every pair of equal claimants* $j, k \in N$*,* $R_j(\mathcal{I}, N, E, c, \alpha) = R_k(\mathcal{I}, N, E, c, \alpha)$*.*

The third property is a requirement of robustness when some agents leave the problem with their allocations (see Thomson, 2011, 2018). In particular, when a subset of claimants leave the problem respecting the allocations to those who remain, then it seems reasonable that claimants who leave will receive the same in the new problem as they did in the original. This is formally formulated in the following axiom.

Axiom 3 (CONS)**.** *Given a rule* R*, it satisfies* consistency*, if for every problem* $(\mathcal{I}, N, E, c, \alpha) \in \mathcal{MBC}$*, and* $N' \subset N$*, it holds that*

$$R_j(\mathcal{I}, N, E, c, \alpha) = R_j(\mathcal{I}', N', E'^R, c|_{N'}, \alpha), \text{ for all } j \in N',$$

where $(\mathcal{I}', N', E'^R, c|_{N'}, \alpha) \in \mathcal{MBC}$, *called the* reduced problem associated with N', $\mathcal{I}' = \{i \in \mathcal{I} :$ *there exists* $k \in N'$*such that* $i \in \alpha(k)\}$, $E'^R = (e_1'^R, \ldots, e_m'^R)$ *so that* $e_i'^R = e_i - \sum_{j \in N \setminus N' : i \in \alpha(j)} R_j(\mathcal{I}, N, E, c, \alpha)$, *for all* $i \in \mathcal{I}'$, *and* $c|_{N'}$ *is the vector whose coordinates correspond to the claimants in* N'.

The fourth property is related to the priority order.

Axiom 4 (PRI)**.** *Given a rule* R*, it satisfies* priority*, if for every problem* $(\mathcal{I}, N, E, c, \alpha) \in \mathcal{MBC}$ *and every pair of homologous claimants* $j, k \in N$*, if* $\sigma(j) < \sigma(k)$*,* $c_j - R_j(\mathcal{I}, N, E, c, \alpha) \leq c_k - R_k(\mathcal{I}, N, E, c, \alpha)$*.*

Theorem 1. CSP^σ *for multi-issue bankruptcy problems with crossed claims satisfies* $PEFF$*,* $CONS$*, and* PRI*.*

Proof. We go axiom by axiom.

- CSP^σ satisfies $PEFF$ and PRI by definition.
- Given $(\mathcal{I}, N, E, c, \alpha) \in \mathcal{MBC}$ and $(\mathcal{I}', N', E'^{CPA}, c|_{N'}, \alpha) \in \mathcal{MBC}$ the reduced problem associated with $N' \subset N$. We assume that when some claimants leave the problem, they respect the priority order defined by σ in the new problem. Therefore, if we follow the sequential procedure when a claimant in N' has to be attended, what she can get is exactly the same as what she got in the original problem, since what the claimants who remained in the original problem keep is exactly what corresponded to them according to the order of priority defined by σ. Therefore, CSP^σ satisfies $CONS$.

□

It is well-known that the random arrival rule does not satisfy $CONS$, so neither does the constrained random arrival rule.

Theorem 2. CRA *for multi-issue bankruptcy problems with crossed claims satisfies* ETE*.*

Proof. The proof is straightforward by definition of CRA. □

In the context of multi-issue bankruptcy problems with crossed claims, the average operator does not preserve the property of Pareto efficiency and, therefore, CRA does not satisfy this basic property, as shown in the following proposition.

Proposition 1. CRA *rule does not satisfies* $PEFF$*.*

Proof. Consider the problem with $M = \{1, 2\}$, $N = \{1, 2, 3\}$, $E = (4, 8)$, $c = (2, 5, 7)$, and $\alpha(1) = \{1\}, \alpha(2) = \{1, 2\}, \alpha(3) = \{2\}$. CSP for each order and CRA are given by

Order / Claimant	1	2	3
123	2	2	6
132	2	1	7
213	0	4	4
231	0	4	4
312	2	1	7
321	2	1	7
CRA	$\frac{8}{6}$	$\frac{13}{6}$	$\frac{35}{6}$

Table 1: CSP and CRA for Proposition 1.

Note that $\frac{8}{6}+\frac{13}{6} < 4$ and $\frac{8}{6} < 2$, therefore we can improve up to $\frac{11}{6}$ the allocation of agent 1 without exceeding any estate. Consequently, CRA does not satisfy $PEFF$. □

A property that is satisfied by most of the rules for bankruptcy problems is resource monotonicity which is also used in the characterization of sequential priority rules (see Thomson, 2019, Th.11.11). This property simply says that if the available resource increases, allocations to claimants do not decrease. In the particular context of multi-issue bankruptcy problems with crossed claims, this property reads as follows.

Axiom 5 (R-MON). *Given a rule R, it satisfies* resource monotonicity*, if for every problem $(\mathcal{I}, N, E, c, \alpha) \in \mathcal{MBC}$ and $E' \geq E$, $R_j(\mathcal{I}, N, E', c, \alpha) \geq R_j(\mathcal{I}, N, E, c, \alpha)$ for all $j \in N$.*

Although this property seems very weak, constrained sequential priority rules do not satisfy it as the following proposition shows.

Proposition 2. *CSP^σ for multi-issue bankruptcy problems with crossed claims does not satisfy $R - MON$.*

Proof. Consider the problem with $\mathcal{I} = \{1, 2, 3\}$, $N = \{1, 2, 3, 4, 5, 6, 7, 8\}$, $E = (9, 12, 7)$, $c = (3, 5, 4, 3, 5, 4, 4, 5)$, and $\alpha(1) = \{1\}, \alpha(2) = \{1\}, \alpha(3) = \{1, 2\}, \alpha(4) = \{1, 2\}, \alpha(5) = \{2\}, \alpha(6) = \{2, 3\}, \alpha(7) = \{2, 3\}, \alpha(8) = \{3\}$. If we take for example the priority order $\sigma = 1\,3\,5\,7\,2\,4\,6\,8$,

$$CSP^\sigma(\mathcal{I}, N, E, c, \alpha) = (3, 2, 4, 0, 5, 0, 3, 4),$$

and there is nothing left.

However, if $E' = (9, 13, 7)$, we have that

$$CSP^\sigma(\mathcal{I}, N, E', c, \alpha) = (3, 2, 4, 0, 5, 0, 4, 3).$$

Note that claimant 8 obtains less when the available resources increase. Therefore, CSP^σ does not satisfy $R - MON$. □

Another monotonicity property is population monotonicity which says that if all claimants agree that a claimant j will obtain her claim, then the remaining claimants should be worse off after claimant j is fully compensated. This property is satisfied by many bankruptcy rules, including the random arrival rule. Moreover, the random arrival rule is characterized by using this property in Hwang and Wang (2009, Th.1). On the other hand, a property that appears recurrently related to the Shapley value and, therefore, to the random arrival rule is that of balanced contributions (Myerson, 1980; Hart and Mas-Colell, 1989). In the context of bankruptcy problems this property was used by Bergantiños and Méndez-Naya (1997) to characterize the random arrival rule. Moreover, in the context of multi-issue bankruptcy problems was introduced by Lorenzo-Freire et al. (2007) and also used to characterize the random arrival rule. This property requires that claimant j impacts to claimant k's allocation what claimant k impacts to claimant j's allocation. Natural extensions of population monotonicity and balanced contributions to the context of multi-issue bankruptcy problems with crossed claims are the following.

Axiom 6 (P-MON)**.** *P-MON Given a rule R, it satisfies* population monotonicity*, if for every problem* $(\mathcal{I}, N, E, c, \alpha)$*, and each* $j \in N$*,*

$$R_k(\mathcal{I}, N, E, c, \alpha) \geq R_k(\mathcal{I}^{-j}, N, E^{-j}, c_{-j}, \alpha), \text{ for all } k \in N \backslash j,$$

where $\mathcal{I}^{-j}$ *is the set of issues for which claimants in* $N\backslash\{j\}$ *have claims;* $E^{-j} = (e_1^{-j}, \ldots, e_m^{-j})$ *so that* $e_i^{-j} = e_i - c_j$ *if* $i \in \alpha(j)$ *and* $e_i^{-j} = e_i$ *otherwise; and* c_{-j} *is the vector of claims from which* $j-th$ *coordinate has been deleted.*

Axiom 7 (BAL)**.** *Given a rule R, it satisfies* balanced impact*, if for every problem* $(\mathcal{I}, N, E, c, \alpha)$*, and every pair of claimants* $j, k \in N$*,*

$$R_j(\mathcal{I}, N, E, c, \alpha) - R_j(\mathcal{I}^{-k}, N, E^{-k}, c_{-k}, \alpha) = R_k(\mathcal{I}, N, E, c, \alpha) - R_k(\mathcal{I}^{-j}, N, E^{-j}, c_{-j}, \alpha),$$

where $\mathcal{I}^{-h}$ *is the set of issues for which claimants in* $N\backslash\{h\}$ *have claims;* $E^{-h} = (e_1^{-h}, \ldots, e_m^{-h})$ *so that* $e_i^{-h} = e_i - c_h$ *if* $i \in \alpha(h)$ *and* $e_i^{-h} = e_i$ *otherwise; and* c_{-h} *is the vector of claims from which* $h-th$ *coordinate has been deleted.*

As mentioned above, this type of properties are used to characterize Shapley-like solutions as the random arrival is, but in the context of multi-issue bankruptcy problems with crossed claims, the constrained random arrival rule does not satisfy them as the following proposition shows.

Proposition 3. *CRA rule satisfies neither* $P-MON$ *nor* BAL*.*

Proof. Consider the situation described by Figure 2 with three issues $\mathcal{I} = \{1, 2, 3\}$, and three claimants $N = \{1, 2, 3\}$, the available resources are given by $E = (4, 5, 7)$, the claims are given by the vector $c = (3, 4, 5)$, and finally, $\alpha(1) = \{1, 2\}, \alpha(2) = \{2, 3\}, \alpha(3) = \{3\}$.

In order to calculate CRA, we need to consider all possible orders of arrival. These calculations are summarized in Table 2 below.

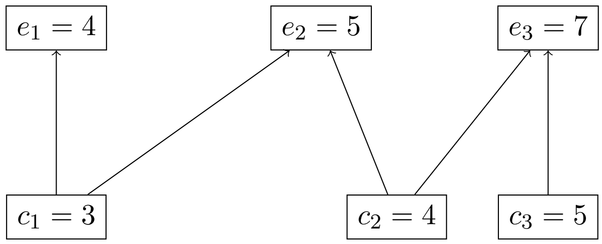

Figure 2: Multi-issue problem with crossed claims with three issues and three claimants.

Order / Claimant	1	2	3
123	3	2	5
132	3	2	5
213	1	4	3
231	1	4	3
312	3	2	5
321	3	2	5
CRA	$\frac{7}{3}$	$\frac{8}{3}$	$\frac{13}{3}$

Table 2: Constrained random arrival rule for Proposition 3.

In Table 3 the calculations to obtain CRA for the two claimants subproblems are shown.

Order	1	2	**Order**	1	3	**Order**	2	3
12	3	2	13	1	3	23	2	5
21	3	2	31	1	3	32	2	5
CRA	3	2		1	3		2	5

Table 3: Constrained random arrival rule for the two claimants subproblems in Proposition 3.

Now, we observe that $CRA_1(\mathcal{I}, N, E, c, \alpha) = \frac{7}{3} < 3 = CRA_1(\mathcal{I}^{-3}, N^{-3}, E^{-3}, c_{-3}, \alpha)$. Therefore, claimant 1 is better off after claimant 3 leaves with her claim. Consequently, CRA does not satisfy $P - MON$.

On the other hand, we have that

$$CRA_1(\mathcal{I}, N, E, c, \alpha) - CRA_1(\mathcal{I}^{-2}, N^{-2}, E^{-2}, c_{-2}, \alpha) = \frac{7}{3} - 1 = \frac{4}{3},$$

and

$$CRA_2(\mathcal{I}, N, E, c, \alpha) - CRA_2(\mathcal{I}^{-1}, N^{-1}, E^{-1}, c_{-1}, \alpha) = \frac{8}{3} - 2 = \frac{2}{3}.$$

Therefore, CRA does not satisfy BAL. $\square$

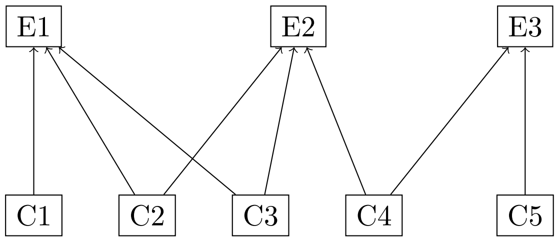

Figure 3: Multi-issue bankruptcy problem with crossed claims in Example 2.

The previous results show the complexity of finding properties that allow axiomatic characterizations of sequential priority rules and the random arrival rule in the context of multi-issue bankruptcy problems with crossed claims, so it is necessary to look for perhaps more specific properties (and likely more technical) to achieve it.

5 An alternative extension of the random arrival rule for MBC

In this section, we introduce a modification of the constrained random arrival rule for multi-issue bankruptcy problems with crossed claims, that we call CRA*. In particular, the CRA* consists of two levels of arrival orders. However, for this rule to be well defined, it is first necessary that the claims are truncated by the minimum of their associated endowments to avoid inconsistencies, i.e., we must take as claims $c_i' = \min\{c_i, \min_{k\in\alpha(i)}\{e_k\}\}, i \in N$, and this must be done before calculating each RA with the updated estates and claims. Once the claims have been truncated, first, all the possible orders of the issues are taken. For the first issue in the order, we calculate the RA rule, then the estates and claims are updated down and the RA rule is calculated for the second issue in the order, and so on until the last issue in the order. Once we have obtained an allocation for each possible order of issues we take their average. Note that CRA* can be seen as a new and different allocation from CRA, but we use the same rationale behind. Moreover, it is also an extension of the random arrival rule since for one issue bankruptcy problems both coincide. The following example illustrates how CRA* works.

Example 2. *Consider the situation described by Figure 3 with* $\mathcal{I} = \{1,2,3\}$*,* $N = \{1,2,3,4,5\}$*,* $E = (9,10,8)$*,* $c = (3,4,3,6,5)$*, and* $\alpha(1) = \{1\}, \alpha(2) = \{1,2\}, \alpha(3) = \{1,2\}, \alpha(4) = \{2,3\}, \alpha(5) = \{3\}$*.*

- *Consider the order of issues* **123***.*

  We start with issue 1. The bankruptcy problem associated with this issue is $e_1 = 9, c_1 = 3, c_2 = 4, c_3 = 3$*. No need to truncate claims.*

Order / Claimant	1	2	3
123	3	4	2
132	3	3	3
213	3	4	2
231	2	4	3
312	3	3	3
321	2	4	3
RA	$2\frac{2}{3}$	$3\frac{2}{3}$	$2\frac{2}{3}$

UPDATE PHASE:

1. *We update the estates and claims:* $e'_1 = 0, e'_2 = 3\frac{2}{3}, e'_3 = 8, c'_1 = \frac{1}{3}, c'_2 = \frac{1}{3}, c'_3 = \frac{1}{3}, c'_4 = 6, c'_5 = 5$.
2. *We truncate the claims by the estates:* $c''_1 = 0, c''_2 = 0, c''_3 = 0, c''_4 = 3\frac{2}{3}, c''_5 = 5$.

Now we continue with issue 2. The bankruptcy problem associated with this issue is $e'_2 = 3\frac{2}{3}, c''_2 = 0, c''_3 = 0, c''_4 = 3\frac{2}{3}$. *Therefore, RA* $= (0, 0, 3\frac{2}{3})$.

UPDATE PHASE:

1. *We update the estates and claims:* $e'_1 = 0, e'_2 = 0, e'_3 = 4\frac{1}{3}, c'_1 = 0, c'_2 = 0, c'_3 = 0, c'_4 = 0, c'_5 = 5$.
2. *We truncate the claims by the estates:* $c''_1 = 0, c''_2 = 0, c''_3 = 0, c''_4 = 0, c''_5 = 4\frac{1}{3}$.

We end with issue 3. The bankruptcy problem associated with this issue is $e'_3 = 4\frac{1}{3}, c''_4 = 0, c''_5 = 4\frac{1}{3}$. *Therefore, RA* $= (0, 4\frac{1}{3})$.
Therefore, the allocation for the order of issues 123 is $(2\frac{2}{3}, 3\frac{2}{3}, 2\frac{2}{3}, 3\frac{2}{3}, 4\frac{1}{3})$.

- *Consider the order of issues* **132**.

  We start with issue 1. This case is identical to order 123. Therefore, RA $= (2\frac{2}{3}, 3\frac{2}{3}, 2\frac{2}{3})$. *The update phase is also the same, but we write it down for the sake of completeness.*

  UPDATE PHASE:

  1. *We update the estates and claims:* $e'_1 = 0, e'_2 = 3\frac{2}{3}, e'_3 = 8, c'_1 = \frac{1}{3}, c'_2 = \frac{1}{3}, c'_3 = \frac{1}{3}, c'_4 = 6, c'_5 = 5$.
  2. *We truncate the claims by the estates:* $c''_1 = 0, c''_2 = 0, c''_3 = 0, c''_4 = 3\frac{2}{3}, c''_5 = 5$.

  Now we continue with issue 3. The bankruptcy problem associated with this issue is $e'_3 = 8, c''_4 = 3\frac{2}{3}, c''_5 = 5$. *Therefore, RA* $= (3\frac{1}{3}, 4\frac{2}{3})$.

  UPDATE PHASE:

1. *We update the estates and claims:* $e'_1 = 0, e'_2 = \frac{1}{3}, e'_3 = 0, c'_1 = 0, c'_2 = 0, c'_3 = 0, c'_4 = \frac{1}{3}, c'_5 = \frac{1}{3}$.
2. *We truncate the claims by the estates:* $c''_1 = 0, c''_2 = 0, c''_3 = 0, c''_4 = 0, c''_5 = 0$.

We end with issue 2, but there is no bankruptcy problem since $c''_2 = 0, c''_3 = 0, c''_4 = 0$.

Therefore, the allocation for the order of issues 132 is $(2\frac{2}{3}, 3\frac{2}{3}, 2\frac{2}{3}, 3\frac{1}{3}, 4\frac{2}{3})$.

- *Consider the order of issues* **213**.

  We start with issue 2. The bankruptcy problem associated with this issue is $e_2 = 10, c_2 = 4, c_3 = 3, c_4 = 6$. *No need to truncate claims.*

Order / Claimant	2	3	4
234	4	3	3
243	4	0	6
324	4	3	3
342	1	3	6
423	4	0	6
432	1	3	6
RA	3	2	5

  UPDATE PHASE:

  1. *We update the estates and claims:* $e'_1 = 4, e'_2 = 0, e'_3 = 3, c'_1 = 3, c'_2 = 1, c'_3 = 1, c'_4 = 1, c'_5 = 5$.
  2. *We truncate the claims by the estates:* $c''_1 = 3, c''_2 = 0, c''_3 = 0, c''_4 = 0, c''_5 = 3$.

  Now we continue with issue 1 and then issue 3, but in these cases it is immediate that $RA = (3, 0, 0)$ *and* $RA = (0, 3)$.

  Therefore, the allocation for the order of issues 213 is $(3, 3, 2, 5, 3)$.

- *Consider the order of issues* **231**. *This case is completely identical to the order of issues 213, therefore, the allocation for the order of issues 231 is* $(3, 3, 2, 5, 3)$.

- *Consider the order of issues* **312**.

  We start with issue 3. The bankruptcy problem associated with this issue is $e_3 = 10, c_4 = 6, c_5 = 5$. *No need to truncate claims.*

Order / Claimant	4	5
45	6	2
54	3	5
RA	4.5	3.5

UPDATE PHASE:

1. *We update the estates and claims:* $e'_1 = 9, e'_2 = 5.5, e'_3 = 0, c'_1 = 3, c'_2 = 4, c'_3 = 3, c'_4 = 1.5, c'_5 = 1.5$.
2. *We truncate the claims by the estates:* $c''_1 = 3, c''_2 = 4, c''_3 = 3, c''_4 = 0, c''_5 = 0$.

Now we continue with issue 1. The bankruptcy problem associated with this issue is $e'_1 = 9, c''_1 = 3, c''_2 = 4, c''_3 = 3$.

Order / Claimant	1	2	3
123	3	4	1.5^a
132	3	2.5^b	3
213	3	4	1.5^a
231	3^c	4	1.5^a
312	3	2.5^b	3
321	3^c	2.5^b	3
RA	3	3.25	2.25

Table 4: (a) Claimant 3 cannot receive 2 units because the sum of the allocations of claimants 2 and 3 would exceed $e'_2 = 5.5$. (b) Claimant 2 cannot receive 3 units because the sum of the allocations of claimants 2 and 3 would exceed $e'_2 = 5.5$. (c) Claimant 1 can receive more than 2 because by receiving claimants 2 and 3 less she can receive more.

UPDATE PHASE:

1. *We update the estates and claims:* $e'_1 = 0.5, e'_2 = 0, e'_3 = 0, c'_1 = 0, c'_2 = 0.75, c'_3 = 0.75, c'_4 = 0, c'_5 = 0$.
2. *We truncate the claims by the estates:* $c''_1 = 0, c''_2 = 0, c''_3 = 0, c''_4 = 0, c''_5 = 0$.

We end with issue 2, but there is no bankruptcy problem.

Therefore, the allocation for the order of issues 312 is $(3, 3.25, 2.25, 4.5, 3.5)$.

- *Consider the order of issues* **321**.

  We start with issue 3. The situation is the same as the order 312. Therefore, $RA = (4.5, 3.5)$.

  UPDATE PHASE:

  1. *We update the estates and claims:* $e'_1 = 9, e'_2 = 5.5, e'_3 = 0, c'_1 = 3, c'_2 = 4, c'_3 = 3, c'_4 = 1.5, c'_5 = 1.5$.
  2. *We truncate the claims by the estates:* $c''_1 = 3, c''_2 = 4, c''_3 = 3, c''_4 = 0, c''_5 = 0$.

Now we continue with issue 2. The bankruptcy problem associated with this issue is $e'_2 = 5.5, c''_2 = 4, c''_3 = 3, c''_4 = 0$. $RA = (3.25, 2.25, 0)$.

UPDATE PHASE:

1. *We update the estates and claims:* $e'_1 = 3.5, e'_2 = 0, e'_3 = 0, c'_1 = 3, c'_2 = 0.75, c'_3 = 0.75c'_4 = 0, c'_5 = 0$.
2. *We truncate the claims by the estates:* $c''_1 = 3, c''_2 = 0, c''_3 = 0, c''_4 = 0, c''_5 = 0$.

We end with issue 1, but there is no bankruptcy problem and $RA = (3, 0, 0)$.

Therefore, the allocation for the order of issues 321 is $(3, 3.25, 2.25, 4.5, 3.5)$.

Finally, CRA is determined by calculating the average of all the allocations obtained.*

Order / Claimant	1	2	3	4	5
123	$2\frac{2}{3}$	$3\frac{2}{3}$	$2\frac{2}{3}$	$3\frac{2}{3}$	$4\frac{1}{3}$
132	$2\frac{2}{3}$	$3\frac{2}{3}$	$2\frac{2}{3}$	$3\frac{1}{3}$	$4\frac{2}{3}$
213	3	3	2	5	3
231	3	3	2	5	3
312	3	3.25	2.25	4.5	3.5
321	3	3.25	2.25	4.5	3.5
CRA*	$2\frac{8}{9}$	$3\frac{11}{36}$	$2\frac{11}{36}$	$4\frac{1}{3}$	$3\frac{2}{3}$

Note that issue 3 is completely distributed, but issues 1 and 2 are not. Moreover, it is easy to check that the allocation given by CRA is not Pareto efficient.*

Formally, CRA* is defined as follows.

Definition 4. *Let* $(\mathcal{I}, N, E, c, \alpha) \in \mathcal{MBC}$, $CRA*(\mathcal{I}, N, E, c, \alpha)$ *is defined as follows:*
Let $\omega : \{1, 2, \ldots, |\mathcal{I}|\} \to \mathcal{I}$ *a one-to-one application, and* $\Omega(\mathcal{I})$ *the set of all those applications. Let* $N^i = \{j \in N : i \in \alpha(j)\}$. *Let* $\xi^i : N^i \to \{1, 2, \ldots, |N^i|\}$ *a one-to-one application, and* $\Xi(N^i)$ *the set of all those applications.*

- *For each* $\omega \in \Omega(\mathcal{I})$*:*
  - *Initialization:*
    1. $e_i^{U(0)} = e_i, i \in \mathcal{I}$, *and* $c_j^{U(0)} = c_j, j \in N$.
    2. $c_j^{UT(0)} = \min\left\{c_j^{U(0)}, \min\{e_i^{U(0)} : i \in \alpha(j)\}\right\}, j \in N$.
  - *From* $k = 1$ *to* $|\mathcal{I}|$*, do*
    1. *For* $\omega(k)$*, we consider the bankrupty problem* $(N^{\omega(k)}, e_{\omega(k)}^{U(k-1)}, \{c_j^{UT(k-1)}\}_{j \in N^{\omega(k)}})$.

2. *We calculate* $m_j^{\xi^{\omega(k)}}, j \in N^{\omega(k)}$ *as follows:*

$$m_j^{\xi^{\omega(k)}} = min\left\{ c_j^{UT(k-1)},\ max\left\{ 0,\ \min_{i\in\alpha(j)}\{e_i^{U(k-1)} - \sum_{\substack{h\in N^{\omega(k)}: i\in\alpha(h)\\ \xi^{\omega(k)}(h)<\xi^{\omega(k)}(j)}} c_h^{UT(k-1)}\} \right\}\right\}.$$

3. *We calculate*

$$RA_j^{\omega(k)} = \frac{1}{|N^{\omega(k)}|!}\sum_{\xi^{\omega(k)}\in\Xi(N^{\omega(k)})} m_j^{\xi^{\omega(k)}}, j \in N^{\omega(k)}.$$

4. *UPDATE PHASE*
   (a) $e_i^{U(k)} = e_i^{U(k-1)} - \sum_{j\in N^{\omega(k)}:i\alpha(j)} RA_j^{\omega(k)}, i \in \mathcal{I}$; *and* $c_j^{U(k)} = c_j^{UT(k-1)} - RA_j^{\omega(k)}, j \in N^{\omega(k)}; c_j^{U(k)} = c_j^{UT(k-1)}, j \in N\backslash N^{\omega(k)}$.
   (b) $c_j^{UT(k)} = \min\{c_j^{U(k)}, \min\{e_i^{U(k)} : i \in \alpha(j)\}, j \in N$.

– *The last step is to calculate the allocation associated with* ω*:*

$$A_j^{\omega} = \sum_{k=1}^{|\mathcal{I}|}\delta(j,\omega(k))RA_j^{\omega(k)}, j \in N,$$

where $\delta(j,\omega(k)) = 1$ *if* $\omega(k) \in \alpha(j)$ *and* 0 *otherwise.*

- *Finally, we calculate CRA* for each* $j \in N$ *as follows:*

$$CRA_j^* = \frac{1}{|\mathcal{I}|!}\sum_{\omega\in\Omega((I))} A_j^{\omega}.$$

The following examples shows that CRA and CRA* do not coincide in general.

Example 3. *Consider the multi-issue bankruptcy problem with crossed claims in Proposition 3. The allocation for each order of issues and CRA* are shown in the following table:*

Order of issues / Claimant	1	2	3
123	3	2	5
132	3	2	5
213	2	3	4
231	2	3	4
312	2	3	4
321	2	3	4
CRA*	$\frac{7}{3}$	$\frac{8}{3}$	$\frac{13}{3}$

We observe that CRA and CRA coincide. Moreover, if claimant 3 leaves the problem with her claim, CRA* of the new problem is* $(3,2)$*, therefore, CRA* does not satisfies P-MON. Furthermore, if claimant 1 leaves the problem with her claim, CRA* of the new problem is* $(2,5)$*; and if claimant 2 leaves the problem with her claim, CRA* of the new problem is* $(1,3)$*, therefore CRA* does not satisfies BAL.*

Example 4. *Consider the multi-issue bankruptcy problem with crossed claims in Proposition 1. The allocation for each order of issues and CRA* are shown in the following table:*

Order of issues / Claimant	1	2	3
12	1	3	5
21	1.5	2.5	5.5
CRA*	1.25	2.75	5.25

We now observe that CRA and CRA do not coincide. Moreover, for this example CRA* is Pareto efficient, but CRA does not.*

Finally, we observe that we have the same problems with the properties as for CRA, therefore, it is not easy to find a characterization of CRA* using the usual properties in the characterizations of the random arrival rule for bankruptcy problems.

6 Conclusions

In many allocation problems the concept of priority is relevant, for example, in the legislation related to the liquidation of a company through bankruptcy in many countries, an order of priority is established to satisfy the claims. First, wage claims are settled, then taxes are paid. Creditor claims are then satisfied, and finally shareholder claims are addressed. Therefore, sequential priority rules, although simple, are not strange in real life. In this work an extension of the sequential priority rules has been introduced in the context of multi-issue bankruptcy problems with crossed claims. Next, by means of the average of all these rules, the extension of the random arrival rule to this new context is defined.

In the analysis of the properties satisfied by the sequential priority rules and the random arrival rule, it has been shown that the natural extensions of the properties used in the characterization of these rules in the context of bankruptcy problems and multi-issue bankruptcy problems are not satisfied by the constrained sequential priority rules and the constrained random arrival rule. Moreover, CRA* does not satisfies either the usual properties in the characterization of the random arrival rule. Therefore, the characterizations of the rules introduced in this work will require properties that are perhaps too technical or very *ad hoc*, which could detract from a simple interpretation.

The reason that these properties are not satisfied is that there is no efficiency in the context of multi-issue problems with crossed claims but rather a weaker concept such as Pareto efficiency. This means that the total quantity distributed can change from one problem to another, so the results from one problem to another are not easily comparable. This is a problem but it also indicates us that these problems are of theoretical interest because they are not a mere and simple extension of bankruptcy problems but rather have a different structure that makes them interesting for further study.

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Design of water quality policies based on proportionality in multi-issue problems with crossed claims

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Abstract

Water pollutants can be classified in three categories, each which includes several groups of substances. In this paper, we present a methodology based on bankruptcy models to determine the emission limits of polluting substances considering they can belong to more than one category. In particular, we model the problem as a multi-issue allocation problem with crossed claims and introduce the constrained proportional awards rule to obtain the emission limits. This rule is based on the concept of proportionality and extends the proportional rule for bankruptcy problems. We also provide an axiomatic characterization of this rule. Moreover, this allocation rule is illustrated by means of a numerical example based on real-world data. Finally, managerial and policy implications of this approach on water pollution control are given.

Keywords: Game theory, bankruptcy problems, multi-issue allocation problems, proportional rule, water pollution control

1 Introduction

Water is necessary for almost any form of life, which is why it is present in almost all the reports prepared by international institutions such as the United Nations (UN) and many of its specialized organizations, such as the World Health Organization (WHO) or Food and Agriculture Organization (FAO). The reason for this is that water is an essential good for economic development, health and the environment.

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Three of the great problems related to water are: access to fresh water, fresh water management and water pollution. In fact, the solution of these problems are directly or indirectly included in many of the *Sustainable Development Goals* (SDG's) promoted by the UN[1]. This paper deals with the third of these problems, water pollution, in particular, the design of water quality policies by means of water pollution control.

We would like to emphasize the importance of the water pollution control. In fact, water pollution is a serious threat to human health, to the survival of ecosystems and thus to the biodiversity of the planet. The contamination of freshwater causes numerous diseases, and reduces the availability of an already scarce resource that is essential for human consumption and for agriculture. Therefore, proper management of water pollution control in a certain region is imperative for the survival of the region and the development of its economic activity (Helmer and Hespanhol, 1997; Goel, 2009).

Goel (2009) defines a water pollutant as follows: "*A water pollutant can be defined as a physical, chemical or biological factor causing aesthetic or detrimental effects on aquatic life and on those who consume the water.*". Nesaratnam (2014) divides water pollutants into several categories: benzenoids, oxygen-demanding wastes, and eutrophic nutrients. Each of these groups of water pollutants comes from different sources, generally, related to human activity and has different effects on water quality.

Toxic benzenoids as *benzene*, *ethylbenzene*, *toluene*, and *xylenes*, including also *phenols* are very poisonous to any living organisms, being able to cause serious diseases in humans. These aromatic hydrocarbons have a low boiling point and are abundant in petroleum representing its most dangerous fraction. Furthermore, hydrocarbons, once incorporated into a given organism, are very stable, being able to pass through many members of the food chain without being altered. They are therefore transferred to all food chain, a situation analogous to that of heavy metals and pesticides (see, Tomlinson (1971) for details about benzenoid compounds). Moreover, these substances, particularly phenols, can also affect negatively to the presence of dissolved oxygen (DO) in water which is essential for the development, inside it, of the life of animals and plants. A body of water is classified as contaminated when the DO concentration falls below the level necessary to maintain a normal biota for such water. The main cause of deoxygenation of water is the presence of substances that are called oxygen-demanding wastes. These are compounds that are easily degraded or decomposed due to bacterial activity in the presence of oxygen. Regarding the pollutants which are oxygen-demanding wastes, we can find the already mentioned *phenols*, *ammoniacal compounds*, *oxidizable inorganic substances* (OIS), and overall *biodegradable organic compounds* (BOC) (Riffat, 2013).

As for the last group of water pollutants, eutrophication is the process by which a body of water becomes excessively enriched with nutrients that induce excessive growth of aquatic plants and algae. The most evident effect of eutrophication is the creation of dense blooms of noxious and smelly phytoplankton that reduce water clarity and damage water quality. However, there are other more dangerous effects affecting life in the aquatic ecosystems. Apart from natural causes, eutrophication is

[1]https://www.un.org/sustainabledevelopment/

caused by the action of man with the discharges of detergents, fertilizers or wastewater containing nitrates or phosphates in an aquatic system. Particularly relevant are nitrogen eutrophics as *nitrates* and *ammonical compounds* (see, for instance, Ansari et al. (2011) for details about the eutrophication problem).

Nowadays, there is a certain concern about other contaminants present in the water of which little is known, these contaminants are called emerging contaminants. Among these many products can be found such as pharmaceuticals and personal care products, nanomaterials, fire retardants, pesticides, plasticizers, surfactants, disinfection byproducts, antibiotic resistant bacteria, microplastics, and genes (see, for instance, Geissen et al., 2015).

The European Union (EU) has promoted and implemented different environmental policies addressed to protect water quality. Thus, EU directives have specified emission limit values for water and set standards on how to monitor, report, and manage the water quality (see, for instance, Directive/2000/60/EC, Directive 2006/118/EC, Directive 2008/105/EC, and Directive 2013/39/EU). Steinebach (2019) analyzes the effectiveness of EU policies in the quality of the national water resources of the member states over a period of 23 years (1990-2012). In the case of Spain, there is also a legal body in order to control different aspects of water management, including the water quality (Royal Decree 9/2008). All the previous legal framework derive the responsibility for the management and control of water towards the regional and local authorities that are closest to the water resource. Consequently, on the one hand, the legislation establishes limits (of immission) to the parameters indicating contamination in body waters, whose values vary according to the use of body waters (bathing, purification, ...). On the other hand, the discharges are those that are carried out directly or indirectly to the body waters, whatever their nature. Finally, local and regional administrations can legislate on maximum concentrations (of emission) of pollutants in discharges made. Therefore, it is of great social and economic interest to generate tools that help local and regional authorities to design policies to control water quality.

The most common in the literature is to find regulations that limit the discharge of certain substances but do not take into account their common and uncommon effects. Therefore, it is possible that several substances whose effects are the same are discharged into the water within their limits, but producing very high concentrations of substances as a whole that cause an undesired effect in the water. We can also find in the literature publications that deal with total discharge limits but without distinguishing by their effects on the water. Therefore, in the literature, we find that either the discharges of a certain substance or the total discharges are limited, but we have not found that the problem of limiting the discharges of substances as a whole is addressed, but differentiating by substances taking into account its effects, perhaps multiple, on water quality. Thus, in this paper, we consider the situation in which a certain authority responsible of the water quality in her region is interested in controlling the water pollution. In particular, they are interested in limiting the concentration of the three categories of water pollutants mentioned above. On the one hand, for each of these categories of pollutants certain levels of concentration are fixed in order to keep a reasonable quality of water. On the other hand, the main substances mentioned above are monitored and there are also maximum concentration limits for them. The relation between the categories of pollutants and the

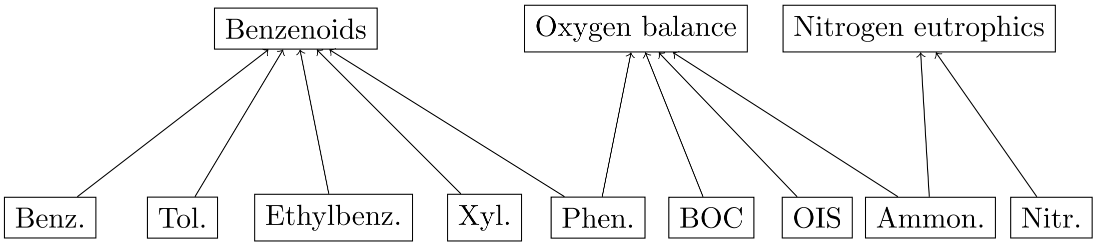

Figure 1: Relationship between families of pollutants and substances.

substances in each category is shown in Figure 1. Thus, the problem to be solved by the authority is how to allocate new thresholds to the substances taking into account the limits fixed for each category of pollutants. Therefore, the authority faces an allocation problem with certain special characteristics. One way to solve the problem is to resort to solutions that can be found in the literature on allocation problems, or based on them to introduce new solutions adapted to the particular problem. In this paper, we will use the allocation model introduced in Acosta et al. (2021) which is the one that best fits the situation described. For this model, we introduce a new solution based on the concept of proportionality that adapts to the structure of the described allocation problem, we carry out an axiomatic analysis of the proposed solution to show that it has good properties and illustrate its application with a numerical example based on real data. Finally, managerial and policy implications that has the approach proposed in this work are drawn.

2 Literature review

The applications of operational research (OR) to environmental management problems have been increasing since the first works in the 1970s, see, for example, the review by Bloemhof-Ruwaarda et al. (1995), and the references therein. This review highlighted the potential of OR to solve environmental management problems or to include environmental elements in optimization problems. More recently, Mishra (2020) insists in the impact of OR in environmental management. In particular, many applications of game theory to environmental management problems can be found in the literature (see, for example, Hanley and Folmer, 1998; Dinar et al., 2008).

On the one hand, ReVelle (2000) reviews the challenging OR problems in environmental management, indicating five areas: (1) water management, (2) water quality management, (3) solid wastes management, (4) cost allocation, and air quality management. On the other hand, Liu et al. (2011) highlight three problems in water management: pollution, water governance and access rights to water in practice. In addition, they propose the use of ethical principles in addressing water resource management problems. These principles are closely related to those that are common in game theory. Therefore, it seems reasonable that game theory plays a relevant role in the analysis of the mentioned problems. Thus, Dinar and Hogarth (2015) provide an exhaustive review on applications of game theory to water (resource) management. In many of these problems there are problems of allocation of water resources to uses, of water resources to regions, of waste discharges or of wastewater treatment costs, among others. In general, allocation problems describe situations in which a resource (or resources) must be distributed among a set of agents. These problems are of great interest in many settings, for this reason the literature on the matter is extensive.

In Helmer and Hespanhol (1997), water pollution control is highlighted as one of the most relevant problems in water resource management. Moreover, different aspects and principles of water quality management are analyzed. As for the allocation problems that we find in the literature on water pollution control, most are related either to the allocation of waste discharges in water or to the allocation of costs in wastewater treatment or a combination of both. Bogardi and Szidarovszky (1971) present a game theoretical model based on oligopoly games to determine the amount of water to be treated by each polluter. Niksokhan et al. (2009) and Nikoo et al. (2011,2012) study how to reallocate the treatment costs of discharges among polluters when they cooperate. For doing that, they combine the use of optimization models, non-cooperative games and cooperative games. Once the reallocation of the treatment costs is determined, a trading discharge permit policy is obtained associated with it. Poorsepahy-Samian et al. (2012) present a methodology for water and pollution discharge permit allocation in a shared river. This methodology is based on linear optimization, cooperative game theory and minimax regret theory to determine the best water and discharge permit allocation strategies. Bai et al. (2019) use the indicators water environmental carrying capacity (WECC)[2] and total emission pollutant control (TEPC)[3] to allocate in a two steps procedure emission pollutant permits to polluters by applying the principle of equal proportion reduction and the reduction potential of agents. Chen et al. (2019) use a nonlinear optimization model to allocate emission pollutant permits to point and non-point pollution sources[4]. Xie et al. (2022) use DEA and non-cooperative game theory to allocate wastewater discharge permits among polluters. In almost all the previous works, the proposed allocation methodologies are illustrated through numerical examples inspired by real-world data. Moreover, in Bai et al. (2019) and Chen et al. (2019), the emission permits are differentiated by more than one pollutant (or pollution parameters of water quality). On the other hand, Ni and Wang (2007), Dong et al. (2012), Gómez-Rúa (2012, 2013), Alcalde-Unzu et al. (2015, 2021) and Li et al. (2021), among others, study different methods to allocate the costs of cleaning a polluted river among the pollutant agents. These allocation methods are mainly analyzed from a game theoretical perspective and, therefore, ethical principles are used as suggested by Liu et al. (2011).

In none of the previous works, the main objective is to determine the emission limits of pollutants, but to allocate emission limits (or permits) to the polluting agents. Nevertheless, from the emission permits, the limit of pollutant emissions

[2] "Water environmental carrying capacity (WECC) is a comprehensive evaluation index for the state of the water environment system and is usually used to reflect the bearing capacity of the water environment system under the impacts of human activities." (Bai et al., 2019)

[3] "Total emission pollutant control (TEPC) is a commonly used water environment management strategy aiming to improve water quality by controlling a total load of water pollutants within the range of a given WECC."(Bai et al., 2019)

[4] See, for example, https://www.watereducation.org/aquapedia-background/point-source-vs-nonpoint-source-pollution.

could be determined. However, in most of them only one of the categories of pollutants or a particular polluting substance or total pollution is taken into account, and only in some are the emission limits of polluting substances differentiated, but in any case without taking into account that they can have more than one negative effect on water quality as shown in Figure 1. In this work, on the contrary, the main objective is to allocate emission limits to each polluting substance, taking into account that they can have more than one negative effect on water quality. Moreover, the problem, that we deal with, would correspond to a pre-analysis phase, which should be taken into account before the problem addressed in the previous works, that is, the emission limits are set and then the emission permits are allocated. The approach of considering pollutant interactions has already been suggested by Endres (1985), and Kuosmanen and Laukkanen (2011) show the importance of this approach for avoiding inefficient environmental policies. Therefore, the approach used in this work is interesting and fills a gap in the literature on the control of pollutant emissions into water.

On the other hand, a particular allocation problem is arisen in situations where there is a perfectly divisible resource over which there is a set of agents who have rights or demands, but the resource is not sufficient to honor them. This problem is known as bankruptcy problem and was first formally analyzed in O'Neill (1982) and Aumann and Maschler (1985). Since then, it has been extensively studied in the literature and many allocation rules have been defined (see Thomson, 2019, for a detailed inventory of rules). In the literature we also find works that apply this bankruptcy problem model to study the water allocation problem (Wickramage et al., 2020) and the problem of allocation of pollution discharge permits in rivers (Aghasian et al., 2019; Moridi, 2019). But also to other environmental problems, see, for example, Giménez-Gómez et al. (2016), Gutiérrez et al. (2018), and Duro et al. (2020) which analyze the CO2 allocation problem.

However, the base bankruptcy model does not always fit all problems, which is the reason why there are different extensions of the classical bankruptcy model. Some of them are the following: Young (1994) and Moulin (2000) study bankruptcy problems in the indivisible goods case. An application of the discrete bankruptcy model to the apportionment problem in proportional electoral systems is given in Sánchez-Soriano et al. (2016). Pulido et al. (2002, 2008) introduce bankruptcy problems with references and claims to study allocation problems in university management. Gozalvez et al. (2012), and Lucas-Estañ et al. (2012) present bankruptcy problems with claims given by a discrete nonlinear function of the resource to analyze radio resource allocation problems in telecommunications. Habis and Herings (2013) and Kooster and Boonen (2019) study bankruptcy problems in which the estate and the claims are stochastic values. An interesting extension of bankruptcy problems are multi-issue allocation problems (Calleja et al., 2005). These describe situations in which there is a (perfect divisible) resource which can be distributed among several issues, and a (finite) number of agents that have claims on each of those issues, such that the total claim is above the available resource. This problem is also solved by means of allocation rules and there are different ways to do it (see, for example, Calleja et al., 2005; Borm et al., 2005; González-Alcón et al., 2007; Izquierdo and Timoner, 2016). However, the situation described in Figure 1 does not fit to a multi-issue allocation problem as referred in the previous paragraph, but to a multi-issue allocation problem with crossed claims as introduced by Acosta et al. (2021). These describe situations in which there are several (perfect divisible) resources and a (finite) set of agents who have claims on them, but only one claim (not a claim for each resource) with which one or more resources are requested. The total claim for each resource exceeds its availability. Therefore, in this work, we use multi-issue allocation problems with crossed claims to allocate emission limits to pollutants. As far as we know, there are no applications of this model of bankruptcy problems to water pollution control.

In allocation problems the concept of proportionality is put into practice with the well-known proportional rule. This rule has been extensively studied in the literature from many different point of views and for many allocation models. Focusing on bankruptcy models and their extensions to the multi-issue case, the proportional rule has been characterized in the context of bankruptcy problems in Chun (1988) and de Frutos (1999). In both papers, non-manipulability plays a central role in the axiomatic characterization of the proportional rule. For multi-issue allocation problems, Ju et al. (2007) and Moreno-Ternero (2009) introduce two different definitions of proportional rule following two different approaches. Moreover, Ju et al. (2007) and Bergantiños et al. (2010) provides characterizations of both proportional rules. Again, in both approaches, non-manipulability is an essential property. In this paper, we introduce a definition of proportional rule, that we call constrained proportional awards rule, for multi-issue allocation problems with crossed claims and provide a characterization of it. Once again, non-manipulability is used. In this paper, in order to solve the allocation problem faced by the authority responsible for the water quality, we introduce the constrained proportional awards rule for multi-issue allocation problems with crossed claims that naturally extends the proportional rule for single issue allocation problems. This rule is characterized axiomatically by using five properties: *Pareto efficiency*, *equal treatment of equals*, *guaranteed minimum award*, *consistency*, and *non-manipulability by splitting*. The first one says that there is no a feasible allocation in which at least one of the claimants receive more. *Equal treatment of equals* states that equal agents must receive the same. *Guaranteed minimum award* establishes that a claimant should not receive less than what she would receive in the worst case, if the issues were distributed separately. *Consistency* requires that if a subset of agents leave the problem respecting what had been allocated to those who remain, then what those agents receive in the new problem is the same as what they received in the original problem. Finally, *non-manipulability by splitting* states that it is not profitable to split one agent in several agents. Therefore, we fill a gap in the literature of proportional distributions in allocation problems in line with the previous studies.

Summarizing, in this paper we address two gaps in the literature, one applied and another theoretical. On the one hand, we propose a methodology based on bankruptcy models to allocate emission limits to pollutants taking into account their effects on water quality can be several and therefore that there are interactions among them. This approach is in agreement with the works of Endres (1985) and Kuosmanen and Laukkanen (2011). As far as we know, this approach is novel in the field of water pollution control policies. On the other hand, we introduce and axiomatically analyze a new solution based on the concept of proportionality for multi-issue allocation problems with crossed claims that are an extension of bankruptcy problems.

The rest of the paper is organized as follows. Section 3 presents multi-issue allocation problems with crossed claims (MAC), and the concept of rule for these problems. In Section 4, the constrained proportional awards rule for multi-issue bankruptcy problems with crossed claims is defined. In Section 5, we present several properties which are interesting in the context of MAC problems. In Section 6, we characterize the constrained proportional awards rule. Section 7 includes an application of the constrained proportional awards rule to the management of control pollution water. Section 8 concludes.

3 Multi-issue allocation problems with crossed claims

Before starting with the description of the model used in this work, some mathematical notation is provided. Given $a \in \mathbb{R}^n$ and $b \in \mathbb{R}^m$, $a \oplus b \in \mathbb{R}^{n+m}$ denotes the concatenation of the two vectors. Given $a, b \in \mathbb{R}^n$, $a \leq b$ means that $a_i \leq b_i, \forall i = 1, \ldots, n$; and $a < b$ means that $a_i \leq b_i, \forall i = 1, \ldots, n$ with at least a strict inequality. Given a set S, $|S|$ denotes its cardinality. Given two sets S, T, $S \subset T$ includes the possibility that $S = T$.

We consider a situation where there is a finite set of issues $I = \{1, 2, ..., m\}$ such that there is a perfectly divisible amount e_i of each issue i. Let $E = (e_1, e_2, ..., e_m)$ be the vector of available amounts of issues. There is a finite set of claimants $N = \{1, 2, ..., n\}$ such that each claimant j claims c_j. Let $c = (c_1, c_2, ..., c_n)$ be the vector of claims. However, each claimant claims to different sets of issues, in general. Thus, let α be a set-valued function that associates with every $j \in N$ a set $\alpha(j) \subset I$. In fact, $\alpha(j)$ represents the issues to which claimant j asks for. Furthermore, $\sum_{j:i\in\alpha(j)} c_j > e_i$, for all $i \in I$, otherwise, those issues could be discarded from the problem because they do not impose any limitation, and so the allocation would be trivial. Therefore, a multi-issue allocation problem with crossed claims (MAC in short) is defined by a 5-tuple (I, N, E, c, α), and the family of all these problems is denoted by $\mathcal{MAC}$.

A rule for $\mathcal{MAC}$ is a mapping R that associates with every $(I, N, E, c, \alpha) \in \mathcal{MAC}$ a unique vector $R(I, N, E, c, \alpha) \in \mathbb{R}^N$ such that:

1. $0 \leq R_j(I, N, E, c, \alpha) \leq c_j$, for all $j \in N$.
2. $\sum_{j\in N:i\in\alpha(j)} R_j(I, N, E, c, \alpha) \leq e_i$, for all $i \in I$.

These two requirements are standard in the literature of allocation problems (see Thomson, 2019), and state that the allocation is feasible. Therefore, the vector $R(I, N, E, c, \alpha)$ represents an allocation to the claimants which is simultaneously feasible for all issues.

Given $(I, N, E, c, \alpha) \in \mathcal{MAC}$, we define the following two sets:

- $\mathcal{I} = \{i \in I : e_i > 0\}$ is the set of active issues;
- $\mathcal{N} = \{j \in N : c_j > 0 \text{ and } e_i > 0, \forall i \in \alpha(j)\}$ is the set of active claimants.

4 The constrained proportional awards rule for $\mathcal{MAC}$ problems

The proportional rule (Aristotle, 4th Century BD) is perhaps the most important rule to solve allocation problems in general,[5] and bankruptcy problems in particular. This rule simply divides the resource in proportion to the claims. Formally, for a one-issue allocation problem (I, N, E, c), where $I = \{1\}$, the proportional rule (PROP) is defined as follows:

$$PROP_j(I, N, E, c) = \frac{c_j}{C}E, \quad j \in N. \tag{1}$$

The question in $\mathcal{MAC}$ problems is what "in proportion to the claims" means. In the context of one-issue allocation problems, "in proportion to the claims" means that all claimants receive the same amount for each unit of claim. How to extrapolate this to the MAC situations. To answer this question, we introduce the constrained proportional awards rule (CPA in short) as the result of an iterative process in which the available amount of at least one of the issues is fully distributed in each step and so on and so forth while possible. This rule is formally defined below.

Definition 1. *Let $(I, N, E, c, \alpha) \in \mathcal{MAC}$, the* constrained proportional awards rule *for (I, N, E, c, α), $CPA(I, N, E, c, \alpha)$, is defined by means of the following iterative procedure:*

- *Initialization (Step 0)*

  1. $\mathcal{I}^1 = \mathcal{I}$, $\mathcal{N}^1 = \mathcal{N}$.
  2. *For each $i \in I$, $e_i^1 = e_i$, and for each $j \in N$, $c_j^1 = c_j$.*

- *General step (Step s)*

  1. $\mathcal{I}^s$, $\mathcal{N}^s$.
  2. *For each $i \in \mathcal{I}^s$, we calculate the greatest $\lambda_i^s \in [0, 1]$, so that $\lambda_i^s \sum_{j \in \mathcal{N}^s : i \in \alpha(j)} c_j^s \leq e_i^s$, and take $\lambda^s = \min\{\lambda_i^s : i \in \mathcal{I}^s\}$.*
  3. *We allocate to each claimant $j \in \mathcal{N}^s$, $a_j^s = \lambda^s c_j^s$, and $a_j^s = 0$ otherwise.*
  4. *We update the active issues, $\mathcal{I}^{s+1}$, and the active claimants, $\mathcal{N}^{s+1}$. If $\mathcal{I}^{s+1} = \varnothing$ or $\mathcal{N}^{s+1} = \varnothing$, then the process ends, and*

  $$CPA_j(I, N, E, c, \alpha) = \sum_{h=1}^{s} a_j^h, \forall j \in N.$$

  Otherwise, the available amounts of issues and the claims are updated:

  $$e_i^{s+1} = e_i^s - \lambda^s \sum_{j \in N : i \in \alpha(j)} c_j^s, \forall i \in I, \text{ and } \quad c_j^{s+1} = c_j^s - \lambda^s c_j^s, \forall j \in N,$$

  and we go to Step $s + 1$

[5]See, for instance, Algaba et al. (2019a) who present two solutions belonging to the family of proportional solutions for the problem of sharing the profit of a combined ticket for a transport system.

The application of the iterative procedure described in Definition 1 generates a succession of problems $(I^s, N^s, E^s, c^h, \alpha)$, $s = 1, 2, \ldots$, such that $I^s = I, N^s = N, E^s \geq E^{s+1}, c^s \geq c^{s+1}, s = 1, 2, \ldots$ Moreover, for each of those problems, an allocation $a^s \in \mathbb{R}^n_{\geq 0}$ is obtained. Therefore, $CPA_j(I, N, E, c, \alpha) = \sum_{s=1}^{+\infty} a_j^s, \forall j \in N$.

Note that if $\mathcal{I}^s \neq \varnothing$ and $\mathcal{N}^s \neq \varnothing$, then for each $i \in \mathcal{I}^s$, $\lambda_i^s > 0$. Moreover, if $\lambda_i^s < 1$, then λ_i^s is such that $\lambda_i^s \sum_{j \in \mathcal{N}^s : i \in \alpha(j)} c_j^s = e_i^s$, therefore, $i \notin \mathcal{I}^{s+1}$ and $\{j \in \mathcal{N}^s : i \in \alpha(j)\} \cap \mathcal{N}^{s+1} = \varnothing$. If $\lambda_i^s = 1$, we have two alternatives: either (1) $\{j \in \mathcal{N}^s : i \in \alpha(j)\} = \varnothing$, in which case it does affect neither the next set of active issues nor the next set of active claimants; or (2) $\{j \in \mathcal{N}^s : i \in \alpha(j)\} \neq \varnothing$, in which case $\{j \in \mathcal{N}^s : i \in \alpha(j)\} \cap \mathcal{N}^{s+1} = \varnothing$, and i will belong to $\mathcal{I}^{s+1}$ or not depending on whether $\sum_{j \in \mathcal{N}^s : i \in \alpha(j)} c_j^s < e_i^s$.

According to the above, $\lambda^s > 0$. If $\lambda^s < 1$, then $\mathcal{I}^{s+1} \subsetneqq \mathcal{I}^s$ and $\mathcal{N}^{s+1} \subsetneqq \mathcal{N}^s$. If $\lambda^s = 1$, then $\mathcal{N}^{s+1} = \varnothing$, and the process ends. Therefore, in each step at least the available amount of one issue is distributed in its entirety, except maybe in the last step. This implies that the process ends in a finite number of steps, at most $|I|$. Thus, $CPA_j(I, N, E, c, \alpha) = \sum_{s=1}^{r} a_j^s, \forall j \in N$, where $r \leq |I|$. Accordingly, CPA is well-defined and always leads to a single point.

Note that when we have a one-issue allocation problem, then it is easy to check that we obtain PROP. Thus, this definition extends PROP to the context of MAC.

From the application of the iterative process to calculate CPA, we can consider the chains of active issues and active claimants in the application of the procedure to calculate $CPA(\mathcal{I}, N, E, c, \alpha) \in \mathcal{MAC}$:

$$\mathcal{I}^1 \supsetneqq \mathcal{I}^2 \supsetneqq \ldots \supset \mathcal{I}^r, \text{ and } \mathcal{N}^1 \supsetneqq \mathcal{N}^2 \supsetneqq \ldots \supsetneqq \mathcal{N}^r$$

Furthermore, we can associate with each pair of sets $\mathcal{I}^s$ and $\mathcal{N}^s$ a number ρ^s, $\rho^s \in [0, 1]$, which represents the proportion of claims obtained by claimants in $\mathcal{N}^s$ but not in $\mathcal{N}^{s+1}$. Moreover, by construction $\rho^s < \rho^{s+1}$. Thus, we have that

$$0 < \rho^1 < \rho^2 < \ldots < \rho^r \leq 1.$$

These $\rho' s$ represent the accumulative proportion of the claims allocated to the claimants, i.e., what part of their claims they have received up to a given step of the iterative procedure. In this way, this procedure is reminiscent of the constrained equal awards rule (CEA) in bankruptcy problems, but instead of using the principle of egalitarianism, the principle of proportionality is used, hence the name of *constrained proportional awards* rule. Therefore, not all claimants receive the same proportion of their claims, but the rule tries to keep the proportionality as much as possible restricted to (1) the relation between the available amounts of issues and the total claims to them, and (2) the goal of allocating as much as possible of all available amounts of issues.

Finally, note that if a problem can be separated into two disjoint problems, then it is the same to calculate CPA for the whole problem as for each of them and then paste the results. This is established in the following proposition.

Proposition 1. *Given* $(I, N, E, c, \alpha) \in \mathcal{MAC}$, *if there are problems* $(I_1, N_1, E_1, c^1, \alpha_1)$, $(I_2, N_2, E_2, c^2, \alpha_2) \in \mathcal{MAC}$, *such that* $I_1 \cup I_2 = I$, $N_1 \cup N_2 = N$, $E_1 \oplus E_2 = E$,

$c^1 \oplus c^2 = c$, *and* $\alpha_1(j) = \alpha(j), \forall j \in N_1$ *and* $\alpha_2(j) = \alpha(j), \forall j \in N_2$, *so that* $\left(\bigcup_{j \in N_1} \alpha(j)\right) \bigcap \left(\bigcup_{j \in N_2} \alpha(j)\right) = \varnothing$, *then*

$$CPA(I, N, E, c, \alpha) = CPA(I_1, N_1, E_1, c^1, \alpha_1) \oplus CPA(I_2, N_2, E_2, c^2, \alpha_2).$$

Proof. The proof follows from the fact that since there are no crossed demands between the two subproblems, they do not affect each other and, therefore, the results are independent of each other. □

5 Properties

In this section, we present several properties which are interesting in the context of MAC problems. These properties are related to efficiency, fairness, consistency, and manipulability.

First, we introduce two concepts related to two claimants comparisons. In MAC situations claimants are characterized by two elements: their claims and the issues to which they claim. Therefore, both should be taken into account when establishing comparisons among them.

Definition 2. *Let* $(I, N, E, c, \alpha) \in \mathcal{MAC}$, *and two claimants* $j, k \in N$, *we say they are* homologous, *if* $\alpha(j) = \alpha(k)$; *and we say that they are* equal, *if they are homologous and* $c_j = c_k$.

Next, we give a set of properties which are very natural and reasonable for an allocation rule in MAC situations.

The first property relates to efficiency. In allocation problems is desirable that the resources to be fully distributed, but in MAC situations this is not always possible (see Acosta-Vega, 2021). Therefore, a weaker version of that is considered in which only is required that there is no a feasible allocation in which at least one of the claimants receive more. This is established in the following axiom.

Axiom 1 (PEFF). *Given a rule* R, *it satisfies* Pareto efficiency, *if for every problem* $(I, N, E, c, \alpha) \in \mathcal{MAC}$, *there is no a feasible allocation* $a \in \mathbb{R}^N_+$ *such that* $a_j \geq R_j(I, N, E, c, \alpha), \forall j \in N$, *with at least one strict inequality.*

Note that $PEFF$ implies that at least the available amount of one issue is fully distributed, and no amount is left of an issue undistributed, if it is possible to do so. However, it does not require that all available amounts of the issues have to be fully distributed. On the other hand, a feasible allocation that satisfies the condition in Axiom 1 is called Pareto efficient.

The second property states that equal claimants should receive the same in the final allocation. This is a basic requirement of fairness and non-arbitrariness. This is defined in the following axiom.

Axiom 2 (ETE). *Given a rule* R, *it satisfies* equal treatment of equals, *if for every problem* $(I, N, E, c, \alpha) \in \mathcal{MAC}$ *and every pair of equal claimants* $j, k \in N$, $R_j(I, N, E, c, \alpha) = R_k(I, N, E, c, \alpha)$.

The third property states the minimum that should be guaranteed to each claimant. In our case, these minimum amounts are determined from the analysis of the problems associated with each issue independently. In particular, the property states that a claimant should not receive less than what she would have received in the worst case, if the rule had been applied to each problem separately to each of the issues. This is established in the following property.

Axiom 3 (GMA). *Given a rule R, it satisfies* guaranteed minimum award*, if for every problem* $(I, N, E, c, \alpha) \in \mathcal{MAC}$,

$$R_j(I, N, E, c, \alpha) \geq \min \left\{ R_j \left(\{i\}, N_i, e_i, c|_{N_i} \right) : i \in \alpha(j) \right\}, \forall j \in N,$$

where $N_i = \{k \in N : i \in \alpha(k)\}$*, and* $c|_{N_i}$ *is the vector whose coordinates correspond to the claimants in* N_i*.*

The fourth property is a requirement of robustness when some agents leave the problem with their allocations (see Thomson, 2011, 2018). In particular, when a subset of claimants leave the problem respecting the allocations to those who remain, then it seems reasonable that claimants who leave will receive the same in the new problem as they did in the original. This is formally given in the following axiom.

Axiom 4 (CONS). *Given a rule R, it satisfies* consistency*, if for every problem* $(I, N, E, c, \alpha) \in \mathcal{MAC}$*, and* $N' \subset N$*, it holds that*

$$R_j(I, N, E, c, \alpha) = R_j(I', N', E'^R, c|_{N'}, \alpha), \text{ for all } j \in N',$$

where $(I', N', E', c|_{N'}, \alpha) \in \mathcal{MAC}$*, called the* reduced problem associated with N'*,* $I' = \{i \in I : \text{ there exists } k \in N' \text{such that } i \in \alpha(k)\}$*,* $E'^R = (e'^R_1, \ldots, e'^R_m)$ *so that* $e'^R_i = e_i - \sum_{j \in N \setminus N' : i \in \alpha(j)} R_j(I, N, E, c, \alpha)$*, for all* $i \in I'$*, and* $c|_{N'}$ *is the vector whose coordinates correspond to the claimants in* N'*.*

The last two properties are related to claimants' ability to manipulate the final allocation by splitting their claims among several new claimants or merging their claims into a single claimant. It seems sensible that if the claimants do this, they will not benefit and receive the same as they did in the original problem. These two possibilities are established in the following axioms.

Axiom 5 (NMS). *Given a rule R, it satisfies* non-manipulability by splitting*, if for every pair of problems* $(I, N, E, c, \alpha), (I, N', E, c', \alpha') \in \mathcal{MAC}$*, such that:*

1. $N \subset N'$*,* $S = \{i_1, \ldots, i_k\}$*, such that* $N' = (N \setminus S) \cup S_{i_1} \cup \ldots \cup S_{i_m}$*, where* S_{i_k} *is the set of agents into which agent* i_k *has been divided including itself.*

2. $c'_j = c_j, \forall j \in N \setminus S$ *and* $\sum_{k \in S_{i_h}} c'_k = c_{i_h}, h = 1, \ldots, m$,

3. $\alpha'(j) = \alpha(j), \forall j \in N \setminus S$ *and* $\alpha'(j) = \alpha(i_h), \forall j \in S_{i_h}, h = 1, \ldots, m$,

it holds

$$\sum_{j \in S_{i_h}} R_j(I, N', E, c', \alpha') = R_{i_h}(I, N, E, c, \alpha), h = 1, \ldots, m.$$

Axiom 6 (NMRM)**.** *Given a rule R, it satisfies* non-manipulability by restricted merging*, if for every pair of problems* $(I, N, E, c, \alpha), (I, N', E, c', \alpha') \in \mathcal{MAC}$*, such that:*

1. $N \subset N'$,
2. $c_j = c'_j, \forall j \in N \backslash \{j_0\}$ *and* $c_{j_0} = \sum_{k \in (N' \backslash N) \cup \{j_0\}} c'_k$,
3. $\alpha(j) = \alpha'(j), \forall j \in N \backslash \{j_0\}$ *and* $\alpha(j) = \alpha'(j_0), \forall j \in (N' \backslash N) \cup \{j_0\}$,

it holds

$$R_{j_0}(I, N, E, c, \alpha) = \sum_{j \in (N' \backslash N) \cup \{j_0\}} R_j(I, N', E, c', \alpha').$$

Note that in NMS we move from the allocation problem with set of claimants N to the problem with set of claimants N', i.e., one of the claimants is splitted into several new claimants, one of whom has the same name as in N. However, in $NMRM$ we move from the problem in N' to the problem in N, i.e., several claimants merge into one claimant who has the same name as in N', but all merged claimants are homologous. Thus, we are only considering the merging of homologous claimants. For this reason we call this axiom non-manipulability by "restricted" merging. Nevertheless, it seems reasonable from a perspective of symmetry of both properties, because when one claimant is splitted into several new claimants, these are homologous in the new problem.

CPA satisfies all properties above mentioned. We establish this in the following theorem.

Theorem 1. *CPA for multi-issue bankruptcy problems with crossed claims satisfies* $PEFF$*,* ETE*,* GMA*,* $CONS$*,* NMS*, and* $NMRM$*.*

Proof. We go axiom by axiom.

- CPA satisfies $PEFF$ and GMA by definition.
- If two claimants are equal, then CPA allocates both the same, since the procedure to calculate the rule treats, in each step, all active equal claimants egalitarianly, so if two claimants are equal, they stop receiving at the same step. Therefore, CPA satisfies ETE.
- Given $(I, N, E, c, \alpha) \in \mathcal{MAC}$ and $(I', N', E'^{CPA}, c|_{N'}, \alpha) \in \mathcal{MAC}$ the reduced problem associated with $N' \subset N$. Let us consider the following sets obtained from the application of CPA to (I, N, E, c, α):

  $$\mathcal{A}^s = \mathcal{N}^s \setminus \mathcal{N}^{s+1}, s = 1, \ldots, r, \text{ and } \mathcal{B}^s = \mathcal{I}^s \setminus \mathcal{I}^{s+1}, s = 1, \ldots, r.$$

  We now consider the following sets: $N' \cap \mathcal{A}^s, s = 1, \ldots, r$. Taking into account the definitions of E'^{CPA}, $\mathcal{N}^s$, and $\mathcal{I}^s$, it is evident that claimants in $N' \cap \mathcal{A}^s$ cannot receive more than ρ^s times their claims, because, otherwise, the available amounts of issues $e_i'^{CPA}$, $i \in \mathcal{B}^s$, would be exceeded. Thus, from the definition of CPA, claimants in $N' \cap \mathcal{A}^s$ have to receive exactly ρ^s times their claims. Therefore, the claimants in $(I', N', E'^{CPA}, c|_{N'}, \alpha)$ receive the same as in (I, N, E, c, α). Hence, CPA satisfies consistency.

- Note that when one claimant splits into several new claimants, CPA for the new problem will have the same number of iterations as in the original one, since the claim for each issue will be obviously the same in each step. Therefore, all split claimants will receive the same proportion of their claims which coincides with the proportion obtained by the split claimant in the original problem. Thus, the aggregate allocation of the split claimants in the new problem coincides with the allocation of the split claimant in the original problem.

- When two homologous claimants merge into a new one claimant, we can make a completely analogous reasoning as in the case of a claimant splits into several new claimants. Therefore, CPA also satisfies NMRM.

□

6 Characterization

In this section, the aim is to achieve a better knowledge of the CPA rule for $\mathcal{MAC}$ by describing it in a unique way as a combination of some reasonable axioms. We characterize the CPA rule by means of $PEFF$, ETE, GMA, $CONS$, and NMS. Therefore, the CPA rule can be considered as a desirable way to distribute a set of issues among their claimants. Before giving the characterization of CPA, we need the following lemmas.

Lemma 1. *Let $(I, N, E, c, \alpha) \in \mathcal{MAC}$, such that $|I| = 1$. If a rule R satisfies $PEFF$, ETE, and NMS, then*

$$R_i(I, N, E, c, \alpha) = \frac{c_1}{\sum_{j \in N} c_j} e, \text{ for all } i \in N.$$

Proof. First note that in this case the function α is irrelevant. Let $R_1, R_2, \ldots, R_n$ be the allocations for claimants in N, respectively. By $PEFF$, and the definition of rule, we know that there are $\beta_i \in [0,1]$, $i \in N$, such that $R_i = \beta_i c_i, i \in N$, and $\sum_{i \in N} \beta_i c_i = e$.

Consider the following chain of problems:

$$(I, N, E, c, \alpha) \longrightarrow (I, N(q), E, c(q), \alpha)$$

where the first problem is the original, the second is the problem in which each claimant i is split into a number of identical claimants k_i, $k_i \in \mathbb{N}_+$, with claims exactly equal to $q \in \mathbb{R}_+$. We now distinguish two cases:

1. $c_1, c_2, \ldots, c_n \in \mathbb{Q}_+$. In this case, there exists $q \in \mathbb{Q}_+$ such that $c_i = k_i q, k_i \in \mathbb{N}_+, i \in N$. Now, by $PEFF$ and ETE, we have that

$$R_j(I, N(q), E, c(q), \alpha) = \beta q, j \in N(q).$$

On the other hand, by NMS, it holds for every $i \in N$ that

$$\beta_i c_i = k_i \beta q = \beta c_i \Rightarrow \beta_i = \beta$$

2. $c_1, c_2, \ldots, c_n \in \mathbb{R}_+$. In this case, for each $\varepsilon > 0$, there exists $q \in \mathbb{R}_+$ such that $c_i = k_i(q)q + \varepsilon_i(q), k_i(q) \in \mathbb{N}_+$, and $\varepsilon_i(q) < \frac{\varepsilon}{n}$, for all $i \in N$.

   Now, by ETE, we have the following equality for the second problem:

$$\left(\sum_{i\in N} k_i(q)\right)\beta(q)q + \sum_{i\in N}\delta_i(q)\varepsilon_i(q) = e,$$

   where $\beta(q) \in [0,1]$, and $\delta_i(q) \in [0,1]$ for all $i \in N$. This equality can be written as follows:

$$\beta(q)\sum_{i\in N}(c_i - \varepsilon_i(q)) + \sum_{i\in N}\delta_i(q)\varepsilon_i(q) = e,$$

   or equivalently,

$$\frac{e}{\sum_{i\in N} c_i} - \beta(q) = \frac{\sum_{i\in N}\{(\delta_i(q) - \beta(q))\varepsilon_i(q)\}}{\sum_{i\in N} c_i},$$

   taking limits on both sides when q goes to zero, we obtain that $\lim_{q\to 0^+}\beta(q) = \frac{e}{\sum_{i\in N} c_i}$.

   On the other hand, by NMS, for each q and for each $i \in N$,

$$\beta_i c_i = k_i(q)\beta(q)q + \delta_i(q)\varepsilon_i(q) = \beta(q)c_i + (\delta_i(q) - \beta(q))\varepsilon_i(q).$$

   Since $\lim_{q\to 0^+}\beta(q) = \frac{e}{\sum_{i\in N} c_i}$, $\beta_i = \frac{e}{\sum_{i\in N} c_i}$, for each $i \in N$.

□

Lemma 2. *For each problem $(I, N, E, c, \alpha) \in \mathcal{MAC}$, and each rule R that satisfies $PEFF$, ETE, GMA and NMS, if for each $N' \subset N$ with $|N'| = |N| - 1$, we have $R_i(I, N, E, c, \alpha) = CPA_i(I', N', E'^R, c|_{N'}, \alpha)$ for all $i \in N'$, then $R(I, N, E, c, \alpha) = CPA(I, N, E, c, \alpha)$.*

Proof. We first prove that if there is $R_i = R_i(I, N, E, c, \alpha) = CPA_i(I, N, E, c, \alpha)$, then the result holds. Indeed, let us consider R in the conditions of the statement, and $R_i = CPA_i(I, N, E, c, \alpha)$. We now consider $N' = N\backslash\{i\}$, since $R_i = CPA_i(I, N, E, c, \alpha)$,

$$(I', N', E'^R, c|_{N'}, \alpha) = (I', N', E'^{CPA}, c|_{N'}, \alpha).$$

By hypothesis, we have that for all $k \in N'$,

$$R_k(I, N, E, c, \alpha) = CPA_k(I', N', E'^R, c|_{N'}, \alpha) = CPA_k(I', N', E'^{CPA}, c|_{N'}, \alpha).$$

Moreover, since CPA satisfies consistency,

$$CPA_k(I, N, E, c, \alpha) = CPA_k(I', N', E'^{CPA}, c|_{N'}, \alpha) \text{ for all } k \in N'.$$

Therefore, $CPA_k(I, N, E, c, \alpha) = R_k(I, N, E, c, \alpha)$ for all $k \in N'$.

Let us consider R in the conditions of the statement and we assume without loss of generality that $\beta_1 = \frac{R_1}{c_1} \leq \beta_2 = \frac{R_2}{c_2} \leq \ldots \leq \beta_{|N|} = \frac{R_{|N|}}{c_{|N|}}$, where for the sake of simplicity, we denote $R_k(I, N, E, c, \alpha)$ by R_k for each $k \in N$.

First, for $\alpha(1)$, for every $i \in \alpha(1)$, we take $\gamma_i > 0$ such that $\gamma_i \sum_{j \in N : i \in \alpha(j)} c_j = e_i$. We now define $\gamma_1 = \min\{\gamma_i : i \in \alpha(1)\}$, and we assume that γ_1 is without loss of generality obtained for issue 1.

Second, $\beta_1 \leq \gamma_1$, otherwise, we would have that

$$\sum_{j:1\in\alpha(j)} \beta_j c_j \geq \beta_1 \sum_{j:1\in\alpha(j)} c_j > \gamma_1 \sum_{j:1\in\alpha(j)} c_j = e_1,$$

which is a contradiction.

Third, by Lemma 1 and GMA, $R_1 \geq \min\{\frac{c_1}{\sum_{j:i\in\alpha(j)} c_j} e_i : i \in \alpha(1)\} = \gamma_1 c_1$. Therefore, $\beta_1 = \gamma_1$. Now, by $PEFF$, $\beta_j = \gamma_1$ for all $j \in N$ such that $1 \in \alpha(j)$.

Fourth, for each $N' \subset N$ with $1 \in N'$ and $|N'| = |N| - 1$, by hypothesis and the definition of CPA, we have that β_1 coincides with the λ^1's of the iterative procedures for calculating each $CPA(I', N', E'^R, c|_{N'}, \alpha)$. This implies that $\beta_1 = \lambda^1 = \min\{\lambda_i^1 : i \in \mathcal{I}'^1\}$, for each $N' = N\backslash\{k\}, k \in N\backslash\{1\}$, where

$$\lambda_i^1 \sum_{j\in\mathcal{N}'^1 : i\in\alpha(j)} c_j^1 = e_i^1 - \delta(i,k)\beta_k c_k, \forall i \in I',$$

where $\delta(i,k) = 1$ if $i \in \alpha(k)$, and 0 otherwise. Since $R_1 = \beta_1 c_1 = CPA_i(I', N', E'^R, c|_{N'}, \alpha)$, $\arg\min\{\lambda_i^1 : i \in \mathcal{I}'^1\} \in \alpha(1)$, otherwise, claimant 1 would obtain more than $\beta_1 c_1$ which is a contradiction. In particular, the minimum will be attained, although possibly among others, at issue 1, because $\beta_j = \gamma_1$ for all $j \in N$ such that $1 \in \alpha(j)$.

On the other hand, by definition of CPA, we have that $\lambda^1 = \min\{\lambda_i^1 : i \in \mathcal{I}^1\}$, where

$$\lambda_i^1 \sum_{j\in\mathcal{N}^1 : i\in\alpha(j)} c_j^1 = e_i^1, \forall i \in \mathcal{I}.$$

This λ^1 can be also obtain by solving the following simple linear program:

$$\begin{array}{rll} \lambda^1 = & \max & \lambda \\ & \text{subject to} & \lambda \sum_{j\in\mathcal{N}^1 : i\in\alpha(j)} c_j^1 \leq e_i^1, \forall i \in \mathcal{I} \\ & & \lambda \geq 0 \end{array}.$$

It is obvious that $\lambda^1 \leq \gamma_1$, because of the definition of γ_1. Now, since R satisfies $PEFF$, γ_1 is a feasible solution for the linear program above and $\lambda^1 \leq \gamma_1$, γ_1 is an optimal solution of the problem. Therefore, the inequality associated with issue 1 is saturated in the optimal solution and by definition of CPA claimant 1 will obtain $\gamma_1 c_1 = \beta_1 c_1 = R_1$, i.e., $R_1(I, N, E, c, \alpha) = CPA_1(I, N, E, c, \alpha)$. □

Theorem 2. *CPA is the only rule that satisfies $PEFF$, ETE, GMA, $CONS$, and NMS.*

Proof. We distinguish three cases, depending on the number of claimants in the problem.

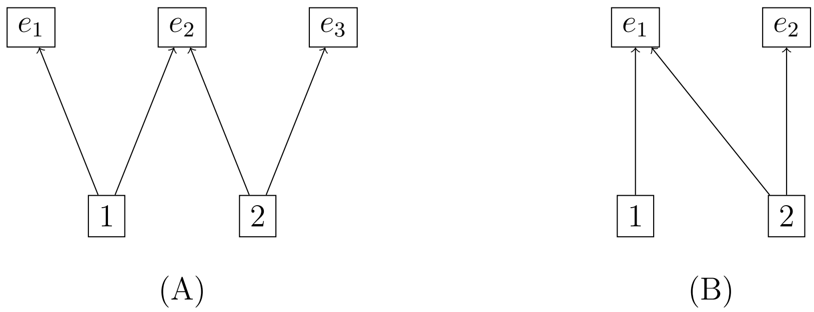

Figure 2: Basic 2-claimants situations when $\alpha(1) \cap \alpha(2) \neq \varnothing$ and $\alpha(1) \neq \alpha(2)$.

1. $|N| = 1$. In this case, all rules that satisfy $PEFF$ coincide with CPA.

2. $|N| = 2$. We distinguish two cases:

   (a) $\alpha(1) \cap \alpha(2) = \varnothing$. In this situation, since the rule satisfies $PEFF$ we can consider two separate problems of only one claimant each. Now by applying the case $|N| = 1$, all rules that satisfy $PEFF$ coincide with CPA.

   (b) $\alpha(1) \cap \alpha(2) \neq \varnothing$. We consider other two cases:

      i. $\alpha(1) = \alpha(2)$. By GMA, Lemma 1, and the definition of rule,

      $$R_1 = \frac{c_1}{c_1 + c_2} e, \text{ and } R_2 = \frac{c_2}{c_1 + c_2} e,$$

      where e is the minimum of the available amounts of the issues.

      ii. $\alpha(1) \neq \alpha(2)$. First note that by Lemma 1, we know that for every single issue we obtain the proportional distribution of the available amount among the corresponding claimants. Therefore, in order to apply GMA, we can consider without loss of generality the two situations shown in Figure 2.

      We next analyze the two situations in Figure 2:

      A. By GMA, $c_1 \geq e_1$, $c_2 \geq e_3$, and $c_1 + c_2 \geq e_2$,

      $$R_1 \geq \min\{e_1, \frac{c_1}{c_1 + c_2} e_2\}, \text{ and } R_2 \geq \min\{\frac{c_2}{c_1 + c_2} e_2, e_3\}$$

      If $\min\{e_1, \frac{c_1}{c_1+c_2} e_2\} = e_1$, then $R_1 = e_1$, and by $PEFF$ $R_2 = \min\{c_2, e_2 - e_1, e_3\}$. If $\min\{e_1, \frac{c_1}{c_1+c_2} e_2\} = \frac{c_1}{c_1+c_2} e_2$, then we have two possibilities:

      - $\min\{\frac{c_2}{c_1+c_2} e_2, e_3\} = e_3$, then $R_2 = e_3$, and by $PEFF$ $R_1 = \min\{c_1, e_2 - e_3, e_1\}$.

      - $\min\{\frac{c_2}{c_1+c_2} e_2, e_3\} = \frac{c_2}{c_1+c_2} e_2$, then $R_1 = \frac{c_1}{c_1+c_2} e_2$ and $R_2 = \frac{c_2}{c_1+c_2} e_2$.

B. By GMA, $c_1 \geq e_1$, $c_2 \geq e_3$, and $c_1 + c_2 \geq e_2$,

$$R_1 \geq \frac{c_1}{c_1 + c_2} e_1, \text{ and } R_2 \geq \min\{\frac{c_2}{c_1 + c_2} e_1, e_2\}$$

Now reasoning as in the previous case,

$$R_1 = \frac{c_1}{c_1 + c_2} e_1, \text{ and } R_2 = \frac{c_2}{c_1 + c_2} e_1$$

or

$$R_1 = \min\{c_1, e_1 - e_2\}, \text{ and } R_2 = e_2$$

3. $|N| = 3$. Let R be a rule that satisfies $PEFF$, ETE, GMA, $CONS$, and NMS, and let $(I, N, E, c, \alpha) \in \mathcal{MAC}$, then we have that

$$R(I, N, E, c, \alpha) = CPA(I, N, E, c, \alpha).$$

Indeed, for each $N' = \{i_1, i_2\} \subset N$ such that $|N'| = 2$, since R satisfies CONS,

$$R_{i_k}(I', N', E'^R, c|_{N'}, \alpha) = R_{i_k}(I, N, E, c, \alpha), k = 1, 2,$$

and since $|N'| = 2$, we have that

$$R_{i_k}(I', N', E'^R, c|_{N'}, \alpha) = CPA_{i_k}(I', N', E'^R, c|_{N'}, \alpha), k = 1, 2.$$

Since we can take all possible $N' = \{i_1, i_2\} \subset N$, by Lemma 2

$$R(I, N, E, c, \alpha) = CPA(I, N, E, c, \alpha).$$

4. $|N| \leq k$. Let us suppose that for each (I, N, E, c, α) with $|N| \leq k$, $R(I, N, E, c, \alpha) = CPA(I, N, E, c, \alpha)$.

5. $|N| = k + 1$. For each $N' \subset N$ such that $|N'| = k$, since R satisfies CONS,

$$R_i(I', N', E'^R, c|_{N'}, \alpha) = R_i(I, N, E, c, \alpha), i \in N'.$$

and since $|N'| \leq k$, we have that

$$R_i(I', N', E'^R, c|_{N'}, \alpha) = CPA_i(I', N', E'^R, c|_{N'}, \alpha), i \in N'.$$

Finally, since we can take all possible $N' \subset N$ with $|N'| = k$, by Lemma 2,

$$R(I, N, E, c, \alpha) = CPA(I, N, E, c, \alpha).$$

$\square$

Proposition 2. *Properties in Theorem 2 are logically independent.*

Proof. We consider the four posibilities:

- (No $PEFF$) The null rule satisfies all properties but $PEFF$.

- (No ETE) Consider an order on the set of claimants and a rule which reimburses each claimant all that can be, in that order, until it is not possible to do it. If we assume that when a claimant splits into several new claimants or some claimants leave, the order in which the claims are attended is preserved, then this rule satisfies $PEFF$, GMA, $CONS$, and NMS, but not ETE.

- (No GMA) Consider a rule that has two phases. In the first phase, each issue is distributed proportionally, but only among those claimants that only demand the corresponding issue. In the second phase, the amounts of each issue are updated down accordingly, and distributed among the rest of the claimants applying CPA. This rule satisfies $PEFF$, ETE, $CONS$, and NMS, but not GMA.

- (No $CONS$) For every problem $(I, N, E, c, \alpha) \in \mathcal{MAC}$, consider the following rule defined in two steps:

    1. First, we allocate to each claimant j, $\min\{PROP_j(\{i\}, N_i, e_i, c|_{N_i}) : i \in \alpha(j)\}$.
    2. Next, we revise down the available amounts of issues and the claims, and we assume without loss of generality that $e'_1 \leq e'_2 \leq \ldots \leq e'_m$. Then we begin to distribute each state proportionally among the claimants, starting from the smallest to the largest quantity available. It is not until one state has been fully distributed or the claimants fully satisfied that we move on to the next updating the claims. We continue until all the states have been distributed as much as possible.

  The allocation to each claimant is the sum of everything that she has obtained in each of the steps of the procedure described.

  By definition this rule satisfies GMA, $PEFF$, and ETE. Moreover, using arguments similar to those used in Theorem 1, it can be shown that this rule satisfies NMS. However, it does not satisfies $CONS$ since this rule does not coincide with CPA, and if we consider reduced problems with $|N'| = 2$, by $PEFF$, ETE, GMA, and NMS, we obtain the allocations prescribed by CPA.

- (No NMS) The CEA rule for MAC satisfies all properties but NMS (Acosta et al., 2021).

□

7 Numerical example based on real-world data

In this section, we consider the applied limits corresponding to a representative local normative of spills to the sewage network established by the Honorary Granada City Council. (BOP Nº 129, 30/05/2000). These emission limits of some pollutants are shown in Table 1 and correspond to inital emission limits for pollutant categories of 5.6 ppm for benzenoids, 1400 ppm for oxygen-demanding wastes and 250 ppm for nitrogen eutrophing nutrients.

Pollutants	Maximum value
Benzene	0.05
Toluene	0.25
Ethylbenzene	0.15
Xylenes	0.15
Phenols	5
BOC	700
OIS	545
Amoniacal compounds	150
Nitrate compounds	100

Table 1: Maximum allowed values (ppm) for some pollutants.

Now suppose that the city council wants to impose more stringent limitations on the concentrations (ppm) of each of the three categories of pollutants mentioned in Section 1 (benzenoids, oxygen-demanding wastes, and eutrophing nutrients), independently of the emission limits for each of the pollutants. In this sense, what the city council intends is to control emissions more by categories of pollutants than by each of the pollutants themselves, respecting, at the same time, the limitations on the emissions of each substance. Suppose the city council sets the following limits for each of the groups of pollutants: 4 ppm for benzenoids, 1000 ppm for oxygen-demanding wastes, and 150 ppm for nitrogen eutrophing nutrients. The emission limits situation is shown in Figure 3

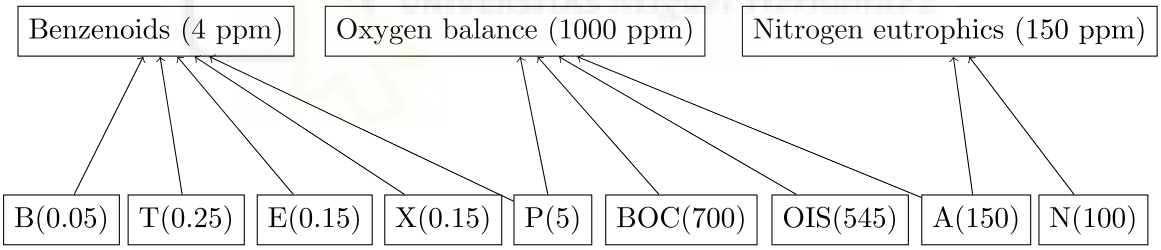

Figure 3: Relationship between families of pollutants and substances.

Associated with the situation described above we consider the following multi-issue allocation problem with crossed claims $MAC = (I, N, E, c, \alpha)$ with

- $I = \{1, 2, 3\}$;
- $N = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$;
- $E = (4, 1000, 150)$;
- $c = (0.05, 0.25, 0.15, 0.15, 5, 700, 545, 150, 100)$; and
- $\alpha(1) = \{1\}$, $\alpha(2) = \{1\}$, $\alpha(3) = \{1\}$, $\alpha(4) = \{1\}$, $\alpha(5) = \{1, 2\}$, $\alpha(6) = \{2\}$, $\alpha(7) = \{2\}$, $\alpha(8) = \{2, 3\}$, and $\alpha(9) = \{3\}$.

It is obvious that the emission limits for each of the three pollutants categories are not sufficient to guarantee the emission limits for each of the substances, therefore, their limits must be recalculated down. To do this, we can now apply the procedure described above to calculate the constrained proportional awards rule for MAC problems.

Step 1. $\mathcal{N}^1 = N$, $\mathcal{I}^1 = I$,

$E^1 = (4, 1000, 150)$, $c^1 = (0.05, 0.25, 0.15, 0.15, 5, 700, 545, 150, 100)$

- $\lambda_1^1 = 0.714$
- $\lambda_2^1 = 0.714$
- $\lambda_3^1 = 0.600$

$\lambda^1 = \min\{0.714, 0.714, 0.600\} = 0.600$

$a_1^1 = 0.03, a_2^1 = 0.15, a_3^1 = 0.09, a_4^1 = 0.09, a_5^1 = 3, a_6^1 = 420, a_7^1 = 327, a_8^1 = 90, a_9^1 = 60$

Step 2. $\mathcal{N}^2 = \{1, 2, 3, 4, 5, 6, 7\}$, $\mathcal{I}^2 = \{1, 2\}$,

$E^2 = (0.64, 160, 0)$, $c^2 = (0.02, 0.1, 0.06, 0.06, 2, 280, 218, 60, 40)$

- $\lambda_1^2 = 0.286$
- $\lambda_2^2 = 0.32$

$\lambda^2 = \min\{0.286, 0.32\} = 0.286$

$a_1^2 = 0.01, a_2^2 = 0.03, a_3^2 = 0.02, a_4^2 = 0.02, a_5^2 = 0.57, a_6^2 = 80, a_7^2 = 62.29, a_8^2 = 0, a_9^2 = 0$

Step 3. $\mathcal{N}^3 = \{6, 7\}$, $\mathcal{I}^2 = \{2\}$,

$E^3 = (0, 17.143, 0)$, $c^3 = (0.014, 0.071, 0.043, 0.043, 1.429, 200, 155.714, 60, 40)$

- $\lambda_2^3 = 0.048$

$\lambda^3 = \min\{0.048\} = 0.048$

$$a_1^2 = 0, a_2^2 = 0, a_3^2 = 0, a_4^2 = 0, a_5^2 = 0, a_6^2 = 9.64, a_7^2 = 7.50, a_8^2 = 0, a_9^2 = 0$$

Step 4. $\mathcal{N}^4 = \varnothing$, $\mathcal{I}^2 = \varnothing$.

$$CPA(\mathcal{I}, N, E, c, \alpha) = (0.036, 0.179, 0.107, 0.107, 3.571, 509.639, 396.79, 90, 60)\,.$$

Therefore, if the local government wants to limit the emissions of the three categories of pollutants below 4 ppm, 1000 ppm, and 150 ppm, respectively, but keeping a fixed limit of emissions for each of the pollutant substances, an alternative would be to limit the emissions of the pollutant substances to the new values in the third column of Table 2.

Thus, we can observe how the constrained proportional awards rule for multi-issue problems with crossed claims can help authorities to design new water quality

Pollutants	Original maximum value	New maximum value
Benzene	0.05	0.036
Toluene	0.25	0.179
Ethylbenzene	0.15	0.107
Xylenes	0.15	0.107
Phenols	5	3.571
BOC	700	509.639
OIS	545	396.79
Amoniacal compounds	150	90
Nitrate compounds	100	60

Table 2: Maximum allowed values (ppm) for some pollutants after limiting the emissions of the three groups of pollutants.

policies, in particular, how the limits of emissions of different substances can be established taking into account the categories of water pollutants which have different effects in the quality of water.

As already mentioned in the introduction, water quality control management usually falls on local authorities as they are the institutions closest to the problem. Normally, local authorities tend to set limits on discharges of pollutants to maintain adequate levels of water quality for use. These limits are usually modified from time to time, however, conditions can change in short periods of time, so having management tools that allow you to quickly adapt to changes is relevant. As shown in the numerical example, the proposed management methodology would respond quite well to this more dynamic type of management that allows the pollutant emission limits to be adapted in a timely manner. In addition, the fact of using the pollutant categories and the constrained proportional awards rule means that the adaptations are staggered taking into account the different levels of abatement in the categories, as can be seen in Table 2. Moreover, if a category of pollutants did not experience reductions in its discharges, the limits on pollutants that only belonged to that category would not change.

8 Discussion and conclusion

Water resource management includes two major issues: water quantity (when there is scarcity) and water quality (when there is degradation). In both cases allocation problems usually arise and one way to address them is through game theory (see Dinar and Hogarth, 2015). The use of game theory to solve this type of problem seems adequate because the solutions are usually based on principles that seek the reasonableness and acceptability of the result for all the parties involved. This is in line with what was proposed in Liu et al. (2011). In the case of water quality, the most common allocation problems are the distribution of costs of water cleaning treatments, and the wastewater (or pollutant) discharge permits (see Section 2). In this paper, we deal with a different allocation problem in the design of water quality policies, the problem of establishing limits of emissions of pollutant substances when an authority wants to guarantee different parameters of water quality which are affected by the main categories of water pollutants.

In the design of water quality policies, it is necessary to take into account, on the one hand, the physical-chemical composition of the water and the target of its composition in accordance with the use that is going to be given to the water and, on the other hand, pollutant discharges and the capacity for water cleaning treatment of these substances (see Figure 4). All these elements can change for different reasons over time, so the design of water quality management policies must be adaptable, flexible and coordinated (see, Beck, 1981; Helmer and Hespanhol, 1997). This means that it is interesting to have a methodology like the one presented here to systematically recalculate the emission limits of pollutants when the conditions change, for example from a dry to a wet period.

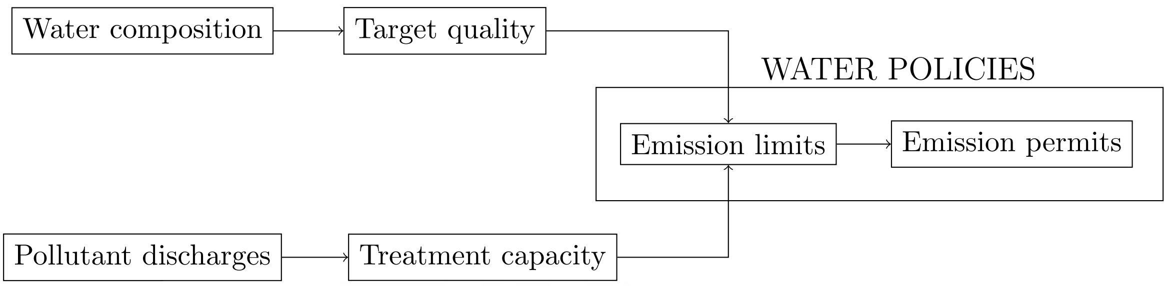

Figure 4: Some relationships between different elements for the design of water quality policies.

In water quality policies, emission limits are usually set for each of the polluting substances (see, for example, Table 1 for the case presented in Section 7). However, these substances can have more than one negative effect on water quality, so they could belong to several categories of pollutants (see, for example, Figure 3 in Section 7). This should lead to thinking about a more complex approach that goes from the emission limits singled out by substances to the limits of the categories of polluting substances according to their effects on water quality. This does not mean that limits are not set for each polluting substance, but that the category structure of pollutants is also taken into account. This is the approach that has been presented in this work for the design of water quality policies. In addition, both Beck (1981) and Helmer and Hespanhol (1997) suggest the use of more complex models for better management of water quality control, given that water is an essential resource for the existence of life.

Moreover, as it has been shown throughout this work, the introduction of greater complexity in the problem of setting emission limits for polluting substances into water has not meant an excessive practical difficulty of calculation, as shown in the numerical example in Section 7, by contrast it does have greater theoretical complexity. Therefore, an applicable methodology in water quality management is proposed, which is supported by an adequate mathematical foundation.

On the other hand, Endres (1985) and Kuosmanen and Laukkanen (2011) propose to design environmental policies taking into account pollutant interactions. In this work, this approach is followed in some way since the problem is approached from the perspective of the categories of pollutants and interactions between the different pollutants are considered. In this sense, the interactions considered in this work are additive and other types of interactions (non-linear relationships) would be interesting to study in the future.

Although it seems reasonable to use the well-known principle of proportionality as it is done in this work, to analyze the use of other solutions to bankruptcy problems based on other principles of fairness different from proportionality, such as the random arrival rule (O'Neill, 1982) or the Talmud rule (Aumann and Maschler, 1985), would be of interest. The first of these rules is related to the well-known Shapley value (Shapley, 1953), that satisfies desirable properties in line with principles of fairness (Algaba et al., 2019c), see also Algaba et al. (2019b) for an update on theoretical and applied aspects of this value, while the second is related to the nucleolus (Schmeidler, 1969). Although the first seems easier to study than the second in the context of multi-issue allocation problems with crossed claims.

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